Tripartite mutual information, entanglement, and scrambling in permutation symmetric systems

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Outline

1. Motivation
2. Tripartite Mutual Information and OTOCs: Brief introduction
3. Permutation symmetric states: reduced density matrices, random symmetric states, entanglement and TMI.
4. OTOC, TMI and entanglement in the symmetric states of the quantum kicked top.

Motivations

- Entanglement within many-body quantum states drives subsystems to thermalization although the full state remain pure and of zero entropy.
- Localized information spreads within typical many-body systems exponentially fast in the "scrambling" time scale, defining a form of quantum chaos.
- Out-of-time-order correlators (OTOC) or commutator growth, operator scrambling and Tripartite-mutual-information (TMI) are being vigorous investigated as measures of such delocalization of information.
- Permutation symmetric systems are interesting to study these issues in, as the rank of subsystems is only linear in their dimensionality rather than exponential. Occur naturally in spin systems with collective variables.

Tripartite mutual information

Mutual Information: Decrease in the uncertainty of X given Y.

$$\begin{aligned} I(X:Y) &= H(X) - H(X|Y) = H(Y) - H(Y|X) \\ &= H(X) + H(Y) - H(XY) \geq 0. \end{aligned}$$

TMI: A measure of tripartite correlation

$$I(X:Y:Z) = I(X:Z) + I(X:Y) - I(X:YZ)$$
$$= H(X) + H(Y) + H(Z) - H(XY) - H(XZ) - H(YZ) + H(XYZ)$$

Can be > 0 or $= 0$ or < 0.

If TMI< 0, MI is monogamous like entanglement measures:

"Synergy" whole $>$ sum of parts.

If TMI> 0 there is **redundant information**, for example $Z = Y$.

TMI's sign

- No definite sign for many-body states or field theories.
- TMI <0 for conformal field theories with classical holographic duals. [P. Hayden, M. Headrick, and A. Maloney, Holographic Mutual Information is Monogamous, Phys. Rev. D (2013), arXiv:1107.2940].
- Typical (generic, random) subsystems of many qubit pure states have small negative TMI. [Tripartite information of highly entangled states, M. Rota, arXiv:1512.03751]

TMI examples and significance

Two examples of 4-party ($ABCD$) pure states each of dimension d::

(1) Can be Absolutely Maximally Entangled ("perfect") if $d \neq 2$ or 6.
$I(A:B:C) = 3\log\ d - 3\log\ d^2 + \log\ d = -2\log d$.
(2): AB and CD are maximally entangled pairwise.
$I(A:B:C) = 3\log d - 4\log d + \log d = 0$

Due to the negative sign of TMI for AME or perfect states, this has been proposed as an alternative indicator of **scrambling**. [P. Hosur, X.-L. Qi, D. A. Roberts, and B. Yoshida, Chaos in quantum channels, arXiv:1511.0402.]

Scrambling via out-of-time-order correlators

Growth of commutators of initially localized operators:

$$F(t) = \left\langle [A(0), A(t)]^{\dagger} [A(0), A(t)] \right\rangle .$$
$$A(t) = e^{iHt} A(0) e^{-iHt}$$

If $F(t) \sim e^{2\lambda t}$ for $t_d < t < t_S$, λ was called a Lyapunov exponent and t_S a scrambling time, beyond which the information is not locally retrievable. [Hayden, Presskill, 2007, Susskind, Sekino, 2008, Maldacena, Shenker, Stanford 2015 ...]

Black holes, SYK models, O(N) field theories to Models of quantum of chaos scramble exponentially fast. [Explicit calculation for the quantum bakers map in AL arXiv:1810.12029]

Permutation Symmetric States: No preferred qubit

Dicke state: basis for N qubit PS states:

$$|m_N\rangle = \frac{1}{c_N(m)} \sum_{\substack{0 \leq i \leq 2^N - 1 \\ w(i) = m}} |binary\ expansion\ of\ \ i\rangle\,,\ 0 \leq m \leq N$$

$w(i)$ =Hamming weight of i=# of 1 in the binary expansion of i.
Normalization constant:

$$c_N(m) = \sqrt{\binom{N}{m}} = \sqrt{\frac{N!}{m!(N-m)!}}.$$

An arbitrary N-qubit PS pure state $\in S_N$:

$$|\psi\rangle = \sum_{m=0}^{N} a_m\,|m_N\rangle\,,\ \ \sum_{m=0}^{N} |a_m|^2 = 1.$$

Partial traces of symmetric states

$$S_N \subset S_Q \otimes S_{N-Q}$$

$$|\psi\rangle = \sum_{m=0}^{N} a_m |m_N\rangle = \sum_{m=0}^{Q} \sum_{n=0}^{N-Q} A_{mn} |m_Q\rangle |n_{N-Q}\rangle$$

$$A_{mn} = \frac{c_Q(m) c_{N-Q}(n)}{c_N(m+n)} a_{m+n}.$$

Reduced density matrix of Q qubits only of rank $Q+1$:

$$\rho_Q = A A^\dagger$$

Maximum entanglement $= \log_2(Q+1)$: achievable with $Q = [N/2]$, only for $N = 2, 3, 4, 6$ [N. Gisin and H. Bechmann-Pasquinucci, Phys. Lett. A 246, 1 (1998), (quant-ph/9804045)]

Random symmetric states in S_N

Sample uniformly from the space of PS states S_N:

$$P(\{a_m\}) = \frac{N!}{\pi^{N+1}} \delta \left(1 - \sum_{m=0}^{N} |a_m|^2 \right) .$$

Properties of ρ_Q^{PS}, where A_Q is constructed from the $N+1$ complex random numbers a_m as

$$(A_Q)_{mn} = \sqrt{\frac{\binom{Q}{m}\binom{N-Q}{n}}{\binom{N}{m+n}}}\, a_{m+n}, \quad \rho_Q^{PS} = A_Q A_Q^\dagger,$$

with $0 \le m \le Q$, $0 \le n \le N-Q$. The normalization of a_m guarantees that $\mathrm{tr}(A_Q A_Q^\dagger) = 1$. Decorated random Hankel matrices.

[Also "Characterizing the entanglement of symmetric many-particle spin-1/2 systems" John K. Stockton, J. M. Geremia, Andrew C. Doherty, and Hideo Mabuchi, quant-ph/0210117]

Compare with random state in $S_Q \otimes S_{N-Q}$

When pure states are uniformly sampled from the bipartite Hilbert space of dimension $(Q+1) \times (N-Q+1)$.

$$\rho_Q^{HS} = \frac{GG^\dagger}{\mathrm{tr}(GG^\dagger)}$$

where$G$ is a $Q+1 \times N-Q+1$ dimensional matrix with complex entries whose real and imaginary parts are independently normally (zero centered) distributed. Eigenvalues are distributed according to the Marcenko-Pastur law.

Average Linear Entropy

$$S_Q^{lin} = 1 - \mathrm{tr}(\rho_Q^2)$$

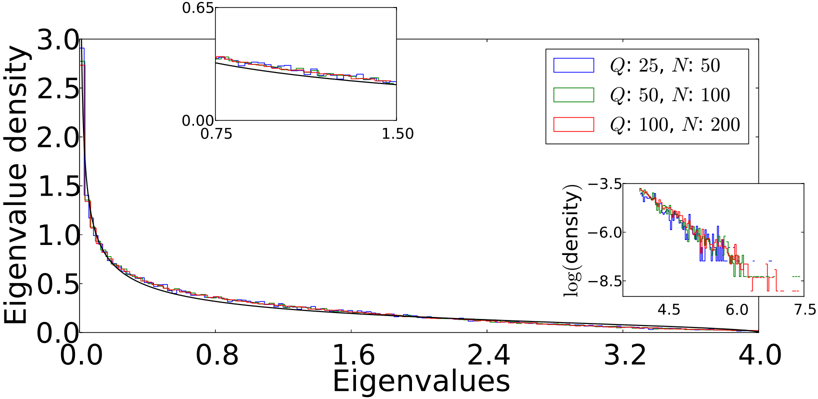
$$\langle S_Q^{lin} \rangle_{PS} = \frac{Q(N-Q)}{(Q+1)(N-Q+1)} \quad \text{(AS,VM, AL, PRE, 2018)}$$

$$\langle S_Q^{lin} \rangle_{HS} = \frac{Q(N-Q)}{1+(Q+1)(N-Q+1)} \quad \text{(E. Lubkin, JMP, 1978)}$$

Symmetric states in S_N have larger average linear entropy than random states in $S_Q \otimes S_{N-Q}$.

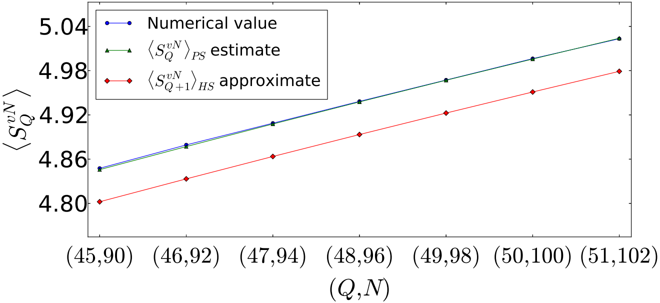
Eigenvalue density of ρ_Q

λ: Eigenvalues of ρ_Q^{PS}: Collapse on scaling $x = (Q+1)\lambda$. $P(x)$. *c.f.* ρ_Q^{HS} density: Marchenko-Pastur $P_{MP}(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}$



PS distribution is NOT Marchenko-Pastur: it has an exponential tail and is finite at 0.

Von Neumann entropy



$$\langle S_Q^{vN} \rangle_{PS} = -\left\langle \sum_{i=1}^{Q+1} \lambda_i \log \lambda_i \right\rangle_{PS} \approx \log(Q+1) - \alpha \frac{Q+1}{N-Q+1}$$

where $1 \leq Q \leq N/2$, $1 \ll N$ and $\alpha \approx 2/3$ is a constant. The leading order correction to this seems to be $1/(N+1)$ for the case of

Von Neumann entropy

- Even if symmetric states cannot have entanglement as large as $\log_2(Q+1)$ the typical states entanglement scale as $\log_2(Q+1)$ from a Levy Lemma argument.
- The deficit is smaller than that of the non-symmetric state. For $Q = N/2$: $\langle S_Q^{vN} \rangle_{HS} \approx \log_2(Q+1) - 0.721$ while $\langle S_Q^{vN} \rangle_{PS} \approx \log_2(Q+1) - 0.66$.
- Symmetric states have marginal, "area-law" scaling of block entanglement.
- This has implications for the Tripartite Mutual Information and entanglement sharing.

TMI in symmetric states of N qubits

Consider $I(Q:Q:Q) = 3S_A(Q) - 3S_{AB}(2Q) + S_{ABC}(3Q)$
For $3Q < N/2$

$$\begin{aligned}\langle I(Q:Q:Q)\rangle_{PS} &\approx 3\log(Q+1) - 3\log(2Q+1) + \log(3Q+1)\\ &= \log\left[\frac{(3Q+1)(Q+1)^3}{(2Q+1)^3}\right] > 0\end{aligned}$$

Marginal, area-law kind of entanglement prohibits negative TMI in typical symmetric states.
Contrast with ensemble of all states on the full 2^N dimensional space, using Page's formula

$$\langle I_3(Q,Q,Q)\rangle_{W,2^Q} \approx -\frac{2^{2Q-N-1}}{\ln 2}\left(2^{4Q} - 3\,2^{2Q} + 3\right) < 0.$$

Scrambling and TMI in a kicked top or lots of spins

One large spin or all-to-all connected spin-model: No disorder unlike SYK, but time-dependent and with a transverse field.

$$H = \left(\frac{\hbar\pi}{2\tau}\right) J_y + \left(\frac{\hbar\kappa}{2j}\right) J_z^2 \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$$

(Kus, Scharf, Haake, 1987; Haake's book.)

$J^{x,y,z} = \sum_{l=1}^{2j} \sigma^{x,y,z}/2$, the unitary or Floquet operator:

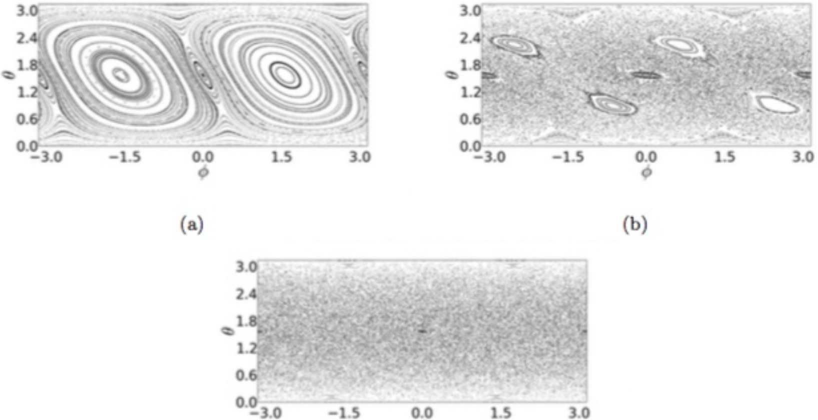
$$U = \exp\left(-i\frac{\kappa}{8j}\sum_{l\neq l'=1}^{2j} \sigma_l^z \sigma_{l'}^z\right) \exp\left(-i\frac{\pi}{4}\sum_{l=1}^{2j} \sigma_l^y\right),$$

(Milburn 2000, Wang, Ghose, Sanders, and Hu, 2004)

Thermodynamic limit is also classical limt $j \to \infty$.

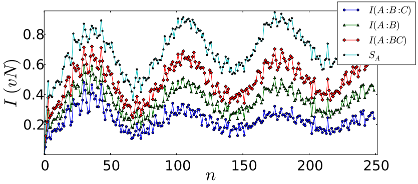
Classical map on the sphere: $X_n^2 + Y_n^2 + Z_n^2 = 1$

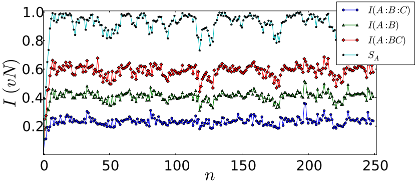
$$
\begin{aligned}
X_{n+1} &= Z_n \cos(\kappa X_n) + Y_n \sin(\kappa X_n) \\
Y_{n+1} &= -Z_n \sin(\kappa X_n) + Y_n \cos(\kappa X_n) \\
Z_{n+1} &= -X_n .
\end{aligned}
$$



Time evolving symmetric states

$$|\psi(n)\rangle = U^n(k,j) \otimes^{2j} |\phi_0\theta_0\rangle$$



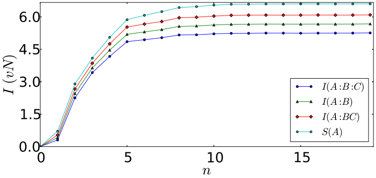


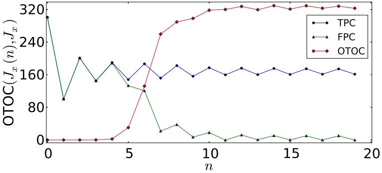
TMI, MI and entanglement between three 1 qubit subsystems for $j = 10$ (i.e., a 20 qubit system). in the kicked top. Left: state intially localized in a regular island of the mixed phase space for $k = 3$, and right: initial state in the chaotic sea.

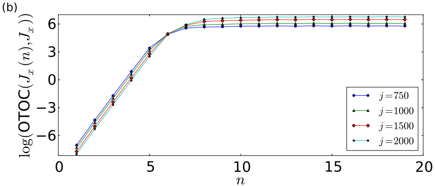
TMI> 0 whether there is chaos or not.

Time evolving symmetric states: TMI vs OTOC

Large collections of spins: $Q = 100$ spins for subsystems.







TMI grows along with the OTOC, but not exponentially Both saturate at the order of the Ehrenfest time $t_{EF} = \log(1/h)/\lambda = \log(2j+1)/\lambda$.

Conclusions

1. Symmetric pure states have marginal entanglement and result in typically positive TMI. They typically violate monogamy for mutual information.
2. The ensemble of random symmetric states has a different limiting distribution than Marchenko-Pastur, but scale is the same. Interesting to prove that the limiting distribution exists and to find it.
3. TMI can be positive even when there is exponentially growing OTOC and scrambling as seen in the quantum kicked top.
4. Futher understanding of TMI in many-body systems, useful for multiparty correlations.
5. Stability of these conclusions with perturbations of symmetric states?