$$e^a{}_\nu \longrightarrow \Lambda^a{}_b\, e^b{}_\nu$$

$$\Rightarrow \quad e \longrightarrow \Lambda e$$

Now $\quad E = e^{-1} \longrightarrow e^{-1}\Lambda^{-1}$

$$= \hat{E}\,\Lambda^{-1}$$

$$\therefore \quad E^\mu{}_a \xrightarrow[\text{Lorentz trs.}]{\text{local}} E^\mu{}_b\,(\Lambda^{-1}(y))^b{}_a$$

---

suppose we have a ~~ten~~ vector field $A_\mu(x)$

Define:- $\hat{A}_a = E^\mu{}_a(x)\,A_\mu(x)$

(we will now treat this as the indep. variable instead of $A_\mu(x)$)

We could take $\hat{A}_a$ as indep. variables.

$\hat{A}_a$ is scalar under general coordinate trs.

$$\hat{A}_a = E^\mu{}_a(x)\,A_\mu(x) \xrightarrow[\text{Lorentz trs.}]{\text{local}} E^\mu_b(x)\,A_\mu(x)\,(\Lambda^{-1}(x))^b{}_a$$

$$= \hat{A}_b(x)\,(\Lambda^{-1}(x))^b{}_a$$

$\therefore$ $\hat{A}_a$ transforms as a covariant vector under local Lorentz trs.

Given A^μ, we define:-

$$\hat{A}^a = e^a_\mu\, A_\mu$$

$\longrightarrow$ scalar under general coord. trs. & contravariant under local Lorentz trs.

$$\hat{A}^a(y) \longrightarrow \Lambda^a{}_b(y)\, \hat{A}^b(y)$$

In general :-

Given $A^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m}$

we define :-

$$\hat{A}^{a_1 \dots a_n}{}_{b_1 \dots b_m} = e^{a_1}{}_{\mu_1} e^{a_2}{}_{\mu_2} \dots e^{a_n}{}_{\mu_n} \times E^{\nu_1}{}_{b_1} \dots E^{\nu_m}{}_{b_m} \times A^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m}$$

Guaranteed ↓
whatever new action we get in terms of the vierbein should be inv. under local Lor. trs. bcos the original action was inv. under local trs.
— though this fact isn't manifest.
Possible confusion comes from derivatives
→ So what are the prop. which ~~gives~~ makes local Lor. trs.?

Q) How to define covariant derivatives of tensors of local Lorentz trs.?

$$\to \mathcal{D}_\mu \hat{A}_b = \underbrace{E^\nu{}_b}\, D_\mu A_\nu$$

Here there is no fundamental gauge field to compensate for the ∂_μ term — but here we can if introduce indep. gauge fields for local L.T. — that will be ~~too~~ many d.o.f.

So the normal way of defining covariant deriv. doesn't work here

guaranteed to have the right trs. laws but need to work this out

↳ invariant under local L.T. bcs A_ν has no knowledge about local L.T.

$$= E^\nu{}_b \left(\partial_\mu A_\nu - \Gamma^\rho{}_{\mu\nu} A_\rho \right)$$

Now $A_\nu = e^a_\nu \hat{A}_a$

We get $\mathcal{D}_\mu \hat{A}_b = E^\nu_b \left\{ \partial_\mu (e^a_\nu \hat{A}_a) - \Gamma^\rho_{\mu\nu} e^a_\rho \hat{A}_a \right\}$

$$e^a_\nu \partial_\mu \hat{A}_a + (\partial_\mu e^a_\nu) \hat{A}_a - \Gamma^\rho_{\mu\nu} e^a_\rho \hat{A}_a$$

Note :- $E^\nu_b D_\mu A_\nu$ has been used bcos that gives $\mathcal{D}_\mu \hat{A}_b$ the right trs. prop. – we need to decide what we want $\mathcal{D}_\mu \hat{A}_b$ to be

$$= \partial_\mu \hat{A}_b + \hat{\omega}_{\mu b}{}^a \hat{A}_a$$

Where

$$\hat{\omega}_{\mu b}{}^a = E^\nu_b \partial_\mu e^a_\nu - E^\nu_b \Gamma^\rho_{\mu\nu} e^a_\rho$$

(It's as if $\hat{\omega}_{\mu b}{}^a$ is a gauge field – Instead of a gauge field here we have a complicated field det. in terms of e^a_μ)

can be written in terms of e^a_μ & its inverse.

Similarly, $\mathcal{D}_\mu \hat{B}^b = e^b_\nu D_\mu B^\nu$
$= \partial_\mu \hat{B}^b + \omega_\mu{}^b{}_a \hat{B}^a$

Ex: S.T. $\omega_\mu{}^b{}_a = e^b_\nu \partial_\mu E^\nu_a + \Gamma^\mu_{\nu\rho} e^b_\nu E^\rho_a$

ω isn't an indep. field — it's det. in terms of $e^a{}_\mu$'s

Define:- $\hat{\omega}_\mu{}^{ba} = \eta^{bc}\, \hat{\omega}_{\mu c}{}^a$

$\omega_\mu{}^{ba} = \omega_\mu{}^b{}_c\, \eta^{ca}$

EX. Check that ① $\omega_\mu{}^{ab} = \hat{\omega}_\mu{}^{ab}$

② $\omega_\mu{}^{ab} = -\,\omega_\mu{}^{ba}$

(To prove this, write $\Gamma^\rho_{\mu\nu}$ in terms $e^a{}_\mu$)

Bcos of ①, As long as we keep our connection fixed, we don't have to use ω & $\hat{\omega}$ — can just write ω (be careful about raising & lowering indices)

EX. S.T. $\mathcal{D}_\mu\left(\hat{A}^{a_1 \ldots a_n}{}_{b_1 \ldots b_m}\right)$

$$= \partial_\mu \hat{A}^{a_1 \ldots a_n}{}_{b_1 \ldots b_m}$$
$$+ \omega_\mu{}^{a_1}{}_{c_1} \hat{A}^{c_1 a_2 \ldots a_n}{}_{b_1 \ldots b_m}$$
$$+ \ldots$$
$$\ldots + \omega_\mu{}^{a_n}{}_{c_n} \hat{A}^{a_1 \ldots a_{n-1} c_n}{}_{b_1 \ldots b_m}$$
$$+ \omega_{\mu b_1}{}^{d_1} \hat{A}^{a_1 \ldots a_n}{}_{d_1 b_2 \ldots b_m} + \ldots$$
$$\ldots + \omega_{\mu b_m}{}^{d_m} \hat{A}^{a_1 \ldots a_n}{}_{b_1 \ldots b_{m-1} d_m}$$

$$\mathcal{D}_\mu \left(\hat{A}^{a_1 \ldots a_n}{}_{b_1 \ldots b_m} \hat{B}^{c_1 \ldots c_k}{}_{d_1 \ldots d_l} \right)$$

$$= \mathcal{D}_\mu \left(\hat{A}^{a_1 \ldots a_n}{}_{b_1 \ldots b_m} \right) \hat{B}^{c_1 \ldots c_k}{}_{d_1 \ldots d_l}$$

$$+ \hat{A}^{a_1 \ldots a_n}{}_{b_1 \ldots b_m} \mathcal{D}_\mu \left(\hat{B}^{c_1 \ldots c_k}{}_{d_1 \ldots d_l} \right)$$

But why should these trs. as tensors of $SO(3,1)$?

→ This would have been obvious had the $\omega_\mu{}^{ab}$'s been gauge fields — but they aren't

---

$\omega_\mu{}^b{}_a = e^b{}_\nu \partial_\mu E^\nu{}_a + \Gamma^\mu{}_{\nu\rho} e^b{}_\nu E^\rho{}_a$

Under local Lorentz trs.,

$$e^b{}_\nu \xrightarrow[\text{to}]{\text{goes}} \Lambda^b{}_d e^d{}_\nu$$

$$E^\nu{}_a \longrightarrow E^\nu{}_c (\Lambda^{-1})^c{}_a$$

Using these, we get

$$\omega_\mu{}^b{}_a \xrightarrow[\text{to}]{\text{goes}} \Lambda^b{}_d e^d{}_\nu \partial_\mu \left(E^\nu{}_c (\Lambda^{-1})^c{}_a \right)$$

$$+ \Gamma^\mu{}_{\nu\rho} \Lambda^b{}_d e^d{}_\nu E^\rho{}_c (\Lambda^{-1})^c{}_a$$

$$\Rightarrow \omega_\mu{}^b{}_a \rightarrow \Lambda^b{}_d \omega_\mu{}^d{}_c (\Lambda^{-1})^c{}_a$$

Ex: show this ⇒

$$+ \Lambda^b{}_c \partial_\mu (\Lambda^{-1})^c{}_a$$

<u>Define</u> :- $\underset{\sim}{\omega}_\mu \Rightarrow$ matrix with ac component $\omega_\mu{}^a{}_c$

(Then the eqn. becomes) $\underset{\sim}{\omega}_\mu \longrightarrow \Lambda\, \underset{\sim}{\omega}_\mu \Lambda^{-1} + \Lambda\, \partial_\mu (\Lambda^{-1})$

$\Rightarrow$ (exactly the way a Non-abelian gauge field transforms)

Here we didn't have the freedom to introduce new gauge fields — Nevertheless $\omega_\mu{}^{ab}$ transforms as if it is a gauge field

$$D_\mu \hat{A}^a \equiv \partial_\mu \hat{A}^a + \omega_\mu{}^a{}_b \hat{A}^b$$

$$\hat{A}^a \longrightarrow \Lambda^a{}_b \hat{A}^b$$

As a consequence $D_\mu \hat{A}^a \longrightarrow \Lambda^a{}_b D_\mu \hat{A}^b \longrightarrow$ transforms covariantly

$$\gamma^\mu = E^\mu{}_a \gamma^a$$

$\gamma^a \longrightarrow$ we'll now think this as fixed matrices

$$\{\gamma^a, \gamma^b\} = \eta^{ab}$$

Previously we had the problem that γ^μ are fixed matrices under general coord. trs.

The only way to define fermions is to erect a locally inertial frame & ψ's on it — can't define ψ in a general coord. (or as a tensor under general coord. trs.) or as a tensor in a Manifold.

Rep. of Lorentz grp

$dx^\mu \rightarrow$ Lorentz grp tensor
or ~~Lorent~~ general coord. tensor

But Lorentz grp has tensors which aren't tensors under g.c.t. (can't be written as products of dx^μ)

$\longrightarrow$ So better write everything in terms of repn. of Lorentz grp.

It's an accident that some of the repn. of the Lorentz grp. can also be

---

~~7/11/07~~

$$\omega_\mu{}^a{}_b = e^a_\nu \partial_\mu E_b^\nu + \Gamma^\nu_{\mu\rho} e^a_\nu E_b^\rho$$

Under local Lorentz trs.:—

$$e^a_\nu \rightarrow \Lambda^a_b(x)\, e^b_\nu(x)$$

$$\omega_\mu{}^a{}_b \longrightarrow \omega'_\mu{}^a{}_b = (\Lambda \partial_\mu \Lambda^{-1})^a{}_b + (\Lambda \omega_\mu \Lambda^{-1})^a{}_b$$

[It's like a non-abelian gauge trs. with gauge grp. SO(3,1)]

$\omega_\mu{}^a{}_b$ transforms as a covariant vector under general coordinate transformation:–

$$\omega_\mu{}^a{}_b = e^a{}_\nu D_\mu E_b^\nu$$

$\longrightarrow$ transforms as a cov. vector under general coord. trs.

[Note: a, b indices don't trs. under general coord. trs. — they trs. only under local Lorentz trs.]

$\mathcal{D}_\mu B^a = \partial_\mu B^a + \omega_\mu{}^a{}_b B^b$ etc.

$\hookrightarrow$ trs. covariantly under ~~both~~ general coord. trs. & contravariantly under local local trs.

(This formalism is needed to deal with fermions — not necessary for tensors under g.c.t.)

$$\mathcal{L} = \bar{\Psi}(-i\gamma^\mu \partial_\mu \Psi - m\Psi)$$

$\hookrightarrow$ (no manifest 'a' index here — so we ask whether it is general coord. trs. invariant)

If Ψ transforms as a scalar, we have

$$\Psi'(x') = \Psi(x)$$

and clearly the above expression cannot be general-coord. invariant.

So replace γ^μ by $\gamma^a E_a^\mu$.

$$\mathcal{L} = \bar{\Psi}(-i\underset{\substack{\Downarrow \\ \gamma^a E_a^\mu}}{\gamma^\mu} \partial_\mu \Psi - m\Psi)$$

$$\Rightarrow \mathcal{L} = \bar{\Psi}(-i\gamma^a E_a^\mu \partial_\mu \Psi - m\Psi)$$

$\longrightarrow$ manifestly general coord. trs. invariant

But what about local Lorentz trs.?

$\longrightarrow$ Under this, $E_a^\mu \longrightarrow E_b^\mu (\Lambda^{-1})^b{}_a$

We should take Ψ as trs. a spinor under local Lor. trs.

$$\Psi'(\cancel{x}) = R(\Lambda)\,\Psi(x)$$

or $$\Psi'_\alpha(x') = \left(R(\Lambda)\right)_{\alpha\beta}\,\Psi_\beta(x)$$

spinor indices

$R(\Lambda)$ ↓ matrix acting on the spinor space for Lor. trs. matrix Λ

$$\mathcal{L} \xrightarrow{\text{becomes}} \Psi^\dagger \left(R(\Lambda)\right)^\dagger \gamma^0 \Big[i\gamma^a E^\mu_\alpha (\Lambda^{-1})^b_a \times \partial_\mu \left(R(\Lambda)\Psi\right) - m R(\Lambda)\Psi \Big]$$

This is not a normal Lorentz trs. that we do in flat space — here we are dealing with an internal trs.

e.g. → $B^a(x) \to \Lambda^a{}_b(x)\, B^b(x)$

"Under Lorentz trs. $\int d^4x\, \mathcal{L}(\Psi(x)) = \int d^4x\, \mathcal{L}(\Psi'(x))$ under a local Lorentz trs. $x' = x$ as it is an internal trs.

We are trying to guess what the trs. of Ψ should be — a priori we have no knowledge about it

Local Lor. trs. ~~acts on~~ doesn't act on x at all — it's like an internal sym. trs., where the relevant grp. is the Lorentz grp.

Local ~~Lorentz~~ trs. is only changing the choice of ~~vieb~~ vierbein & not the coordinates.

$\therefore \mathcal{L} \to \psi^\dagger (R(\Lambda))^\dagger \gamma^0 \Big(-i\gamma^a E_a^{\ \mu} (\Lambda^{-1})^b{}_a \, \partial_\mu (R(\Lambda)\psi)$
$- m R(\Lambda)\psi \Big)$

$= \psi^\dagger R(\Lambda)^\dagger \gamma^0 \left(-i\gamma^a E_b^{\ \mu} (\Lambda^{-1})^b{}_a (\partial_\mu R(\Lambda)) \psi \right.$ term 1

$+ \psi^\dagger (R(\Lambda))^\dagger \gamma^0 (-i\gamma^a E_b^{\ \mu} (\Lambda^{-1})^b{}_a R(\Lambda) \partial_\mu \psi$ term 2

$- m \psi^\dagger R(\Lambda)^\dagger \gamma^0 R(\Lambda) \psi$ term 3

(At this pt. we are just <u>trying out</u> our guess that $\psi'(x) = R_{\alpha\beta} \psi_\beta (x)$ — we may have to change our guess to make $\mathcal{L}$ local-Lorentz inv.) — don't even know whether there is any trs. which will make $\mathcal{L}$ local Lor.-inv.

$\boxed{\text{Term 3}}$:-

(Need to use the prop. of $R(\Lambda)$)

Consider <u>ordinary</u> Lorentz trs. :-

$\psi'(x') = R(\Lambda_0) \psi(x)$

(not x-dep. for an ordinary L.T.)

$m\bar{\psi}\psi$ is invariant under an ordinary Lorentz trs. & hence

$m \bar{\psi}'(x') \psi'(x') = m \bar{\psi}(x) \psi(x)$

$\Rightarrow m \psi^\dagger(x) R(\Lambda_0)^\dagger \gamma^0 R(\Lambda_0) \psi(x)$

$= m \psi^\dagger(x) \gamma^0 \psi(x)$

By demanding $R(\Lambda_0)^\dagger \gamma^0 R(\Lambda_0) = \gamma^0$, we determine $R(\Lambda_0)$

So we learn that

$$R(\Lambda_0)^\dagger \gamma^0 R(\Lambda_0) = \gamma^0 \quad \text{—— (A)}$$

So at a given pt. x,

$$R(\Lambda(x))^\dagger \gamma^0 R(\Lambda(x)) = \gamma^0$$

bcos at each pt., $R(\Lambda(x))$ satisfies property (A).

So, $$-m \psi^\dagger R(\Lambda(x))^\dagger \gamma^0 R(\Lambda(x)) \psi$$

$$= -m \bar{\psi}(x) \psi(x)$$

Term (2) :-

$$\psi^\dagger (R(\Lambda))^\dagger \gamma^0 \cancel{R(\Lambda)} \gamma^a (\Lambda^{-1})^b{}_a \partial_\mu \psi \left(-i E^\mu{}_a\right)$$

insert $R(\Lambda) R(\Lambda)^{-1}$

Now, $R(\Lambda)^{-1} \gamma^a R(\Lambda) = \Lambda^a{}_c \gamma^c$

To check whether the above relation is true $\Rightarrow$

$\mathcal{L} = \bar{\psi}(-i\gamma^\mu \partial_\mu \psi - m\psi)$ is inv. under ordinary L.T.

$$\mathcal{L}'(x') = \bar{\psi}'(x') \left(-i\gamma^\mu \partial'_\mu \psi'(x')\right)$$

$\underline{\text{known}}$ $\bar{\psi}(x) \left(-i\gamma^\mu \partial_\mu \psi(x)\right)$

under $\begin{cases} \psi'(x') = \Lambda_0 \psi(x) \\ x'^\mu = (\Lambda_0)^\mu{}_\rho x^\rho \end{cases}$

$$\mathcal{L}'(x') = \psi^{+}(x)\Big\{R(\Lambda_0)^{\dagger}\gamma^0(-i\gamma^{\mu})(\Lambda^{-1})^{\nu}{}_{\mu} \times R(\Lambda_0)\Big\}\frac{\partial}{\partial x^{\nu}}\psi(x)$$

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial}{\partial x^{\nu}} = (\Lambda^{-1})^{\nu}{}_{\mu}\frac{\partial}{\partial x^{\nu}}$$

this bracket should be equal to $(-i\gamma^0\gamma^{\nu})$

- This is a specific prop. of $R(\Lambda_0)$ irrespective of $\psi(x)$

So we must have

$$R(\Lambda_0)^{\dagger}\gamma^0\gamma^a R(\Lambda_0)(\Lambda^{-1})^b{}_a = \gamma^0\gamma^b$$

for any Λ_0 [γ^a's are fixed matrices for flat space]

[So taking γ^a, we can define γ^{μ}'s for curved space by $\gamma^{\mu} = \gamma^a E^{\mu}_a$]

γ^a's don't trs. under g.c.t. — γ^{μ}'s do transform

So, term 2 $= \psi^{\dagger}\gamma^0\gamma^b\partial_{\mu}\psi(-iE^{\mu}_b)$

$= \bar{\psi}\gamma^b\partial_{\mu}\psi(-iE^{\mu}_b)$

Hence, to summarize, we have,

$$\mathcal{L} \xrightarrow[\text{Lorentz trs.}]{\text{local}} \mathcal{L} + \psi^{\dagger}R(\Lambda)^{\dagger}\gamma^0(-i\gamma^{a})E^{\mu}_b(\Lambda^{-1})^b{}_a \times \partial_{\mu}R(\Lambda)\psi$$

extra term

[this is term 1]

Simplify the extra term:-

$$\psi^\dagger R^\dagger \gamma^0 (-i\gamma^a) E^\mu_{\ b} (\Lambda^{-1})^b_{\ a} (R_{(\Lambda)} R^{-1}_{(\Lambda)}) \partial_\mu R_{(\Lambda)} \psi$$

insert ~~...~~ this here

So,

$$\mathcal{L} \longrightarrow \mathcal{L} + \bar{\psi} (-i\gamma^b E^\mu_{\ b}) R(\Lambda)^{-1} \partial_\mu R(\Lambda) \psi$$

Q> What is wrong?, why the extra term?

$\longrightarrow$ We have not used a covariant derivative for the local Lor. trs. which is like a gauge trs.

$\mathcal{L}$ has the global Lor. trs.; but when we try to make it local, the sym. is broken — need to modify $\mathcal{L}$ to make it invariant

Need to replace $\partial_\mu \psi$ by $D_\mu \psi$.

Q> What is $D_\mu \psi$?

$$\longrightarrow (D_\mu \psi)_\alpha = \partial_\mu \psi_\alpha + (\Omega_\mu)_{\alpha\beta} \psi_\beta$$

$\Downarrow$

Spinor representation of ω_μ

Need to use the spinor repn. of ω_μ^{ab} (a, b: VECTOR INDICES)

$\rightarrow (\Omega_\mu)_{\alpha\beta}$ ($\alpha\beta$: spinor indices)

$$C_m / C_m = \rho_r / \rho_m \, \lambda$$

$$\left(\frac{\rho_r}{\rho_m}\right)_{t=t_0} \lambda(t_0) = \left(\frac{\rho_r}{\rho_m}\right)_{t=t_{eq}} \lambda(t_{eq})$$

$$\Rightarrow \lambda(t_{eq}) = \left(\frac{\rho_r}{\rho_m}\right)_{t_0} \lambda(t_0)$$

$$\therefore RHS = 4 \left(\frac{C_0}{C_m}\right)^{1/4} \left(\frac{\rho_r}{\rho_m}\right)^{3/4}_{t_0} \left(\lambda(t_0)\right)^{3/4}$$

$$C_m = \rho_m \lambda^3$$

$$(C_m)^{-1/4} = (\rho_m)^{-1/4} \lambda^{-3/4}$$

$$\frac{\rho_m}{\rho_r} = 2.6 \times 10^4$$

$$\therefore RHS = 4 \left(\frac{C_0}{\rho_m}\right)^{1/4}_{t_0} \left(\frac{\rho_r}{\rho_m}\right)^{\cancel{3/4}}_{t_0}$$

$$= 4 \times \frac{10^{28}}{(2.6)^{1/4} \times 10} \, \frac{1}{3 \times (2.6)^{3/4} \times 10}$$

$$\simeq 10^{26}$$

$$e^N > 10^{26}$$

$$N > 26 \ln 10$$

$$> 60$$

$C_0^{1/4} = 10^{15}\,GeV$

$= 10^{28}\,K$

$\frac{\rho_m}{\rho_r} = 2.6 \times 10^4$

$\rho_m = \rho_r \times 2.6 \times 10^4$

$\rho_m \propto T$

$\rho_R(t_1) = C_0 \qquad \lambda(t_1) = \left(\frac{C_R}{C_0}\right)^{1/4}$

End of cosm. const. dominated era:—

$$d(t_f) \simeq \sqrt{\frac{3}{8\pi G C_0}}\ \frac{e^N}{\lambda(t_1)}$$

$$\lambda_{in} = \lambda(t_1)$$

$$\rho_{in} = C_0$$

$$d(t_{eq}) = d(t_f) + \sqrt{\frac{3}{8\pi G C_R}}\ \lambda(t_{eq})$$

t : matter dominated era :—

$$d(t) = d(t_{eq}) + 2\sqrt{\frac{3}{8\pi G C_m}}\ \sqrt{\lambda(t)}$$

$$d(t_{rec}) = d(t_f) + 2\sqrt{\frac{3}{8\pi G C_m}}\ \sqrt{\lambda(t_{rec})}$$

$$+ \sqrt{\frac{3}{8\pi G C_R}}\ \lambda(t_{eq})$$

$$\sqrt{\frac{3}{8\pi G C_m}}\ \sqrt{\lambda(t_{eq})}$$

$$\boxed{\begin{aligned} C_R &= C_m \lambda_{eq} \\ \lambda(t_1) &= \left(\frac{C_m \lambda_{eq}}{C_R}\right)^{1/4} \end{aligned}}$$

$$\simeq d(t_f) + 2\sqrt{\frac{3}{8\pi G C_m}}\ \sqrt{\lambda(t_{rec})} \quad \Big| \quad \lambda(t_{eq}) \ll \lambda(t_{rec})$$

$$\Delta = 2\left(d(t_0) - d(t_{rec})\right) = 2\cdot 2\sqrt{\frac{3}{8\pi G C_m}}\ \sqrt{\lambda(t_0)} \quad \Big/ \quad \lambda(t_0) \gg \lambda(t_{rec})$$

We want $\dfrac{\Delta}{d(t_{rec})} < 1$

$$\Rightarrow \Delta < d(t_{rec})$$

$$\Rightarrow 4\sqrt{\frac{3}{8\pi G C_m}}\ \sqrt{\lambda(t_0)} < \sqrt{\frac{3}{8\pi G C_0}}\ \frac{e^N}{\lambda(t_1)}$$

$$\Rightarrow e^N > 4\sqrt{\frac{C_0}{C_m}}\ \lambda(t_1)\sqrt{\lambda(t_0)}$$

$$RHS = 4\sqrt{\frac{C_0}{C_m}}\ \left(\frac{C_m}{C_0}\right)^{1/4} (\lambda_{eq})^{1/4} \sqrt{\lambda(t_0)}$$

$$= 4\left(C_0/C_m\right)^{1/4} (\lambda_{eq})^{1/4} \sqrt{\lambda(t_0)}$$

Steps to get Ω_μ :-

In Non-abelian gauge theory, we have gauge fields A_μ^a.

Suppose T^a are the generators in some representation 'R'.

$$D_\mu \phi = \partial_\mu \phi + i A_\mu^\alpha T^\alpha \phi$$ if ϕ is in repn 'R'

Suppose X is a field in a different repn.

Then $D_\mu X = \partial_\mu X + i A_\mu^\alpha \underbrace{R(T^\alpha)}_{\text{generator in the repn for } X} X$

(Make sure generators are normalised in the same way in all the repns :)

If $[T^a, T^b] = i f^{abc} T^c$

then $[R(T^a), R(T^b)] = i f^{abc} R(T^c)$

Then if $D_\mu \phi$ transforms covariantly,

then $D_\mu X$ also " " .

No need to worry about $Tr[T^a T^b] =$

$$D_\mu B^a = \partial_\mu B^a + \omega_\mu{}^a{}_b B^b$$

$$= \partial_\mu B^a + i \sum_{c<d} \omega_\mu^{cd} (T_{cd})^a{}_b B^b$$

where $(T_{cd})^a{}_b \equiv i[\delta^a{}_c \eta_{bd} - \delta^a{}_d \eta_{bc}]$

$\therefore\ \ \kappa \xleftrightarrow[\text{for}]{\text{stands}} (cd)$ with $c < d$,

[We have 6 such pairs $= {}^4C_2 = \frac{4!}{2 \times 2}$]

$A_\mu^{\ \kappa} \xleftrightarrow[\text{for}]{\text{stands}} \omega_\mu^{\ cd}$,

and $(T^\kappa)^a{}_b \xleftrightarrow[\text{for}]{\text{stands}} (T_{cd})^a{}_b$

Now, $[T_{cd}, T_{ef}] = i\, f_{(cd)(ef)(gh)}\, T_{gh}$ ← real

(ensuring $c<d$, $e<f$, $g<h$ for each pair)

So $(\Omega_\mu)_{\alpha\beta} = i\, \omega_\mu^{\ cd} \{R(T_{cd})\}_{\alpha\beta}$

↳ (prescription to define the cov. deriv. for ψ)

We know that

$[R(T_{cd}), R(T_{ef})] = i\, f_{(cd)(ef)(gh)}\, R(T_{gh})$

↓ Lorentz generators in the spinor repr.

$R(T_{cd}) = K(\gamma_c \gamma_d - \gamma_d \gamma_c)$

↓ Need to fix the normalisation 'K' by using the commutation relation $[R(T_{cd}), R(T_{ef})] = i\, f_{(cd)(ef)(gh)}\, R(T_{gh})$

Ex. Find K

$\therefore$ $D_\mu \psi = \partial_\mu \psi + i K \omega_\mu^{\ cd} (\gamma_c \gamma_d - \gamma_d \gamma_c) \psi$

So the lagrangian now is

$$\mathcal{L} = \bar{\psi}\left(-i \underbrace{\gamma^\mu}_{\gamma^a E^\mu_{\ a}} D_\mu \psi - m \psi\right)$$

Ex. Check explicitly that $\mathcal{L}$ is local Lorentz invariant.

$\mathcal{L}$ is also inv. under g.c.t. as $\omega_\mu^{\ cd}$ is a vector under g.c.t. (so $E^\mu \omega_\mu^{\ cd}$ is a scalar) & γ^a's are inv. under g.c.t.

Only diff. from Non-abelian gauge theory is that there we use an indep. gauge field A_μ^k (an indep. entity), while here $\omega_\mu^{\ ad}$'s are constructed out of .... — however it trs. ~~as~~ as if it were a gauge field

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If we fix T^a, T^b → in vector repn. of the gauge grp. & have some $\mathrm{Tr}[T^a, T^b] = \#$, $R(T^a)$, $R(T^b)$ will be det. by

$$[R(T^a), R(T^b)] = i f^{abc} R(T^c) \quad \text{— (A)}$$

& $\mathrm{Tr}(R(T^a) R(T^b))$ will also be det. by (A).

In $\mathrm{Tr}(F_{\mu\nu} F^{\mu\nu})$ we need only $\mathrm{Tr}[T^a, T^b] \#$ which is fixed by our choice.

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Generalisation of Schwarzschild Black holes to Black holes carrying charge

Schwarzschild metric:-

$$ds^2 = -\Psi(\rho)\,dt^2 + \chi(\rho)\,d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$(y^1, y^2, y^3) = (\rho\sin\theta\cos\phi,\ \rho\sin\theta\sin\phi,\ \rho\cos\theta)$$

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} \longrightarrow R\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}$$

(spherical sym.)

(Action of Maxwell's th. coupled to Einstein's th.)

$$S = -\frac{1}{16\pi G}\int \sqrt{-\det g}\; R\, d^4x - \frac{1}{4}\int d^4x \sqrt{-\det g}\; F_{\mu\nu}F^{\mu\nu}$$

It's better not to try to figure out the form of A_μ to get $F_{\mu\nu}$ bcos A_μ's are gauge-dep $\Rightarrow$ so try to find an ansatz for $F_{\mu\nu}$ itself.

~~$F_{ti} = \partial_0 A_i - \partial_i A_0$~~

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$A_0 = f(\rho) \quad A_i = g(\rho)$

Won't work — Don't think $F_{\mu\nu}$ as fns of A_μ - try to find its overall form

$$F_{ti} = \tilde{f}(\rho)\; y^i$$

bcos y^i trs. as a vector under rotn. - so F_{ti} will have the same prop.

$$F_{ij} = \epsilon_{ijk}\, y^k\, \tilde{g}(\rho)$$

No time dep. - so fn. of 'ρ' only

Ex. Show that in t, ρ, θ, ϕ coordinates

$$F_{t\rho} = f(\rho) \qquad f(\rho) = \rho\,\tilde{f}(\rho)$$

$$F_{\theta\phi} = g(\rho)\sin\theta \qquad \text{All other components } 0$$

We now need to determine $\Psi(\rho)$, $\chi(\rho)$, $f(\rho)$ and $g(\rho)$.

We have :—

① Maxwell's eqn.

② Einstein's eqn.

③ Bianchi identity

Can't use ~~every~~ div. of $T_{\mu\nu}$ zero bcos that follows as a consequence of e.o.m.

bcos it's not a priori guaranteed that the ansatz for $F_{\mu\nu}$ we have taken will satisfy Bianchi identity — of course $F_{\mu\nu}$ in the form $\partial_\mu A_\nu - \partial_\nu A_\mu$ ~~guarantees~~ would have guaranteed that Bianchi identity is satisfied

In flat space

$$T^{\mu\nu}_{e.m.} = \eta^{\alpha\rho}\eta^{\mu\sigma}\eta^{\kappa\tau} F_{\rho\kappa}F_{\sigma\tau} - \frac{1}{4}\eta^{\mu\nu}\eta^{\rho\sigma}\eta^{\kappa\tau}F_{\rho\kappa}F_{\sigma\tau}$$

$$T^{\mu\nu}_{matter} = \sum_n m_n \int d^4x\, \delta^{(4)}(x^\rho - x^\rho_n(\tau)) \frac{dx^\alpha}{d\tau}\frac{dx^\mu}{d\tau}$$

$$T^{00}_{em} = g^{0\rho}g^{0\sigma}g^{\kappa\tau}F_{\rho\kappa}F_{\sigma\tau} - \frac{1}{4}g^{00}g^{\rho\sigma}g^{\kappa\tau} \times F_{\rho\kappa}F_{\sigma\tau}$$

$$= g^{00}g^{00}g^{\kappa\tau}F_{0\kappa}F_{0\tau} + \frac{1}{4\psi}\left[g^{00}g^{\kappa\tau}F_{0\kappa}F_{0\tau} + g^{ii}g^{\kappa\tau}F_{i\kappa}F_{i\tau}\right]$$

$$= \frac{1}{\psi^2}g^{ii}(F_{0i})^2 + \frac{1}{4\psi}\left[-\frac{1}{\psi}g^{ii}(F_{0i})^2 + g^{ii}g^{00}(F_{i0})^2 + g^{ii}g^{kk}F_{ik}F_{ik}\right]$$

$$= \frac{1}{\psi^2}g^{\rho\rho}(F_{t\rho})^2 + \frac{1}{4\psi}\left[-\frac{1}{\psi}g^{\rho\rho}(F_{t\rho})^2 - \frac{g^{\rho\rho}}{\psi}(F_{t\rho})^2 + g^{\theta\theta}g^{\phi\phi}F_{\theta\phi}F_{\theta\phi}\times 2\right]$$

$$= \frac{f^2}{\psi^2 \chi} + \frac{1}{4\psi}\left[-\frac{f^2}{\psi\chi} - \frac{f^2}{\psi\chi} + 2\frac{g^2\cancel{\sin^2\theta}}{\rho^4\cancel{\sin^2\theta}}\right]$$

$$= \frac{f^2}{\psi^2\chi} - \frac{f^2}{2\psi^2\chi} + \frac{g^2}{2\psi\rho^4}$$

$$T^{00}_{em} = + \frac{f^2}{2\,4^2 \chi} + \frac{g^2}{2\,4\,\rho^4}$$

$$T^{\rho\rho}_{em} = g^{\rho\rho} g^{\rho\rho} g^{\kappa\tau} F_{\rho\kappa} F_{\rho\tau}$$

$$- \tfrac{1}{4}\, g^{\rho\rho} g^{\delta\sigma} g^{\kappa\tau} F_{\delta\kappa} F_{\sigma\tau}$$

$$= \frac{1}{\chi^2}\left[g^{tt} F_{\rho t} F_{\rho t}\right] - \frac{1}{4\chi}\left[g^{tt} g^{\rho\rho} F_{t\rho} F_{t\rho} \times 2 + g^{\theta\theta} g^{\phi\phi} F_{\theta\phi} F_{\theta\phi} \times 2\right]$$

$$= \frac{-1}{\chi^2 4} f^2 - \frac{1}{4\chi}\left[-\frac{2f^2}{4\chi} + \frac{2\, g^2 \cancel{\sin^2\theta}}{\rho^4 \cancel{\sin^2\theta}}\right]$$

$$= \frac{-f^2}{\chi^2 4} + \frac{f^2}{2\,4\,\chi^2} - \frac{g^2}{2\chi\rho^4}$$

$$= -\frac{f^2}{2\,4\,\chi^2} - \frac{g^2}{2\chi\rho^4}$$

$$T^{\phi\phi}_{em} = g^{\phi\delta} g^{\phi\sigma} g^{\kappa\tau} F_{\delta\kappa} F_{\sigma\tau} - \frac{1}{4} g^{\phi\phi} g^{\delta\sigma} g^{\kappa\tau} F_{\delta\kappa} F_{\sigma\tau}$$

$$= g^{\phi\phi} g^{\phi\phi} g^{\theta\theta} F_{\phi\theta} F_{\phi\theta}$$

$$- \frac{1}{4} g^{\phi\phi}\left[-\frac{2f^2}{4\chi} + \frac{2g^2}{\rho^4}\right]$$

$$= \frac{g^2 \sin^2\theta}{\rho^6 \sin^4\theta} - \frac{1}{2\rho^2 \sin^2\theta}\left[-\frac{f^2}{4\chi} + \frac{g^2}{\rho^4}\right]$$

$$= \frac{g^2}{\rho^6 \sin^2\theta} + \frac{f^2}{2\rho^2 \sin^2\theta\, 4\chi} - \frac{g^2}{2\rho^6 \sin^2\theta}$$

$$= + \frac{g^2}{2\rho^6 \sin^2\theta} + \frac{f^2}{2\rho^2 \sin^2\theta \, 4\chi}$$

$$T_{em}^{00} = g^{00} g^{00} g^{\phi\phi} F_{0\phi} F_{0\phi} - \frac{1}{4} g^{00} [g^{\delta\sigma} g^{\kappa\tau} F_{\delta\kappa} F_{\sigma\tau}]$$

$$= \frac{g^2 \cancel{\sin^2\theta}}{\rho^6 \cancel{\sin^2\theta}} - \frac{1}{4\rho^2} \left[-\frac{2f^2}{4\chi} + \frac{2g^2}{\rho^4} \right]$$

$$= \frac{g^2}{\rho^6} + \frac{f^2}{2\rho^2 4\chi} - \frac{g^2}{2\rho^6}$$

$$= \frac{g^2}{2\rho^6} + \frac{f^2}{2\rho^2 4\chi}$$

$$T_{em}^{0i} = 0$$

$(i \neq j)$ $$T_{em}^{ij} = g^{j\delta} g^{i\sigma} g^{\kappa\tau} F_{\delta\kappa} F_{\sigma\tau}$$

$$- \cancel{\frac{1}{4} g^{ij}}^{\to 0} [\ --\ ]$$

$$= g^{jj} g^{ii} g^{\kappa\tau} F_{j\kappa} F_{i\tau}$$

$$= \cancel{g^{jj} g^{ii} g^{00} F_{j0}}^{\to 0} F_{i0}$$

$$+ \cancel{g^{jj} g^{ii} g^{ll} F_{jl}}^{\to 0} F_{il}$$

$$= 0$$

To solve :-

$$R_{\mu\nu} = 4\pi G \left[g^{\alpha\sigma} g_{\mu\nu} T_{\alpha\sigma} - 2 T_{\mu\nu} \right]$$

~~tt - component~~

$$\cancel{R_{tt} = 4\pi G \left[g^{\alpha\sigma} g_{tt} T_{\alpha\sigma} - 2 T_{tt} \right]}$$

$$\cancel{\Rightarrow \frac{R_{tt}}{4\pi G} = g_{tt} \left[-\frac{f^2}{2\psi^3 \chi} - \frac{g^2}{\rho^4 \psi^2} - \frac{f^2}{2\chi^3 \psi} \right.}$$

$$T_{tt} = \frac{1}{2\psi^2} \left[\frac{f^2}{\psi \chi} + \frac{g^2}{\rho^4} \right] = \frac{f^2}{2\psi^4 \chi} + \frac{g^2}{2\rho^4 \rho^3}$$

$$T_{\rho\rho} = g^{\rho\rho} g^{\rho\rho} T^{\rho\rho}$$

$$= -\frac{1}{2\chi^2} \left[\frac{f^2}{\chi^2 \psi} + \frac{g^2}{\chi \rho^4} \right]$$

$$= -\frac{f^2}{2\chi^4 \psi} - \frac{g^2}{2\chi^3 \rho^4}$$

$$T_{\phi\phi} = \frac{1}{2\rho^2 \sin^2\theta} \left[\frac{g^2}{\rho^6 \sin^2\theta} + \frac{f^2}{\rho^2 \sin^2\theta\, \psi \chi} \right]$$

$$= \frac{g^2}{2\rho^6 \sin^4\theta} + \frac{f^2}{2\rho^4 \sin^4\theta\, \psi \chi}$$

$$T_{\theta\theta} = \frac{1}{2\rho^2} \left[\frac{g^2}{\rho^6} + \frac{f^2}{\chi \psi \rho^2} \right]$$

$$= \frac{g^2}{2\rho^8} + \frac{f^2}{2\rho^4 \chi \psi}$$

$$\frac{R_{tt}}{4\pi G} = g_{tt}\left(g^{\alpha\sigma}T_{\alpha\sigma}\right) - 2T_{tt}$$

$$= -\frac{\cancel{2}}{\psi^2 \times \cancel{2}}\left[\frac{f^2}{\psi\chi} + \frac{g^2}{\rho^4\psi}\right]$$

$$-\frac{1}{\psi}\left[-\frac{f^2}{2\psi^5\chi} - \frac{g^2}{2\rho^4\psi^4} - \frac{f^2}{2\chi^5\psi^2} - \frac{g^2}{2\chi^4\rho^4} + \frac{g^2}{2\rho^{10}} + \frac{f^2}{2\rho^6\chi\psi} + \frac{g^2}{2\rho^8\sin^6\theta} + \frac{f^2}{2\rho^6\sin^6\theta\,\psi\chi}\right]$$

$$T_{\rho\rho} = -\frac{f^2}{2\psi} - \frac{g^2\chi}{2\rho^4}$$

$$T_{\phi\phi} = \frac{g^2\sin^2\theta}{2\rho^2} + \frac{f^2 g^2\sin^2\theta}{2\psi\chi}$$

$$T_{tt} = \frac{f^2}{2\chi} + \frac{g^2\psi}{2\rho^4}$$

$$T_{\theta\theta} = \frac{g^2}{2\rho^2} + \frac{\rho^2 f^2}{2\chi\psi}$$

Bianchi identities give us

$$g(\rho) = \text{constant}$$