CONVEX RESOURCE THEORY OF NON-MARKOVIANITY

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MOTIVATION

Control and manipulation of characteristic quantum traits of any physical system are more often than not hindered by **decoherence**.

As a result, the system monotonically relaxes to thermal equilibrium, or generally, to a non-equilibrium steady state.

The one-way information flow characterized by the monotonic relaxation towards the stationary states is a direct consequence of the Born-Markov approximation.

For very large stationary environments, the **Born-Markov approximation** leads to the CP (Complete Positive)-divisibility of the dynamics.

However, when the system-environment coupling is not sufficiently weak, Born-Markov approximation is not valid.

Consequently the CP-divisibility may break down, leading to the observation of **Non-Markovian backflow of information**.

Non-Markovian backflow of information can act as a resource in various quantum mechanical scenario.

Information theory can be understood as examples of inter-conversion of various resources.

Since non-Markovianity can be converted into other resources via information backflow, there is a natural demand for a resource theoretic structure of this phenomenon.

Here we construct a convex resource theory of Non-Markovianity in the similar footings of that of entanglement, coherence, Athermality, or asymmetry.

Consequently, we develop a consistent framework of **Resource Theory of non-Markovianity** (RTNM) by characterizing its fundamental components.

The framework is valid for arbitrary finite dimensional single or any-partite system, satisfying all the basic ingredients of Resource Theory.

POINTS TO REMEMBER

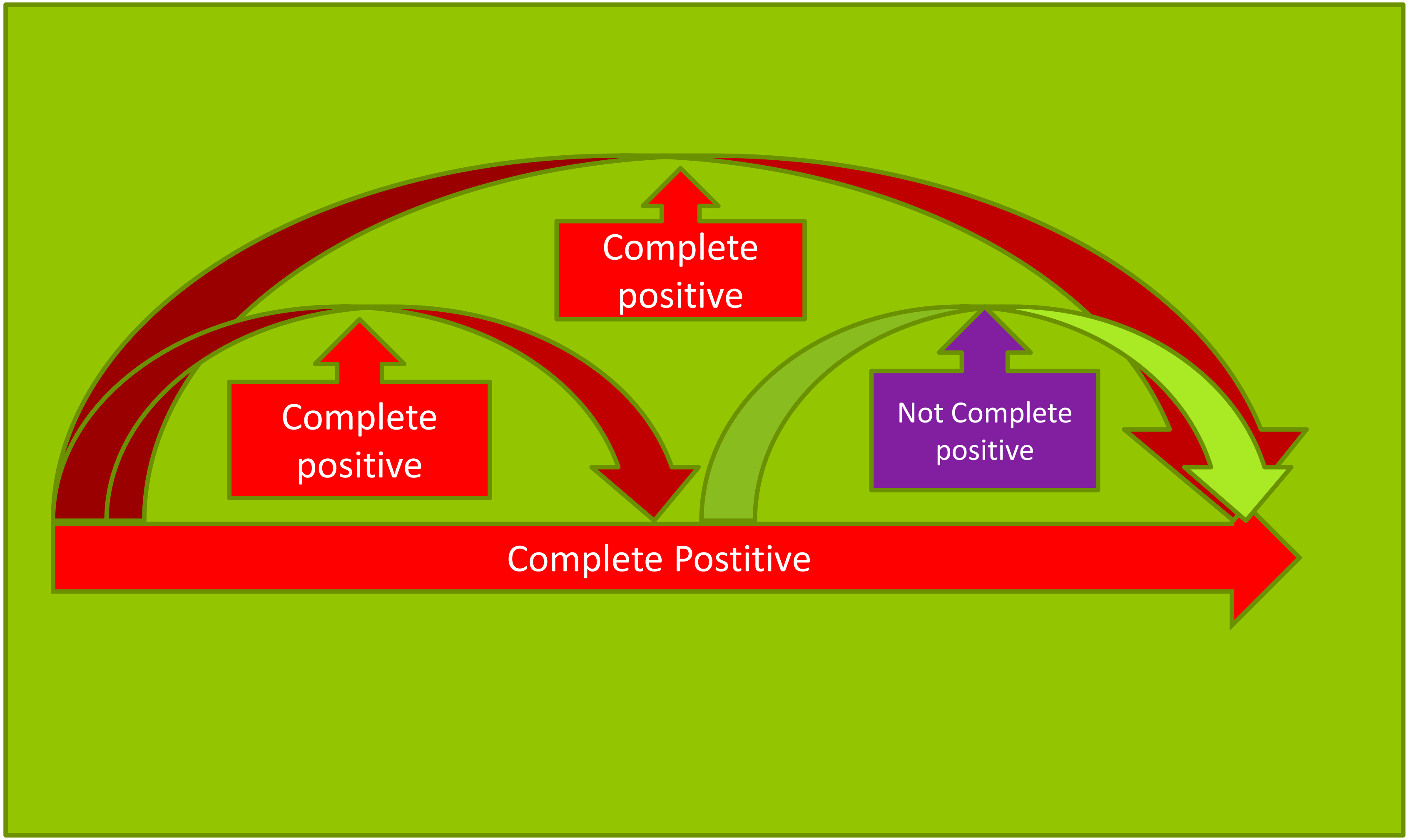
The definition of non-Markovianity we consider here, is based on divisibility breaking of a channel.

The classical definition of non-Markovianity is more general.

Though the notion of NM can be more general, its characterization in terms of indivisibility is sufficient for information backflow.

We are constrained within the class of quantum operations having a Lindblad type generator.

Non-Markovianity in terms of divisibility breaking of a channel



The characterization and quantification of the non-Markovianity is a fundamental aspect of open quantum dynamics.

RHP measure : In this approach, the non-Markovian behaviour is attributed to the deviation from divisibility and the quantification of non-Markovianity is done based on the amount of the deviation.

$$g_N(t) = \lim_{\epsilon \to 0^+} \frac{\|I_d \otimes \phi(t+\epsilon, t)|\Phi^+\rangle\langle\Phi^+|\|_1 - 1}{\epsilon}$$

$|\Phi^+\rangle$ is the maximally entangled state of $d \times d$ dimension.

LIMITATION

We are constrained within the quantum operations having Lindblad type generators of the form

$$\boldsymbol{\rho}(\boldsymbol{t_2}) = \boldsymbol{\Lambda}(\boldsymbol{t_2}, \boldsymbol{t_1})[\boldsymbol{\rho}(\boldsymbol{t_1})]$$

$$\int_0^T \Gamma(t)dt \geq 0$$

For all T

Where $\boldsymbol{\Lambda}(\boldsymbol{t}+\boldsymbol{\epsilon}, \boldsymbol{t}) = \mathbf{exp}[\int_{\boldsymbol{t}}^{\boldsymbol{t}+\boldsymbol{\epsilon}} \boldsymbol{L}(\boldsymbol{t}')\boldsymbol{dt}']$

$$\dot{\rho}(t) = L[\rho(t)]$$

$$= \sum_i \Gamma_{\mathrm{i}}(t)[A_i\rho(t)A_i^+ - \frac{1}{2}\{A_i^+A_i, \rho(t)\}]$$

BASIC STRUCTURE OF A CONVEX RESOURCE THEORY

- **Free States:** Set of States containing no resources
- **Free operations:** Allowed operations under which the corresponding resource does not increase and the free states are converted to free states.
- The set of free states is **convex**.
- **Measure:** Bona Fide measure of resource.

HOW TO CONSTRUCT FREE STATES ?

- Non-Markovianity is a phenomenon of quantum operations, independent of quantum states of the system.
- To construct a geometric picture, we use Channel-State duality to project the channels to a state.
- All the operations are represented by the corresponding Choi states.
- The Choi states corresponding to the Markovian (Divisible) operations are considered to be the free states.

The set of Free states:

$$\boldsymbol{F} = \{\boldsymbol{C}^{\boldsymbol{M}}(\boldsymbol{t} + \boldsymbol{\epsilon}, \boldsymbol{t}) \mid \|\, \boldsymbol{C}^{\boldsymbol{M}}(\boldsymbol{t} + \boldsymbol{\epsilon}, \boldsymbol{t})\, \|_{\boldsymbol{1}} = \boldsymbol{1} \,\forall\, \boldsymbol{t}, \boldsymbol{\epsilon}\}$$

Here $C^M(t+\epsilon, t) = I \otimes \Lambda_M(t+\epsilon, t)|\phi^+\rangle\langle\phi^+|$ is the Choi state corresponding to a divisible operation $\Lambda_M(t+\epsilon, t)$.

Serious problem

Divisible operations are not convex, therefore the set $\boldsymbol{F}$ is not a convex set.

An Important Proposition

In the limit $\epsilon \rightarrow 0$, the set of free Choi states $C^M(\mathrm{t}+\epsilon,\mathrm{t})$ form a convex set.

Proof: Consider two Markovian Channels

$$\Lambda_M^1 = \exp[\int_t^{t+\epsilon} L_M^1(t')dt'] \quad and \quad \Lambda_M^2 = \exp[\int_t^{t+\epsilon} L_M^2(t')dt'].$$

Invoking a small time interval approximation $\epsilon \rightarrow 0$, we have

$\Lambda_M^i \approx \mathrm{I} + \epsilon L_i(t)$ with i=1,2. Using the convexity of divisible Lindblad operations we find that

$$\boldsymbol{\Lambda_M = p\Lambda_M^1 + (1-p)\Lambda_M^2}$$

is also a Markovian channel .

Therefore, we retrieve convexity of the set of Markovian Choi states under this small time interval approximation. ■

THREE BASIC PROPOSITIONS

Resourceful states cannot be generated under tensor product, partial trace and permutations of spatially separated subsystems.

The set of free states F is a compact set.

Free operations cannot generate resourceful states.

GEOMETRIC MEASURE OF NON-MARKOVIANITY

- A proper measure of any resource can be constructed by the minimum contractive distance between a resourceful state and the set of free states.
- Here we have to consider the distance between a Markovian and a non-Markovian operation. Choi-Jamilkowski isomorphism reduces this problem to finding distance between corresponding Choi states.

The Measure can be

$$M(t+\epsilon, t) = \inf_{C_M} D(C_N \,||\, C_M)$$

Modified Measure of Non-Markovianity

- We consider the right derivative of M as the measure of non-Markovianity.

$$D_T(t) = \lim_{\epsilon \to 0} \frac{M(t+\epsilon, t) - 0}{\epsilon}$$

- Next step is to check whether this measure is appropriate.

PROPOSITION

$D_T(t)$ is a Bona Fide measure of non-Markovianity

D_T is faithful → $D_T \geq 0$ and D_T is zero iff for divisible operations.

D_T is convex → $D_T\left(\sum_i p_i\, C_N^i\right) \leq \sum_i p_i\, D_T(C_N^i)$

D_T is a monotonically decreasing function under free operations.

THEOREM

Let Λ_N be the dynamical map corresponding to an arbitrary non-Markovian operation N and $g^N(t)$ be the corresponding RHP measure. Then the non-Markovianity measure $D_T(t)$ is lower bounded by $g^N(t)$.

$$\boxed{\boldsymbol{D_T(t) \geq g^N(t)}}$$

It can be shown that $\boldsymbol{g^N(t)}$ is also a monotone under free operations.

Implication: Though $\boldsymbol{D_T(t)}$ contains optimization over all Markovian Choi states, it is lower bounded by optimization free measure of non-Markovianity.

Robustness of Non-Markovianity

Finally, we construct the concept of robustness of NM (RONM), in the similar footings of entanglement, coherence and asymmetry.

In accordance with the definitions of robustness for other quantum resources, we define RONM as the minimum amount of Markovian noise needed to be added to a NM evolution to make the resulting evolution Markovian.

$$\boldsymbol{R_N(t) = \inf_{s}\{s \geq 0 : \frac{C_N(t) + s\tau_N\ (t)}{1+s} = \delta_M(t) \in F\}}$$

where τ_N is an arbitrary element from the set of all Choi states A.

Interesting Implications

1. $R_N(t)$ is a Bona Fide measure of non-Markovianity.
2. $R_N(t)$ can also be expressed as:

$$\boxed{\boldsymbol{R_N(t) = \frac{(\,\|\, C_N\,(t)\,\|_1\, - 1\,)}{2}}}$$

3. Relation between $R_N(t)$ and RHP measure:

$$\boxed{\boldsymbol{R_N(t) = \frac{1}{2}\int g_N(t')dt'}}$$

CONCLUSIONS

- In this work, we have constructed a convex RTNM under the constraint of small time interval within the temporal evolution, which satisfies all the properties of Resource Theory.
- We have defined the divisible operations as the free operations, Choi states corresponding to the divisible operations as the free states and constructed a bona fide measure of NM.
- We have defined RONM and connected it directly to the RHP measure of non-Markovianity.
- The convex structure that we have constructed, enables us to address the following very important issues

  (A) **Linear Witnesses of Non-Markovianity**

  (B) **Non-Linear Witnesses**

  (C) **Catalytic non-Markovianity**

  (D) **Bound Non-Markovianity** !!!

Thank You