Estimation of Optimal Singlet Fraction (OSF) and Entanglement Negativity (EN)

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Outline

- Structural Physical Approximation to partial transposition
- Determination of minimum eigenvalue for SPAed state
- Relation between minimum eigenvalue and average fidelity
- Estimation of entanglement negativity
- Estimation of singlet fraction

S. Adhikari, Phys. Rev. A **97**, 042344(2018)
S. Adhikari, arxiv:1808.09745, Accepted in Europhysics Letters

Structural Physical Approximation (SPA)

- SPA is a transformation from positive maps to completely positive maps.
- SPA has been exploited to approximate unphysical operation such as partial transpose.
- The idea is to form a mixture of the positive map P with a CP map O that transforms all quantum states onto maximally mixed state.
- It can be used directly in experiment to detect the entanglement directly without full tomographic reconstruction of the bipartite state whose entanglement is to be determined.

P. Horodecki and A. Ekert, Phys. Rev. Lett. **89**, 127902-1(2002).
J. Fiurasek, Phys. Rev. A **66**, 052315 (2002).
J. K. Korbicz, et.al., Phys. Rev. A **78**, 062105 (2008).

SPA to Partial Transposition

Any arbitrary two qubit density matrix is given by

$$\rho_{12} = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{12}^* & t_{22} & t_{23} & t_{24} \\ t_{13}^* & t_{23}^* & t_{33} & t_{34} \\ t_{14}^* & t_{24}^* & t_{34}^* & t_{44} \end{pmatrix}, \sum_{i=1}^{4} t_{ii} = 1 \quad (1)$$

where $(*)$ denotes the complex conjugate.
In general, partial transposition operation $id \otimes \mathbf{T}$ can be approximated as

$$[\widetilde{id \otimes \mathbf{T}}](\rho_{12}) = [(1 - q^*)(id \otimes \mathbf{T}) + \frac{q^*}{4} I_A \otimes I_B](\rho_{12}) \quad (2)$$

where $q^* = \frac{16\nu}{1+16\nu}$ and $\nu = -min_{Q>0} Tr[Q(id \otimes \mathbf{T})|\psi^+\rangle\langle\psi^+|]$,
$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
P. Horodecki and A. Ekert, Phys. Rev. Lett. **89**, 127902-1(2002).

SPA to Partial Transposition

Structural physical approximation of $\rho_{12}^{T_2}$:

$$\widetilde{\rho_{12}} = [\frac{1}{3}(I \otimes \widetilde{T}) + \frac{2}{3}(\widetilde{\Theta} \otimes D)]\rho_{12} \quad (3)$$

where $D(\rho) = \frac{I_2}{2}$.
In terms of Pauli matrices

$$D(.) = \frac{1}{4} \sum_{i=0,x,y,z} \sigma_i(.)\sigma_i \quad (4)$$

$\tilde{T}$: SPA to transpose operation.
$\tilde{\theta}$: SPA to inversion operation.

$$\tilde{T}(\rho) = \sum_{k=1}^{4} Tr(M_k \rho)|v_k\rangle\langle v_k| \quad (5)$$

SPA to Partial Transposition

where $|v_1\rangle \propto |0\rangle + \frac{ie^{\frac{2\pi i}{3}}}{i+e^{\frac{-2\pi i}{3}}}|1\rangle$,$|v_2\rangle \propto |0\rangle - \frac{ie^{\frac{2\pi i}{3}}}{i-e^{\frac{-2\pi i}{3}}}|1\rangle$,
$|v_3\rangle \propto |0\rangle + \frac{ie^{\frac{2\pi i}{3}}}{i-e^{\frac{-2\pi i}{3}}}|1\rangle$, $|v_4\rangle \propto |0\rangle - \frac{ie^{\frac{2\pi i}{3}}}{i+e^{\frac{-2\pi i}{3}}}|1\rangle$ and $M_k = \frac{1}{2}|v_k^*\rangle\langle v_k^*|$

$$\tilde{\theta}(.) = \sigma_y \tilde{T}(.)\sigma_y \tag{6}$$

where $\sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$
H. T. Lim et.al., Phys. Rev. Lett. **107**, 160401(2011).

Matrix form of ρ_{12} after performing SPA

$$
\begin{aligned}
\widetilde{\rho_{12}} &= [\frac{1}{3}(I \otimes \widetilde{T}) + \frac{2}{3}(\widetilde{\Theta} \otimes \widetilde{D})]\rho_{12} \\
&= \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12}^* & E_{22} & E_{23} & E_{24} \\ E_{13}^* & E_{23}^* & E_{33} & E_{34} \\ E_{14}^* & E_{24}^* & E_{34}^* & E_{44} \end{pmatrix}
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
&E_{11} = \frac{1}{9}(2 + t_{11}), E_{12} = \frac{1}{9}(-it_{12} + t_{12}^*), E_{13} = \frac{1}{9}(t_{13} - i(t_{13}^* + t_{24}^*)), \\
&E_{14} = \frac{1}{9}(-it_{14} + t_{23}), E_{22} = \frac{1}{9}(2 + t_{22}), E_{23} = \frac{1}{9}(t_{14} + it_{23}), \\
&E_{24} = \frac{-i}{9}(t_{13}^* + t_{24}^*), E_{33} = \frac{1}{9}(2 + t_{33}), E_{34} = \frac{1}{9}(-it_{34} + t_{34}^*), \\
&E_{44} = \frac{1}{9}(2 + t_{44})
\end{aligned} \tag{8}
$$

Determination of minimum eigenvalue of $\widetilde{\rho_{12}}$ using witness operator

Interesting fact: Minimum eigenvalue of $\widetilde{\rho_{12}}$ detect whether the state ρ_{12} is entangled or not?
Consider the operator $O = \widetilde{\rho_{12}} - \frac{1}{9}\rho_{12}^{T_2}$.
Expectation value of the operator:

$$\begin{aligned} \langle\phi|O|\phi\rangle &= Tr[(\widetilde{\rho_{12}} - \frac{1}{9}\rho_{12}^{T_2})|\phi\rangle\langle\phi|] = \frac{2}{9} \\ \Rightarrow Tr[|\phi\rangle\langle\phi|\widetilde{\rho_{12}}] &- \frac{1}{9}Tr[|\phi\rangle\langle\phi|^{T_2}\rho_{12}] = \frac{2}{9} \\ \Rightarrow Tr[W^{(1)}\rho_{12}] &= Tr[\widetilde{\rho_{12}}|\phi\rangle\langle\phi|] - \frac{2}{9} \end{aligned} \quad (9)$$

where $|\phi\rangle = \alpha|00\rangle + \beta|11\rangle$ and $W^{(1)} = \frac{1}{9}|\phi\rangle\langle\phi|^{T_2}$

Determination of minimum eigenvalue of $\widetilde{\rho_{12}}$ using witness operator

Minimum eigenvalue of $\widetilde{\rho_{12}}$: λ_{min}
Eigenvector corresponding to the minimum eigenvalue: $|\phi\rangle$
Eigenvalue equation is given by

$$\widetilde{\rho_{12}}|\phi\rangle = \lambda_{min}|\phi\rangle \tag{10}$$

Using (10) in (9), we have

$$Tr(W^{(1)}\rho_{12}) = \lambda_{min} - \frac{2}{9} \tag{11}$$

For all separable state ρ_{12}^{s}, we have

$$Tr(W^{(1)}\rho_{12}^{s}) \geq 0 \Rightarrow \lambda_{min} \geq \frac{2}{9} \tag{12}$$

Determination of minimum eigenvalue of $\widetilde{\rho_{12}}$ using witness operator

Different form of $Tr(W^{(1)}\rho_{12})$:

$$Tr(W^{(1)}\rho_{12}) = Tr[(|\phi\rangle\langle\phi| - \frac{2}{9}I)\widetilde{\rho_{12}}] \tag{13}$$

Minimum eigenvalue of $\widetilde{\rho_{12}}$ is given by

$$\lambda_{min} = Tr[(|\phi\rangle\langle\phi| - \frac{2}{9}I)\widetilde{\rho_{12}}] + \frac{2}{9} \tag{14}$$

- If $W^{(1)}$ detect an entangled state ρ_{12} then

$$Tr[(|\phi\rangle\langle\phi| - \frac{2}{9}I)\widetilde{\rho_{12}}] < 0 \tag{15}$$

Determination of minimum eigenvalue of $\widetilde{\rho_{12}}$ using witness operator

Let $\mathbf{V} \equiv |\phi\rangle\langle\phi| - \frac{2}{9}I$.
In terms of Pauli matrices, $\mathbf{V}$ can be expressed as

$$\begin{aligned} \mathbf{V} &= \frac{9}{28}[\frac{7}{9}I \otimes I + (\alpha^2 - \beta^2)(I \otimes \sigma_z + \sigma_z \otimes I) \\ &+ 2\alpha\beta(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) + \sigma_z \otimes \sigma_z] \end{aligned} \quad (16)$$

Properties of the operator V:

- $\mathbf{V}$ is a hermitian operator.
- $Tr(\mathbf{V}\widetilde{\rho_{12}}^{sep}) \geq 0$.
- $\mathbf{V}$ has at least one negative eigenvalues.

$\mathbf{V}$ satisfies all properties of a witness operator.
Minimum eigenvalue of $\widetilde{\rho_{12}}$ can be determined as

$$\lambda_{min} = Tr[V\widetilde{\rho_{12}}] + \frac{2}{9} \quad (17)$$

Number of measurements needed to determine the value of $Tr(\mathbf{V}\widetilde{\rho_{12}})$

Question: Can we realize the operator **V** in experiment?

- **First Step:** Construction of approximate entanglement witness operator

Approximate entanglement witness operator $\widetilde{\mathbf{V}}$:

$$\widetilde{\mathbf{V}} = p\mathbf{V} + (1-p)I, 0 \leq p \leq 1 \tag{18}$$

The Hermitian operator $\widetilde{\mathbf{V}}$ is positive semi-definite if $p_{min} = \frac{8}{15}$.
$\widetilde{\mathbf{V}}$ can be re-expressed as

$$\widetilde{\mathbf{V}} = \frac{8}{15}\mathbf{V} + \frac{7}{15}I, \tag{19}$$

Number of measurements needed to determine the value of $Tr(\mathbf{V}\widetilde{\rho_{12}})$

- **Second step:** Normalize the approximate entanglement witness operator **V**.

Normalized approximate entanglement witness operator:

$$\widetilde{\mathbf{V}} = \frac{2}{9}\mathbf{V} + \frac{7}{36}I, \tag{20}$$

(i)$\widetilde{\mathbf{V}}$ is Hermitian
(ii) it is positive semi-definite
(iii) $Tr(\widetilde{\mathbf{V}}) = 1$
$\widetilde{\mathbf{V}}$ can be treated as a quantum state.

- **Third Step:** Express $Tr(\mathbf{V}\widetilde{\rho_{12}})$ into the trace of the product of two quantum states.

To achieve the third step, write $Tr(\mathbf{V}\widetilde{\rho_{12}})$ in terms of $Tr(\widetilde{\mathbf{V}}\widetilde{\rho_{12}})$ as

$$Tr(\mathbf{V}\widetilde{\rho_{12}}) = \frac{15}{8}Tr(\widetilde{\mathbf{V}}\widetilde{\rho_{12}}) - \frac{7}{8} \tag{21}$$

Number of measurements needed to determine the value of $Tr(\mathbf{V}\widetilde{\rho_{12}})$

NOTE: It can be shown that $Tr(\widetilde{\mathbf{V}}\widetilde{\rho_{12}})$ is equal to the average fidelity for two mixed quantum states $\widetilde{\mathbf{V}}$ and $\widetilde{\rho_{12}}$,

$$Tr(\widetilde{\mathbf{V}}\widetilde{\rho_{12}}) = F_{avg}(\widetilde{\mathbf{V}}, \widetilde{\rho_{12}}) \tag{22}$$

Using (21) and (22), we have

$$Tr(\mathbf{V}\widetilde{\rho_{12}}) = \frac{15}{8} F_{avg}(\widetilde{\mathbf{V}}, \widetilde{\rho_{12}}) - \frac{7}{8} \tag{23}$$

Result: C. J. Kwong et.al. have shown that the average fidelity between two mixed quantum states can be estimated experimentally by Hong-Ou-Mandel interferometry with only two detectors.

C. J. Kwong et.al., arxiv:1606.00427

- The quantity $Tr(\mathbf{V}\widetilde{\rho_{12}})$ needs only two measurements to estimate it in experiment.

Relation between minimum eigenvalue and average fidelity

The minimum eigenvalue can be expressed in terms of $F_{avg}(\widetilde{\mathbf{V}}, \widetilde{\rho_{12}})$ as

$$\lambda_{min} = \frac{15}{8} F_{avg}(\widetilde{\mathbf{V}}, \widetilde{\rho_{12}}) - \frac{47}{72} \quad (24)$$

Since the minimum eigenvalue of the quantum state $\widetilde{\rho_{12}}$ can be determined using two measurements and minimum eigenvalue is responsible for the detection of entanglement so we can say that the presence of entanglement in ρ_{12} can be detected using two measurements only.
S. Adhikari, Phys. Rev. A **97**, 042344(2018).

Application of SPA in the determination of entanglement negativity

Entanglement negativity

$$N^D(\rho) = 2\sum_i max(0, -\nu_i) \tag{25}$$

where $\nu_i's$ are the negative eigenvalues of the partial transpose ρ^Γ of the density matrix ρ.

G. Vidal, and R.F. Werner, Phys. Rev. A **65**, 032314 (2002).

Lemma:- For any two Hermitian 4×4 matrices F_1 and F_2, the inequality given below holds true

$$\sum_{i=1}^{4} \lambda_i(F_1)\lambda_{5-i}(F_2) \leq Tr(F_1 F_2) \leq \sum_{i=1}^{4} \lambda_i(F_1)\lambda_i(F_2) \tag{26}$$

where $\lambda_i(F_1)$ and $\lambda_i(F_2)$ denote the eigenvalues of the matrices F_1 and F_2 respectively. The eigenvalues are arranged in descending order i.e. $\lambda_1(.) > \lambda_2(.) > \lambda_3(.) > \lambda_4(.)$.

We use the above stated lemma for $F_1 = |\phi\rangle\langle\phi|$ and $F_2 = \rho_{AB}^{T_B}$, where $|\phi\rangle = \alpha|00\rangle + \beta|11\rangle$.

J. B. Lasserre, IEEE Trans. on Automatic Control **40**, 1500 (1995).

Lower bound of entanglement negativity

The left hand inequality of (26) will become

$$\lambda_4(\rho_{AB}^{T_B}) \leq Tr(|\phi\rangle\langle\phi|\rho_{AB}^{T_B}) \tag{27}$$

The amount of entanglement contained in ρ_{AB} can be determined by entanglement negativity $N(\rho_{AB})$ which is given by

$$N(\rho_{AB}) = -2\lambda_4(\rho_{AB}^{T_B}) \tag{28}$$

Inserting (28) in (27), we have

$$-\frac{N(\rho_{AB})}{2} \leq Tr(|\phi\rangle\langle\phi|\rho_{AB}^{T_B}) \tag{29}$$

Note: Value of the R.H.S of inequality (29) cannot be determined experimentally.

Lower bound of entanglement negativity

The relation between $\rho_{AB}^{T_B}$ and $\widetilde{\rho_{AB}}$ is given by

$$Tr(|\phi\rangle\langle\phi|\rho_{AB}^{T_B}) = 9\,Tr[|\phi\rangle\langle\phi|\widetilde{\rho_{AB}}] - 2 \tag{30}$$

Using (29) and (30), we get

$$Tr[|\phi\rangle\langle\phi|\widetilde{\rho_{AB}}] \geq \frac{1}{18}[4 - N(\rho_{AB})] \tag{31}$$

Minimum eigenvalue of $\widetilde{\rho_{AB}}$: μ_{min}
Eigenvalue equation is given by

$$\widetilde{\rho_{AB}}|\phi\rangle = \mu_{min}|\phi\rangle \tag{32}$$

Using the eigenvalue equation (32) in (31), we get

$$N(\rho_{AB}) \geq 4 - 18\mu_{min} \tag{33}$$

The minimum eigenvalue μ_{min} of the quantum state $\widetilde{\rho_{AB}}$ can be determined by the formula

$$\mu_{min} = \frac{15}{8} F_{avg}(\widetilde{\mathbf{W}}, \widetilde{\rho_{AB}}) - \frac{47}{72} \tag{34}$$

Lower bound of entanglement negativity

The approximated entanglement witness operator can be expressed in terms of entanglement witness operator $\mathbf{W} = |\phi\rangle\langle\phi| - \frac{2}{9}I$ as

$$\widetilde{\mathbf{W}} = \frac{2}{9}\mathbf{W} + \frac{7}{36}I, \qquad (35)$$

The average fidelity $F_{avg}(\widetilde{\mathbf{W}}, \widetilde{\rho_{AB}})$ can be determined experimentally with only two measurement by using Hong-Ou-Mandel interferometer and hence the minimum eigenvalue μ_{min}.
Thus we can determine the value of lower bound of entanglement negativity experimentally with only two measurement.

Determination of entanglement negativity

- We have obtained the analytical lower bound of entanglement negativity.
- We have shown that the analytic lower bound can be achieved experimentally.
- But the inequality (33) only tells us that the entanglement negativity can take value greater than the bound obtained but it does not determine the actual amount of entanglement in an arbitrary two-qubit state.

Determination of entanglement negativity

- The quantity $4 - 18\mu_{min}$ is less than or equal to $N(\rho_{AB})$. This suggest that if we add a positive quantity Q to the quantity $4 - 18\mu_{min}$ then it may be equal to $N(\rho_{AB})$.
- By adding a positive quantity Q to the R.H.S of (33), we get

$$N(\rho_{AB}) = 4 - 18\mu_{min} + Q, Q > 0 \tag{36}$$

- To search for the quantity Q, we keep in mind the following facts: (i) the inequality $\mu_{min} < \frac{2}{9}$ holds for all entangled state ρ_{AB} [?] and (ii) $Tr[(I - |\phi\rangle\langle\phi|)\widetilde{\rho_{AB}}] = 1 - \mu_{min} > 0$. Using these two facts, we can always choose $Q = (\frac{2}{9} - \mu_{min})Tr[(I - |\phi\rangle\langle\phi|)\widetilde{\rho_{AB}}]$.

Determination of entanglement negativity

With the above choice of Q, (36) can be re-written as

$$\begin{aligned} N(\rho_{AB}) &= 4 - 18\mu_{min} + (\frac{2}{9} - \mu_{min})(1 - \mu_{min}) \\ &= (\frac{2}{9} - \mu_{min})(19 - \mu_{min}), \\ & \frac{1}{6} \leq \mu_{min} < \frac{2}{9} \end{aligned} \tag{37}$$

The normalized $N(\rho_{AB})$ is then given by

$$N^N(\rho_{AB}) = K(\frac{2}{9} - \mu_{min})(19 - \mu_{min}), \frac{1}{6} \leq \mu_{min} < \frac{2}{9} \tag{38}$$

Here K is a normalization constant.

Determination of entanglement negativity

K can be determined by using the fact that $\mu_{min} = \frac{1}{6}$ for maximally entangled state.
The normalized entanglement negativity is given by

$$N^N(\rho_{AB}) = \frac{108}{113}(\frac{2}{9} - \mu_{min})(19 - \mu_{min}), \quad \frac{1}{6} \leq \mu_{min} < \frac{2}{9} \tag{39}$$

Since $N^N(\rho_{AB})$ expressed in terms of μ_{min} so the normalized entanglement negativity can be determined experimentally with two measurement using Hong-Ou-Mandel interferometry.
S. Adhikari, arxiv:1808.09745, Accepted in Europhysics Letters

Remarks

- We have presented an entanglement estimation protocol in which we need only single copy of the quantum state and no requirement of QST.
- This result contradict the fact that it is impossible to detect the exact value of the entanglement with only single-copy measurements without QST.
  D. Lu et.al., Phys. Rev. Lett. **116**, 230501 (2016).
  Carmeli et.al., Phys. Rev. Lett. **116**, 230403 (2016).
- The above contradiction can be explained by observing the fact that there does not exist any quantum operation that can achieve non-physical operation such as partial transpose map with unit fidelity in an experiment.

Remarks

- By seeing this fact, it is necessary to have approximate quantum operation that can approximate partial transpose for the possible realization in experiment. SPA is such an approximation of PT operation that can be realized in experiment with fidelity less than unity.
- Further, the parameter needed for the estimation of entanglement is the minimum eigenvalue of the SPA of the given state and this parameter is related with the average fidelity. So, the minimum eigenvalue can be estimated approximately and hence the negativity.

Realization of optimal singlet fraction

The singlet fraction is defined as

$$F(\rho_{12}) = max[\langle\phi^+|\rho_{12}|\phi^+\rangle \langle\phi^-|\rho_{12}|\phi^-\rangle \quad (40)$$
$$\langle\psi^+|\rho_{12}|\psi^+\rangle \langle\psi^-|\rho_{12}|\psi^-\rangle] \quad (41)$$

where $\{|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle\}$ are the maximally entangled Bell states.
Result:

- Verstraete and Vershelde suggested the optimal trace preserving protocol for maximizing the singlet fraction of a given state. The optimal singlet fraction is given by

$$F^{opt}(\rho_{12}) = \frac{1}{2} - Tr(X_{opt}\rho_{12}^{T_B}) \quad (42)$$

X_{opt} is given by

$$X_{opt} = (A \otimes I_2)|\psi^+\rangle\langle\psi^+|(A^\dagger \otimes I_2), \quad (43)$$

F. Verstraete and H. Verschelde, Phys. Rev. Lett. **90**, 097901 (2003).

Realization of optimal singlet fraction

where I_2 represent an identity matrix of order 2, $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, -1 \leq a \leq 1$.

PROBLEMS:

- $\rho_{12}^{T_B}$ cannot be realized in the laboratory
- The parameter in X_{opt} is state dependent and hence to construct X_{opt}, we need to know the state under investigation

Optimal singlet fraction in terms of minimum eigenvalue

To overcome the above mentioned problems, let us consider the operator $B \equiv X_{opt}(\widetilde{\rho_{12}} - \frac{1}{9}\rho_{12}^{T_2})$, where X_{opt} is given in (43).
Calculate Tr(B) and after simplification it reduces to

$$F^{opt}(\rho_{12}) = \frac{1}{2} - [9\,Tr(X_{opt}\widetilde{\rho_{12}}) - (a^2+1)] \tag{44}$$

- $F^{opt}(\rho_{12}) > \frac{1}{2}$ if and only if $9\,Tr(X_{opt}\widetilde{\rho_{12}}) - (a^2+1) < 0$. This condition leads to

$$Tr(X_{opt}\widetilde{\rho_{12}}) < \frac{a^2+1}{9}, -1 \leq a \leq 1 \tag{45}$$

In $\{|00\rangle, |11\rangle\}$ subspace, the operator $\frac{1}{a^2+1}X$ can be expressed as

$$\frac{1}{a^2+1}X_{opt} = \frac{1}{2}|\chi\rangle\langle\chi| \tag{46}$$

where $|\chi\rangle = \frac{1}{\sqrt{a^2+1}}(a|00\rangle + |11\rangle)$.

Optimal singlet fraction in terms of minimum eigenvalue

We note that the vector $|\chi\rangle$ and the eigenvector $|\phi\rangle$ of the operator $\widetilde{\rho_{12}}$ are parallel vectors and thus there exist a real scalar k such that

$$|\chi\rangle = k|\phi\rangle \tag{47}$$

Using it can be easily shown that the vector $|\chi\rangle$ is also a eigenvector corresponding to the minimum eigenvalue λ_{min}. Hence, the resource state ρ_{12} is useful for teleportation iff

$$\lambda_{min} < \frac{2}{9} \tag{48}$$

The singlet fraction $F^{opt}(\rho_{12})$ given by can be re-expressed in terms of the minimum eigenvalue λ_{min} as

$$F^{opt}_{(a,\lambda_{min})}(\rho_{12}) = \frac{1}{2} - \frac{9(a^2+1)}{2}[\lambda_{min} - \frac{2}{9}], \quad -1 \leq a \leq 1 \tag{49}$$

Optimal singlet fraction in terms of minimum eigenvalue

Further, we find that if the minimum eigenvalue λ_{min} is restricted to lie in the interval $[\frac{1}{6}, \frac{2}{9})$ then the singlet fraction $F^{opt}_{(a,\lambda_{min})}(\rho_{12})$ lies in the interval

$$\frac{1}{2} < F^{opt}_{(a,\lambda_{min})}(\rho_{12}) < \frac{1}{2} + \frac{a^2+1}{4}, -1 \leq a \leq 1 \quad (50)$$

The optimal singlet fraction can be achieved by putting $a = \pm 1$ in (49) and it is given by

$$F^{opt}_{(\pm 1,\lambda_{min})}(\rho_{12}) = \frac{1}{2} - 9[\lambda_{min} - \frac{2}{9}], \frac{1}{6} \leq \lambda_{min} < \frac{2}{9} \quad (51)$$

Without any loss of generality, we can take $a = 1$ and thus we have $X^{opt} = |\psi^+\rangle\langle\psi^+|$. Afterward, we denote $F^{opt}_{(\pm 1,\lambda_{min})}(\rho_{12})$ as simply $F^{opt}_{\lambda_{min}}(\rho_{12})$. Therefore,

$$F^{opt}_{\lambda_{min}}(\rho_{12}) = \frac{1}{2} - 9[\lambda_{min} - \frac{2}{9}], \frac{1}{6} \leq \lambda_{min} < \frac{2}{9} \quad (52)$$

Optimal singlet fraction in terms of average fidelity

NOTE: The problem of finding the optimal singlet fraction reduces to finding the minimum eigenvalue of SPA-PT of any arbitrary state ρ_{12}. The expression for optimal singlet fraction in terms of average fidelity $F_{avg}(\widetilde{\mathbf{V}}, \widetilde{\rho_{12}})$can be obtained in two steps.

- The optimal singlet fraction in terms of λ_{min} can be re-written as

$$F^{opt}_{\lambda_{min}}(\rho_{12}) = \frac{1}{2} - 9[\lambda_{min} - \frac{2}{9}], \frac{1}{6} \leq \lambda_{min} < \frac{2}{9} \quad (53)$$

- Substituting the expression of λ_{min} in terms of average fidelity, we get

$$\begin{aligned} F^{opt}_{\lambda_{min}}(\rho_{12}) &= \frac{1}{2} - \frac{135}{8}[F_{avg}(\widetilde{\mathbf{V}}, \widetilde{\rho_{12}}) - \frac{7}{15}], \\ & \frac{59}{135} \leq F_{avg}(\widetilde{\mathbf{V}}, \widetilde{\rho_{12}}) < \frac{7}{15} \end{aligned} \quad (54)$$

Since $F_{avg}(\widetilde{\mathbf{V}}, \widetilde{\rho_{12}})$ can be determined experimentally so $F^{opt}_{\lambda_{min}}(\rho_{12})$ can be realized using Hong-Ou-Mandel interferometry with only two detectors.

Optimal Teleportation Fidelity for Two Qubit System

For two qubit system, optimal teleportation fidelity $f_{opt}(\rho_{12})$ and optimal singlet fraction $F^{opt}_{\lambda_{min}}(\rho_{12})$ are related by

$$\begin{aligned} f_{opt}(\rho_{12}) &= \frac{2F^{opt}_{\lambda_{min}}(\rho_{12})+1}{3} \\ &= \frac{2}{3} - \frac{135}{12}[F_{avg}(\widetilde{\mathbf{V}}, \widetilde{\rho_{12}}) - \frac{7}{15}] \\ & \frac{59}{135} \leq F_{avg}(\widetilde{\mathbf{V}}, \widetilde{\rho_{12}}) < \frac{7}{15} \end{aligned} \quad (55)$$

We can say a teleportation scheme is quantum if teleportation fidelity is greater than $\frac{2}{3}$. We can find that the teleportation fidelity given in (55) is always greater than $\frac{2}{3}$.