Pairwise correlation inequalities and joint measurability

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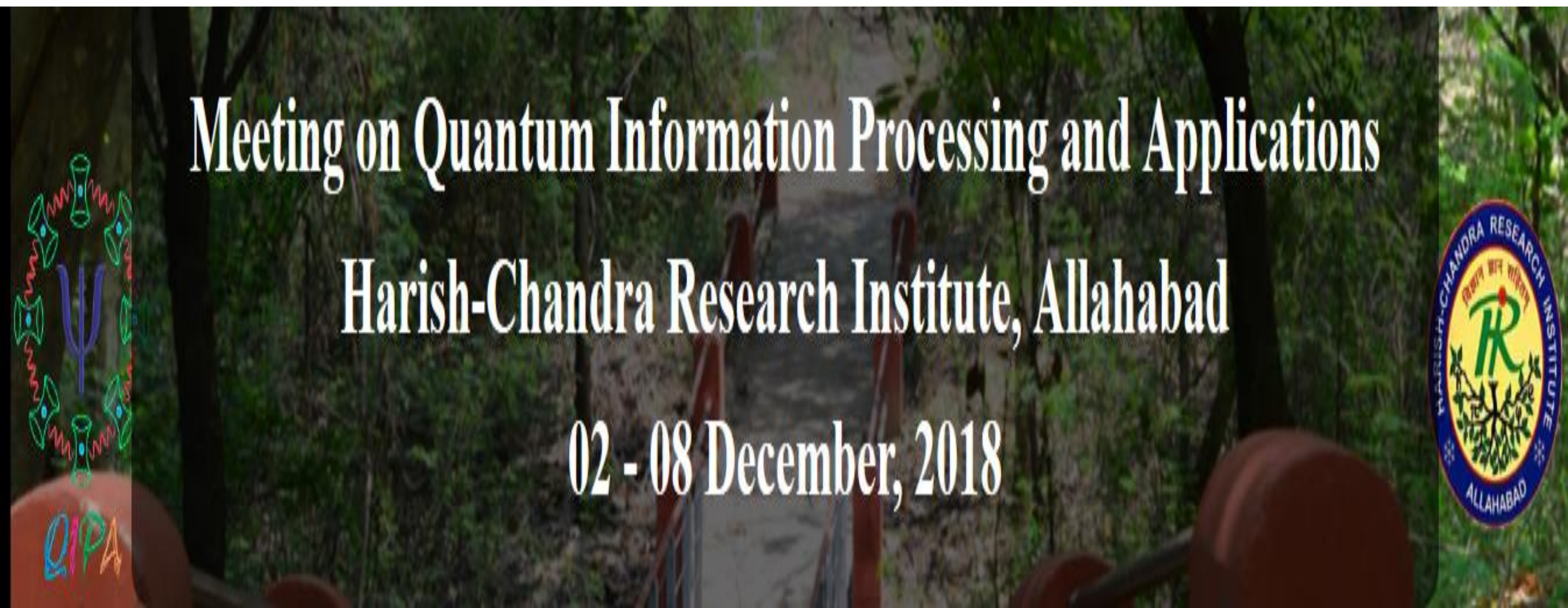

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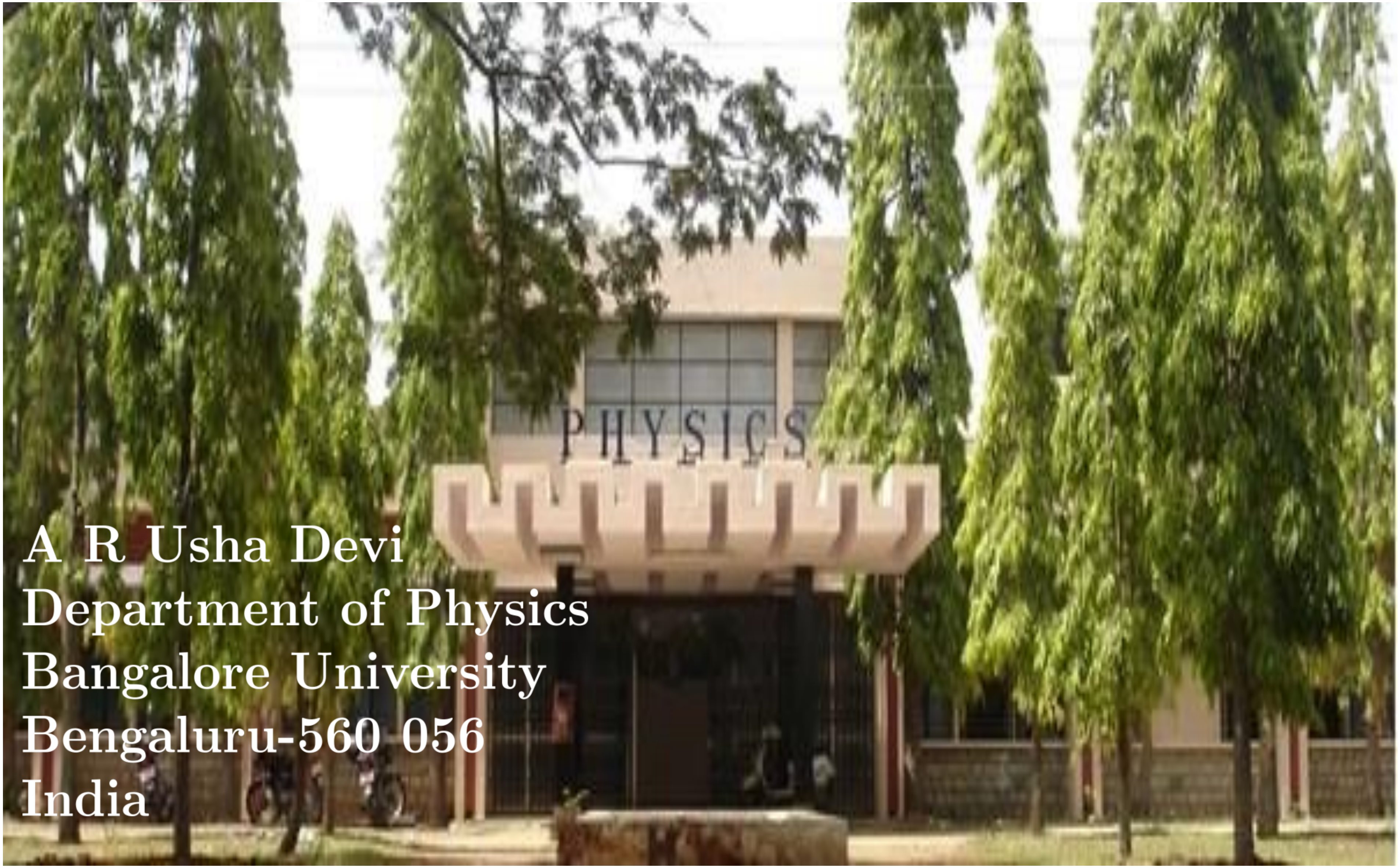

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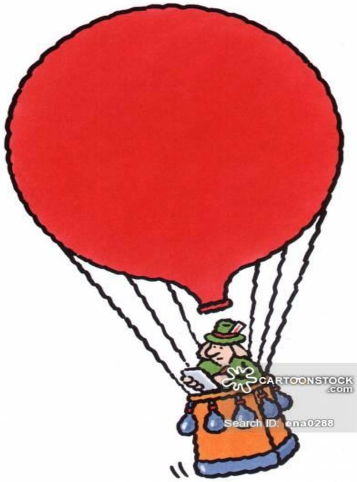
Outline

- **Bell-CHSH inequality and EPR Steering**
- **Positive Operator Valued (POV) observables and unsharp measurements**
- **Compatible (jointly measurable) POVMs**
- **N-term correlation inequality and joint measurability**
- **Equivalence between steering and joint measurability**



Bell's Inequality

Predictions of quantum mechanics cannot be squared with the belief, called local realism that physical systems have realistic properties whose pre-existing values are revealed by measurements. The predictions of quantum mechanics for spatially separated systems are at odds with any version of local realism

Bell-CHSH Correlation Inequality:

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$$\left|\langle A_1 B_1\rangle + \langle A_1 B_2\rangle + \langle A_2 B_1\rangle - \langle A_2 B_2\rangle\right| \leq 2$$

Quantum mechanical violation:

$$\left|\langle A_1 B_1\rangle + \langle A_1 B_2\rangle + \langle A_2 B_1\rangle - \langle A_2 B_2\rangle\right|_{\mathrm{QM}}^{\max} = 2\sqrt{2}.$$

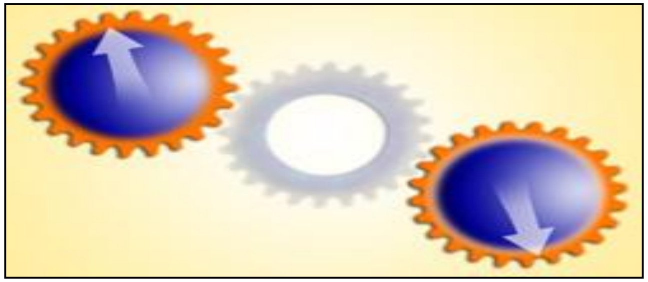
Quantum Tsirelson Bound

Algebraic Maximum of CHSH: 4 → Generalized Probability Theory (beyond quantum theory

Non-local steering

(E. Schrodinger, Proc. Camb. Phil. Soc. 31, 555563 (1935))

EPR Steering → **The ability to non-locally alter the states of one part of a composite system by performing measurements on another space-like separated part.**



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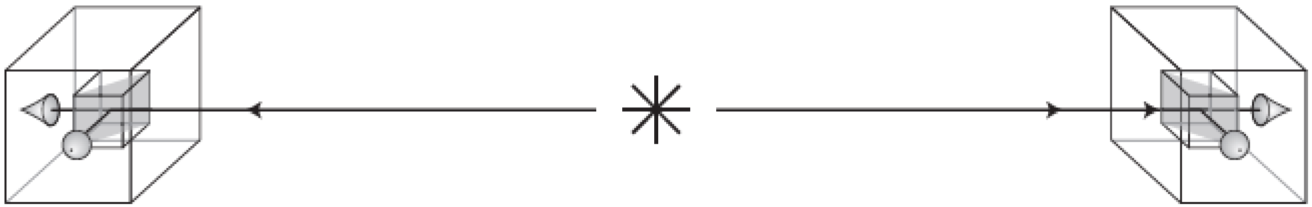
Non-local EPR Steering

(N. Brunner, News and views, nature physics, 6, 842 (2010))

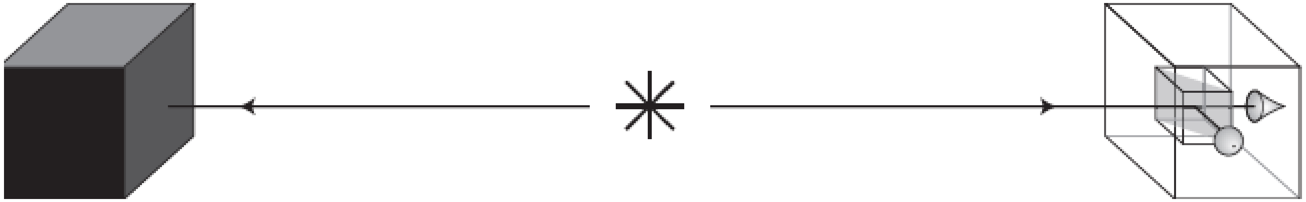
Alice



Bob

Trusted devices -- Bell-CHSH scenario

Alice



Bob

Bob cannot trust Alice! Verifies if a 'Steering Inequality' is violated

Non-local steering

(E. Schrodinger, Proc. Camb. Phil. Soc. 31, 555563 (1935))

- Suppose Alice prepares a bipartite quantum state ρ_{AB} and sends a subsystem to Bob.
- If the state is entangled, and Alice chooses suitable local measurements, on her part of the state, she can affect Bob's quantum state remotely.
- How would Bob convince himself that his state is indeed steered by Alice's local measurements?

EPR Steering

- It could be that Alice is not honest; she does not prepare a composite state ρ_{AB} at all; but she chooses the states ρ_λ with probability $g(\lambda)$ from a chosen ensemble and sends it to Bob.
- Bob asks Alice to perform local measurements of the observables $\mathbf{X}_k = \sum_{a_k} a_k \, \Pi_{x_k}(a_k)$, on her part of the state and communicate the outcomes a_k in each experimental trial.
- Alice might merely communicate a fake outcome a_k to have occurred with the probability $p(a_k|k) = \sum_\lambda g(\lambda)\, p(a_k|k,\lambda)$, $\sum_{a_k} p(a_k|k,\lambda) = 1$.

Assemblage:

$$\rho_{a_k|k} = \mathrm{Tr}_A \left[\rho_{AB}\, E_k(a_k) \otimes I\right]$$

Contd...

- **If Bob's conditional reduced states (unnormalized)**

$$\varrho^B_{a_k|x_k} = \mathrm{Tr}_A[\Pi_{x_k}(a_k) \otimes \mathbb{1}_B\, \rho_{AB}]$$

admit a Local Hidden State (LHS) decomposition viz.,

$$\varrho_{a_k|x_k} = \sum_\lambda p(\lambda)\, p(a_k|x_k, \lambda)\, \rho^B_\lambda$$

where

$$0 \leq p(\lambda) \leq 1, \quad \sum_\lambda p(\lambda) = 1$$

and

$$0 \leq p(a_k|x_k, \lambda) \leq 1, \; \sum_{a_k} p(a_k|x_k, \lambda) = 1$$

($(p_\lambda, \rho^B_\lambda)$ denote Bob's LHS ensemble)

then Bob can declare that Alice is not able to steer his state through local measurements at her end.

- **Incompatibility of Alice's local measurements too plays a crucial role in revealing steerability.**

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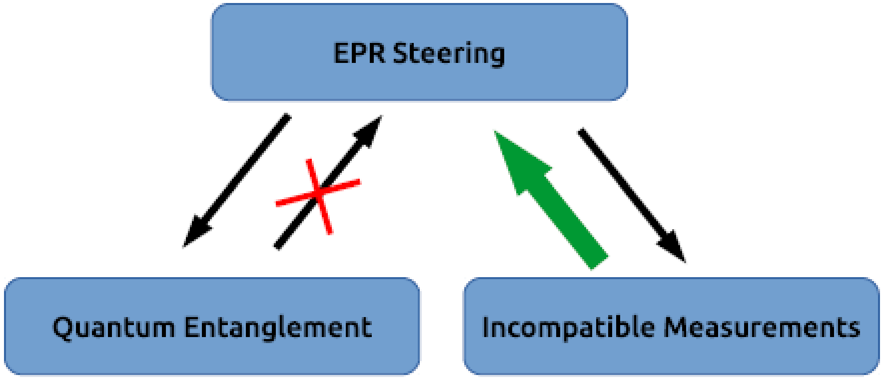
Steering and Joint Measurability are synonymous



Steering implies both entanglement and incompatible measurements at Bob's end

M. T. Quintino, T. Vértesi, and N. Brunner, Phys. Rev. Lett. 113, 160402 (2014)
R. Uola, T. Moroder, and O. Gühne, Phys. Rev. Lett. 113, 160403 (2014)

But Bell non-locality and joint measurability not synonymous (except in the CHSH case).

Measurement incompatibility (Non-joint measurability)

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- **Heisenberg's uncertainty relation points towards incompatibility of non-commuting physical observables. The measurements are called *incompatible* because they are not *jointly measurable*.**
- **It has been realized that incompatible measurements are necessary to witness the violation of a Bell inequality/EPR steering inequality/non-contextual inequality and so on.**

PHYSICAL REVIEW A 91, 012115 (2015)

Joint measurability, steering, and entropic uncertainty

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(Received 10 October 2014; published 22 January 2015)

There has been a surge of research activity recently on the role of joint measurability of unsharp observables in nonlocal features, *viz.*, violation of Bell inequality and EPR steering. Here, we investigate the entropic uncertainty relation for a pair of noncommuting observables (of Alice's system) when an entangled quantum memory of Bob is restricted to record outcomes of jointly measurable positive operator valued measures. We show that with this imposed constraint of joint measurability at Bob's end, the entropic uncertainties associated with Alice's measurement outcomes—conditioned by the results registered at Bob's end—obey an entropic steering inequality. Thus, Bob's nonsteerability is intrinsically linked to his inability to predict the outcomes of Alice's pair of noncommuting observables with better precision, even when they share an entangled state. As a further consequence, we prove that in the joint measurability regime, the quantum advantage envisaged for the construction of security proofs in quantum key distribution is lost.

- **In the classical world physical observables commute with each other and they can all be jointly measured. But in the quantum scenario, measurement of observables, which do not commute are usually declared to be *incompatible* in the quantum scenario.**

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- **Commuting observables can be measured jointly using projective valued (PV) measurements and their statistical outcomes can be discerned classically.**

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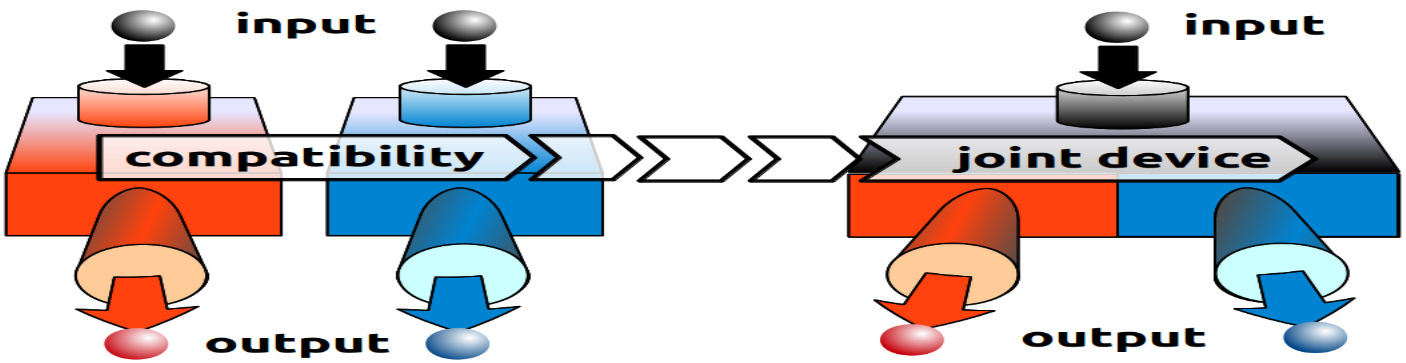
- **Commuting observables can be measured jointly using projective valued (PV) measurements and their statistical outcomes can be discerned classically.**
- **A joint measurement of commuting observables $\Rightarrow$ by performing one measurement, we can produce the results for each of the two observables.**



PV measurements of a pair of commuting observables

- **In the classical world physical observables commute with each other and they can all be jointly measured. But in the quantum scenario, measurement of observables, which do not commute are usually declared to be *incompatible* in the quantum scenario.**
- **Commuting observables can be measured jointly using projective valued (PV) measurements and their statistical outcomes can be discerned classically.**
- **A joint measurement of commuting observables $\Rightarrow$ by performing one measurement, we can produce the results for each of the two observables.**
- **But quantum mechanics places restrictions on how *sharply* two noncommuting observables can be measured jointly.**

Are joint unsharp measurements possible?

Extended framework: Joint measurements of Positive Operator Valued (POV) observables

- **The orthodox notion of *sharp* projective valued (PV) measurements of self adjoint observables gets broadened to include *unsharp* measurements of POV observables.**

Extended framework: Joint measurements of Positive Operator Valued (POV) observables

1. P. Busch, Phys. Rev. D **33**, 2253 (1986).
2. T. Heinosaari, D. Reitzner, and P. Stano, Found. Phys. **38**, 1133 (2008).
3. P. Busch, P. Lahti, and P. Mittelstaedt, *The Quantum Theory of Measurement*, No. v. 2 in Environmental Engineering, Springer, 1996.
4. M. M. Wolf, D. Perez-Garcia, and C. Fernandez, Phys. Rev. Lett. **103**, 230402 (2009).
5. N. Stevens and P. Busch, Phys. Rev. A **89**, 022123 (2014)
6. M. T. Quintino, T. Vertesi, and N. Brunner, Phys. Rev. Lett. **113**, 160402 (2014)
7. R. Uola, T. Moroder, and O. Gühne, Phys. Rev. Lett. **113**, 160403 (2014).

Positive Operator Valued (POV) observables and unsharp measurements

Def: POV observable $\mathbb{E}$ is a collection $\{E(x)\}$ of positive self-adjoint operators

$$0 \leq E(x) \leq 1$$

called *effects*.

The *effects* satisfy the condition

$$\sum_x E(x) = \mathbb{1}$$

($\mathbb{1}$ is the identity operator)

When a quantum system is prepared in the state ρ, measurement of the POV observable $\mathbb{E}$ gives an outcome x with probability

$$p(x) = \mathrm{Tr}[\rho\, E(x)]$$

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Joint Measurability: Global Positive Operator Valued Measure (POVM)

Consider two POV observables

$$\mathbb{E}_i = \{E_i(x_i)\}, \quad i = 1, 2$$

These two POVMs are jointly measurable if there exists a *global* POVM

$$\mathbb{G} = \{G(x_1, x_2);\ 0 \leq G(\lambda) \leq \mathbb{1}, \sum_\lambda G(\lambda) = \mathbb{1}, \quad \lambda = \{x_1, x_2\}\}$$

such that the POV observables $\mathbb{E}_i$ can be realized as the marginals:

$$E_1(x_1) = \sum_{x_2} G(x_1, x_2), \quad E_2(x_2) = \sum_{x_1} G(x_1, x_2)$$

- In general, if the effects $E_i(x_i)$ can be expressed as

$$E_i(x_i) = \sum_{\lambda} p(x_i|i, \lambda)\, G(\lambda) \quad \forall \;\; i$$

where $\sum_{x_i} p(x_i|i, \lambda) = 1$, the fuzzy observables $\mathbb{E}_i$ are jointly measurable.

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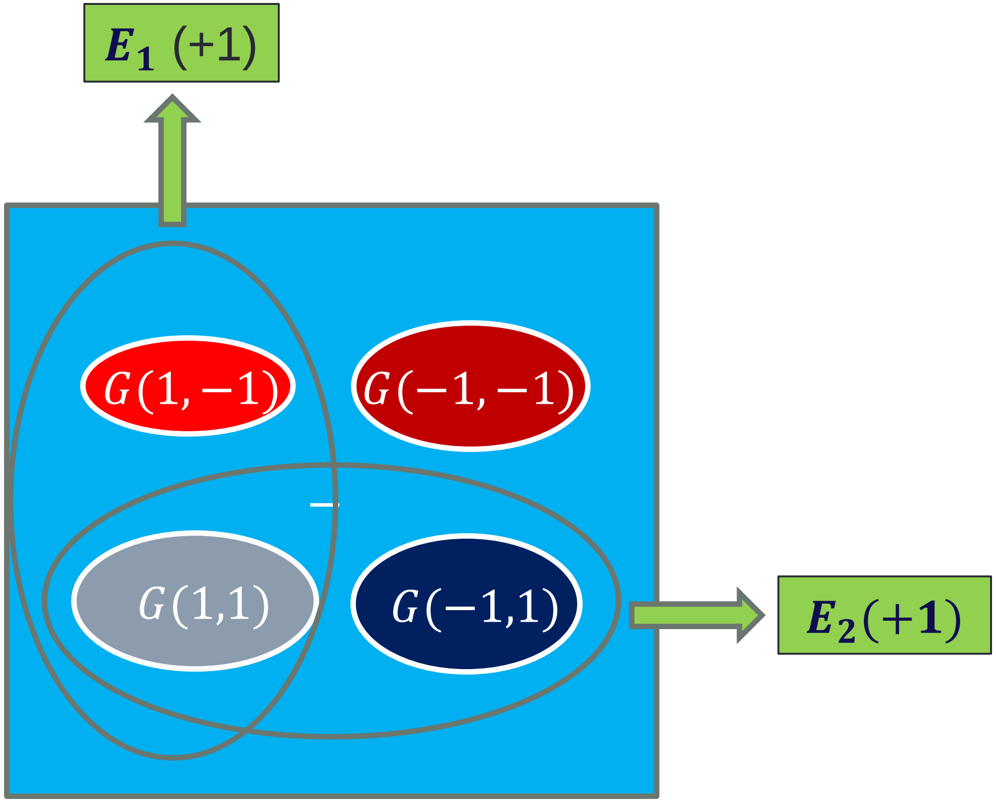
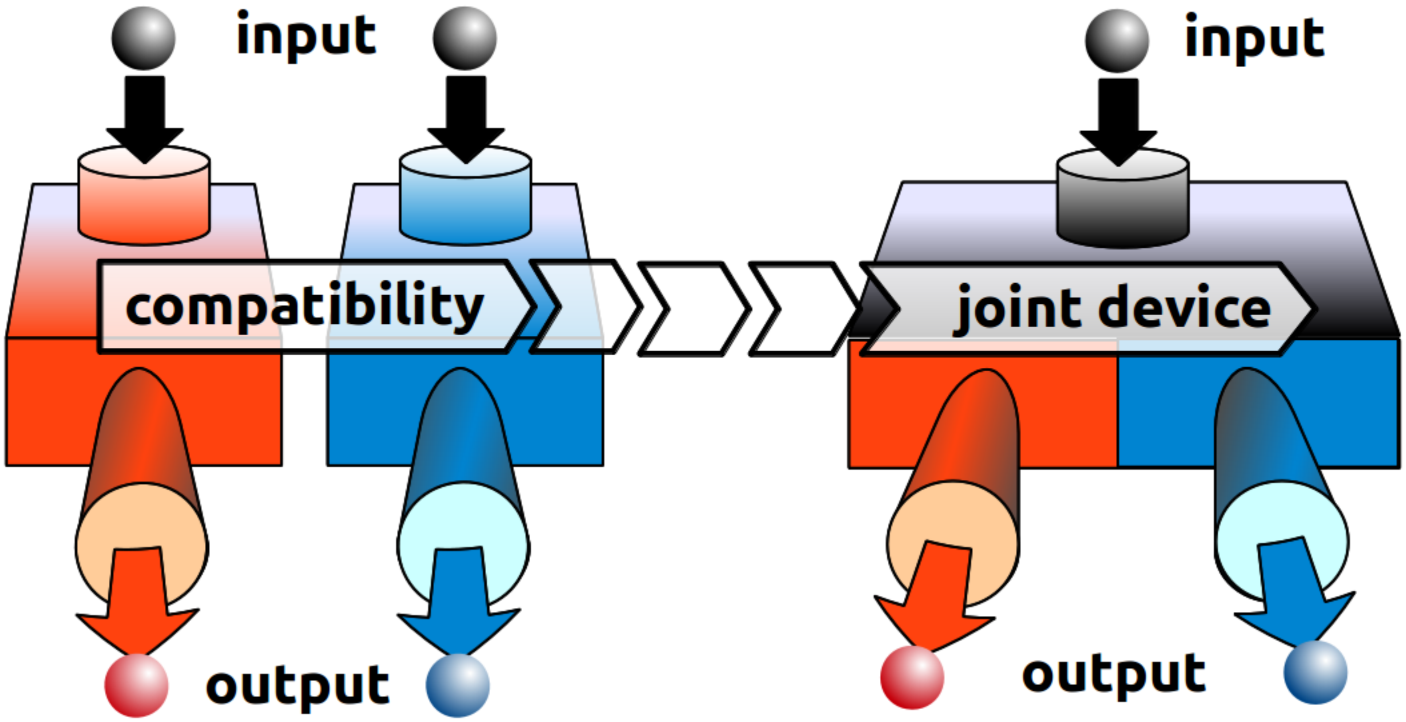
- Think of G as a **common measurement device** with four LEDs (corresponding to four outcomes); two of the LEDs correspond to the measurement outcome $+1$ for the binary POVMs E_1 and similarly for E_2.





Unsharp measurements of a pair of compatible POVMs

Courtesy: Heinosari et al., J. Phys. A: Math. Theor. 49 (2016) 123001

Fuzzy measurements of noisy qubit observables

Positive operator valued *fuzzy* spin observables:

Unsharp x-spin $\longrightarrow$ $\{E_x(+1), E_x(-1)\}$

Unsharp z-spin $\longrightarrow$ $\{E_z(+1), E_z(-1)\}$

$$
\begin{aligned}
E_x(\pm 1) &= \frac{1}{2}[\mathbb{1} \pm \eta\,\sigma_x] \\
E_z(\pm 1) &= \frac{1}{2}[\mathbb{1} \pm \eta\,\sigma_z]
\end{aligned}
$$

$$0 \leq \eta \leq 1$$

$\eta \longrightarrow$ sharpness parameter

- **PV measurements** $\longrightarrow \eta = 1 \Rightarrow$ **sharp measurement**

- **Joint measurability of** $\sigma_x,\ \sigma_z$ **requires** $\eta \leq \frac{1}{\sqrt{2}}$
- **Global POVM for pairwise joint measurabililty:**

$$G(x,z) = \frac{1}{4}\left[\mathbb{1} + \frac{x}{\sqrt{2}}\,\sigma_x + \frac{z}{\sqrt{2}}\,\sigma_z\right], \quad x,z = \pm 1.$$

- **Joint measurability of three orthogonal spin components** $\sigma_x,\ \sigma_y, \sigma_z$ **implies** $\eta \leq \frac{1}{\sqrt{3}}$
- **Three orthogonal qubit orientations are pairwise measurable but not tripplewise measurable iff** $\frac{1}{\sqrt{3}} \leq \eta \leq \frac{1}{\sqrt{2}}$.

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Number of POVMs	Orientation of $\hat{n}_k$	$\eta_{\rm opt}$
	Orthogonal axes	
$N = 3$	$\hat{n}_k \cdot \hat{n}_l = 0,\ k \neq l = 1, 2, 3$	$\frac{1}{\sqrt{3}}$
$N = 2$	$\hat{n}_1 \cdot \hat{n}_2 = 0$	$\frac{1}{\sqrt{2}}$
	Trine axes	
$N = 3$	$\hat{n}_k \cdot \hat{n}_l = -\frac{1}{2};\ k \neq l = 1, 2, 3$	$\frac{2}{3}$
$N = 2$	$\hat{n}_1 \cdot \hat{n}_2 = -\frac{1}{2}$	0.732

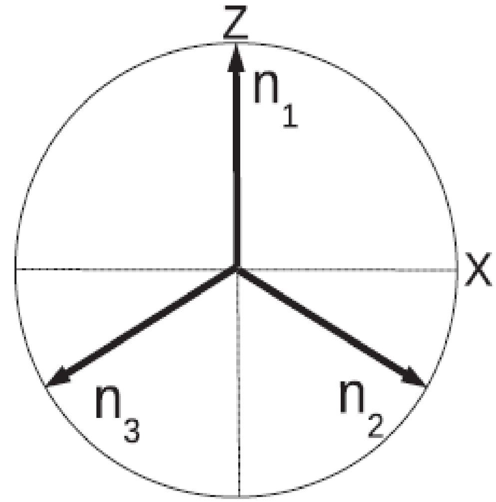

Table 1: Optimal value $\eta_{\rm opt}$ of the unsharpness parameter (evaluated using the necessary and sufficient conditions, below which the joint measurability of the qubit POVMs $\{E_{x_k}(a_k) = \frac{1}{2}(\mathbb{1} + \eta\, a_k\, \vec{\sigma} \cdot \hat{n}_k)\}$ for different orientations $\hat{n}_k$ are compatible.

Necessary condition for joint measurability of N dichotomic POVMs with arbitrary qubit orientations $\hat{n}_k,\ k = 1, 2, \ldots, N$

$$\eta \leq \frac{1}{N} \max_{x_1, x_2, \ldots, x_N} \left| \vec{m}_{x_1, x_2, \ldots, x_N} \right|$$

$$\vec{m}_{x_1, x_2, \ldots, x_N} = \sum_{k=1}^{N} x_k \, \hat{n}_k; \quad x_k = \pm 1$$

Sufficient condition:

$$\eta \leq \frac{2^N}{\sum_{\mathbf{a}} |\vec{m}_{\mathbf{a}}|}.$$

See: Ravi Kunjwal and Sibasish Ghosh, Phys. Rev. A 89, 042118 (2014)
Y. C. Liang, R. W. Spekkens, and H. M. Wiseman, Phys. Rep. 506, 1 (2011).

Joint measurability of equatorial qubit observables

- **The POVMs**

$$\{E_{\theta_k}(a_k = \pm 1) = \frac{1}{2}(\mathbb{1} + \eta\, a_k\, \sigma_{\theta_k})\}$$

with

$$\sigma_{\theta_k} = \sigma_x \cos(\theta_k) + \sigma_y \sin(\theta_k)$$

$$\theta_k = k\,\pi/N, \;\; k = 1, 2, \ldots, N$$

correspond geometrically to the points on the circumference of the circle in the equatorial half-plane of the Bloch sphere, separated successively by an angle $\theta = \pi/N$.

- **Joint measurability of $\{E_{\theta_k}(a_k)\}$**

$$\Rightarrow \eta_{\text{opt}} = \frac{1}{N}\sqrt{N + 2\sum_{k=1}^{[\frac{N}{2}]}(N - 2k)\,\cos\left(\frac{k\,\pi}{N}\right)}.$$

Number of POVMs	η_{opt}
3	0.6666
4	0.6532
5	0.6472
6	0.6439
10	0.6392
20	0.6372
50	0.6367
100	0.6366

- In the large N limit, the degree of incompatibility (i.e., the cut-off value of the unsharpness parameter) approaches $\eta_{\text{opt}} \to 0.6366$ and thus the POVMs associated with the set of all qubit observables σ_θ, $0 \leq \theta \leq \pi$ in the equatorial plane of the Bloch sphere are jointly measurable in the range $0 \leq \eta_{\text{opt}}^{(\infty)} \leq \frac{2}{\pi} \approx 0.6366$.

- Joint measurability of qubit POVMs corresponding to points on the entire Bloch sphere $\Rightarrow \eta_{\text{opt}} = \frac{1}{2}$.

- Existence of joint probabilities in turn implies that the set of all Bell inequalities are satisfied (A. Fine, Phys. Rev. Lett. 48, 291 (1982))
- Legitimate joint probabilities result when measurements are *compatible.*

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PHYSICAL REVIEW A **87**, 052125 (2013)

Degree of complementarity determines the nonlocality in quantum mechanics

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(Received 9 July 2012; published 20 May 2013)

Bohr's complementarity principle is one of the central concepts in quantum mechanics which restricts joint measurement for certain observables. Of course, later development shows that joint measurement could be

PHYSICAL REVIEW A **89**, 022123 (2014)

Steering, incompatibility, and Bell-inequality violations in a class of probabilistic theories

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(Received 5 December 2013; published 24 February 2014)

We show that connections between a degree of incompatibility of pairs of observables and the strength of violations of Bell's inequality found in recent investigations can be extended to a general class of probabilistic physical models. It turns out that the property of universal uniform steering is sufficient for the saturation of a generalized Tsirelson bound, corresponding to maximal violations of Bell's inequality. It is also found that a limited form of steering is still available and sufficient for such saturation in some state spaces where universal uniform steering is not given. The techniques developed here are applied to the class of regular polygon state spaces, giving a strengthening of known results. However, we also find indications that the link between incompatibility and Bell violation may be more complex than originally envisaged.

$$|\langle A_1 B_1\rangle + \langle A_1 B_2\rangle + \langle A_2 B_1\rangle - \langle A_2 B_2\rangle| \le 2$$

$$|\langle A_1 B_1\rangle + \langle A_1 B_2\rangle + \langle A_2 B_1\rangle - \langle A_2 B_2\rangle| \le \frac{2}{\eta_{opt}}$$

Tsirelson bound of $2\sqrt{2}$

$\eta_{opt} = 1 \longrightarrow$ **classical**

$\eta_{opt} = \frac{1}{2} \longrightarrow$ **GPT**

$\eta_{opt} = \frac{1}{\sqrt{2}} \longrightarrow$ **quantum**

4-term CHSH inequality and joint measurability

Fuzziness scale **GPT** **QM** **Classical**

- **Do classical features emerge in general, when one merely confines to measurements of *compatible* unsharp POVMs?**
- **Is it possible to classify physical theories based on the *fuzziness* necessiated by joint measurability?**

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Some pertinent observations

- **Recall that measurement of a single *grand* POVM leads to results of the measurements of all compatible POVMs. In other words, joint measurability entails a joint probability distribution for all compatible observables in the quantum framework. Also, there is Fine's result (A. Fine, Phys. Rev. Lett. 48, 291 (1982)): *Existence of joint probabilities in turn implies that the set of all Bell inequalities are satisfied.***

- **Bell non-locality is not revealed when only compatible measurements are employed — even with an entangled state.**

- **Wolf et. al., (Phys. Rev. Lett. 103, 230402 (2009)) have shown that incompatible measurements of a pair of POVMs with dichotomic outcomes are necessary and sufficient for the violation of Clauser-Horne-Shimony-Holt (CHSH) Bell inequality.**

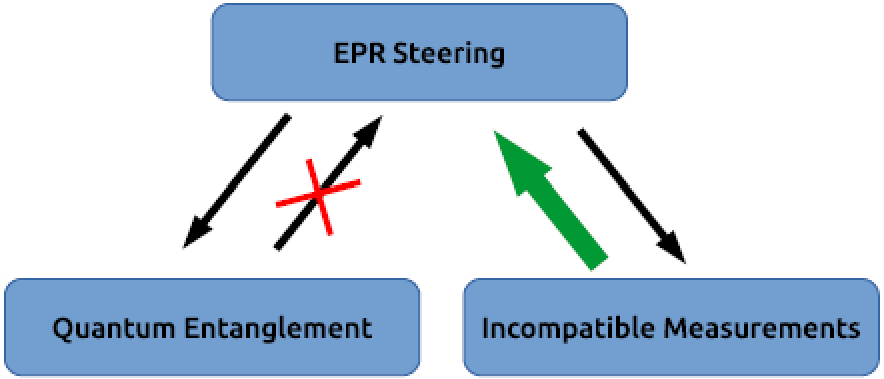
- **Equivalence between steering and joint measurability:**

  A set of POVMs are not compatible if and only if they can be employed for the task of non-local quantum steering.

  M. T. Quintino, T. Vértesi, and N. Brunner, Phys. Rev. Lett. 113, 160402 (2014)

  R. Uola, T. Moroder, and O. Gühne, Phys. Rev. Lett. 113, 160403 (2014))

Steering and Joint Measurability are synonymous



Steering implies both entanglement and incompatible measurements at Bob's end

M. T. Quintino, T. Vértesi, and N. Brunner, Phys. Rev. Lett. 113, 160402 (2014)
R. Uola, T. Moroder, and O. Gühne, Phys. Rev. Lett. 113, 160403 (2014)

But Bell non-locality and joint measurability not synonymous (except in the CHSH case).

N-term pairwise correlation inequalities & joint measurability?

Classical Moment Problem

➡ **Addresses the issue of finding a probability distribution given a set of moments.**

This brings forth the fact ➡

A given sequence of real numbers qualifies to be moment sequence of a legitimate probability distribution if and only if the associated moment matrix is positive.

Existence of joint probability distribution ⟺ **Moment matrix is positive**

J.A Sholat and J.D. Tamarkin, *The problem of moments,* AMS (1943)

N.J. Akhiezer, *The Classical Moment Problem*, Hofuer Publishing Co., (1965)

Classical Moment problem

Momen Inversion

↓

Are the probabilities moment determinate?

Moment Matrix Positivity

↓

Is the given sequence of "moments" admit *legitimate* probabilities?

H. S. Karthik, H. Katiyar, A. Shukla, T. S. Mahesh, A. R. Usha Devi and A. K. Rajagopal, Phys. Rev. A 87, 052118 (2013).

Joint measurability of observables ensures joint probabilities

&

Moment matrix positivity implies existence of joint probabilities

Joint measurability of observables ensures joint probabilities

&

Moment matrix positivity implies existence of joint probabilities

N-term pairwise correlation inequalities
&
joint measurability?

- **We construct chained N term pairwise correlation inequalities based on the positivity of a sequence of moment matrices.**
- **Question: Is moment matrix positivity necessary and sufficient for joint measurability?**

Chained correlation inequality

- **Consider N classical random variables X_k with outcomes $x_k = \pm 1$.**
- **Construct 4×4 moment matrices $M_k = \langle \xi_k \, \xi_k^T \rangle$ containing only pairwise moments of a set of three random variables each.**

$$\xi_k = \begin{pmatrix} 1 \\ x_1 \, x_k \\ x_k \, x_{k+1} \\ x_1 \, x_{k+1} \end{pmatrix}, \; k = 2, 3, \ldots N-1$$

and $\langle \cdot \rangle$ denotes expectation value.

Chained correlation inequality ...

The 4×4 moment matrix M_k has the form:

$$M_k = \begin{pmatrix} 1 & \langle X_1 \, X_k \rangle & \langle X_k \, X_{k+1} \rangle & \langle X_1 \, X_{k+1} \rangle \\ \langle X_1 \, X_k \rangle & 1 & \langle X_1 \, X_{k+1} \rangle & \langle X_k \, X_{k+1} \rangle \\ \langle X_k \, X_{k+1} \rangle & \langle X_1 \, X_{k+1} \rangle & 1 & \langle X_1 \, X_k \rangle \\ \langle X_1 \, X_{k+1} \rangle & \langle X_k \, X_{k+1} \rangle & \langle X_1 \, X_k \rangle & 1 \end{pmatrix}.$$

$k = 2, 3, \ldots, N-1 \Rightarrow N-2$ moment matrices

Chained correlation inequality . . .

The moment matrix is real, symmetric and positive semidefinite by construction.

Chained correlation inequality . . .

- **The eigenvalues $\mu_i^{(k)}; i = 1, 2, 3, 4$ of the moment matrix:**

$$\begin{aligned}
\mu_1^{(k)} &= 1 + \langle X_1\, X_k\rangle - \langle X_k\, X_{k+1}\rangle - \langle X_1\, X_{k+1}\rangle \\
\mu_2^{(k)} &= 1 - \langle X_1\, X_k\rangle + \langle X_k\, X_{k+1}\rangle - \langle X_1\, X_{k+1}\rangle \\
\mu_3^{(k)} &= 1 - \langle X_1\, X_k\rangle - \langle X_k\, X_{k+1}\rangle + \langle X_1\, X_{k+1}\rangle \\
\mu_4^{(k)} &= 1 + \langle X_1\, X_k\rangle + \langle X_k\, X_{k+1}\rangle + \langle X_1\, X_{k+1}\rangle.
\end{aligned}$$

- **Positivity of the moment matrix implies that the eigenvalues $\mu_i^{(k)}$ are positive.**

Chained correlation inequality ...

For a set of $N-2$ moment matrices $M_2, M_3, \ldots, M_{N-2}$, positivity condition $\displaystyle\sum_{k=2,3,\ldots,N-1} \mu_i^{(k)} \geq 0$, for the sum of eigenvalues $\mu_i^{(k)}$, $i = 1,2,3,4$ leads to four chained inequalities for pairwise moments:

$$\sum_{k=2}^{N-1} \langle X_k\, X_{k+1}\rangle + \langle X_1\, X_N\rangle - \langle X_1\, X_2\rangle \leq N-2$$

$$2\sum_{k=1}^{N-2} \langle X_1\, X_k\rangle - \sum_{k=2}^{N-1}\langle X_k\, X_{k+1}\rangle + \langle X_1\, X_N\rangle - \langle X_1\, X_2\rangle \leq N-2$$

$$\sum_{i=1}^{N-1} \langle X_i\, X_{i+1}\rangle - \langle X_1\, X_N\rangle \leq N-2$$

$$-\sum_{k=2}^{N-1} \langle X_k\, X_{k+1}\rangle - 2\sum_{k=2}^{N-2}\langle X_1\, X_{k+1}\rangle - \langle X_1\, X_N\rangle + \langle X_1\, X_2\rangle \leq N-2.$$

S. Wehner, Tsirelsen bounds for generalized Clauser-Horne-Shimony-Holt Inequalities, Phys. Rev. A 73, 022110 (2006)

PHYSICAL REVIEW A 73, 022110 (2006)

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Tsirelson bounds for generalized Clauser-Horne-Shimony-Holt inequalities

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(Received 18 November 2005; published 14 February 2006; publisher error corrected 28 February 2006)

Quantum theory imposes a strict limit on the strength of nonlocal correlations. It only allows for a violation of the Clauser, Horne, Shimony, and Holt (CHSH) inequality up to the value $2\sqrt{2}$, known as Tsirelson's bound. In this paper, we consider generalized CHSH inequalities based on many measurement settings with two possible measurement outcomes each. We demonstrate how to prove Tsirelson bounds for any such generalized CHSH inequality using semidefinite programming. As an example, we show that for any shared entangled state and observables $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$ with eigenvalues ± 1 we have $|\langle X_1Y_1\rangle+\langle X_2Y_1\rangle+\langle X_2Y_2\rangle+\langle X_3Y_2\rangle+\cdots+\langle X_nY_n\rangle-\langle X_1Y_n\rangle| \leqslant 2n\cos[\pi/(2n)]$. It is well known that there exist observables such that equality can be achieved. However, we show that these are indeed optimal. Our approach can easily be generalized to other inequalities for such observables.

2N-term CHSH-Bell inequality

N measurements each by Alice and Bob

Classical Bound:

$$\left|\sum_{i=1}^{n}\langle X_iY_i\rangle+\sum_{i=1}^{n-1}\langle X_{i+1}Y_i\rangle-\langle X_1Y_n\rangle\right|\leqslant 2n-2.$$

Quantum Tsirelsen Bound:

$$\left|\sum_{i=1}^{n}\langle X_iY_i\rangle+\sum_{i=1}^{n-1}\langle X_{i+1}Y_i\rangle-\langle X_1Y_n\rangle\right|\leqslant 2n\cos\left(\frac{\pi}{2n}\right)$$

Semidefinite Programming (SDP)

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C. Budroni, T. Moroder, M. Kleinmann, O. Günhe, Bounding temporal quantum correlations, Phys. Rev. Lett. 111, 020403 (2014)

PRL 111, 020403 (2013) PHYSICAL REVIEW LETTERS week ending 12 JULY 2013

Bounding Temporal Quantum Correlations

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(Received 15 March 2013; published 10 July 2013)

Sequential measurements on a single particle play an important role in fundamental tests of quantum mechanics. We provide a general method to analyze temporal quantum correlations, which allows us to compute the maximal correlations for sequential measurements in quantum mechanics. As an application, we present the full characterization of temporal correlations in the simplest Leggett-Garg scenario and in the sequential measurement scenario associated with the most fundamental proof of the Kochen-Specker theorem.

DOI: 10.1103/PhysRevLett.111.020403 PACS numbers: 03.65.Ud, 03.65.Ta

Classical Bound:

$$S_N = \sum_{i=1}^{N-1} \langle X_i X_{i+1} \rangle - \langle X_1 X_N \rangle \leq N - 2.$$

Quantum Tsirelsen Bound:

$$S_N \leq N \cos\left(\frac{\pi}{N}\right)$$

Semidefinite Programming (SDP)

Chained correlation inequality ...

(S. Wehner, Phys. Rev. A 73, 022110 (2006); C. Budroni, T. Moroder, M. Kleinmann, and O. Gühne, Phys. Rev. Lett. 111, 020403 (2013))

$$\mathcal{S}_N = \sum_{i=1}^{N-1} \langle X_i \, X_{i+1} \rangle - \langle X_1 \, X_N \rangle \leq N - 2.$$

- **For** $N = 3$**, we have** $\langle X_1 \, X_2 \rangle + \langle X_2 \, X_3 \rangle - \langle X_1 \, X_3 \rangle \leq 1$ **(3 term Leggett-Garg Inequality).**
- **For** $N = 5$**, we have** $\langle X_1 \, X_2 \rangle + \langle X_2 \, X_3 \rangle + \langle X_3 \, X_4 \rangle \langle X_4 \, X_5 \rangle - \langle X_1 \, X_5 \rangle \leq 3$ **(5 term LGI)**

Chained correlation inequality ...

- **We replace the classical random variables by a set of N dichotomic qubit observables**

$$X_k = \vec{\sigma} \cdot \hat{n}_k, k = 1, 2, \ldots, N$$

and the classical probability distribution by an arbitrary single qubit density matrix.

- **The pairwise moments**

$$\langle X_k \, X_l \rangle \equiv \langle X_k \, X_l \rangle_{\rm seq}$$

are obtained from sequential measurements of the observables – in the order in which they are written.

- **We obtain the chained inequality**

$$\mathcal{S}_N = \sum_{i=1}^{N-1} \langle X_i \, X_{i+1} \rangle_{\rm seq} - \langle X_1 \, X_N \rangle_{\rm seq} \leq N - 2.$$

Chained correlation inequality ...

C. Budroni, T. Moroder, M. Kleinmann, O. Günhe,
Bounding temporal quantum correlations,
Phys. Rev. Lett. 111, 020403 (2014)

- The Tsirelson bound (quantum bound), $N \cos\left(\frac{\pi}{N}\right)$ can be reached, when the system is prepared in a maximally mixed state $\rho = I/2$; and sequential projective measurements of qubit observables $\vec{\sigma} \cdot \hat{n}_k$, with unit vectors $\hat{n}_k$ equally separated by an angle π/N in a plane, one obtains pairwise correlations

$$\langle X_k \, X_{k+1} \rangle = \hat{n}_k \cdot \hat{n}_{k+1} = \cos\left(\frac{\pi}{N}\right)$$

and

$$\langle X_1 \, X_{k+1} \rangle = \hat{n}_1 \cdot \hat{n}_N = -\cos\left(\frac{\pi}{N}\right)$$

leading to the the Tsirelsen bound $N \cos\left(\frac{\pi}{N}\right)$.

Question: With optimal jointly measurable (unsharp) POVMs do we recover Tsirelson bound for N term inequalities?

This happens if joint measurability is both necessary and sufficient for violation of the N-term correlation inequality.

Measurement incompatibility ⟺ **Sequential correlation inequality??**

Chained correlation inequality ...

- **Consider fuzzy qubit POVMs**

$$\{E_k(x_k) = \frac{1}{2}(I + \eta\, x_k\, \vec{\sigma}\cdot\hat{n}_k), k = 1, 2, \ldots N\}$$

with successive unit vectors $\hat{n}_k$ separated by π/N in a plane.

- **We obtain**

$$\langle X_i\, X_{i+1}\rangle_{POVM} = \eta\,\langle X_i\, X_{i+1}\rangle_{\text{sharp}} = \eta\,\cos(\pi/N)$$

and

$$\langle X_1\, X_N\rangle_{POVM} = \eta\,\langle X_1\, X_N\rangle_{\text{sharp}} = -\eta\,\cos(\pi/N)$$

.

Joint measurability of qubit POVMs in the equatorial plane

- **The POVMs**

$$\{E_k(x_k) = \frac{1}{2}(I + \eta\, x_k\, \vec{\sigma}\cdot\hat{n}_k)$$

are all jointly measurable/compatible if the *unsharpness* parameter is less than the optimal value $0 \leq \eta \leq \eta_{\rm opt}$.

Number of POVMs	$\eta_{\rm opt}$
3	0.6666
4	0.6532
5	0.6472
6	0.6439
10	0.6392
20	0.6372
50	0.6367
100	0.6366
$\vdots$	$\vdots$
$N \to \infty$	$2/\pi$

Joint measurability of qubit POVMs in the equatorial plane

- **The POVMs**
$$\{E_k(x_k) = \frac{1}{2}(I + \eta\, x_k\, \vec{\sigma}\cdot\hat{n}_k)$$
are all jointly measurable/compatible if the *unsharpness* parameter is less than the optimal value $0 \leq \eta \leq \eta_{\rm opt}$.

No. of POVMs employed	Classical bound N-2	Quantum bound $N\ \cos\left(\frac{\pi}{N}\right)$	Maximum achievable value $\mathcal{S}_N^Q(\eta_{\rm opt})$
3	1	1.5	1
4	2	2.83	1.85
5	3	4.05	2.62
6	4	5.20	3.35
10	8	9.51	6.08
20	18	19.75	12.59
50	48	49.90	31.77
100	98	99.95	63.62

- **Incompatible measurements are necessary, but are not sufficient to violate the chained N term correlation inequality**

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Note

- Wolf et. al., (Phys. Rev. Lett. 103, 230402 (2009)): Incompatible measurements of a pair of POVMs with dichotomic outcomes are necessary and sufficient for the violation of standard 4 term Clauser-Horne-Shimony-Holt (CHSH) Bell inequality.

PHYSICAL REVIEW A 87, 052125 (2013)

Degree of complementarity determines the nonlocality in quantum mechanics

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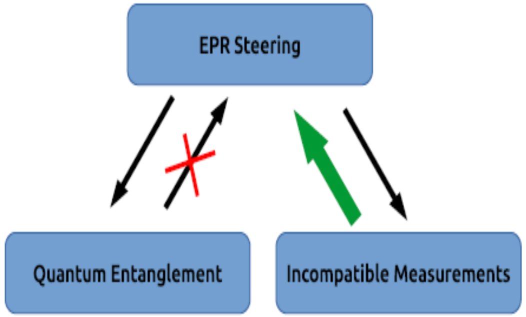
(Received 9 July 2012; published 20 May 2013)

Bohr's complementarity principle is one of the central concepts in quantum mechanics which restricts joint measurement for certain observables. Of course, later development shows that joint measurement could be possible for such observables with the introduction of a certain degree of unsharpness or fuzziness in the measurement. In this paper, we show that the optimal degree of unsharpness, which guarantees the joint measurement of all possible pairs of dichotomic observables, determines the degree of nonlocality in quantum mechanics as well as in more general no-signaling theories.

$$\eta_{\mathrm{opt}} = \frac{1}{\sqrt{2}} \Longleftrightarrow$$ **No violation of 4 term Bell-CHSH inequality**

Only a Special Case of correlation inequality resulting in Tsirelson Bound as a result of joint measurability

Steering and Joint Measurability are synonymous



M. T. Quintino, T. Vértesi, and N. Brunner, Phys. Rev. Lett. 113, 160402 (2014)
R. Uola, T. Moroder, and O. Gühne, Phys. Rev. Lett. 113, 160403 (2014)

N-term steering inequality & joint measurability

- Find a steering protocol such that incompatibility of equatorial qubit measurements is both necessary and sufficient

S. J. Jones and H. M. Wiseman, Phys. Rev. A. 84, 012110 (2011).

Linear steering inequality

- **Expectation value of a qubit observable**

$$\mathbf{S}_{\text{plane}} = \frac{1}{\pi}\int_0^{\pi} d\theta\, \alpha_\theta\, \sigma_\theta, \qquad -1 \leqslant \alpha_\theta \leqslant 1.$$

(where $\sigma_\theta = \sigma_x \cos(\theta) + \sigma_y \sin(\theta)$ $\rightarrow$ **equatorial qubit observable) is upper bounded by** $\frac{2}{\pi} \approx 0.6366.$

- **The Wiseman-Jones steering inequality:**

$$\frac{1}{\pi}\int_0^{\pi} d\theta\, \langle \sigma_\theta^A\, \sigma_\theta^B \rangle \leq \frac{2}{\pi}$$

$$\langle \sigma_\theta^B \rangle_{a_\theta} = \sum_{b_\theta = \pm 1} b_\theta\, p(b_\theta | a_\theta; \theta)$$

- **Violation of the inequality in any bipartite quantum state** ρ_{AB} **demonstrates non-local EPR steering phenomena.**

Linear steering inequality in the finite setting

With N evenly spaced equatorial measurements of σ_{θ_k} by Bob, conditioned by dichotomic outcomes a_k of Alice's measurements $\sigma^A_{\theta_k}$ $\Rightarrow$

$$\frac{1}{N}\sum_{k=1}^{N}\langle\sigma^A_{\theta_k}\,\sigma^B_{\theta_k}\rangle \quad \leq \quad f(N)$$

$$f(N) \quad = \quad \frac{1}{N}\left(\left|\sin\left(\frac{N\pi}{2}\right)\right| + 2\sum_{k=1}^{[N/2]}\sin\left[(2k-1)\frac{\pi}{2N}\right]\right).$$

- **$f(N)$ is the maximum eigenvalue of the observable $\frac{1}{N}\sum_{k=1}^{N}\sigma_{\theta_k}$.**
- **Largest value of $f(N)$ is $f(2)\approx 0.7071$ for $N=2$.**
  Smallest value $f(\infty)=0.6366$ when $N\to\infty$.

S. J. Jones and H. M. Wiseman, Phys. Rev. A. 84, 012110 (2011)

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ARTICLE

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Experimental proof of nonlocal wavefunction collapse for a single particle using homodyne measurements

Maria Fuwa[1], Shuntaro Takeda[1], Marcin Zwierz[2,3], Howard M. Wiseman[3] & Akira Furusawa[1]

A single quantum particle can be described by a wavefunction that spreads over arbitrarily large distances; however, it is never detected in two (or more) places. This strange phenomenon is explained in the quantum theory by what Einstein repudiated as 'spooky action at a distance': the instantaneous nonlocal collapse of the wavefunction to wherever the particle is detected. Here we demonstrate this single-particle spooky action, with no efficiency loophole, by splitting a single photon between two laboratories and experimentally testing whether the choice of measurement in one laboratory really causes a change in the local quantum state in the other laboratory. To this end, we use homodyne measurements with six different measurement settings and quantitatively verify Einstein's spooky action by violating an Einstein-Podolsky-Rosen-steering inequality by 0.042 ± 0.006. Our experiment also verifies the entanglement of the split single photon even when one side is untrusted.

Exptl verification of violation of 6 term finite setting steering inequality

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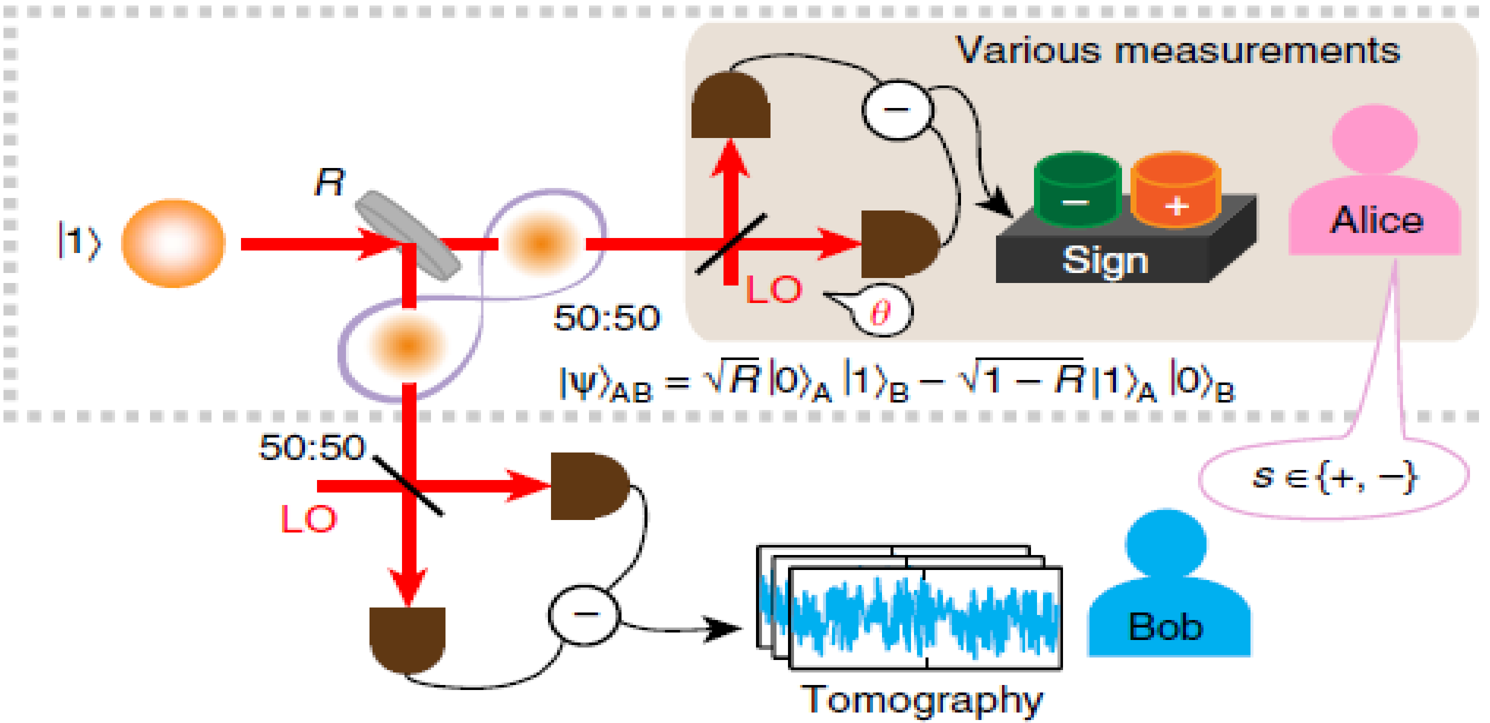
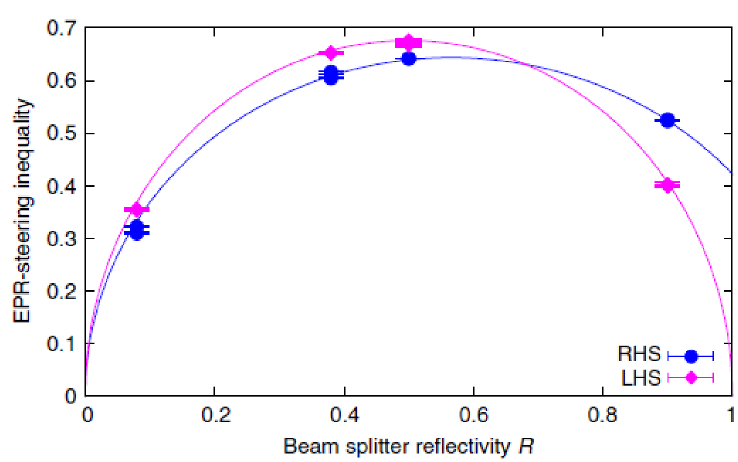
Blackbox to Bob in an EPR-steering test





PHYSICAL REVIEW A 95, 052105 (2017)

N-term pairwise-correlation inequalities, steering, and joint measurability

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Chained inequalities involving pairwise correlations of qubit observables in the equatorial plane are constructed based on the positivity of a sequence of moment matrices. When a jointly measurable set of positive-operator-valued measures (POVMs) is employed in the first measurement of every pair of sequential measurements, the chained pairwise correlations do not violate the classical bound imposed by the moment matrix positivity. We find that incompatibility of the set of POVMs employed in first measurements is only necessary, but not sufficient, in general, for the violation of the inequality. On the other hand, there exists a one-to-one equivalence between the degree of incompatibility (which quantifies the joint measurability) of the equatorial qubit POVMs and the optimal violation of a nonlocal steering inequality, proposed by Jones and Wiseman [S. J. Jones and H. M. Wiseman, Phys. Rev. A **84**, 012110 (2011)]. To this end, we construct a local analog of this steering inequality in a single-qubit system and show that its violation is a mere reflection of measurement incompatibility of equatorial qubit POVMs, employed in first measurements in the sequential unsharp-sharp scheme.

DOI: 10.1103/PhysRevA.95.052105

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Implications of joint measurability on the the finite setting N term linear steering inequality:

$$\eta_{\text{opt}} \leq f(N)$$

- A striking agreement between the degree of incompatibility η_{opt} and $f(N)$.
- This is another clear example of the intrinsic connection between steering and measurement incompatibility.

Connection between joint measurability and time-like steering in single system is discussed in

- **H. S. Karthik, J. Prabhu Tej, A. R. Usha Devi, and A. K. Rajagopal, J. Opt. Soc. Am. B. 32, A34 (2015)**
- **M. Pusey, J. Opt. Soc. Am. B. 32, A56 (2015).**

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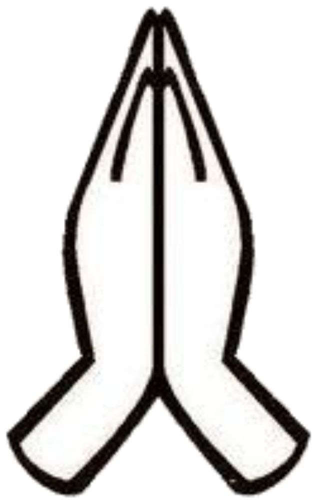

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