$\therefore$ ① reduces to

$$f_1(t', F(t', \vec{x}')) = f_1(t', \vec{x}')$$

For $dx'^k dx'^l$:-

③ $$f_{3ij}(t', \vec{F}) \frac{\partial F^i}{\partial x'^k} \frac{\partial F^j}{\partial x'^l} = f_{3kl}(t', \vec{x}')$$

Homogeneity :-

Given any two points : $(t, \vec{x})$ & $(t, \vec{y})$, there is an isometry F such that

$$\vec{F}(t, \vec{x}) = \vec{y}$$

This implies $f_1(t', \vec{y}) = f_1(t', \vec{x}')$ for any pair of points $(\vec{x}, \vec{y})$ in space

[follows from ①]

$$\Rightarrow \boxed{f_1(t', \vec{x}') = f_1(t')} \quad (\text{indep. of } \vec{x})$$

(Isotropy is not going to give any further restriction on f_1 bcos anyway it is indep. of $\vec{x}$)

$$f_{3ij}(t', \vec{F}) \frac{\partial F^i}{\partial x'^k} \frac{\partial F^j}{\partial x'^l} = f_{3kl}(t', \vec{x}')$$

Isometries associated to the space we consider here are homogeneity & isometry

Fix t' (think of it as a parameter without thinking 4D spacetime) & think of f_{3ij} as a 3-dim. metric.

Then, $x^i = F^i(\vec{x}')$ is an isometry of the 3-dimensional metric $f_{3ij}(t', \vec{x}')$ for every t'.

$$ds_3^2 \equiv f_{3ij}(t', \vec{x}')\, dx'^i dx'^j$$

& make a coord. trs. $x'^i = F^i(\vec{x}')$

then this is an isometry of the 3-d metric

if $\boxed{f_{ij}(t', \vec{F})\, \frac{\partial F^i}{\partial x'^k} \frac{\partial F^j}{\partial x'^l} = f_{3kl}(t', x'^{i})}$

is satisfied.

The statement ~~of~~ about the 4-d metric is converted to the statement about the 3-d metric — the condn. can be restated in the 3-d language without any reference to time.
→ Reinterpretation
This could be done bcos we have the notion of COSMIC TIME
→ any isometry trs. should never change that time

List of homogeneous and isotropic metrics in D = 3

(1) flat space :-

$$f_{3ij}\, dx^i dx^j = \hat{f}(t)\left\{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right\}$$

$\hat{f}(t)$ → arbitrary fn. of time

$\vec{x} \longrightarrow \vec{x} + \vec{a}$: establishes homogeneity

SO(3) rotation around origin → ~~isometry~~ establishes isotropy

(2) Surface of a 3-dimensional sphere S^3 :-

$$\sum_{i=1}^{4} (Z^i)^2 = R^2$$

$$ds_4^2 = dZ^i dZ^i$$

SO(4)

Given any 2 pts. on the 2D sphere S^2, there is a rot. about origin which takes first pt. to the other pt.

Ricci scalar is -ve & const. → +ve curvature metric

③ Constant -ve curvature metric :

$$Z_4^2 - (Z_1^2 + Z_2^2 + Z_3^2) = R^2$$

$$ds^2 = -(dZ_4)^2 + \sum_{i=1}^{3} (dZ_i)^2$$

Ricci scalar is const. but +ve → in our notation

(A homogeneous metric should have a const. 3-d curvature.
But a const. 3-d Ricci scalar doesn't imply homogeneous metric bcos it only gives one prop. of the space)

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$$ds^2 = -f_1(t)\, dt^2 + f_{3ij}(\vec{x}, t)\, dx^i dx^j$$

$$(i, j = 1, 2, \ldots, 3)$$

Isometries:-

$$t = t'$$
$$x^i = f^i(\vec{x}')$$

$$ds_3^2 = f_{3ij}(\vec{x}, t)\, dx^i dx^j$$

List of homogeneous & isotropic metrics in $D = 3$

① flat space:-

$$f_{3ij}\, dx^i dx^j = \hat{f}(t)\{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\}$$

$$= \hat{f}(t)\{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\}$$

② ~~Sph~~ 3-dimensional sphere S^3:-

$$(z_1)^2 + (z_2)^2 + (z_3)^2 + (z_4)^2 = R^2 \quad \text{—①}$$

$$ds_4^2 = dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 \quad \text{—②}$$

4D flat metric

embedding space metric

Analogous to sphere in 3d

$z_4 = R\cos\psi$

$z_3 = R\sin\psi\cos\theta$

$z_1 = R\sin\psi\sin\theta\cos\phi$

$z_2 = R\sin\psi\sin\theta\sin\phi$

θ, ϕ, ψ can be taken as the 3 indep. coordinates on S^3

Ex. Check that

$$ds_3^2 = ds_4^2\Big|_{\text{sphere}} = R^2\left\{d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

R is const, & so we don't vary R

$r = \sin\psi$

$dr = \cos\psi\, d\psi = \sqrt{1-r^2}\, d\psi$

$$ds_3^2 = R^2\left\{\frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right\}$$

Now, ds_4^2 is invariant under

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \longrightarrow U\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

where $U^T U = \mathbb{1} \rightarrow 4\times 4$ orthogonal matrix

(If (z_1, z_2, z_3, z_4) satisfies $z_1^2 + z_2^2 + z_3^2 + z_4^2 = R^2$, then (z_1', z_2', z_3', z_4') also satisfies $\sum_{i=1}^{4} z_i'^2 = R^2$)

Any rotn. is an isometry, analysing the eqns 1 & 2

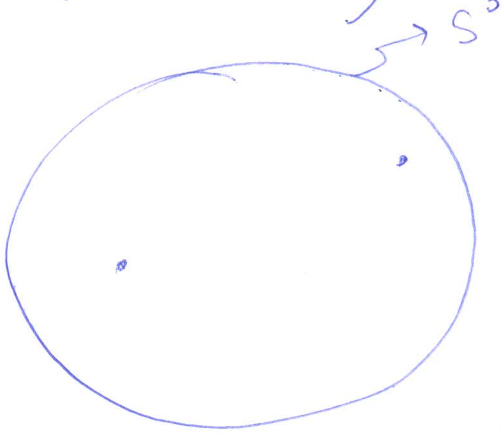
$\rightarrow$ induces an isometry on the sphere

(Take 2 pts. on the sphere → Rotate it by 'U'.
→ Distance is preserved bcos distance here is compared with metric of the embedding space ⇒ S^3 is isotropic)

$$\cancel{R}\begin{pmatrix} \sin\psi' \sin\theta' \cos\phi' \\ \sin\psi' \sin\theta' \sin\phi' \\ \sin\psi' \cos\theta' \\ \cos\psi' \end{pmatrix} = \cancel{R}\, U \begin{pmatrix} \sin\psi \sin\theta \cos\phi \\ \sin\psi \sin\theta \sin\phi \\ \sin\psi \cos\theta \\ \cos\psi \end{pmatrix}$$

(Bcos of orthonormality, one eqn. is trivial)

(Appling the above eqns., we will see that rotn. is an isometry of of ds_3^2)



① ((Given any 2 pts., we can find an isometry which takes us from one to another — this is by $SO(4)$))
→ Homogeneity is satisfied

② $\{x_0^\mu + \delta x^\mu\}$ $\quad$ $A \cdot \{x_0^\mu + \delta y^\mu\}$

A'

$$g_{\mu\nu}(x_0)\, \delta x^\mu \delta x^\nu = g_{\mu\nu}(x_0)\, \delta y^\mu \delta y^\nu$$

x_0

(Taking A to A' is an isometry → Homogeneity)

That isometry must be such that even for $\{x_0^\mu + \lambda\, \delta x^\mu\}$, $\{x_0^\mu + \lambda\, \delta y^\mu\}$, the same isometry should take from $\{x_0^\mu + \lambda\, \delta x^\mu\}$ to $\{x_0^\mu + \lambda\, \delta y^\mu\}$ → for isotropy

It's some kind of rotn. about x_0

the isometry should keep x_0^μ fixed → take $\lambda = 0$

Choose a point :- $\psi' = 0$

If it is isotropic ~~abo~~ around the North pole, it is isotropic about any other pt. — Here it is rot^n that preserves $\psi'=0$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

pt. on the sphere & on the z-axis

$$U = \begin{pmatrix} U_3 & 0 \\ 0 & 1 \end{pmatrix}$$

U_3 : 3×3 orthogonal matrix

This proves the space is also isotropic

③ Hyperboloid

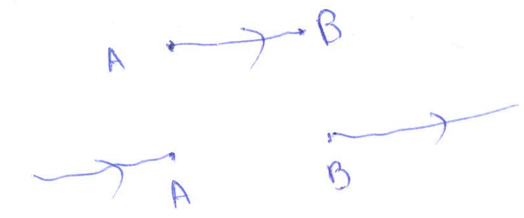
$$z_4^2 - \left(z_1^2 + z_2^2 + z_3^2\right) = R^2$$

$$ds^2 = -(dz_4)^2 + \sum_{i=1}^{3} (dz_i)^2$$

$$|z_4| > R \xrightarrow[\text{means}]{\text{this}} z_4 > R \text{ or } z_4 < -R$$

so we have 2 disconnected branches just like hyperbola

(We choose one particular branch bcos our 3-d space is connected & we will never know about the disconnected part)



(these 2 paths aren't deformable to one another if A & B are identified — our universe may be like this (not simply connected))

Choose the branch :- $z_4 > R$ → Three dim space

Coordinates :-

$$z_4 = R\cosh\psi$$
$$z_3 = R\sinh\psi\cos\theta$$
$$z_2 = R\sinh\psi\sin\theta\cos\phi$$
$$z_1 = R\sinh\psi\sin\theta\sin\phi$$

Ex: $ds_3^2 = ds_4^2\Big|_{\text{hyperboloid}} = R^2\left\{d\psi^2 + \sinh^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right\}$

The particular surface on which the induced metric is introduced has an induced Euclidean metric (it need not have been the case for other surfaces)

at the end whatever metric we get should describe an Euclidean space

this is imp.

Take:- $r = \sinh\psi$

Then $ds_3^2 = R^2\left\{\frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right\}$

4-dim. Lorentz trs. maps a point on the hyperboloid to another point on the hyperboloid and preserves the form of the metric $\rightarrow SO(3,1)$ group of isometries

(The 3d metric is derived from the 4d metric. So if the trs. preserves the 4d metric & the eqn. $z_4^2 - \sum_{i=1}^{3} z_i^2 = R^2$, it should preserve the 3d metric as well)

Homogeneous $\rightarrow$ (bcos by Lor. Trs. we can take one Lor. vector to another $\rightarrow$ one pt. on the surface of hyperboloid to another)

Isotropy $\rightarrow$ Choose $t=0$, $z = R\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

(Find a set of isometries which preserves this pt. & take us from one to another of any 2 pts. equidistant from z)

Isometry follows from :- $U = \begin{pmatrix} U_3 & 0 \\ 0 & 1 \end{pmatrix}$

Homogeneous and isotropic cosmological metric :—

$$ds^2 = -f_1(t)\,dt^2 + \hat{f}(t)\left\{\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right\}$$

$$k = \begin{cases} 0 \longrightarrow \text{flat space (Zero Curvature)} \\ 1 \longrightarrow \text{sphere (positive ")} \\ -1 \longrightarrow \text{Hyperboloid (Negative ")} \end{cases}$$

Under $t \rightarrow F^0(t)$, (we don't violate any of the assumptions we made earlier - we don't violate the idea of cosmic time & hence the cosmological principle - of course $t \rightarrow F(t, \vec{x})$ <u>isn't</u> allowed)

$$-f_1(F^0(t'))\left(\frac{dF^0}{dt'}\right)^2 dt^2 = dt'^2$$

if we choose F^0 such that

$$\frac{dF^0}{dt'} = \left(\sqrt{f_1(F_0(t'))}\right)^{-1}$$

$$\therefore\ ds^2 = -dt'^2 + \underbrace{\hat{f}\left(F^0(t')\right)}_{\left(\lambda(t')\right)^2} \left\{ \frac{dr^2}{1-kr^2} + \cdots \right\}$$

$$\Rightarrow ds^2 = -dt'^2 + \left(\lambda(t')\right)^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right\}$$

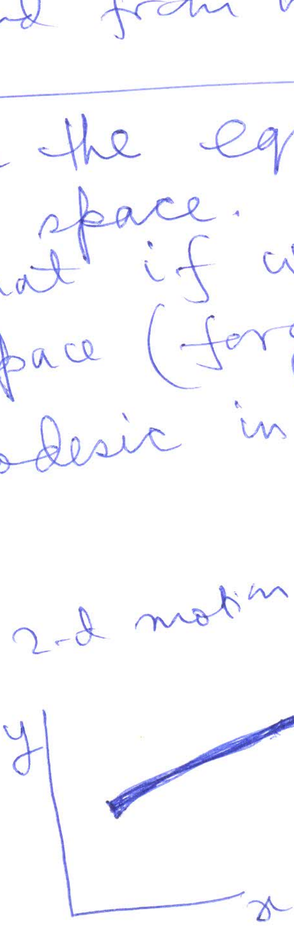
Here, $\lambda(t)$ is known as the <u>scale factor</u>.

(Homogeneity & isotropy don't restrict $\lambda(t)$)

If we know the avg. matter dist. in universe, we can find how λ evolves in time from Einstein's eqns.
If data about λ is available, matter content can be found from Einstein's eqns.

Ex. Consider the eqn for geodesic in the 4-d space.
Show that if we project it onto the 3-dim. space (forget $x^0 = t$), then it is a geodesic in the 3-dimensional space.

e.g. → 2-d motion



projection of the full trajectory in the space part

To get the time evolution, we need to draw it in 3-d with time along z-axis

$$ds^2 = -dt^2 + (\lambda(t))^2 g_{ij}(\vec{x})\, dx^i dx^j$$ → COMOVING COORDINATES

↳ for any metric of this form the statement holds — only in this case we can talk about the 3-d geodesic without reference to t

---

We have to show the x^i-eqn are the ones in which terms like Γ^i_{00} or Γ^i_{0j} are absent in

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} = 0$$

Take 2 points P & Q in this 3-d space:—

$\dot{P}$ $\dot{Q}$

Consider a light ray that starts at P at $t = t_1$ & ends at Q at $t = t_2$.

Given P, Q and t_1, we want to calculate t_2.

→ Firstly, $ds^2 = 0$ — b'cos light travels along null geodesic

$$dt^2 = (\lambda(t))^2 g_{ij}(\vec{x})\, dx^i dx^j$$

$\therefore$

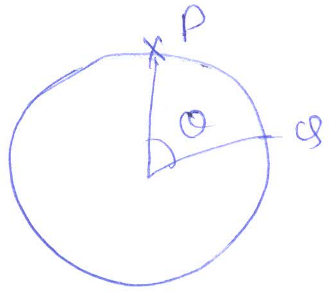
$$\Rightarrow \int_{t_1}^{t_2} \frac{dt}{\lambda(t)} = \int_P^Q \sqrt{g_{ij}(\vec{x})\, dx^i dx^j}$$

$$= \int_P^Q \sqrt{g_{ij}(\vec{x}) \frac{dx^i}{du} \frac{dx^j}{du}}\; du$$

where u is the parameter labelling the path.

The path connecting P & Q must be a geodesic connecting P, Q

as is ~~foun~~ proved in the exercise ← null geodesic in 4-d Min. space gives us null geodesic in 3-d space



(∵ sphere is homogeneous, we choose our coord. such that P is at the North Pole — then the θ-coord. of Q is the answer for the geodesic length from P to Q)

P emits another signal at $t_1 + \Delta t_2$

Q receives it at $t_2 + \Delta t_2$.

$$\int_{t_1+\Delta t_1}^{t_2+\Delta t_2} \frac{dt}{\lambda(t)} = \int_P^Q \sqrt{g_{ij}(\vec{x})\frac{dx^i}{du}\frac{dx^j}{du}}\, du$$

$$\Rightarrow \frac{\Delta t_2}{\lambda(t_2)} = \frac{\Delta t_1}{\lambda(t_1)}$$

for small Δt_1, Δt_2

$$\Rightarrow \boxed{\Delta t_2 = \Delta t_1 \frac{\lambda(t_2)}{\lambda(t_1)}}$$

Comoving coordinates ⟹ when we define the cosmic time & the 3-d metric takes the form $g_{ij}(\vec{x})\lambda^2$ ~~with~~ where g_{ij} is indep. of t

t_1, P

$\Delta t_1 = \frac{1}{\nu}$ → intrinsic frequency of the source

Proper time of the light source sitting at P is the cosmic time

Observed frequency :-

$$\frac{1}{\Delta t_2} = \frac{1}{\Delta t_1} \frac{\lambda(t_1)}{\lambda(t_2)} = \nu \frac{\lambda(t_1)}{\lambda(t_2)}$$

Same source placed at $\mathcal{G}$ will have frequence $\Delta t = \frac{1}{\nu}$

$$\therefore \boxed{\nu_{obs} = \nu_{intrinsic} \frac{\lambda(t_1)}{\lambda(t_2)}} \text{ — (A)}$$

t_1 :— current time t_0 (of the observer)

We can't change t_1 → not in our hand

We can change t_2 by looking at diff. stars at diff. distances.

One observes a series of lines & calculating the ratio of the lines, which is an invariant for a particular intrinsic spectrum, we can do the comparison.

(A) is a linear relation — ratio is same - - -

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$$\nu' = \frac{\lambda(t_1)}{\lambda(t_2)} \nu$$

$\lambda(t_1) \to$ source time

ν' → observed frequency

$\lambda(t_2)$ → observer time t_0

ν → actual frequency

$$\boxed{\nu' = \frac{\lambda(t_1)}{\lambda(t_0)} \nu}$$

$t_0 \longrightarrow$ denotes the present time

(One observes ~~...~~ several spectral lines & calculating their ratios, which are the same always, finds the red-shift)

Det. $\frac{\lambda(t_1)}{\lambda(t_0)}$ by averaging over many stars which are nearly at the same distance from us — this is bcos observing a single star, the peculiar vel. will come in our obsvn.

<u>Expt.</u> :-

$\nu' < \nu$ for faraway objects

$\lambda(t_1) < \lambda(t_0)$

$t_1 < t_0$

$\Rightarrow \lambda(t)$ is increasing with time.

$\Rightarrow$ Universe is expanding.

$$ds^2 = -dt^2 + (\lambda(t))^2 g_{ij}(\vec{x})\, dx^i dx^j$$

physical distance $\Rightarrow$ distance measured in the metric $g_{ij}(\vec{x}) \times \lambda(t)$

($\therefore$ as $\lambda \uparrow$, phys. dist. $\uparrow$ for any any 2 points in the comoving coord.)

Hubble constant $H_0 = \frac{1}{\lambda(t_0)} \frac{d}{dt} \lambda(t) \Big|_{t=t_0}$

↳ measures rate of change of λ at present time — we normalise it by $\frac{1}{\lambda(t_0)}$

$$\frac{\lambda(t)}{\lambda(t_0)} = 1 + H_0(t_0)(t-t_0) - \frac{1}{2} q_0(t_0)\left(H_0(t_0)\right)^2 (t-t_0)^2 + \cdots$$

$$q_0(t_0) = -\frac{1}{H_0(t_0)^2 \lambda(t_0)} \frac{d^2\lambda(t)}{dt^2}$$

$\hookrightarrow$ deceleration parameter

It was believed that universe is decelerating & so historically, q_0 was introduced

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Red shift parameter

$$z(t_1) = \frac{\lambda(t_0)}{\lambda(t_1)} - 1$$

($\rightarrow$ this is due to expansion of universe. For farway objects, grav. red-shifts are small compared to cosmological red-shift)

It is the measurement of time ~~that~~ — z is directly measurable — z can be converted to scale factor, which can be converted to time

---

(For consistency of our eqn., our energy-mom. tensor should have same isometries as $g_{\mu\nu}$)

$T_{\mu\nu}$, being a source for $g_{\mu\nu}$, must have the same isometries as $g_{\mu\nu}$.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

($R_{\mu\nu}$, R are form-invariant bcos $g_{\mu\nu}$ is form-invariant — $R_{\mu\nu}$ & R are constructed from $g_{\mu\nu}$ $\Longrightarrow$ $T_{\mu\nu}$ must also have similar prop.)

General coord. trf:—

$$T'_{\mu\nu}(x') \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} = T_{\rho\sigma}(x)$$

Isometry $\Rightarrow$ $T'_{\mu\nu}(x') = T_{\mu\nu}(x')$

$$\Rightarrow T_{\mu\nu}(x') \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} = T_{\rho\sigma}(x)$$

$$ds^2 = -dt^2 + (\chi(t))^2 \tilde{g}_{ij}(\vec{x})\, dx^i dx^j$$

If the tensor in consideration is related to $g_{\mu\nu}$, it should have the same isometries — e.g. for EM field, the energy-mom. tensor T_{em} should have the same isometry — of course A_μ ~~can have diff.~~ i may not obey the isometry, while T_{em} obeys it, it is very diff. to have such a situation — so better A_μ also obeys it or say the pot. A_μ obeys it

$$t' = t$$
$$x'^i = F^i(\vec{x})$$

$$T_{00}(x') = T_{00}(x)$$

$$T_{00}(t, \vec{x}') = T_{00}(t, \vec{x}) \quad \text{— (1)}$$

$$T_{0i}(t, \vec{x}) = T_{0j}(t, \vec{x}') \frac{\partial x'^j}{\partial x^i} \quad \text{— (2)}$$

$$T_{ij}(t, \vec{x}) = T_{kl}(t, \vec{x}') \frac{\partial x'^k}{\partial x^i} \frac{\partial x'^l}{\partial x^j} \quad \text{— (3)}$$

(1) implies $T_{00}(t, \vec{x}') = \rho(t)$

(2) & (3) imply $\longrightarrow$
Regarding t as a fixed parameter
$T_{0i}(t, \vec{x})$ is a form-invariant covariant rank 1 tensor in 3-dimension.
$T_{ij}(t, \vec{x})$ is a form invariant covariant rank 2 tensor in 3-dimensions.
(Form-inv. tensors are very hard to find)

$k=0$ case :- (flat space)

A const. vector field cannot be rotationally invariant

$$V(\vec{x}) = \vec{V}(\vec{x}+\vec{a})$$
$$\Rightarrow \vec{V} \text{ is constant}$$

$$\Rightarrow \vec{V}=0 \text{ (isotropy)}$$

$$V_{ij}(\vec{x}) = V_{ij} = \text{constant}$$
$$x'^i = R_{ij}\, x^j$$
$$\underset{\displaystyle V_{ij}}{\overset{\shortparallel}{V'_{ij}}} = R_{ik} R_{jl} V_{kl} \Rightarrow \boxed{V = c\,\delta_{ij}}$$

$$V = c\,\tilde{g}_{ij}$$
(bcos in flat space $\tilde{g}_{ij} = \delta_{ij}$)

(Analysis for $k=\pm 1$ cases also give the same result, though the calculations are more involved)

Soln :- $T_{00}(t,\vec{x}) = \rho(t)$

most general form one find when $T_{\mu\nu}$ respects the same isometries as $g_{\mu\nu}$

$$T_{0i}(t,\vec{x}) = 0$$
$$T_{ij}(t,\vec{x}) = f(t)\,\tilde{g}_{ij} = \underbrace{f(t)\,\lambda(t)^{-2}}_{p(t)}\, g_{ij}$$

called pressure bcos it mimics pressure

$$D_\mu T^{\mu\nu} = 0$$

Ex. Show that this gives :-
$$\frac{d}{dt}(\rho\lambda^3) = -3p\lambda^2\dot{\lambda}$$

Note :- No one gives us $f(t)$ - has to be determined from dynamics - of we may be able to find $f(t_0)$ by obsv^n.

Ex. $$ds^2 = -dt^2 + \lambda(t)^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right\}$$

Check that :-

$$R_{00} = \frac{3\ddot{\lambda}}{\lambda}, \quad R_{ij} = -(\lambda\ddot{\lambda} + 2\dot{\lambda}^2 + 2k)\tilde{g}_{ij}$$

$$R = g^{00}R_{00} + g^{ij}R_{ij} = -6\frac{\ddot{\lambda}}{\lambda} - 6\left(\frac{\dot{\lambda}}{\lambda}\right)^2 - 6k/\lambda^2$$

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

Ex. 00 - component gives

$$-8\pi G \rho = -3\left(\frac{\dot{\lambda}}{\lambda}\right)^2 - 3k/\lambda^2 \quad \text{--- (1)}$$

& ij - component gives

$$-8\pi G p = 2\frac{\ddot{\lambda}}{\lambda} + \left(\frac{\dot{\lambda}}{\lambda}\right)^2 + \frac{k}{\lambda^2} \quad \text{--- (2)}$$

(only one eqn. bcos both LHS & RHS are prop. to $\tilde{g}_{ij}$)

---

(1) & (2) aren't independent bcos $\longrightarrow$

$D_\mu (R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}) = 0$ is an identity

Ex. Take $\frac{d}{dt}$ (1) and substitute $\ddot{\lambda}$ from (2)

$\xrightarrow[\text{us}]{\text{gives}}$ $$\frac{d}{dt}(\rho\lambda^3) = -3p\lambda^2\dot{\lambda}$$

---

aren't indep. bcos (2) can be seen to follow from (1) & $\frac{d}{dt}(\rho\lambda^3) = -3p\lambda^2\dot{\lambda}$. Remember we have the Bianchi identity

So, we can work with the eqns

$$\boxed{\frac{d}{dt}(\rho\lambda^3) = -3p\lambda^2\dot{\lambda}}$$

$$\boxed{3\left(\frac{\dot\lambda}{\lambda}\right)^2 + 3k/\lambda^2 = 8\pi G\rho}$$

Energy-mom should follow as a consequence of Einstein eqns by using Bianchi identities → it's no surprise ② isn't indep.

($\lambda(t)$, $\rho(t)$, $p(t)$ → 3 unknown fns
So 2 first order diff. eqns for 3 unknowns — so time evolution isn't det. completely)

Unknown functions ⇒ $\rho(t)$, $p(t)$, $\lambda(t)$

→ Need some more information.

(this is given by the prop. of energy-mom. tensor — not that we need to know it as a fn. of t)

Additional information needed : Equation of state

gives us

functional relationship between p & ρ

(This eqn. of state is indep. of background)

$$T_{00} = -\rho(t)\, g_{00}$$

$$T_{ij} = p(t)\, g_{ij}$$

(T_{00} transforms exactly as g_{00}
T_{ij} " " " g_{ij}
→ So relation betn. ρ & p doesn't depend on the coord. frame we have chosen)

We can show that relation betn. ρ & p won't change under
$t \to f(t)$
$g_{ij} \to \tilde{g}_{ij}(\vec{x}\,')$)

Possible sources in cosmology

1. Non relativistic matter. → their vel. << c

2. Radiation

   Rel. matter can be included in radiation – behaves as light – indistinguishable from props. of radiation

3. Cosmological constant
   (can be either thought of a source in $T_{\mu\nu}$, or as a term modifying Einstein's eqn)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$$

Λ → cosmological constant

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(1) $\Rightarrow \quad p = 0$ (Dark matter is included here)

(2) $\Rightarrow \quad p = \rho/3$

(3) $\Rightarrow \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu}\right)$

$$\therefore \quad T_{\mu\nu}^{\cos} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} \Rightarrow \rho = \frac{\Lambda}{8\pi G}$$

$$p = -\frac{\Lambda}{8\pi G}$$

Recall:-
$T_{00} = -\rho(t) g_{00}$
$T_{ij} = p(t) g_{ij}$

(This is sometimes called dark energy)

(when we talk about NR matter, we essentially mean temp. T = 0
Imagine int. betn. matter & radiation is so small that they aren't equilibriating - - -)

Now, $T^{\mu\nu} = \sum_n m_n \int du \left(\sqrt{-\det g}\right)^{-1} \frac{dx^\mu}{du} \frac{dx^\nu}{d\tau} \delta^{(4)}(x - X(u))$

Non-relativistic matter :-

$$\left|\frac{dx^i}{d\tau}\right| << \frac{dx^0}{d\tau}$$

$(x^0 = ct)$

$$|T^{ij}| << T^{00}$$

So, $T_{ij} = 0$ for non-relativistic matter

$$\Rightarrow p = 0$$