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Resource Theory
of
POVM based coherence

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MARIE CURIE ACTIONS

Outline

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- Resource theory (RT) of coherence
  - Free states and free operations (incoherent states/operations)
  - Measures of coherence
- Generalization to POVM based coherence
  - Naimark extensions
  - Universality
  - Free operations, free states (POVM incoherent)
- Features of POVM based coherence
- Examples
- Discussion

Resource theories[1] (RT)

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- Ingredients:
  - Define free operations Λ_I
  - Free states $I = \{\rho_I\}$
- Examples:
  - Entanglement
  - Purity
  - Coherence

- Conditions:
  - Free operations map free states to free states: $\Lambda_I(\rho_I) = \tilde{\rho}_I$
  - A monotone $C(\rho)$ of the resource should fulfill conditions:

positivity	$C(\rho) \geq 0$
monotonicity	$C\left(\Lambda_I(\rho)\right) \leq C(\rho)$
convexity	$C\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i C(\rho_i)$

[1]F.G. Brandao, G. Gour, PRL **115**, 070503 (2015)
Z.-W. Liu, X. Hu, S. Lloyd, PRL **118**, 060502 (2017)
E. Chitambar, G. Gour, arXiv:1806.06107 (2018)

Resource theory of coherence[2]

- Free states (diagonal states):

$$\rho_I = \sum_i p_i |i\rangle\langle i|$$

with $\langle i \,|\, j\rangle = \delta_{ij}$

- Free operations Λ_I (free CPTP maps):

$$\Lambda_I(\rho_I) = \tilde{\rho}_I, \quad \forall \rho_I \in I$$

- A maximal coherent state:

$$|+\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle$$

- Dephasing operation (resource destroying map):

$$\Delta(\rho) = \rho_I \in I = \sum_i \langle i|\,\rho\,|i\rangle\, |i\rangle\langle i|$$

[2] J. Aberg, arXiv:quant-ph/0612146 (2006)
T. Baumgratz, M. Cramer, M.B. Plenio, PRL **113**, 140401 (2014)

Coherence theory:[3] free operations

- Types of free operations (operational motivation):

- Maximally incoherent operations (MIO) $\Lambda_I(I) \subseteq I$
- Incoherent operations (IO) $K_{In}\rho_I K_{In}^\dagger \sim \tilde{\rho}_I \;\; \forall n$
- Selective incoherent operations (SIO) $\langle i| K_{In}\Delta(\rho)K_{In}^\dagger |i\rangle = \langle i| K_{In}\rho K_{In}^\dagger |i\rangle \;\; \forall i, n, \rho$

$\vdots$

- Physical incoherent operations (PIO)

$\vdots$

$$\mathrm{PIO} \subset \mathrm{SIO} \subset \mathrm{IO} \subset \mathrm{MIO}$$

Compare to RT of entanglement:

$$\mathrm{LOCC} \subset \mathrm{SEP} \subset \mathrm{MSEP}$$

[3] A. Streltsov, G. Adesso, M.B. Plenio, RMP **89**, 041003 (2017)

Coherence monotone/measure[3]

Monotone:

- Positivity/Faithfulness $C(\rho) \geq 0,\ C(\rho_I) = 0$
- Monotonicity $C\left(\Lambda_I(\rho)\right) \leq C(\rho)$
- Convexity $C\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i C(\rho_i)$

+Measure:

- Uniqueness for pure states $C(|\psi\rangle\langle\psi|) = S(\Delta(|\psi\rangle\langle\psi|))$
- Additivity $C(\rho \otimes \sigma) = C(\rho) + C(\sigma)$

With:

$$S(\rho) = -\mathrm{tr}(\rho \log_2 \rho)$$

$$\Delta(\rho) = \sum_i \langle i|\rho|i\rangle\, |i\rangle\langle i|$$

[3]A. Streltsov, G. Adesso, M.B. Plenio, RMP **89**, 041003 (2017)

Relative entropy of coherence

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Def: Relative entropy of coherence

$$C_{\text{rel}}(\rho) \equiv \min_{\sigma \in I} S(\rho||\sigma) = S(\rho||\Delta(\rho)) = S(\Delta(\rho)) - S(\rho)$$

with

$$\Delta(\rho) = \sum_i \langle i| \rho |i\rangle \, |i\rangle\langle i| = \rho_I \in I$$

$$S(\rho||\sigma) = \text{tr}(\rho \log_2 \rho) - \text{tr}(\rho \log_2 \sigma)$$

$$S(\rho) = -\text{tr}(\rho \log_2 \rho)$$

- C_{rel} is a measure of coherence; in case of MIO operations the resource theory becomes reversible (coherence cost = distillable coherence)

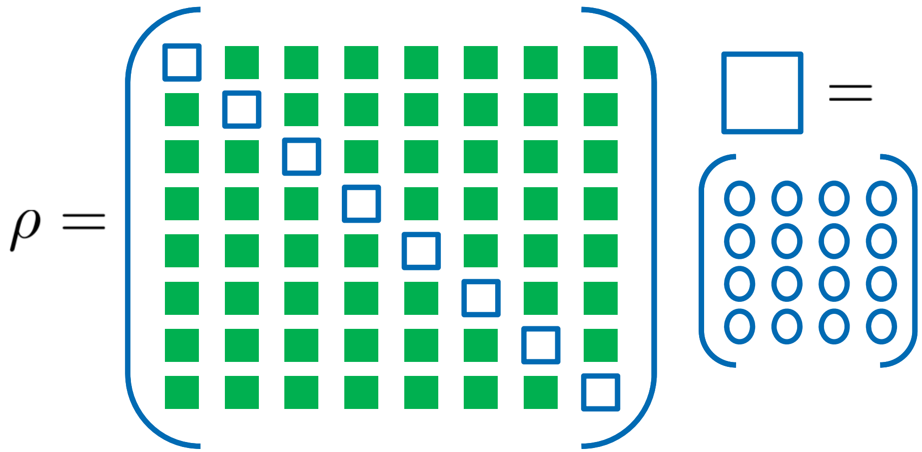
→ C_{rel} is a universal measure

- Block dephasing operation

$$\Delta_B(\rho) = \sum_i P_i \rho P_i = \rho_{BI}$$

- Free states

$$BI = \{\rho_{BI} | \rho_{BI} = \Delta_B(\rho), \forall \rho\}$$



- Free operations (CPTP maps): → Here: MIO operations

$$\Lambda_{BI}(\rho_{BI}) = \tilde{\rho}_{BI}, \quad \forall \rho_{BI} \in BI$$

$$\mathrm{rank}(P_i) \geq 1,$$
$$P_i P_j = \delta_{ij} P_i$$
$$\sum_i P_i = \mathbb{1}$$
$$\mathbb{P} = \{P_i\}$$

- Relative entropy of block-coherence (superposition[2])

$$C_{B\mathrm{rel}}(\rho, \mathbb{P}) \equiv \min_{\sigma \in BI} S(\rho || \sigma)$$
$$= S(\rho || \Delta_B(\rho)) = S(\Delta_B(\rho)) - S(\rho)$$

[2] J. Aberg, arXiv:quant-ph/0612146 (2006)

POVM, Naimark extensions

- Every POVM

$$\mathbb{E} = \{E_i\}, \quad E_i \geq 0, \quad \sum_i E_i = \mathbb{1}$$

on $\mathcal{H}$ can be extended to a projective measurement

$$\mathbb{P}' = \{P_i'\}, \quad P_i' P_j' = \delta_{ij} P_i', \quad \sum_i P_i' = \mathbb{1}$$

on $\mathcal{H}'$ for sufficiently large $d' = \dim(\mathcal{H}') > \dim(\mathcal{H}) = d$
such that $\forall i, \rho$ it holds

$$\mathrm{tr}(E_i \rho) = \mathrm{tr}(P_i'(\rho \oplus 0)) \quad \Rightarrow \quad E_i \oplus 0 = (\mathbb{1} \oplus 0) P_i' (\mathbb{1} \oplus 0)$$

- Specific: Canonical Naimark extension[4]

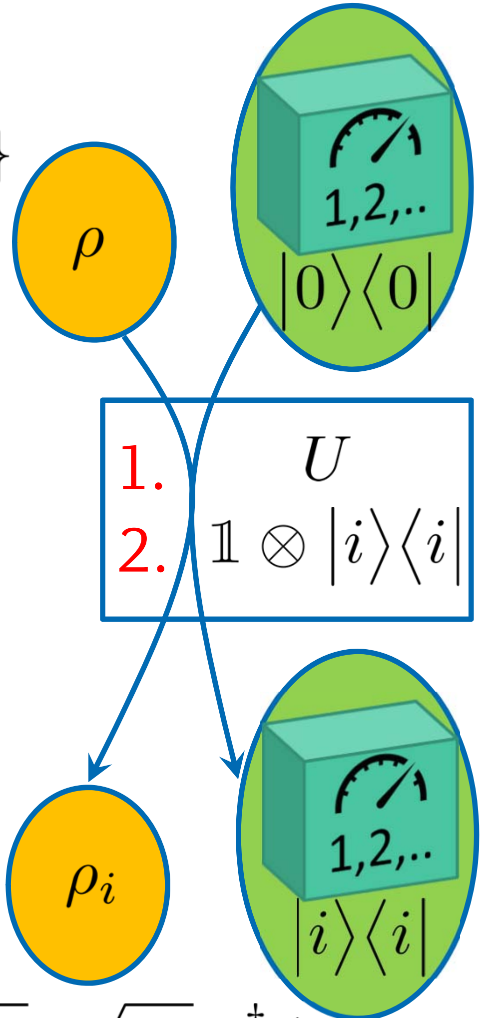
$$\mathrm{tr}(E_i \rho) = \mathrm{tr}\left(P_i'(\rho \otimes |0\rangle\langle 0|)\right) \quad \Rightarrow$$

Construction e.g. using Stinespring dilation

[4] C. Sparaciari, M.G. Paris, PRA **87**, 012106 (2013)

Canonical Naimark extension

- Physical motivation: Coupling to a von Neumann measurement apparatus $\mathbb{P}' = \{P_i'\}$ implements the POVM $\mathbb{E} = \{E_i\}$



$$\begin{aligned}\mathrm{tr}(E_i\rho) &= \mathrm{tr}\left(\mathbb{1} \otimes |i\rangle\langle i| U \rho \otimes |0\rangle\langle 0| U^\dagger\right) \\ &= \mathrm{tr}\left(U^\dagger \mathbb{1} \otimes |i\rangle\langle i| U \rho \otimes |0\rangle\langle 0|\right) \\ &= \mathrm{tr}\left(P_i' \rho \otimes |0\rangle\langle 0|\right), \\ &= \mathrm{tr}\left(\Pi_S P_i' \Pi_S \rho \otimes |0\rangle\langle 0|\right)\end{aligned}$$

$$E_i \otimes |0\rangle\langle 0| = \Pi_S P_i' \Pi_S, \quad \Pi_S = \mathbb{1} \otimes |0\rangle\langle 0|$$

Naimark extensions are not unique; lower dimensional extension are possible

$$\rho_i = U_i \sqrt{E_i} \rho \sqrt{E_i} U_i^\dagger / p_i$$

$$\tilde{\rho} = \sum_i p_i \rho_i$$

POVM based coherence via Naimark extension

- <u>Def (POVM based coherence):</u>

A canonical Naimark extension of the POVM $\mathbb{E}$ on $\mathcal{H}$ defines a set of projectors $\mathbb{P}'$ on $\mathcal{H}'$.

A POVM-based coherence measure is given by the block-coherence measure of the Naimark extension

$$C(\rho, \mathbb{E}) := C(\rho \otimes |0\rangle\langle 0|, \mathbb{P}')$$

assuming $C(\rho', \mathbb{P}') = C(U\rho' U^\dagger, U\mathbb{P}' U^\dagger)$.

Remember:

$$\mathrm{tr}(E_i \rho) = \mathrm{tr}\left(P_i' \rho \otimes |0\rangle\langle 0|\right)$$

Resource theory of POVM coherence

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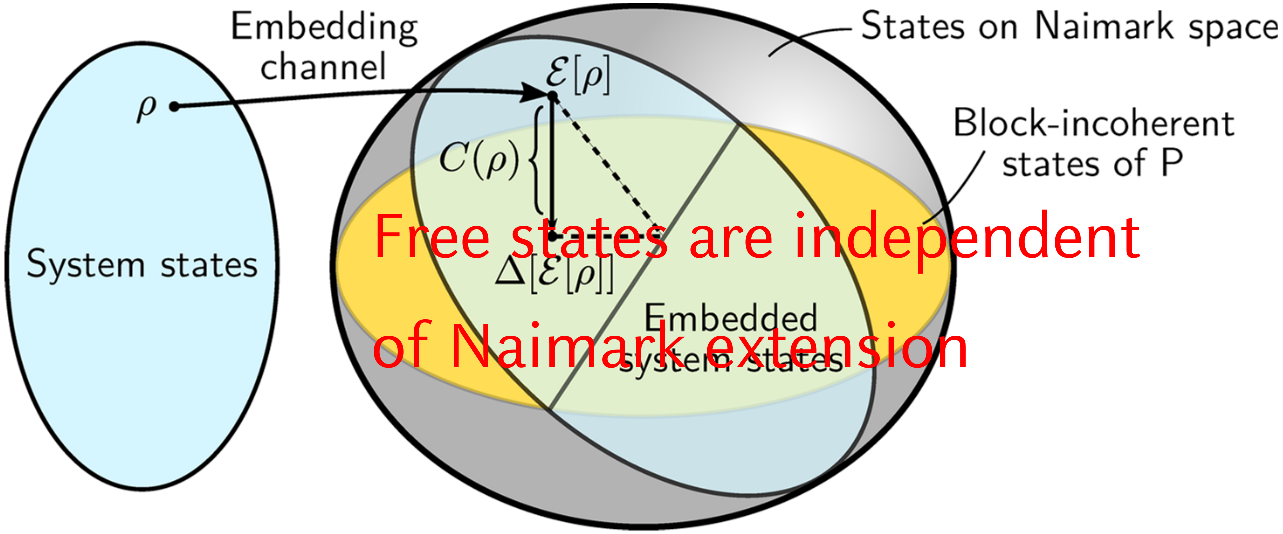
(1) Free states:

$$PI = \{\rho : \rho \otimes |0\rangle\langle 0| \in BI'\}$$

- We restrict the resources to the Hilbert space $\mathcal{H}$ on which ρ (system) acts.

$$BI' = \{\rho'_{BI} : \rho'_{BI} = \sum_i P'_i \rho' P'_i, \forall \rho'\}$$

- This restriction may lead to PI being an empty set
- System + extension act on $\mathcal{H}'$
- For POVM $\mathbb{E} = \{E_i\}$:



Free states are independent of Naimark extension

(2) Free states:

$$\rho_{PI} = \sum_i E_i \rho_{PI} E_i$$

$$\rho_{PI} = \sum_i p_i P_i,$$

with $P_i E_k = P_i$

Resource theory of POVM coherence

Free states

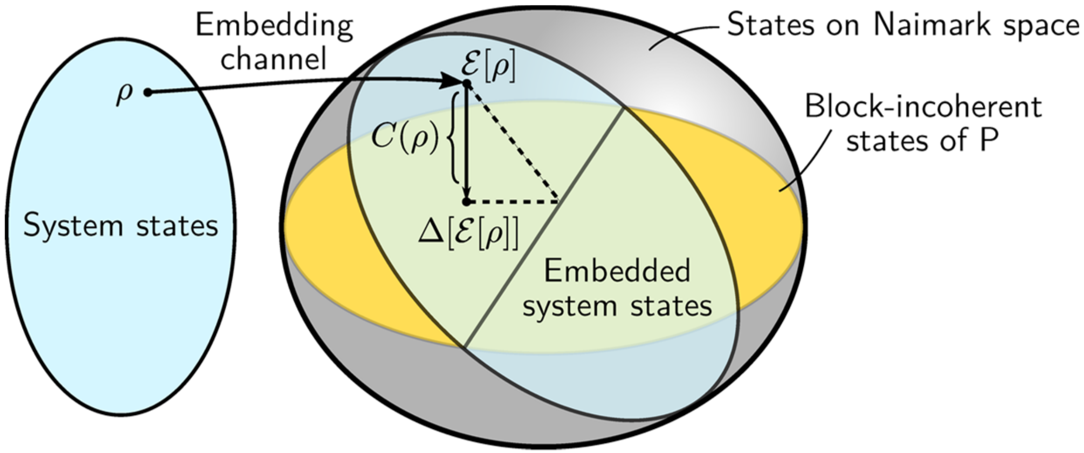
$$PI = \{\rho : \rho \otimes |0\rangle\langle 0| \in BI'\}$$

Free operations (MPIO)*

$$\{\Lambda'_{PI}\} = \{\Lambda'_{PI} : \Lambda'_{PI} \in \{\Lambda'_{BI}\} \wedge \forall \rho \; \Lambda'_{PI}(\rho \otimes |0\rangle\langle 0|) = \rho' \otimes |0\rangle\langle 0|\}$$

- We restrict the resources and operations to the Hilbert space $\mathcal{H}$ on which ρ (system) acts.
- System + extension act on $\mathcal{H}'$

$$(\Lambda_{PI} \otimes \mathbb{1})(\rho \otimes |0\rangle\langle 0|) = \Lambda'_{PI}(\rho \otimes |0\rangle\langle 0|) \text{ holds } \forall \rho$$



- MPIO's are independent of Naimark extension
- Nontrivial Conversions even if $PI = \{\}$

*MPIO: maximally POVM incoherent operation

POVM incoherent maps via SDPs

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SDP feasibility problem:

$$\begin{aligned} \text{find}: \quad & \hat{Q}^\dagger \text{sh}(J) Q = \hat{\Lambda}_{PI} \\ \text{subject to}: \quad & J \geq 0, \quad \text{tr}_1(J) = \mathbb{1}/d' \\ & \text{sh}(J)\hat{\Delta} = \hat{\Delta}\text{sh}(J)\hat{\Delta} \\ & \text{sh}(J)\hat{\Pi} = \hat{\Pi}\text{sh}(J)\hat{\Pi} \end{aligned}$$

- Choi matrix: J
- Shuffle operation: $\text{sh}(.)$
- Map matrix representation: $\hat{.}$
- Von Neumann projection matrix: $\hat{\Delta}$ $\qquad \Delta(\rho) = \sum_i P_i \rho P_i$
- Projection map matrix on space $\mathbb{1} \oplus 0$: $\hat{\Pi}$

Relative entropy of POVM coherence

- The POVM coherence theory on $\mathbb{E}$ uses a subset of free states and free operations of the coherence theory based on $\mathbb{P}'$
- Measures for $\mathbb{P}'$ are measures of coherence for $\mathbb{E}$

Relative entropy of POVM based coherence:

$$
\begin{aligned}
C_{\mathrm{rel}}(\rho, \mathbb{E}) &:= C_{\mathrm{Brel}}(\rho \otimes |0\rangle\langle 0|, \mathbb{P}') \\
&= H(\{p_i\}) + \sum_i p_i S(\rho_i) - S(\rho)
\end{aligned}
$$

with $p_i = \mathrm{tr}(E_i \rho)$

$$\rho_i = \sqrt{E_i}\rho\sqrt{E_i}/p_i \qquad \tilde{\rho}_i = U_i\sqrt{E_i}\rho\sqrt{E_i}U_i^\dagger/p_i$$

$$\bar{\rho} = \sum_i p_i \rho_i \qquad \tilde{\rho} = \sum_i p_i \tilde{\rho}_i$$

(unital map) (General CPTP map)

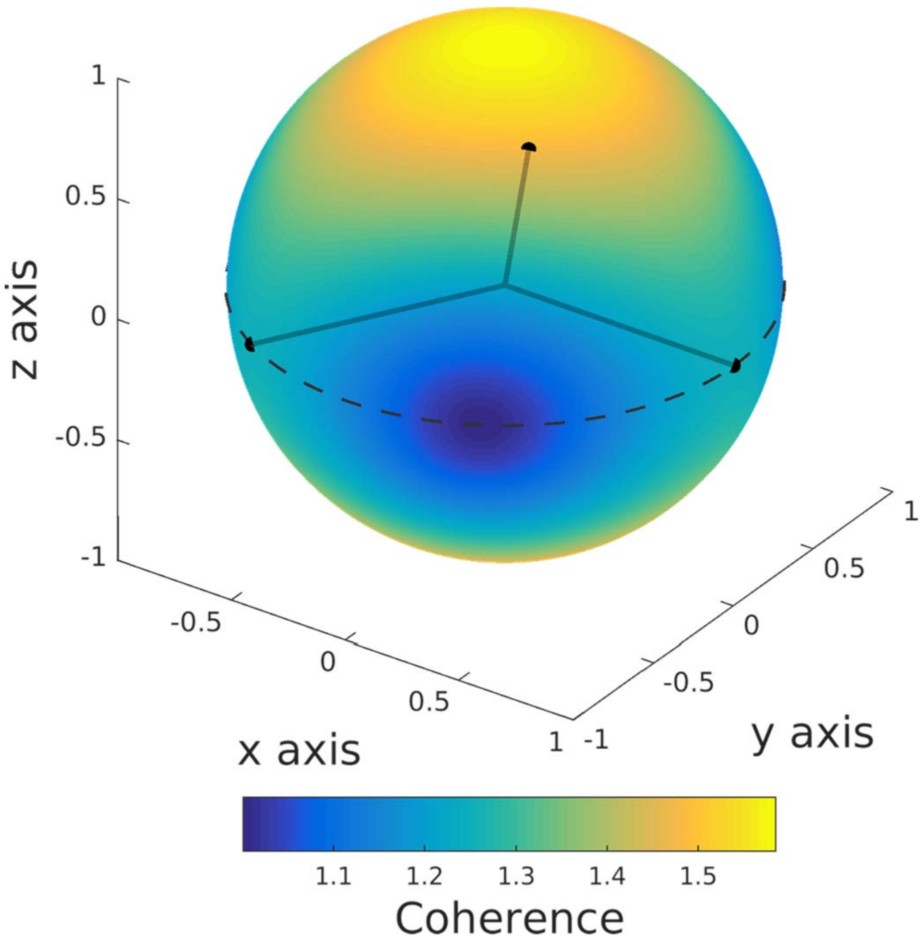
Properties of POVM based coherence

- Free states for POVM $\mathbb{E} = \{E_i\}$ $\rho_{PI} = \sum_i E_i \rho_{PI} E_i$
- Free operations $\{\Lambda_{PI}\}$
- Quantifiers: MBIO coherence measures work
(e.g. relative entropy of block-coherence)
- Independent of Naimark extension
- Accounts for coherence present in a general measurement
- Differences to usual coherence theory:
  - POVM measurement map: $\tilde{\rho} = \sum_i U_i \sqrt{E_i} \rho \sqrt{E_i} U_i^\dagger$
    (Even for $U_i = \mathbb{1}$:
    not dephasing, not incoherent, no statement about $\tilde{\rho}$)

The qubit trine POVM[5]

- The trine POVM $\mathbb{E} = \{2/3|\phi_j\rangle\langle\phi_j|\}$ $|\phi_j\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i2/3\pi j}|1\rangle)$
- Rank 1 POVM elements
- No incoherent local states
- POVM coh. $C_{\mathrm{rel}}(|\psi\rangle\langle\psi|, \mathbb{E})$

of a pure qubit state on the Bloch sphere



- Max. $C_{\mathrm{rel}}(|1\rangle\langle 1|, \mathbb{E}) = \log_2 3$
- Min. $C_{\mathrm{rel}}(\mathbb{1}/2, \mathbb{E}) = \log_2 3 - 1$

[5]R. Josza et al., QIC **3**, 405 (2003)

The qubit trine POVM: conversion

- Optimal state conversions by incoherent operations Λ_{PI}
- Example: the pure state $|\xi\rangle = \cos(\pi/8)\,|0\rangle + \sin(\pi/8)\,|1\rangle$
- Calculated via SDP[6]:

$$\max_{\Lambda_{PI}} F(\Lambda_{PI}(|\xi\rangle), |\psi\rangle)$$

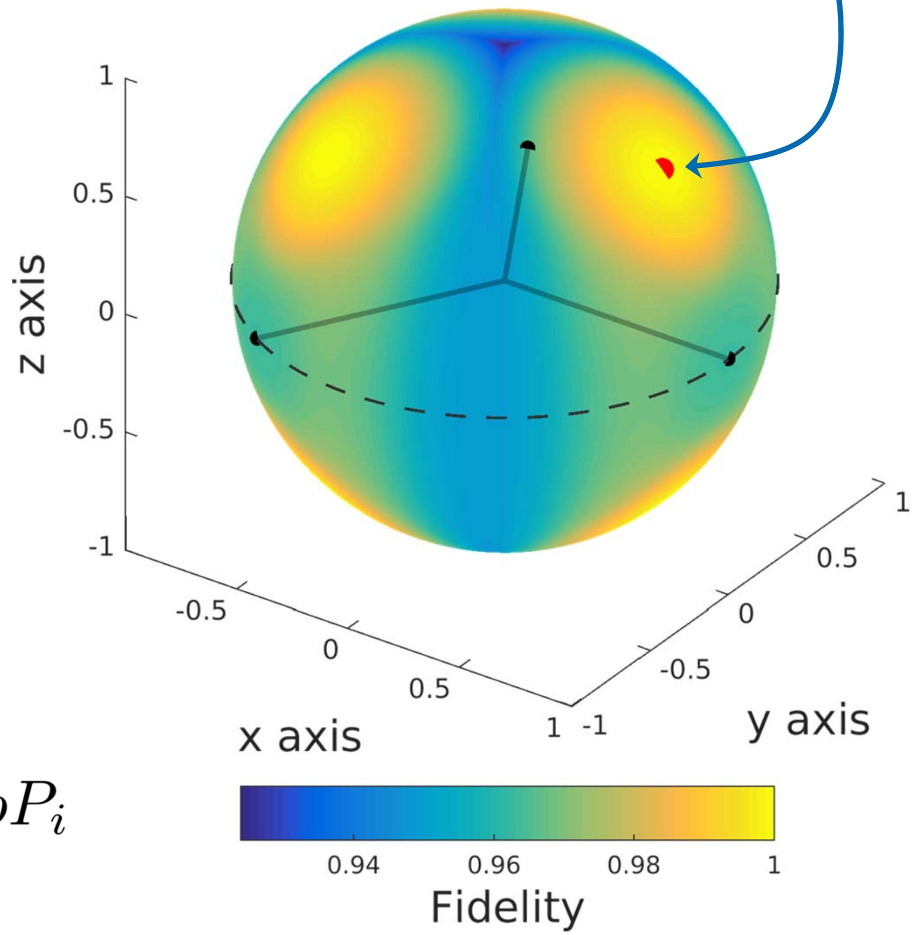
- For this POVM the map

$$\tilde{\rho} = \sum_i \sqrt{E_i}\rho\sqrt{E_i} = \Lambda_{PI}(\rho)$$

is incoherent. In general this is:

  - Neither incoherent
  - Nor dephasing

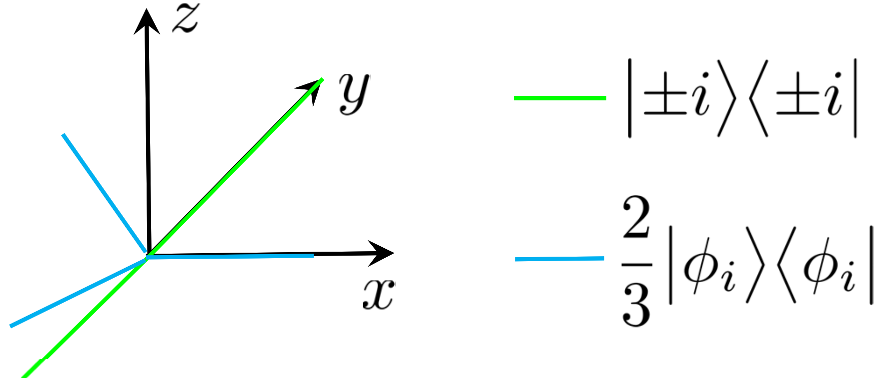
Compare to: $\rho_{BI} = \sum_i P_i \rho P_i$



[6]M. Piani, PRL **117**, 080401 (2016)

One parameter qubit POVM

- POVM $\mathbb{E}$ with $0 \leq p \leq 1$
  - $p = 0, \quad \mathbb{E} = \{|\pm\rangle\langle\pm|\}$
  - $p = 1, \quad \mathbb{E} = \{2/3|\phi_i\rangle\langle\phi_i|\}$
  - $E_i = \alpha_i(\mathbb{1} + \vec{m}_i \cdot \vec{\sigma})$

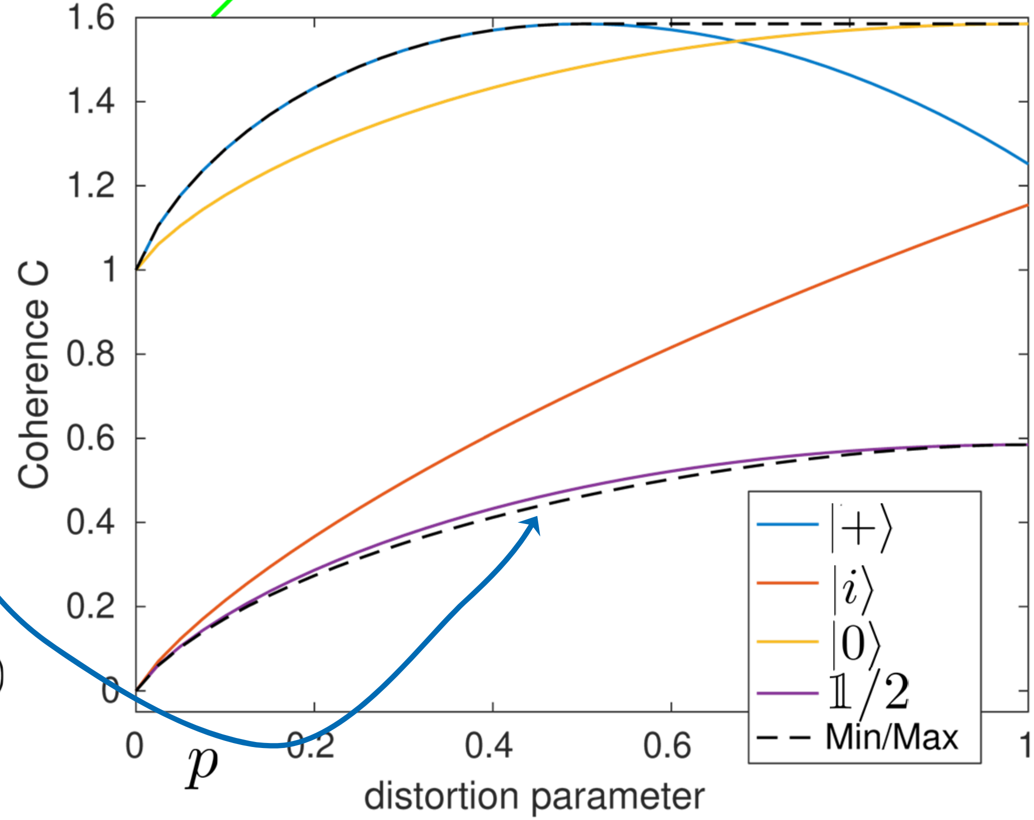


- The state $\mathbb{1}/2$ has not smallest coherence for $p \sim 1/2$
- The map $\sum_i \sqrt{E_i}\rho\sqrt{E_i}$ can increase coherence

$t = p/(3-p) \quad \alpha_1 = p/3,$

$\vec{m}_1 = (1, 0, 0), \quad \alpha_2 = \alpha_3 = 1/2(1 - p/3)$

$\vec{m}_{2,3} = (-t, \pm\sqrt{1-t^2}, 0)$



Conclusion

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- POVM coherence theory recovers usual coherence theory in case of projective measurements
- POVM coherence quantifies amount of coherence with respect to a projective measurement basis
- In many cases no local incoherent states/dephasing map
- Local measurement map is not dephasing
- Outlook/questions:
  - What are the conditions for a reversible theory under MPIO (cost=distillable POVM coherence)?
  - Other measures, more restrictive incoherent operations?
  - Role of classical randomness in POVMs?
  - Reversible Theory under MIO (cost=distillable)?

Thank you !!!
(see ArXiv: 1812.00018)