24/10/07

Gravitational Radiation

(Similar to the case of EM radiation - moving charge gives rise to EM rad.)

(We take weak field approxmn. — not unreasonable bcos we want to study grav. field away from the source)

We use weak field approximation:-

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$h_{\mu\nu} \to$ small

Einstein's eqns :-

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

$\Rightarrow$ [contracting with $g^{\mu\nu}$] $\to$ $-R = -8\pi G T^{\lambda}{}_{\lambda}$

$$\Rightarrow R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}{}_{\lambda}\right)$$

(For weak field approxmn. $\to$
① sources shouldn't be strong
② sufficiently far away from any source

Here let us first assume the source is weak & see how far we can go)

Keep terms linear in h on LHS & zeroth order in h on RHS $\to$ bcos we already have the energy-mom tensor

Ex: Check that we get

$$\Box h_{\mu\nu} - \frac{\partial^2}{\partial x^\lambda \partial x^\mu} h^\lambda{}_\nu - \frac{\partial^2}{\partial x^\lambda \partial x^\nu} h^\lambda{}_\mu + \frac{\partial^2}{\partial x^\mu \partial x^\nu} h^\lambda{}_\lambda = -8\pi G S_{\mu\nu}$$

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^\lambda{}_\lambda$$

All indices are raised & lowered by $\eta_{\mu\nu}$

Define:- $R^{(1)}_{\mu\nu} = \frac{1}{2}\Big[\Box h_{\mu\nu} - \frac{\partial^2}{\partial x^\lambda \partial x^\mu} h^\lambda{}_\nu - \frac{\partial^2}{\partial x^\lambda \partial x^\nu} h^\lambda{}_\mu + \frac{\partial^2}{\partial x^\mu \partial x^\nu} h^\lambda{}_\lambda\Big]$

Now, $T_{\mu\nu} = S_{\mu\nu} - \frac{1}{2} S^\lambda{}_\lambda \eta_{\mu\nu}$

$D_\mu T^{\mu\nu} = 0 \xrightarrow[\text{field limit}]{\text{weak}} \partial_\mu T^{\mu\nu} = 0$

$\Downarrow$

indices raised & lowered by η

(We want to solve) $\rightarrow$

$$\boxed{R^{(1)}_{\mu\nu} = -8\pi G S_{\mu\nu}}$$

(But this eqn also implies) $\rightarrow$

$$R^{(1)}_{\mu\nu} - \frac{1}{2} R^{(1)} \eta_{\mu\nu} = -8\pi G T_{\mu\nu}$$

$g_{\mu\nu} \to \eta_{\mu\nu}$

$\Downarrow$

$\eta^{\mu\nu} R^{(1)}_{\mu\nu}$

Consistency requires

$$\partial_\mu \left(R^{(1)\mu\nu} - \frac{1}{2} R^{(1)} \eta^{\mu\nu}\right) = 0$$

Ex: Check this using the explicit form of $R^{(1)}_{\mu\nu}$.

The fact that this happens is no surprise bcos this is essentially a consequence of Bianchi identity in the weak-field limit

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This is just to make sure that what we are doing is internally consistent

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Maxwell's eqns aren't sufficient to determine the time evolution of A_μ completely — this is bcos of gauge invariance — we fix this problem by choosing a gauge
→ not a physical ambiguity

Time evolution of $h_{\mu\nu}$ is ambiguous due to general coordinate trs. freedom.

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

If $x'^\mu = x^\mu + \epsilon^\mu(x)$,
then $g'_{\mu\nu}(x) - g_{\mu\nu}(x) = \left(\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu\right) + O(\epsilon^2)$

(In order for h & h' both to be small, we require ϵ_μ to be small)

Choose $\epsilon \sim h$ so that weak field → weak field under general coordinate transformation

Then $h'_{\mu\nu} = h_{\mu\nu} + \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) \cancel{+O(h\epsilon^2)} + O(h^2, h\epsilon, \epsilon^2, \ldots)$

EX. Check that LHS of linearized eqns is unchanged

We need to check that the op. on the LHS of the linearized eqn. acting on $(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)$ gives zero

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~~[struck out]~~ we can choose ϵ_μ such that first deriv. of ϵ_μ is zero at $t=0$ – so at $t=0$ h & h' coincide – time evolution of h aren't uniquely determined

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(Need 4 gauge-fixing condns)

Convenient choice of gauge

$$g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0 \qquad \boxed{\lambda = 0,1,2,3}$$

$$\xrightarrow[\text{field}]{\text{weak}} \frac{\partial}{\partial x^\mu} h^\mu{}_\lambda - \frac{1}{2}\frac{\partial}{\partial x^\lambda} h^\mu{}_\mu = 0$$

Ex. Check that in weak field approximation this becomes

$$\frac{\partial}{\partial x^\mu} h^\mu{}_\lambda - \frac{1}{2}\frac{\partial}{\partial x^\lambda} h^\mu{}_\mu = 0$$

The eqn of motion reduces to

$$\Box h_{\mu\nu} = -16\pi G S_{\mu\nu}$$

component by component it is the same as Maxwell's eqns

$\partial^{\mu\nu}$ $\partial_\mu \partial_\nu$

Conservation law for $T_{\mu\nu}$:-

$$\partial_\mu T_{\mu\nu} = 0$$

$$\Rightarrow \partial_\mu S^\mu{}_\lambda - \tfrac{1}{2}\partial_\lambda S^\mu{}_\mu = 0$$

Apply the op. of the LHS of gauge-fixing condition on both sides of

$$\Box h_{\mu\nu} = -16\pi G S_{\mu\nu}$$

As a result of consvn. eqn. for $S_{\mu\nu}$, the gauge-fixing condn is automatically satisfied

Otherwise we would have put some more constraints so as to pick out only those solns which satisfy gauge condns — only these solns would then be relevant

$\Box h_{\mu\nu} = -16\pi G S_{\mu\nu}$ — Ⓐ is a linear eqn

If we can solve for

$\Box h_{\mu\nu} = 0$ — Ⓑ which is a homogeneous eqn,

any soln of Ⓑ can be added to the soln of Ⓐ — we add solns of Ⓑ to adjust the boundary condns

$$\Box h_{\mu\nu} = 0$$

Try

$$h_{\mu\nu} = e_{\mu\nu} e^{ik\cdot x} + e^{*}_{\mu\nu} e^{-ik\cdot x}$$

could be complex → means allowing for a phase

$$\Box h_{\mu\nu} = 0 \Rightarrow -\eta^{\mu\nu} k_\mu k_\nu \left(e_{\mu\nu} e^{ik\cdot x} + e^{*}_{\mu\nu} e^{-ik\cdot x}\right) = 0$$

$$\Rightarrow k^2 = 0 \quad \text{or} \quad e_{\mu\nu} = 0$$

For a non-trivial soln, $k^2 = 0$

$$\Rightarrow k^0 = \pm |\vec{k}|$$

$\Rightarrow$ gravity wave travels at the speed of light

[Classically we can say this much — notion of mass comes only when we quantise the system]

(Now we have to see which of the $e_{\mu\nu}$ satisfies the gauge condn.) →

$$i\left[k_\mu e^\mu{}_\lambda - \frac{1}{2} k_\lambda e^\mu{}_\mu\right] = 0$$

(the complex conjugate part should be indep. zero)

↓

gives us 4 condns on $e_{\mu\nu}$

To start with we had 10 ~~complex~~ $e_{\mu\nu}$'s.

So we have $(10-4) = 6$ $e_{\mu\nu}$'s

① In the Maxwell case we had $k^2 = 0$

$k \cdot \epsilon = 0$ (gauge condn)

② $\epsilon^\mu \to \epsilon^\mu + c k^\mu$

$(\epsilon^\mu + c k^\mu) k_\mu = 0$

→ Residual gauge freedom

A_μ has 4 components

By ① we go from 4 to 3.

By ② we go from 3 to 2 d.o.f.

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$$

let us take $\epsilon_\mu = i \zeta_\mu e^{ik.x}$

$$\therefore e_{\mu\nu} \to e_{\mu\nu} - k_\mu \zeta_\nu - k_\nu \zeta_\mu$$

$$\equiv e'_{\mu\nu}$$

Q) Does $e'_{\mu\nu}$ satisfy the gauge condn if $e_{\mu\nu}$ does?

<u>Ex:</u> Check that it does if $k^2 = 0$.

There is an analog of Gauss law for this case — this vanishing in an indirect conseqence of the Gauss law.

$6-4=2$ independent $e_{\mu\nu}$'s

$\Rightarrow$ 2 polarizations of the gravitational wave.

Take $\vec{k}=(0,0,k^3)$, $k^0=k^3$

Ex. Show that the gauge conditions give:

$$e_{01}=-e_{31}$$
$$e_{02}=-e_{32}$$
$$e_{22}=-e_{11}$$
$$e_{03}=-\tfrac{1}{2}(e_{00}+e_{33})$$

Ex. Show that using $\Delta e_{\mu\nu}=-k_\mu \xi_\nu - k_\nu \xi_\mu$ we can remove all components except e_{11} & e_{12} & those determined by gauge condition.

We will work out the analog of circularly polarized waves for gravity b'cos these have <u>definite helicities</u> ($+1$ or -1) rather than the plane-polarized waves described by e_{11} & e_{12}

Circularly polarized wave picks up a phase $e^{ih\theta}$ under a rotation by θ around the direction of propagation.

h :- a constant $\rightarrow$ helicity

e.g. $\rightarrow$ Maxwell field along x^3-dim. *

$$(\epsilon^1 \pm i\epsilon^2) e^{ik.x}$$

$$\xrightarrow[\text{rotation by } \theta \text{ about 3-axis}]{\text{under}} e^{\pm i\theta} (\cdots)$$

So helicity of cir. pol. Maxwell field is $+1$ or -1

Ex. Check that

$$(e_{11} \mp i e_{12}) e^{ik.x} \quad \text{describe}$$

circularly polarized wave with $h = \pm 2$.

$\rightarrow$ picks up phase of $e^{\pm 2i\theta}$ under rotation by θ about the 3-axis.

(in order to prove this, we need to ~~do~~ use gauge condn.)

[* Circularly polarized waves are the eigenstates of the rotn. op. (rotn. about the 3-axis) - it just picks up the phase $e^{ih\theta}$ — all plane waves must form repn. of the rotn. grp. .. $\rightarrow$ under rotn. op. they go into linear combns of plane waves

Gauss law condn. $\Rightarrow$ The LHS generates gauge equiv. configurations

25/10/07

Energy-momentum tensor of gravity waves

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G \, T^{matter}_{\mu\nu}$$

(clearly, we can't add $T_{\mu\nu}^{gravity}$ on the RHS bcos all info regarding gravitational wave is on the LHS — One might think thus $T_{\mu\nu}^{gravity}$ is zero — we need only $T_{\mu\nu}^{matter}$
→ But this is not the most convenient way to do things)

Now, $$D_\mu \left(T^{matter}\right)^{\mu\nu} = 0$$

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left[\Box h_{\mu\nu} - \frac{\partial^2}{\partial x^\lambda \partial x^\mu} h^\lambda{}_\nu - \frac{\partial^2}{\partial x^\lambda \partial x^\nu} h^\lambda{}_\mu + \frac{\partial^2}{\partial x^\mu \partial x^\nu} h^\lambda{}_\lambda\right]$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

$$= R^{(1)}_{\mu\nu} - \frac{1}{2} R^{(1)} \eta_{\mu\nu} + \text{terms quadratic \& higher order in } h$$

Einstein's eqn.

$$R^{(1)}_{\mu\nu} - \frac{1}{2} R^{(1)} \eta_{\mu\nu} = -8\pi G \left(T_{\mu\nu} + \text{terms quadratic \& higher order in } h\right)$$

$T_{\mu\nu}^{grav}$

This defn. is useful when spacetime is asymptotically flat

$T_{\mu\nu}^{grav}$ can be defined even if weak field limit isn't applicable
[$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$]

$$\partial_\mu \left(R^{(1)\mu\nu} - \frac{1}{2} R^{(1)} \eta^{\mu\nu}\right) = 0$$

$$\eta^{\mu\rho}\eta^{\nu\sigma} R^{(1)}_{\rho\sigma}$$

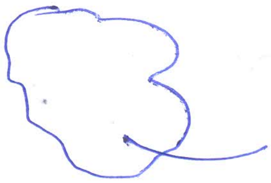
$$\partial_\mu \left(T^{\text{matter}} + T^{\text{grav}} \right)^{\mu\nu} = 0$$

indices raised & lowered by η

(consequence of the old eqn (not a new eqn)
Bianchi identity)

Ex: Check this explicitly

(Thus we can forget about the geometric description of gravity altogether & think it as a field described by $h_{\mu\nu}$ (rank-2 tensor)
So in asym. flat spacetime gravity can be thought of as a rank-2 field in a flat spacetime with a complicated energy-mom. tensor)



Time dependent source of gravity

General feature
→ produces gravity waves at infinity

$$\Box h_{\mu\nu} = \text{source term}$$

(Bcos gravity waves carry energy & bcos total energy of gravity waves + matter is conserved, we expect energy to be carried away by the gravity waves)

Energy momentum tensor of gravity waves

$$h_{\mu\nu} = e_{\mu\nu} e^{ik\cdot x} + e^*_{\mu\nu} e^{-ik\cdot x}$$

Gauge condition + residual gauge invariance

$\rightarrow$ e_{11} & e_{12} are the independent components

Ex. $\left\langle T^{\text{grav}}_{\mu\nu} \right\rangle = \frac{k_\mu k_\nu}{8\pi G}\left(|e_{11}|^2 + |e_{12}|^2\right)$

for wave moving along x^3

average over space or time

$\because$ we have to invert the metric, there will be an infinite series involving all powers of $h_{\mu\nu}$ in $T_{\mu\nu}$

(terms are like $e_{\mu\nu} \times e^*_{\mu\nu}$, $(e_{\mu\nu})^2 e^{2ik\cdot x}$, $(e^*_{\mu\nu})^2 e^{-2ik\cdot x}$ $\rightarrow$ vanish on averaging)

assuming weak field, we keep upto quadratic terms

$$\left\langle T^{\text{grav}}_{\mu\nu} \right\rangle = \frac{k_\mu k_\nu}{8\pi G}\left(|e_{11}|^2 + |e_{12}|^2\right)$$

$$= \frac{k_\mu k_\nu}{16\pi G}\left(|e_+|^2 + |e_-|^2\right)$$

[where $e_\pm = e_{11} \pm i e_{12}$]

Detection of gravity waves

① Direct detection :-

$e_{22} = -e_{11}$

(information about distance betn 2 pts. is encoded in $h_{\mu\nu}$)

Gravity is a measure of distance

change of distance betn 2 pts. — Detection of oscillation of $h_{\mu\nu}$

$$h_{11} = e_{11} e^{ik.x} + e^*_{11} e^{-ik.x}$$

$$h_{22} = -\left(e_{11} e^{ik.x} + e^*_{11} e^{-ik.x}\right)$$

$$= -h_{11}$$

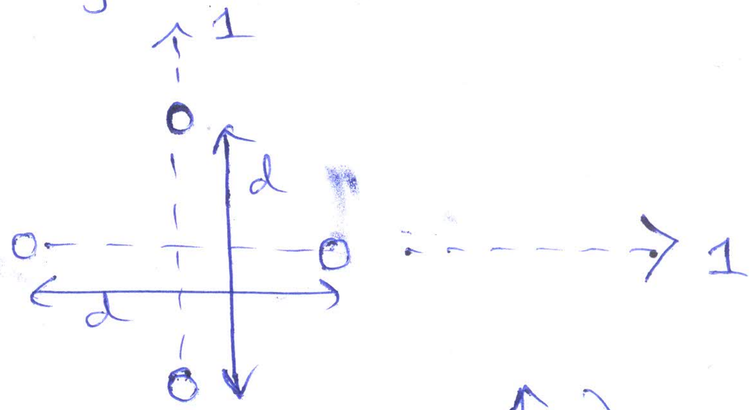
[So if distan in one dirn. is expanding, distance in the other dirn. is ~~const~~ contracting, bcos

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

For $h_{\mu\nu} > 0$, distance betn. 2 pts. is more than that measured by $\eta_{\mu\nu}$

. . . . .]

Take 4 objects



(In absence of gravity waves the 2 distances are identical)

In presence of gravity waves, there is an oscillatory behavior of the diff. of 2 distances — though measurement of distances is difficult, change in distance (though small) can be detected by interference exp.

The diff. can be ↑ by increasing d bcos

$$d = (\eta_{\mu\nu} + h_{\mu\nu}) \Delta x^\mu \Delta x^\nu$$

For same change in $h_{\mu\nu}$, we can get bigger changes for larger Δx^{μ} → change can be seen easily

→ But so far gravity waves haven't been detected !!

Most sources of grav. waves are in outer space

→ e.g. ① explosion of a star can produce grav. disturbances

② Main candidate → Black holes

whole star or planet falling into a black hole

In the geometric picture, particle moves in a geodesic in a curved metric, but it changes the geometry in the process (affects gravity)

— in the non-geometric picture, the effect of presence of matter on gravity is made more concrete by saying accln of particle ~~produces~~ grav. radiation. (a convenient way to interpret Einstein's eqns by speaking of grav. waves)

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② Indirect Detection

$$\Box h_{\mu\nu} = -16.\pi G S_{\mu\nu}$$

(For a given time-dep. source, we can figure out from this eqn how much energy is lost by rad. like in E.M)

⇓ tells us

what gravitational field is produced

⇓

How much gravitational wave is present at ∞

⇓

Rate of loss of energy from the source

to figure out the loss of energy by particle, we have to identify the rad. part — like over & above the Coulomb field, we have time-dep. part in E.M

(Look at the asym. region & identify the rad. part — Integrate the flux over a sphere at ∞ to get the rate of loss of energy) — We have to subtract out the, say, Schwarzschild part to get the radiation part, as we need to subtract out the Coulomb part in case of EM —

Now if we can exp. measure the energy lost by the source, we can compare it with the above theoretical prediction

$\longrightarrow$ Indirect measurement

(Kind of source one uses) $\rightarrow$

Binary pulsars

Pulsar $\rightarrow$ emits pulses (radio waves) at regular intervals

Sometimes we see the period itself changes with a certain period

$\downarrow$ interpretation

pulsar

2 objects rotate about each other

$\rightarrow$ distance of pulsar from us changes periodically acc. to the period of rotn.

Period of the binary pulsar ~~is measurable~~ can be measured accurately.

(On the other hand, this is an accelerated system — expect it to emit grav. radn.)

Due to being in circular orbit the system will emit gravity waves

$\longrightarrow$ lose energy (calculable)

(From various other exp.'s, if we know the masses of the 2 objects, we can calculate the energy loss from the period of rotn.)

There is a 3rd periodicity → due to loss of energy distance betn. the 2 ↓ & freq. of rotn. ↑ (period ↓)

Rate of change of period (which is slow) exp. obsvd. (over a long time) agrees with theory

→ In a sense a verification of Einstein's eqns.

Tetrad (Vierbein) formalism of gravity

(This formalism can be used for ordinary gravity but in addition)

→ can be used for ordinary gravitational system (there it's not necessary bcos we already have a formalism)

→ essential when we try to couple gravity to fermion fields

Coupling gravity to classical field theories

Replace d^4x by $d^4x\sqrt{-\det g}$

(to make the measure general coord. inv.)

$\eta_{\mu\nu} \xrightarrow[\text{by}]{\text{replace}} g_{\mu\nu}$

$\partial_\mu \longrightarrow D_\mu$

gives a consistent coupling of grav. to ordinary field th.

Examples

Also add the Einstein–Hilbert action $\Rightarrow$

$$-\frac{1}{16\pi G}\int\sqrt{-\det g}\,R$$

① ϕ^4 theory

$$\int d^4x\left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4!}\phi^4\right]$$

$$\longrightarrow\int d^4x\,\sqrt{-\det g}\left[-\frac{1}{2}g^{\mu\nu}D_\mu\phi\,D_\nu\phi-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4!}\phi^4\right]$$

② **Maxwell** :-

$$-\frac{1}{4}\int d^4x\,\sqrt{-\det g}\;g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}$$

$$F_{\mu\nu}=D_\mu A_\nu-D_\nu A_\mu$$

③ YM theory :-

$$-\frac{1}{2g_0^2}\int d^4x\,\sqrt{-\det g}\;g^{\mu\rho}g^{\nu\sigma}\,\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma})$$

Remember

Write down the starting action & manipulate to add the effect of gravity by replacing ∂_μ by D_μ $\Rightarrow$

e.g. $(\partial_\mu-ieA_\mu)\phi$

$\downarrow$

$(D_\mu-ieA_\mu)\phi$

8) What about fermion fields?

$\longrightarrow S_\psi = \int \bar{\psi} \, (i\gamma^\mu \partial_\mu - m) \, \psi \; d^4x$

(We know how to get covariant deriv. acting on <u>tensor</u> fields — but what about <u>spinors</u>?)

Try $\quad \psi'_\alpha(x') = \psi_\alpha(x)$

Then $\partial'_\mu \psi'_\alpha(x') = \dfrac{\partial x^\rho}{\partial x'^\mu} \partial_\rho \psi_\alpha(x)$

forget that the subset of Lorentz trs. doesn't keep ψ_α - inv.
$\downarrow$
trs. as a spinor

Treat ψ_α as a scalar field

Now $\quad d^4x \longrightarrow d^4x \sqrt{-\det g}$

$\bar{\psi}'(x')(i\gamma^\mu \partial'_\mu - m) \, \psi'(x')$

$= \bar{\psi}(x) \left(i\gamma^\mu \dfrac{\partial x^\rho}{\partial x'^\mu} \partial_\rho \psi(x) - m\psi(x) \right)$

(One might have thought that trs. of γ^μ will cancel this trs. — but γ^μ's are fixed matrices fixed once & for all)

(Recall:- In Lorentz trs., we had ψ & $\bar{\psi}$ transforming $S^{-1}\gamma^\mu S$ mixes the γ^μ's
$\rightarrow$ they don't actually transform
But here we have no S (ψ_α's are treated as scalars)

<u>Clue</u> $\rightarrow$ somehow we would like to treat γ^μ's as vectors

(Introducing a field $E^{\mu}{}_{a}$ we make it generally cov. inv.)

$$\underbrace{\gamma^{\mu} \to E^{\mu}{}_{a}}_{\text{fields}}\, \gamma^{a}$$

$\gamma^a \to$ fixed matrices
$a = 0, 1, 2, 3$

$$E'^{\mu}{}_{a}(x') = \frac{\partial x'^{\mu}}{\partial x^{\sigma}} E^{\sigma}{}_{a}(x) \quad \text{(trs. as a rank 1 tensor)}$$

(we are introducing 4 vector fields
$\to$ so they must be expressible in terms of the metric (new fields can't be introduced as they don't exist in the flat metric limit)

$\Psi_\alpha \to$ scalar under general coord. trs.
This will be made true for all tensor fields also $\xrightarrow{\text{by}}$

$A_\mu \longrightarrow$ converted into a scalar
$$\hat{A}_a = E^{\mu}{}_{a} A_\mu$$

(all basic fields will be scalars under general coord. trs. <u>except $E^{\mu}{}_{a}$</u>
$\to$ only field that trs.

— So tensor fields are treated on same footing as Ψ_α except there is no analog of $E^{\mu}{}_{a}$ for fermions.

— Decouple the Lor. trs. prop. by introducing $E^{\mu}{}_{a}$

(Diss. op.) $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ – (linear)

Even for static matter, $T_{\mu\nu}^{grav.}$ is non-zero — due to self-coupling grav. acts as its own source.

$T_{\mu\nu}^{grav.}$ isn't dep. on ~~the fact that~~ time-indep.

<u>Maxwell</u> $\partial_\mu F^{\mu\nu} = J^\mu$

↳ no source from field

<u>Non-abelian</u>

source from field by taking non-linear terms on RHS

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e.g. $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$

Take non-linear terms on RHS

→ gravity produces gravity

26/10/07 ~~Gravita~~ Tetrad/Vierbein formalism

$g_{\mu\nu}(x)$

y: some point in this space

Can choose a set of coordinates

$\xi^a(x;y)$ so that in this coordinate system

the metric is η_{ab} $(a, b = 0, 1, 2, \ldots, 3)$

This coordinate won't make the metric flat at a generic pt. → only at a specific pt. y

$$g_{\mu\nu}(x) = \frac{\partial \xi^a(x;y)}{\partial x^\mu} \frac{\partial \xi^b(x;y)}{\partial x^\nu} g^{\xi}_{ab}(x;y)$$

η_{ab} at $x = y$

away from $x = y$ we don't know much about this

$$g_{\mu\nu}(y) = \underbrace{\frac{\partial \xi^a(x;y)}{\partial x^\mu}\bigg|_{x=y}}_{(\text{fn. of } y \text{ only})} \underbrace{\frac{\partial \xi^b(x,y)}{\partial x^\nu}\bigg|_{x=y}}_{e_\nu^{\ b}(y)} \eta_{ab}$$

|| define $e_\mu^{\ a}(y)$

$$\boxed{g_{\mu\nu}(y) = e_\mu^{\ a}(y)\, e_\nu^{\ b}(y)\, \eta_{ab}}$$

(knowing $e_\mu^{\ a}$, the metric can be reconstructed from this info — so instead of having the metric as the basic d.o.f., we can use $e_\mu^{\ a}$ as the basic d.o.f.)

→ We could use $e_\mu^{\ a}(y)$ for different values of y as the basic set of fields describing gravity.

$e_\mu^{\ a}$ has 16 comp. while $g_{\mu\nu}$ has 10 comp. — so $e_\mu^{\ a}$ contains more info than a th. of gravity should

Under a General coordinate transformation

~~We want~~ $\qquad x \longrightarrow x' \qquad\qquad y \to y'$

$$\cancel{e'^{a}_{\mu}(y') = \frac{\partial \xi^a}{\partial x'^\mu}\Big|_{x'=y'}}$$

$\frac{\partial f(x)}{\partial x}\Big|_{x=y} = \frac{\partial f(y)}{\partial y}$

We want to use

$$\boxed{\xi'^{a}(x', y') = \xi^{a}(x, y)}$$

$\frac{\partial f^\rho(x')}{\partial x'^\mu}\Big|_{x'=y'} = \frac{\partial f^\rho(y')}{\partial y'^\mu}$

Then $e'^{a}_{\mu}(y') = \frac{\partial \xi'^{a}(x', y')}{\partial x'^{\mu}}\Big|_{x'=y'}$

$$= \left\{\frac{\partial \xi^a(x,y)}{\partial x^\rho}\,\frac{\partial x^\rho}{\partial x'^\mu}\right\}\Big|_{x'=y'} \qquad \frac{\partial x^\rho}{\partial x'^\mu} = \frac{\partial f^\rho(x')}{\partial x'^\mu}$$

$\Downarrow$ $x = y$

$$= \frac{\partial \xi^a(x,y)}{\partial x^\rho}\Big|_{x=y} \frac{\partial y^\rho}{\partial y'^\mu}$$

$$= \frac{\partial y^\rho}{\partial y'^\mu}\, e^{a}_{\rho}(y) \qquad \frac{\partial f^\rho(y')}{\partial y'^\mu}$$

trs. as a covariant vector
$\Downarrow$
no surprise bcos ξ^a is like a scalar

$x^\rho = f^\rho(x')$

(Physically, e^a_μ has more comp. bcos choice of e^a_μ isn't unique — freedom is that of Lorentz trs.)

The extra freedom in the choice of $e^a_\mu(y)$

$\xleftrightarrow[\text{to}]{\text{related}}$ freedom of Lorentz trs. on $\xi^a(x;y)$

$$\xi^a(x;y) \longrightarrow \Lambda^a{}_b(y)\, \xi^b(x;y)$$

$\Downarrow$
Lorentz trs. matrix

$$\boxed{\Lambda \eta \Lambda^T = \eta}$$

$$g_{\mu\nu}(x) \to \Lambda^a{}_c(y)\, \Lambda^b{}_d(y)\, \frac{\partial \xi^c(x;y)}{\partial x^\mu} \frac{\partial \xi^d(x,y)}{\partial x^\nu}\, g^{\xi}_{ab}(x,y)$$

$\swarrow$

η_{cd}

Why ξ^a's are treated as scalars?

$\to$ For $x=y$, choose ξ^a

Now perform $x \to x'$ so that $y \to y'$

Now, $y \neq y'$ (numerically), but they describe the same pt.

But ξ^a make the metric flat at $x=y$ (or $x'=y'$)

So $x \to x'$ shouldn't affect the value of ξ^a at y (or equiv. y' in the x'-system) $\Rightarrow$ Hence ξ^a should be SCALARS.

$$e_a^{\mu}(y) \longrightarrow \Lambda^a{}_b(y)\, e^b{}_\mu(y) \quad \text{— (A)}$$

$$e^a_\mu(y) \longrightarrow \Lambda^a{}_b(y)\, e^b_\mu(y)$$

$$g_{\mu\nu} \to \Lambda^a{}_c(y)\, e^c{}_\mu(y)\, \Lambda^b{}_d(y)\, e^d{}_\nu(y)\, \eta_{ab}$$

Using $\eta_{ab} \Lambda^a{}_c(y) \Lambda^b{}_d(y) = \eta_{cd}$

we get

$$g_{\mu\nu} \to \eta_{cd}\, e^c_\mu(y)\, e^d_\nu(y) = g_{\mu\nu}$$

[$\therefore$ the trs. (A) are the set of trs. under which $g_{\mu\nu}$ goes to η_{ab}]

$\Lambda^a{}_c(y)$ has 6 parameters.

$\longrightarrow$ like a gauge symmetry

(bcu it is a Lorentz trs., but dep. on the specific pt. y)

Need to impose 6 gauge condns which ~~makes~~ gives 10 indep. components

— we'll however work with 16 comp. & gauge fix when necessary (just like in Q.F.T)

Note :- ~~The original action should also have this gauge inv. though may not be manifest~~

The action is gauge-inv.

New defn

$E_a{}^\mu$:- matrix inverse of e

$$E^\mu{}_a \, e_\mu{}^b = \delta_a{}^b$$

think of this as a matrix eqn.

The fact that left & right inverses are the same implies

$$\Rightarrow E^\mu{}_a \, e^a{}_\nu = \delta^\mu{}_\nu$$

$$\boxed{\begin{aligned} & e^b{}_\mu E^\mu{}_a = \delta_a{}^b \quad \text{Right inverse} \\ & \text{so } E^\mu{}_a e^a{}_\nu = \delta^\mu{}_\nu \quad \text{left inverse} \end{aligned}}$$

Ex: S.T. $g^{\mu\nu} = E^\mu{}_a E^\nu{}_b \eta^{ab}$ (use the fact that this can be seen as a matrix eqn)

trs. as a cov. vector under a local Lor. trs.

$E^\mu{}_a$ local Lorentz trs.