What kind of trajectory does it describe in 3d subspace?

→ Forget about time. The geodesic in 4d is also a ~~trajectory~~ geodesic in 3d. That geodesic is a curve/line for which $\theta = 0$, $\phi = 0$ at every pt.

⇒ The light travels along the spacetime curve

$$-dt^2 + \lambda(t)^2 \frac{dr^2}{1-kr^2} = 0$$

$$\Rightarrow \int_{t_1}^{t_0} \frac{dt}{\lambda(t)} = \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}}$$

$t_1 \Rightarrow$ initial time (when the light was emitted from the source)

$$= \begin{cases} r_0 & \text{for } k = 0 \\ \sin^{-1} r_0 & \text{for } k = 1 \\ \sinh^{-1} r_0 & \text{for } k = -1 \end{cases}$$

① t_1 can be found in terms of r_0, H_0, Ω_m, Ω_Λ, Ω_r.

Suppose the source emits a total energy E per unit time. (per unit cosmic time)

Energy/unit area measured by the observer: $F = \frac{E}{4\pi r_0^2 \lambda_0^2} \frac{\lambda(t_1)}{\lambda_0}$

The energy of each photon decreases by $\frac{\lambda(t_1)}{\lambda_0}$

② F is observed.

F is known for a Standard Candle

$\Rightarrow$ relation between r_0, t_1, H_0, Ω_m, Ω_Λ, Ω_r

We do integral over θ & ϕ for a fixed r $(=r_0)$ & so we get 4π for $k=0, 1, -1$:- $r_0^2 \int \sin^2\theta \, d\theta \int d\phi = 4\pi r_0^2$

using ① & ②

Using this, we can determine t_1 for a given source as a fn. of the Cosmological parameters.

using $F_1 = \frac{E}{4\pi r_0^2 \lambda_0^2} \frac{\lambda}{\lambda_0}$ & ① .....

$$\frac{\lambda(t_1)}{\lambda(t_0)} = \underbrace{1 - H_0(t_1 - t_0) - \frac{1}{2} q_0^2 H_0^2 (t_0 - t_1)^2 + \cdots}$$

observed for a given source

known as a fn. of the cosmological parameters & luminosity observation.

[But we don't know the cosmological parameters.
But every obsn. is some eqn. of cosmological parameters
$\longrightarrow$ can be det. from these obsns]

For H_0 the analysis can be simplified

$$\frac{\lambda(t_0)}{\lambda_0} = 1 - H_0(t_0 - t_1) + O\{(t_0 - t_1)^2\}$$

To calculate H_0, we need to study the relations for small $(t_0 - t_1)$.

$$\int_{t_1}^{t_0} \frac{dt}{\lambda(t)} = \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}}$$

gives us

$$\frac{1}{\lambda_0}(t_0 - t_1) \simeq r_0$$ for t_1 close to t_0

(Similary, ~~the~~ we get) $\rightarrow$ $$F \simeq \frac{F}{4\pi r_0^2 \lambda_0^2}$$

(for t_1 close to t_0)

$$\therefore \quad t_0 - t_1 = \lambda_0 r_0 = \sqrt{\frac{E}{4\pi F}}$$

Approximating everything by the 1st order term

$(t_0 - t_1)$ is a dimensionful quantity so the question is whether the dimensionless quantity $H_0(t_0 - t_1)$ is small or large

H_0 can be ~~ob~~ calculated by looking at <u>nearby objects</u>, as it is related to first deriv.

$\boxed{H_0(t_0 - t_1) \text{ is the relevant quantity}}$ which determines the expansion is good or not

$(t_0 - t_1)$ should be small w.r.t. some unit of measurement of time, say the time in which universe doubles its size

---

~~Other~~

Other measurements

① Age of the universe.

$$\frac{\dot{\lambda}}{\lambda} = \sqrt{\frac{8\pi G}{3}\rho - k/\lambda^2}$$

$$= \sqrt{\frac{8\pi G}{3}\rho_c}\left\{\frac{\rho}{\rho_c} - \frac{k}{\lambda^2}\frac{3}{8\pi G\rho_c}\right\}^{1/2}$$

where $\rho_c = \frac{3H_0^2}{8\pi G}$

$$\Rightarrow \dot{\lambda}/\lambda = H_0\left\{\rho/\rho_c - \frac{k}{\lambda^2 H_0^2}\right\}^{1/2}$$

where $\rho = \rho_m + \rho_\Lambda + \rho_r$

$$\Rightarrow \dot{\lambda}/\lambda = H_0\left\{\frac{\rho_{m0}}{\rho_c}\frac{\lambda_0^3}{\lambda^3} + \rho_{\Lambda 0}/\rho_c + \rho_{r0}/\rho_c\,\lambda_0^4/\lambda^4 - \frac{k}{\lambda^2 H_0^2}\right\}^{1/2}$$

$$\Rightarrow \dot{\lambda}/\lambda = H_0\left\{\Omega_m \lambda_0^3/\lambda^3 + \Omega_\Lambda + \Omega_r\frac{\lambda_0^4}{\lambda^4} - k/\lambda^2 H_0^2\right\}^{1/2}$$

Define :- $\lambda = \lambda_0 u$

$$\Rightarrow \frac{\dot{u}}{u} = H_0\left\{\Omega_m u^{-3} + \Omega_\Lambda + \Omega_r u^{-4} - \frac{k}{\lambda_0^2 H_0^2 u^2}\right\}^{1/2}$$

Origin of time :- $u = 0$ at $t = 0$

Integrating, we get

$$\int_0^{t_0} dt = H_0^{-1} \int_0^1 \frac{du}{u\left\{\Omega_m u^{-3} + \Omega_\Lambda + \Omega_r u^{-4} - \frac{k}{\lambda_0^2 H_0^2} u^{-2}\right\}^{1/2}}$$

$$\parallel$$

$$t_0$$

↳ Age of the universe

If we knew an indep. way of det. t_0, that will give info. about $\Omega_m, \Omega_\Lambda, \Omega_r, H_0$

→ But unfortunately we don't have any such way

→ but we can put a bound on t_0 by knowing the oldest object.

---

e.g. → Globular clusters — we know how they evolve & so can find their age.

$$\boxed{t_0 > t_{\text{oldest object}}}$$

Previously, H_0 wasn't det. properly. There wasn't much problem about not considering Ω_Λ bcos ~~Ω_m dominated over~~ $\Omega_m u^{-3}$ dominated over Ω_Λ at earlier times

---

Measurement of Cosmic Microwave background radiation

$T \simeq 2.73\,K$ (so energy content of CMBR is very small)

$$\nu(t) = \nu(t_0)\frac{\lambda(t_0)}{\lambda(t)}$$

Most of contribution to the integral comes from u close to 1 → so age of oldest star or galaxy is close to the age of the universe

$\because \lambda(t_0) > \lambda(t)$, $\therefore \nu(t) > \nu(t_0)$

$$T(t) = T_0 \frac{\lambda(t_0)}{\lambda(t)}$$

temperature

Scaling up the current blackbody radiation distribution

$\left(\frac{h\nu}{kT}\right)$ enters the dist. law — so scaling up ν means we get a higher T

At $\frac{\lambda(t_0)}{\lambda(t)} \sim 10^3$

$T(t) \sim$ ionization temperature hydrogen

Above this temperature matter was ionized.

assuming matter is mostly hydrogen

has strong interaction with radiation

Ionized matter much more strogly with radiation bcos it is charged

(as a result)

matter & radiadian were in thermal equilibrium

(Radiation stopped int. with matter, except occassionally, when universe cooled & protons & e^-'s combined to form H-atom)

Microwave ~~ra~~ radiation we see today consists of the photon emitted from matter at the ionization temp.

(So if we observe CMBR, fluctuation in CMBR will give us an idea about the " " matter at the time of decoupling)

Fluctuation in microwave background over angular coordinate $\xrightarrow[\text{us}]{\text{gives}}$ fluctuation in spatial distribution of matter at the time of matter-radiation decoupling

(We have a th. of matter fluctuation at the time of ~~earlier times~~ decoupling & it tells us) →

fluctuations over wavelength of

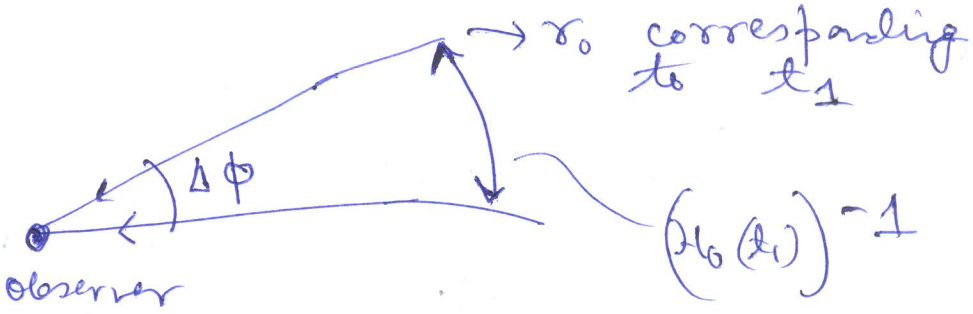
$$\left(\mathcal{H}(t_1)\right)^{-1} \rightarrow \text{decoupling}$$

$$\parallel$$

$$\left.\frac{\dot{\lambda}(t_1)}{\lambda(t_1)}\right|_{t_1}$$

How will these fluctuations look today?

Recall:-
$$\int_{t_1}^{t_0} \frac{dt}{\lambda(t)} = \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}}$$



Evolve the geodesic backward to r_0

$\Delta\phi$:- angular scale of fluctuation in CMBR

⇓ converted to an oscillation of CMBR in the angular scale

→ can be observed

$\Delta\phi$ known from obsvn

r_0 known from the eqn

We know distance betn the pts. by extrapolating back in time

$\xrightarrow{\text{gives us}}$ an equation involving the cosmological parameters

---

the main quantity that enters into the calculation is $\Omega_{tot} = \Omega_m + \Omega_\Lambda + \Omega_r$

$$\simeq 1 \longrightarrow \text{found by us through the calculation}$$

① This CMBR measurement is very accurate

② Supernovae also give q_0 & that is also very reliable

These 2 together gives us

$$\Omega_m = 0.26$$
$$\Omega_\Lambda = 0.74$$
$$\Omega_r \simeq 10^{-5}$$

Direct measurements of Ω_m isn't very reliable.

10/10/07 Gravitational action principle

In a conventional dynamical system, we can get EOMs by writing an action & demanding that 1st order variation of the action is zero.

We want to do the same for gravity

Look for an action S such that extremizing S we get Einstein's eqn. + other possible field eqns.

$$\delta S = 0 \Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

$\therefore$ S must be such that $\delta S \propto \int d^4x\, \delta g_{\mu\nu}(x) \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + 8\pi G T_{\mu\nu} \right]$

$$\delta S \propto \int d^4x \sqrt{-\det g} \Big[\delta g_{\mu'\nu'}(x)\, g^{\mu\mu'}(x)\, g^{\nu\nu'}(x) \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + 8\pi G T_{\mu\nu} \right)$$

gets contribution from metric, Maxwell field, condensates

+ other eqns of motion

multiplying RHS by ....

Volume element is $\sqrt{-g}\, d^4x$

$\times\ \delta$(other dynamical variables)$\Big]$

First concentrate on the other eqns of motion:-

The trajectory of a particle, $X^\mu(s)$ is obtd. by extremising ds^2

Particle motion

Eqn of motion $\longrightarrow$ geodesic eqn

$$S_{particle} = c \int^1 ds \sqrt{g_{\mu\nu}(X(s)) \frac{dX^\mu}{ds} \frac{dX^\nu}{ds}}$$

This integral is not defined in Minkowski space since ds^2 isn't +ve-definite. $\therefore$ we take variation in Euclidean space & continue to Minkowski

$$\delta S = c \int_0^1 ds\, \delta X^\mu(s) \times (\text{geodesic eqn})$$

How about variation of S with respect to $g_{\mu\nu}(x)$?

change $g_{\mu\nu}(x) \longrightarrow g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$

Given that $S_{particle}$ is a part of S, we can ask what is the variation of $S_{particle}$ under a variation of the metric?

$\delta S_{particle} + S_{particle}$

$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$

$$= c \int_0^1 ds \sqrt{g_{\mu\nu}(x) \frac{dX^\mu}{ds}\frac{dX^\nu}{ds} + \delta g_{\mu\nu}(X) \frac{dX^\mu}{ds}\frac{dX^\nu}{ds}}$$

$$= c \int_0^1 ds \sqrt{g_{\mu\nu} \frac{dX^\mu}{ds} dX^\nu/ds} \left\{ 1 + 2 \frac{\delta g_{\rho\sigma}(X(s)) \frac{dX^\rho}{ds}\frac{dX^\sigma}{ds}}{g_{\kappa\sigma}(x) \frac{dX^\kappa}{ds}\frac{dX^\sigma}{ds}} \right\}$$

$$\therefore \delta S_{particle} = c/2 \int d^4x \sqrt{-\det g(x)}\, \delta g_{\rho\sigma}(x) \int_0^1 ds \left(\sqrt{-\det g(x)}\right)^{-1} \prod_{\mu=0}^{3} \delta(x^\mu - X^\mu(s))$$

$$\left(g_{\rho\sigma}(X) \frac{dX^\rho}{ds}\frac{dX^\sigma}{ds} \right)^{-1/2} \frac{dX^\rho}{ds}\frac{dX^\sigma}{ds}$$

$$\cancel{= c/2 \int d^4x\, \delta g_{\rho\sigma}(x) \sqrt{-\det g(x)}}$$

$$= c/2 \int d^4x\, \delta g_{\rho\sigma}(x) \sqrt{-\det g(x)} \int_0^1 ds \Big[\left(\sqrt{-\det g(X(s))}\right)^{-1} \delta^{(4)}(x - X(s)) \left(\frac{d\tau}{ds}\right)^{-1} \frac{dX^\rho}{ds}\frac{dX^\sigma}{ds} \Big]$$

$\frac{1}{m} T^{\rho\sigma}$

$$= c/2m \int d^4x\, \delta g_{\rho\sigma}(x) \sqrt{-\det g(x)}\, T^{\rho\sigma}(x)$$

Take $S_{particle} = m \int_0^1 ds \sqrt{g_{\mu\nu}(x) \frac{dX^\mu}{ds}\frac{dX^\nu}{ds}}$

$$\delta S_{particle} = 1/2 \int d^4x\, \delta g_{\mu\nu}(x)\, T^{\mu\nu}(x)$$

In a sense, the particle action already gives the correct contribution to $T_{\rho\sigma}$ under variation of the metric.

Consider Maxwell eqn :-

$$D_\mu F^{\mu\nu} = 0$$

Now let us do the same thing for the Maxwell action.

In free space it is obtained by varying the action:

$$S_{maxwell} = -\frac{1}{4}\int d^4x\, F_{\mu\nu} F_{\rho\sigma}\, \eta^{\mu\rho}\eta^{\nu\sigma}$$

NOTE $\delta A^\mu \neq g^{\mu\nu}\delta A_\nu$

$$\delta S_{max} = -\tfrac{1}{2}\int d^4x \sqrt{-g}\, g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu}\underbrace{[\partial_\rho \delta A_\sigma - \partial_\sigma \delta A_\rho]}_{2\,\partial_\rho(\delta A_\sigma)}$$

$$\delta S_{maxwell} \propto \int d^4x\, \partial_\mu F^{\mu\rho}(x)\, \delta A_\rho(x)$$

$$\Rightarrow \delta S_{max} = -\int d^4x \sqrt{-g}\, \delta A_\sigma\, \partial_\rho[\sqrt{-g}\, g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu}]$$

Q) What action gives the curved space Maxwell's eqn?

Ans. $\longrightarrow$

$$S_{maxwell} = -\frac{1}{4}\int d^4x \sqrt{-\det g}\; g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho}(x) F_{\nu\sigma}(x)$$

This action correctly reproduces the EOM. $\therefore$ we should not modify its dependence on A_μ.

Ex. Check that:

$$\delta S_{maxwell} \propto \int \partial_\mu F^{\mu\nu}(x)\, \delta A_\nu(x) \sqrt{-\det g(x)}$$

under $A_\mu \longrightarrow A_\mu + \delta A_\mu$ $\Rightarrow D_\rho F^{\rho\sigma} = 0$

Now we study S_{max} under $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$

Q) What is the $\delta S_{maxwell}$ under $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$?

$$S_{max} = -\tfrac{1}{4}\int d^4x \sqrt{-g}\, g^{\mu\rho} g^{\nu\rho} F_{\mu\nu} F_{\rho\sigma}$$

$\longrightarrow$ no dependence on $g_{\mu\nu}$

$\longrightarrow$ $g^{\mu\nu}$: Matrix elements of g^{-1}

$$\because \delta(g^{-1}g) = 0, \therefore \delta g^{-1} = -g^{-1}\,\delta g\, g^{-1}$$

$$\boxed{\begin{aligned} &g g^{-1} = 1 \\ &\Rightarrow \delta g\, g^{-1} + g\,\delta g^{-1} = 0 \\ &\Rightarrow \delta g^{-1} = -g^{-1}\,\delta g\, g^{-1} \end{aligned}}$$

$$\therefore \delta g^{\mu\nu} = -g^{\mu\rho}\,\delta g_{\rho\sigma}\, g^{\sigma\nu}$$

Also, $\delta\left(\sqrt{-\det g}\right) = \frac{1}{2}\left(\sqrt{-\det g}\right)^{-1}\delta(-\det g)$

Again $-\det(g + \delta g) = -\det g \det(1 + g^{-1}\delta g)$

$$= -\det g \times [1 + \mathrm{tr}(g^{-1}\delta g)]$$

$$= -\det g\,(1 + g^{\mu\nu}\delta g_{\mu\nu})$$

$$\therefore -\delta(-\det g) = -(\det g)\, g^{\mu\nu}\,\delta g_{\mu\nu}$$

$$\Rightarrow \delta\sqrt{-\det g} = \frac{1}{2}\sqrt{-\det g}\; g^{\mu\nu}\,\delta g_{\mu\nu}$$

Ex: Check that

$$\delta S_{Maxwell} = \frac{1}{2}\int d^4x \sqrt{-\det g}\ \delta g_{\mu\nu}(x)\ T^{\mu\nu}_{Maxwell}(x)$$

$$\delta S_{max} = -\frac{1}{4}\int d^4x \sqrt{-g}\, F_{\mu\rho}F_{\nu\rho}\left\{ \frac{g^{\kappa\tau}}{2}\,\delta g_{\kappa\tau}\, g^{\mu\rho} g^{\nu\sigma} - g^{\mu\kappa}\,\delta g_{\kappa\tau}\, g^{\tau\rho} g^{\nu\sigma} - g^{\mu\rho} g^{\nu\kappa}\,\delta g_{\kappa\tau}\, g^{\tau\sigma}\right\}$$

$$\delta S_{matter} = \frac{1}{2}\int d^4x\ \delta g_{\mu\nu}(x)\ \sqrt{-\det g(x)}\ T^{\mu\nu}_{matter}$$

Now Gravitational eqn of motion:-

$$\left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right) + 8\pi G T_{\mu\nu} = 0$$

or $$R^{\mu\nu} - \frac{1}{2}R g^{\mu\nu} + 8\pi G T^{\mu\nu} = 0$$

Need S_{grav} such that

$$\delta S_{grav} = \frac{1}{16\pi G}\left(R^{\mu\nu} - \frac{1}{2}R g^{\mu\nu}\right).$$

Then $$\delta S_{grav} + \delta S_{matter}$$

$$= \frac{1}{16\pi G}\left\{R^{\mu\nu} - \frac{1}{2}R g^{\mu\nu} + 8\pi G T^{\mu\nu}_{matter}\right\}$$

Ex: Show that

$$\delta \int \sqrt{-\det g}\, R = -\int d^4x \sqrt{-\det g}\left(R^{\mu\nu} - \frac{1}{2}R g^{\mu\nu}\right)\delta g_{\mu\nu}(x)$$

$$\Rightarrow S_{grav} = -\frac{1}{16\pi G}\int \sqrt{-\det g}\ R$$

All the equations of motion can be derived by varying $S_{grav} + S_{matter}$ with respect to the dynamical variables

10/10/07

Cosmological horizon :- Maximum distance light would have travelled since $t=0$, $\lambda \to 0$

Two points separated by more than this distance ~~wo~~ could not have communicated ever.

Assume that light travels radially outward from $r=0$ at $t=0$.

Then $ds^2 = \frac{\lambda^2}{1-kr^2}dr^2 - dt^2 \Rightarrow dt^2 = \lambda(t)^2 \frac{dr^2}{1-kr^2}$

$$\therefore \int_0^t \frac{dt}{\lambda(t)} = \underbrace{\int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}}}_{\text{Comoving horizon}}$$

$\therefore$ physical distance as seen today

$$= \lambda(t) \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}} = \lambda(t) \int_0^t \frac{dt'}{\lambda(t')} = d_H(t)$$

$\therefore$ if we know the current cosmological parameters, we can extrapolate $\lambda(t')$ backwards using

$$(\dot{\lambda}/\lambda)^2 + k/\lambda^2 = \frac{8\pi G}{3}\rho$$

where $\rho = c\lambda^{-\alpha}$

with $\alpha = 3$ matter

$\alpha = 4$ radiation

$\alpha = 0$ cosmol. constant

Assuming that only one component dominates at any epoch, we have

$\rho = c\lambda^{-\alpha}$

$$\therefore (\dot{\lambda}/\lambda)^2 + k/\lambda^2 = \frac{8\pi G}{3} c\lambda^{-\alpha}$$

Also assume $k = 0$ (since current obsvn is $k/\lambda^2 \ll \rho_{matter} \left(= \frac{\rho_{m0}}{\lambda^3}\right)$ $\Rightarrow$ in the past k/λ^2 will never dominate over ρ_{m0}/λ^3)

$$\therefore \quad \dot{\lambda}^2 = \frac{8\pi G}{3} c\, \lambda^{-\alpha+2}$$

$$\Rightarrow \quad \lambda^{(\alpha-2)/2}\, \dot{\lambda} = \sqrt{\frac{8\pi G}{3}}$$

$$\Rightarrow \quad \frac{2}{\alpha} \lambda^{\alpha/2} = \sqrt{\frac{8\pi G}{3}}\,(t-K)$$

$$\Rightarrow \quad \lambda = \left(\frac{\alpha}{2}\sqrt{\frac{8\pi G}{3}}\right)^{2/\alpha} (t-K)^{2/\alpha}$$

for $\alpha \neq 0$

$$\therefore \quad \lambda = \lambda_0 (t-K)^{2/\alpha}$$

Suppose this epoch begins at t_i & ends at t_f.
For a matter-dominated universe, $t_f \sim$ today.

$$d_H(t_i) = \lambda(t_i) \underbrace{\int_0^{t_i} \frac{dt'}{\lambda(t')}}_{\text{a different phase}}$$

$$\Rightarrow \quad \int_0^{t_i} \frac{dt'}{\lambda(t')} = \frac{d_H(t_i)}{\lambda(t_i)}$$

$$d_H(t_f) = \lambda(t_f) \left\{ \int_0^{t_i} \frac{dt'}{\lambda(t')} + \int_{t_i}^{t_f} \frac{dt'}{\lambda(t')} \right\}$$

$$= \frac{\lambda(t_f)}{\lambda(t_i)} d_H(t_i) + \int_{t_i}^{t_f} \frac{dt'}{\lambda_0 (t-K)^{2/\alpha}} \times \lambda(t_f)$$

$$= \frac{\lambda(t_f)}{\lambda(t_i)} d_H(t_i) + \frac{\lambda(t_f)\,(t-K)^{-2/\alpha+1}}{\lambda_0(-2/\alpha+1)} \Bigg|_{t_i}^{t_f}$$

$$\Rightarrow d_H(t_f) = \frac{\lambda(t_f)}{\lambda(t_i)} d_H(t_i) + \left(\frac{\alpha}{\alpha-2}\right)\frac{1}{\lambda_0}\left\{(t_f-K)^{1-2/\alpha} - (t_i-K)^{1-2/\alpha}\right\}$$

$$= \frac{\cancel{\lambda_0}(t_f-K)^{2/\alpha}}{\cancel{\lambda_0}(t_i-K)^{2/\alpha}} d_H(t_i) + \left(\frac{\alpha}{\alpha-2}\right)\frac{1}{\cancel{\lambda_0}}\left\{(t_f-K)^{1-2/\alpha} - (t_i-K)^{1-2/\alpha}\right\} \times \cancel{\lambda_0}(t_f-K)^{2/\alpha}$$

$\Rightarrow$ No dependence on λ_0

Now $\Rightarrow$ $$d_H(t_f) = (t_f-K)\left[\left\{\frac{\alpha}{\alpha-2}\left(1-\left(\frac{t_i-K}{t_f-K}\right)^{1-2/\alpha}\right)\right\} + \frac{d_H(t_i)}{t_i-K}\left(\frac{t_i-K}{t_f-K}\right)^{1-2/\alpha}\right]$$

$\alpha = 3 \Rightarrow 1-2/\alpha = 1/3$

$\alpha = 4 \Rightarrow 1-2/\alpha = 1/2$

Typically an epoch runs for a long time, i.e., $t_i << t_f$

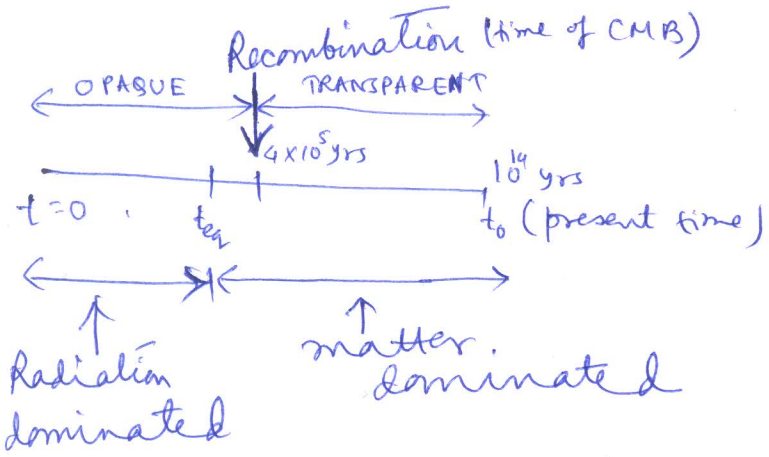
$$\Rightarrow \frac{t_i-K}{t_f-K} << 1$$

$\therefore$ if $\frac{d_H(t_i)}{t_i-K}$ is finite,

$$\boxed{d_H(t_f) \approx (t_f-K)\left(\frac{\alpha}{\alpha-2}\right)}$$

[if $d_H(t_i)$ is very large, the second term might also contribute]

Standard Big-Bang Cosmology



Before recombination, photons scattered by charged matter
$\Rightarrow$ coupled

CMB is "untouched" since after recombination

$\therefore$ CMB corresponds to time of ~~rem~~ recombination.

$\therefore$ we work in time after CMB was emitted for matter-dominated universe.

From $t=0$ to t_{recomb}, $\alpha = 4$

$$\lambda(0) = 0, \quad \lambda = \lambda_0 t^{1/2}$$

$$d_H(t_i) = 0$$

$$\therefore \quad d_H(t_f) = (t_f)\left(\frac{4}{4-2}\right) = 2t_f \rightarrow \text{Radiation dominated universe}$$

Take this as the initial condition for matter dominated ~~p~~ part.

$$d_H(t_i) = 2t_{eq} \qquad \therefore \quad \frac{d_H(t_i)}{t_i - K} \sim 1$$

$\therefore$ our approximation is valid

$\Rightarrow$ only first term dominates.

At the boundary :-

$$\lambda_0 (t_{eq})^{1/2} = \lambda_0 (t_{eq} - K)^{1/3}$$

$$\Rightarrow t_{eq} - K = (t_{eq})^{3/2}$$

$$\Rightarrow K = t_{eq} \left[1 - (t_{eq})^{1/2} \right]$$

$$\frac{d_H(t_{recomb})}{d_H(\text{now})} = \frac{4 \times 10^5}{10^{14}} \rightarrow \text{small number}$$

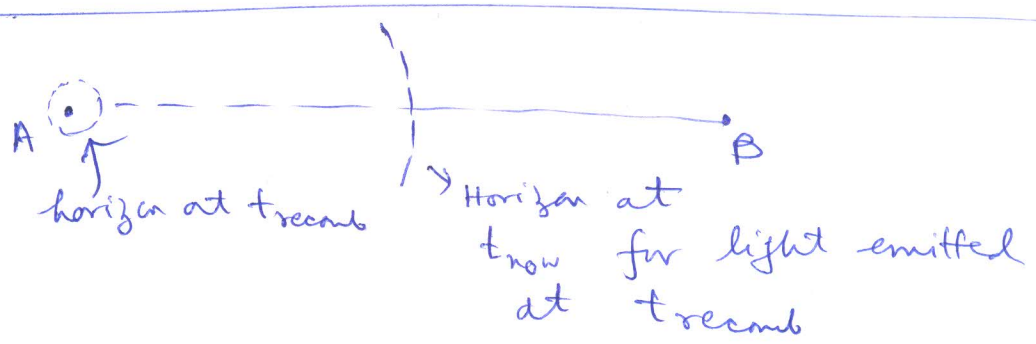
$$d_H(\text{now}) = (t_f - K) \times 3$$

Using $d_H(t_f) \simeq (t_f - K)\left(\frac{\alpha}{\alpha - 2}\right)$

$\therefore$ horizon today $\sim 10^{14}$ light yrs.

$\therefore$ if we consider light from two most distant stars at opposite sides of the sky, their separation $\sim 2 \times$ horizon.

So at the time of recombination, what would have been the horizon?



$\therefore$ regions at A & B can <u>never communicate</u>.

So temperature at A need not be the same as temperature at B.

But observed $\Rightarrow$ CMB is isotropic to order 10^{-5}

$\Rightarrow$ <u>HORIZON PROBLEM!</u> (Assuming the universe need not have started out uniform)

Only solution $\Rightarrow$ the two points have been in the horizon since t_{recom}.

$\therefore$ our estimate of the horizon is a horrible underestimate.

No solution in standard big bang cosmology

bcos $\Rightarrow$ $d_H(t_i)$ is large at some time

$\Rightarrow$ second term dominates

Ex: How does d_H in the cosmological constant dominated universe depend on?

[Here, cosm. const. could at some pt. be dominated by K/a^2]

$\Rightarrow$ $d_H(\text{cosm. const}) \propto e^t$

$\therefore$ perhaps there was an inflationary phase when a phase transition causes the cosmological constant to dominate.

Observer at rest w.r.t. CMB $\equiv$ observer has fixed comoving coordinates

(if not at rest, CMB will be anisotropic!)

Also, in comoving coordinates, the average over velocities of all objects will be zero.

11/10/07 For cosmological constant dominated universe, k/λ^2 term may dominate over ρ_Λ in the past

DESCRIPTION OF THE HORIZON PROBLEM IN COMOVING COORDINATES

$$\text{Horizon} = d(t) = \int \frac{dt'}{\lambda(t')} \equiv \text{Comoving coordinates}$$

For a given epoch dominated by only one component of ρ, we have

$$\rho = C\lambda^{-\alpha} \quad \left[\begin{array}{ll} \alpha = 3 & : \text{matter} \\ \alpha = 4 & : \text{radiation} \\ \alpha = 0 & : \text{cosm. const.} \end{array}\right]$$

$$\left(\dot{\lambda}/\lambda\right)^2 = \frac{8\pi G}{3} C\lambda^{-\alpha}$$

$$\Rightarrow \dot{\lambda}\,\lambda^{\frac{\alpha-2}{2}} = \left(\frac{8\pi G C}{3}\right)^{1/2}$$

$$\therefore \frac{2\lambda^{\alpha/2}}{\alpha} = \sqrt{\frac{8\pi G C}{3}}\,(t-k)$$

$$\Rightarrow \lambda = \left\{\sqrt{\frac{8\pi G}{3}}\left(\frac{\alpha}{2}\right)(t-k)\right\}^{2/\alpha}$$

$$\Rightarrow (t-k)^{1-2/\alpha} = \left\{2/\alpha\left(\frac{8\pi G C}{3}\right)^{-1/2}\right\}^{1-2/\alpha} \times \lambda^{\frac{\alpha}{2}(1-2/\alpha)}$$

$$\Rightarrow (t-k) = 2/\alpha\left(\frac{8\pi G C}{3}\right)^{-1/2}\lambda^{\alpha/2}$$

If t_{in} is the beginning of the epoch, then

$$d(t) - d(t_{in}) = \int_{t_{in}}^{t} dt'/\lambda(t')$$

$$= \left\{\frac{\alpha}{2}\left(\frac{8\pi G C}{3}\right)^{1/2}\right\}^{-2/\alpha} \int_{t_{in}}^{t} \frac{dt'}{(t'-k)^{2/\alpha}}$$

$$= \left\{ \frac{\alpha}{2} \left(\frac{8\pi G c}{3} \right)^{1/2} \right\}^{-2/\alpha} \int_{t_{in}}^{t} \frac{dt'}{(t'-k)^{2/\alpha}}$$

$$= \left\{ \frac{\alpha}{2} \left(\frac{8\pi G c}{3} \right)^{1/2} \right\}^{-2/\alpha} \times \frac{(t'-k)^{1-2/\alpha}}{(1-2/\alpha)} \Bigg|_{t_{in}}^{t}$$

$$= \left(\frac{\alpha}{\alpha-2} \right) \left\{ \frac{\alpha}{2} \sqrt{\frac{8\pi G c}{3}} \right\}^{-2/\alpha} \left\{ (t-k)^{1-2/\alpha} - (t_{in}-k)^{1-2/\alpha} \right\}$$

$$= \left(\frac{\alpha}{\alpha-2} \right) \left\{ \frac{\alpha}{2} \sqrt{\frac{8\pi G c}{3}} \right\}^{-2/\alpha} \times \left\{ \frac{2}{\alpha} \left(\frac{8\pi G c}{3} \right)^{-1/2} \right\}^{1-2/\alpha} \left[\lambda(t)^{\alpha/2-1} - \lambda(t_{in})^{\alpha/2-1} \right]$$

$$= \left(\frac{\cancel{\alpha}}{\alpha-2} \right) \left\{ \frac{2}{\cancel{\alpha}} \sqrt{\frac{3}{8\pi G c}} \right\} \left[\lambda(t)^{\frac{\alpha-2}{2}} - \lambda(t_{in})^{\frac{\alpha-2}{2}} \right]$$

$$\left(\because \lambda(t)^{\alpha/2-1} \gg \lambda(t_{in})^{\alpha/2-1} \right)$$

$$\simeq \frac{2}{\alpha-2} \sqrt{\frac{3}{8\pi G c}} \, \lambda(t)^{\frac{\alpha-2}{2}}$$

In the radiation dominated era, $\lambda(t_{in}) = 0$

$$t_{in} = t_{eq} \quad (\rho_{matter} = \rho_{radiation})$$

$$d(t_{eq}) = \frac{2}{4-2} \sqrt{\frac{3}{8\pi G c}} \, \lambda(t_{eq})^{\frac{4-2}{2}}$$

$$= \sqrt{\frac{3}{8\pi G c_r}} \, \lambda(t_{eq})$$

$$\therefore \boxed{d_{eq} = \sqrt{\frac{3}{8\pi G \rho_r}} \, \lambda_{eq}}$$

$$\left.\begin{aligned}\rho_r &= C_r/\lambda^4 \\ \rho_m &= C_m/\lambda^3\end{aligned}\right\} \Rightarrow \left(\frac{\rho_r}{\rho_m}\right)_0 = \underbrace{\left(\frac{\rho_r}{\rho_m}\right)_{eq}}_{=1} \times \frac{\lambda_{eq}}{\lambda_0}$$

$$\therefore \boxed{\frac{\lambda_{eq}}{\lambda_0} = \left(\frac{\rho_r}{\rho_m}\right)_0} \Rightarrow \frac{\lambda_{eq}}{\lambda_0} \approx 10^4$$

In the matter-dominated era,

$$d(t) - d(t_{eq}) = 2\sqrt{\frac{3}{8\pi G C}}\ \lambda(t)^{1/2}$$

Temperature :- $T^4 \propto \rho$ & $\rho \propto 1/\lambda^4 \Rightarrow T \propto 1/\lambda$

$$\Rightarrow T_{eq} \cong 3 \times 10^4\ K$$

For recombination :- $kT <$ ionisation energy of H_2

$$\Rightarrow T \cong 3000K$$

$\therefore$ recombination took place in matter dominated universe.

$$d(t_{rec}) = 2\sqrt{\frac{3}{8\pi G C_m}}\ \left(\lambda(t_{rec})\right)^{1/2} + d(t_{eq})$$

$$= 2\sqrt{\frac{3}{8\pi G C_m}}\ \left(\lambda(t_{rec})\right)^{1/2} + \sqrt{\frac{3}{8\pi G C_r}}\ \lambda_{eq}$$

$$\frac{C_r}{\lambda_{eq}^4} = \frac{C_m}{\lambda_{eq}^3} \Rightarrow C_r = C_m \lambda_{eq}$$

$$\sqrt{\frac{3}{8\pi G C_m}}\ \frac{\lambda_{eq}}{(\lambda_{eq})^{1/2}}$$

$$\therefore d(t_{rec}) = \left(\frac{3}{8\pi G C_m}\right)^{1/2} \left(2\lambda(t_{rec})^{1/2} - \lambda_{eq}^{1/2}\right)$$

$$\Rightarrow d(t_{rec}) \approx 2\sqrt{\frac{3}{8\pi G \rho_m}}\ \lambda(t_{rec})^{1/2}$$

Horizon of light emitted at t_{rec}

$$= d(t_{now}) - d(t_{rec})$$

$$= 2\sqrt{\frac{3}{8\pi G \rho_m}}\left\{ [\lambda(t_0)]^{1/2} - [\lambda(t_{rec})]^{1/2} \right\}$$

$$\cong 2\sqrt{\frac{3}{8\pi G \rho_m}}\ (\lambda(t_0))^{1/2}$$

$\therefore$ distance betn. 2 diametrically-opposite located sources of CMB photons

$$\equiv \Delta = 4\sqrt{\frac{3}{8\pi G \rho_m}}\ \lambda(t_0)^{1/2}$$

$$\frac{\Delta}{d(rec)} = 2\left(\frac{\lambda(t_0)}{\lambda(t_{rec})}\right)^{1/2} \gg 1$$

$\therefore$ isotropy of the CMB cannot be a result of equilibrium due to interaction.

Unless there is a LARGE additive correction to $\lambda(t_{rec})$. Since the matter dominated universe is well-understood, this cannot occur till atleast t_{eq}. In fact, this can be pushed back to t_{in} (beginning of radiation dominated era : will not have $\lambda = 0$ anymore).

PROPOSED SOLUTION: INFLATION

Before the radiation dominated era, the universe was dominated by a large cosmological constant. By a phase transition, the energy density of this cosmological constant got completely converted to radiation!

[QFT: cosmological const. arises from the potential of a scalar field]

Evolution in cosmological constant dominated phase :-

$$\left(\lambda'/\lambda\right)^2 = \frac{8\pi G}{3} C_0 \Rightarrow \lambda = K_0 \exp\left\{\sqrt{\frac{8\pi G C_0}{3}}\, t\right\}$$

Suppose this phase lasts from some t_i to t_f

Then
$$d(t_f) - d(t_i) = \int_{t_i}^{t_f} \frac{dt'}{K_0\, e^{\sqrt{8\pi G/3 \times C_0}\; t'}}$$

$$= -\frac{1}{K_0}\sqrt{\frac{3}{8\pi G C_0}}\,\frac{1}{\lambda(t_f)}\left\{e^{-\sqrt{\frac{8\pi G C_0}{3}}(t_i - t_f)} - 1\right\}$$

$\because \lambda(t_f) = K_0 e^N$

$$= \sqrt{\frac{3}{8\pi G C_0}}\,\frac{1}{\lambda(t_f)}\left(e^N - 1\right)$$

$$\left[\text{where } N = +\sqrt{\frac{8\pi G C_0}{3}}\,(t_i - t_f)\right]$$

Take this as the initial conds. for the radiation-dominated era :-

$$\lambda_{in} = \lambda(t_f)$$

$$d_{in} = d_{t_f} = \sqrt{\frac{3}{8\pi G C_0}}\left(e^N - 1\right)\frac{1}{\lambda(t_{in})} + d(t_i)$$

found using matching boundary conditions

How large does N have to be to solve the problem?

i.e.

$$\frac{\Delta}{d(t_{rec})} < 1$$

$$\rho_{in} = C_0$$

Ex: Using current knowledge of ρ_{matter}, ρ_{rad} & knowledge about ρ_{in} $\left(= C_0 \sim (10^{15}\ \text{GeV})^4\right)$, Calculate the min. value of N. [1 GeV = 10^{13} K]