Quantum Field Theory 1

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1 Second quantization

Quantum field theory

– a technique for dealing with a quantum system of many particles

– necessary when the number of particles is not conserved, e.g.

• in a system of photons

• in high energy collision of two electrons which can produce electron positron pairs besides two electrons

– also a convenient tool when the number of particles is large but conserved

We begin with a familiar system where the number of particles is conserved.

First consider the simplest case: A single particle in 3 dimensions moving under the influence of a potential $U(\vec{r})$.

Schrodinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \widehat{h}\psi$$

$$\widehat{h}\,\psi = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi + U(\vec{r})\,\psi$$

$\widehat{h}$: Hamiltonian

One way to find the general solution is to first find the eigenstates of $\widehat{h}$.

$$\widehat{h}\,u_n(\vec{r}) = e_n u_n(\vec{r})$$

e_n: eigenvalues, u_n: eigenfunctions, $e_n \geq 0$

$\{u_n\}$: complete basis of states, normalized as

$$\int d^3r\, u_n^*(\vec{r})\, u_m(\vec{r}) = \delta_{mn}$$

General solution:

$$\psi(t,\vec{r}) = \sum_n a_n(t) u_n(\vec{r}), \qquad a_n(t) = a_n(0)\, e^{-ie_n t/\hbar}$$

Now consider a system of N particles of same mass, each moving under the same potential, and with no mutual interaction

$\vec{r}_i$: position of the i'th particle

$\vec{\nabla}_i$: the gradient operators with respect to the i'th particle coordinate

The wave function $\psi(t,\vec{r}_1,\cdots,\vec{r}_N)$ satisfies Schrodinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = \widehat{H}_N\,\psi, \qquad \widehat{H}_N = \sum_{i=1}^{N}\widehat{h}_i, \qquad \widehat{h}_i = -\frac{\hbar^2}{2m}\vec{\nabla}_i^2 + U(\vec{r}_i)$$

Choose basis of states:

$$W_{n_1,\cdots,n_N}(\vec{r}_1,\cdots,\vec{r}_N) = u_{n_1}(\vec{r}_1)\cdots u_{n_N}(\vec{r}_N)$$

General solution:

$$\psi(t;\vec{r}_1,\cdots,\vec{r}_N) = \sum_{n_1,\cdots,n_N} a_{n_1,\cdots,n_N}(t)\, W_{n_1,\cdots,n_N}(\vec{r}_1,\cdots,\vec{r}_N)$$

$$a_{n_1,\cdots,n_N}(t) = e^{-i(e_{n_1}+\cdots+e_{n_N})\,t/\hbar}\, a_{n_1,\cdots,n_N}(0)$$

Note: If each $u_n(\vec{r})$ is normalized to 1, then so is $W_{n_1,\cdots,n_N}$

Now consider the case when the particles are identical and bosonic.

ψ must be symmetric under $\vec{r}_i \leftrightarrow \vec{r}_j$ for every pair i, j

We can still expand ψ in the basis $\{W_{n_1,\cdots,n_N}\}$ but it is redundant since only symmetric functions need to be expanded

For this reason it will be natural to choose the basis states to be also symmetric under the exchange of any pair of particles.

Define new basis:

$$u_{n_1,\cdots,n_N}(\vec{r}_1, \cdots, \vec{r}_N) \equiv \frac{1}{\sqrt{N!}} \sum_{\text{Permutations of } \vec{r}_1, \cdots \vec{r}_N} u_{n_1}(\vec{r}_1) \cdots u_{n_N}(\vec{r}_N)$$

Note: This basis is also automatically symmetric under the exchange of the n_i's

e.g. $u_{1,2}$ and $u_{2,1}$ should not be counted as separate basis states

For this reason we can label the subscripts in a fixed order e.g. in the order of increasing energy and/or other quantum numbers

e.g. $u_{1,2,4}$, $u_{1,1,2,4}$, $u_{1,2,2}$ etc. but not $u_{2,1,4}$ or $u_{2,1,2}$

$$u_{n_1,\cdots,n_N}(\vec{r}_1,\cdots,\vec{r}_N) \equiv \frac{1}{\sqrt{N!}} \sum_{\text{Permutations of } \vec{r}_1,\cdots\vec{r}_N} u_{n_1}(\vec{r}_1)\cdots u_{n_N}(\vec{r}_N)$$

Now let us check the normalization of the basis states.

$$\int d^3r_1\cdots d^3r_N\, u_{n_1,\cdots,n_N}(\vec{r}_1,\cdots,\vec{r}_N)^*\, u_{l_1,\cdots,l_N}(\vec{r}_1,\cdots,\vec{r}_N)$$

•: Unless $l_i = n_i$ for every i, the result vanishes

•: If $l_i = n_i$ for every i and all the n_i's are different, then the result is 1

– the $1/N!$ in the overall normalization cancels the $N!$ contributions, each giving 1

– all cross terms vanish, e.g.

$$\int d^3r_1 d^3r_2 u^*_{n_1}(\vec{r}_1)u^*_{n_2}(\vec{r}_2)u_{n_2}(\vec{r}_1)u_{n_1}(\vec{r}_2) = 0$$

if $n_1 \neq n_2$.

The situation is more complicated if $l_i = n_i$ for every i, and some of the n_i's are equal, e.g. $u_{1,1,2}$

In this case some of the terms in the expression for $u_{n_1,\cdots,n_N}$ are equal since permuting identical indices will not change the term.

To understand what happens in the case, it is useful to use a slightly different way of labelling the basis states.

Instead of saying what states are occupied using the labels $n_1,\cdots,n_N$, possibly with some labels repeated several times, we specify how many times the label 1 appears, how many times the label 2 appears etc.

e.g. $u_{1,1,2}$ will be described as $(2, 1, 0, 0, \cdots)$ – occupancy number

Occupancy number m_i tells us how many particles occupy the i-th state u_i

Occupancy number $(m_1, m_2, \cdots)$ state means that the n'th state appears m_n times

For an N particle state $\sum_{n=1}^{\infty} m_n = N$

Now let us consider the state

$$u_{n_1,\cdots,n_N}(\vec{r}_1, \cdots, \vec{r}_N) \equiv \frac{1}{\sqrt{N!}} \sum_{\text{Permutations of } \vec{r}_1,\cdots\vec{r}_N} u_{n_1}(\vec{r}_1) \cdots u_{n_N}(\vec{r}_N)$$

with occupancy numbers $(m_1, m_2, \cdots)$

Inside the summation not all $N!$ terms are different

– they are grouped into $N!/(m_1!m_2!\cdots)$ groups, each group containing $m_1!m_2!\cdots$ identical terms:

$$u_{n_1,\cdots,n_N}(\vec{r}_1, \cdots, \vec{r}_N) = \frac{1}{\sqrt{N!}} m_1!m_2!\cdots \left(W_{(1)} + W_{(2)} + \cdots W_{(p)}\right), \quad p = \frac{N!}{m_1!m_2!\cdots}$$

Different terms $W_{(i)}$ are orthonormal since at least for one of the $\vec{r}_i$'s the state label differs

The total norm is:

$$\frac{1}{N!} \times (m_1!m_2!\cdots)^2 \times p = m_1!m_2!\cdots$$

Therefore the state $u_{n_1,\cdots,n_N}(\vec{r}_1, \cdots, \vec{r}_N)$ is not normalized to 1.

We shall keep it this way.

Example:

$$\begin{aligned} u_{1,1,2} &= \frac{1}{\sqrt{3!}}[u_1(\vec{r}_1)u_1(\vec{r}_2)u_2(\vec{r}_3) + \text{permutations of } \vec{r}_1, \vec{r}_2, \vec{r}_3] \\ &= \frac{1}{\sqrt{6}}[2\, u_1(\vec{r}_1)u_1(\vec{r}_2)u_2(\vec{r}_3) + 2\, u_1(\vec{r}_1)u_1(\vec{r}_3)u_2(\vec{r}_2) + 2\, u_1(\vec{r}_2)u_1(\vec{r}_3)u_2(\vec{r}_1)] \end{aligned}$$

$$\int d^3r_1 d^3r_2 d^3r_3 u^*_{1,1,2} u_{1,1,2} = \frac{1}{6} \times 2^2 \times 3 = 2$$

– agrees with $m_1!m_2!\cdots = 2! = 2$

Next we shall analyze the action of N particle Hamiltonian on these basis states.

$$\widehat{H}_N = \sum_{i=1}^{N} \widehat{h}_i$$

$$\widehat{h}_i\, u_{n_1}(\vec{r}_1)\cdots u_{n_N}(\vec{r}_N) = e_{n_i} u_{n_1}(\vec{r}_1)\cdots u_{n_N}(\vec{r}_N)$$

$$\widehat{H}_N\, u_{n_1}(\vec{r}_1)\cdots u_{n_N}(\vec{r}_N) = \left(\sum_{i=1}^{N} e_{n_i}\right) u_{n_1}(\vec{r}_1)\cdots u_{n_N}(\vec{r}_N)$$

Note: The eigenvalue of $\widehat{H}_N$ does not change under permutation of the $\vec{r}_i$'s on both sides.

$$\Rightarrow \widehat{H}_N\, u_{n_1,\cdots,n_N}(\vec{r}_1,\cdots,\vec{r}_N) = \left(\sum_{i=1}^{N} e_{n_i}\right) u_{n_1,\cdots,n_N}(\vec{r}_1,\cdots,\vec{r}_N)$$

If the state has occupancy numbers $(m_1, m_2, \cdots)$ then the energy eigenvalue is

$$\sum_{n=1}^{\infty} m_n e_n$$

We shall now forget about this system and consider another quantum system

– a collection of infinite number of one dimensional harmonic oscillators with angular frequencies $\omega_1, \omega_2, \cdots$

Choose ω_n's such that:

$$\omega_n = e_n/\hbar$$

$\Rightarrow$ for each single particle energy eigenstate of the original theory, we have a harmonic oscillator.

If there are k degenerate energy eigenstates in the single particle system for some energy e, we introduce k independent distinguishable harmonic oscillators of angular frequency $e/\hbar$.

We can introduce creation and annihilation operators $a_n, a_n^\dagger$ for each harmonic oscillator:

$$[a_n, a_p] = 0, \qquad [a_n^\dagger, a_p^\dagger] = 0, \qquad [a_n, a_p^\dagger] = \delta_{np}$$

Hamiltonian of this system:

$$\widehat{H} = \sum_{n=1}^{\infty} \hbar\,\omega_n \left(a_n^\dagger a_n + \frac{1}{2} \right) = \sum_{n=1}^{\infty} \hbar\,\omega_n\, a_n^\dagger\, a_n + C$$

C: An overall constant which will have no physical significance

(We shall only be interested in differences in energy eigenvalues of this system)

Ground state $|0\rangle$, also called the vacuum state, is defined as

$$a_n|0\rangle = 0 \quad \text{for every } n$$

Other states are created by applying arbitrary combinations of $a_n^\dagger$ on the vacuum state.

Goal: We shall show that this quantum theory is related to the system considered earlier under certain identifications.

Identification of states:

$$u_{n_1,n_2,\cdots,n_N} \leftrightarrow a^\dagger_{n_1}\cdots a^\dagger_{n_N}|0\rangle$$

Note that like $u_{n_1,n_2,\cdots,n_N}$, the right hand side is automatically symmetric under $n_i \leftrightarrow n_j$

Occupation number representation:

$$(a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}\cdots|0\rangle$$

Next compute inner product of the states $(a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}\cdots|0\rangle$ and $(a_1^\dagger)^{m_1'}(a_2^\dagger)^{m_2'}\cdots|0\rangle$

$$\langle 0|(a_1)^{m_1}(a_2)^{m_2}\cdots(a_1^\dagger)^{m_1'}(a_2^\dagger)^{m_2'}\cdots|0\rangle$$

We can analyze this by taking a_i's to the right using the commutator between a_i and $a_i^\dagger$.

- Unless $m_1 = m_1'$, $m_2 = m_2'$ etc. the result vanishes

Any excess a_n or $a_n^\dagger$ will annihilate either $|0\rangle$ or $\langle 0|$.

Ex. Check that:

$$\langle 0|(a_1)^{m_1}(a_2)^{m_2}\cdots(a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}\cdots|0\rangle = m_1!m_2!\cdots$$

Example:

$$\langle 0|(a_1)^2(a_1^\dagger)^2|0\rangle = \langle 0|a_1a_1a_1^\dagger a_1^\dagger|0\rangle = \langle 0|a_1(a_1^\dagger a_1+1)a_1^\dagger|0\rangle$$

Now replace every $a_1a_1^\dagger$ in this expression by $a_1^\dagger a_1 + 1$ and use

$$\langle 0|a_1^\dagger = 0, \qquad a_1|0\rangle = 0, \qquad \langle 0|0\rangle = 1$$

Result: $2 = 2!$

Conclusion: $a^\dagger_{n_1}\cdots a^\dagger_{n_N}|0\rangle$ has the same norm as $u_{n_1,\cdots,n_N}$ in the first theory.

This gives a map between the Hilbert spaces of the two theories.

Let us compare the energy eigenvalues in the two quantum systems.

$$\widehat{H} a^\dagger_{n_1} \cdots a^\dagger_{n_N} |0\rangle = \left(\sum_{n=1}^{\infty} \hbar\omega_n a^\dagger_n a_n \right) a^\dagger_{n_1} \cdots a^\dagger_{n_N} |0\rangle$$

Use

$$\left[\sum_{n=1}^{\infty} \hbar\omega_n a^\dagger_n a_n, a^\dagger_p \right] = \hbar\, \omega_p\, a^\dagger_p$$

$$\begin{aligned} \widehat{H} a^\dagger_{n_1} \cdots a^\dagger_{n_N} |0\rangle &= \hbar(\omega_{n_1} + \cdots + \omega_{n_N})\, a^\dagger_{n_1} \cdots a^\dagger_{n_N} |0\rangle \\ &= (e_{n_1} + \cdots + e_{n_N})\, a^\dagger_{n_1} \cdots a^\dagger_{n_N} |0\rangle \end{aligned}$$

Note: we have dropped the additive constant C

Example: Using $\widehat{H} a^\dagger_n = a^\dagger_n \widehat{H} + \hbar\omega_n a^\dagger_n$,

$$\begin{aligned} &\widehat{H}\, a^\dagger_{n_1} a^\dagger_{n_2} |0\rangle = \left(a^\dagger_{n_1} \widehat{H} + \hbar\omega_{n_1} \right) a^\dagger_{n_2} |0\rangle \\ = \ & a^\dagger_{n_1} a^\dagger_{n_2} \widehat{H} |0\rangle + a^\dagger_{n_1}\, \hbar\omega_{n_2} a^\dagger_{n_2} |0\rangle + \hbar\omega_{n_1} a^\dagger_{n_1} a^\dagger_{n_2} |0\rangle = (\hbar\omega_{n_1} + \hbar\omega_{n_2}) a^\dagger_{n_1} a^\dagger_{n_2} |0\rangle \end{aligned}$$

This agrees with the energy eigenvalue of $u_{n_1, n_2, \cdots, n_N}$

Eigenstates of $\widehat{H}_N$ get mapped to eigenstates of $\widehat{H}$ with the same eigenvalue.

Later we shall see that every operator in the first theory can be mapped to an operator in the second theory.

However the converse is not true.

Specifying the first system requires fixing a value of N – the total number of particles

This is not the case in the second system, e.g.

$a_n^\dagger|0\rangle$ is a single particle state, $|0\rangle$ is a zero particle state,

$a_{n_1}^\dagger a_{n_2}^\dagger|0\rangle$ is a two particle state etc.

2 Second quantization

We have considered two quantum systems.

First system: N identical non-interacting bosons, each of mass m moving in three dimensions under some potential $U(\vec{r})$.

$$i\hbar\frac{\partial\psi}{\partial t} = \widehat{H}_N\,\psi, \qquad \widehat{H}_N = \sum_{i=1}^{N}\widehat{h}_i, \qquad \widehat{h}_i = -\frac{\hbar^2}{2m}\vec{\nabla}_i^2 + U(\vec{r}_i)$$

Basis states:

$$u_{n_1,\cdots,n_N}(\vec{r}_1,\cdots,\vec{r}_N) \equiv \frac{1}{\sqrt{N!}}\sum_{\text{Permutations of }\vec{r}_1,\cdots\vec{r}_N} u_{n_1}(\vec{r}_1)\cdots u_{n_N}(\vec{r}_N)$$

$u_n(\vec{r})$: single particle energy eigenstates:

$$\widehat{h}\,u_n(\vec{r}) = e_n\,u_n(\vec{r})$$

Second system: A collection of infinite number of harmonic oscillators, one for each energy eigenstate of the first system for $N=1$, with angular frequency $\omega_n = e_n/\hbar$

If there are k degenerate energy eigenstates in the single particle system for some energy e, we introduce k independent distinguishable harmonic oscillators of angular frequency $e/\hbar$.

$a_n, a_n^\dagger$: annihilation and creation operators of the n-th harmonic oscillator, with commutation relations:

$$[a_n, a_p] = 0, \qquad [a_n^\dagger, a_p^\dagger] = 0, \qquad [a_n, a_p^\dagger] = \delta_{np}$$

Hamiltonian of this system (after throwing away a constant term):

$$\widehat{H} = \sum_{n=1}^{\infty}\hbar\,\omega_n\,a_n^\dagger\,a_n = \sum_{n=1}^{\infty} e_n\,a_n^\dagger\,a_n$$

Correspondence:

$$u_{n_1,n_2,\cdots,n_N} \leftrightarrow a^\dagger_{n_1} \cdots a^\dagger_{n_N}|0\rangle, \qquad a_n|0\rangle = 0 \quad \text{for all } n$$

Under this correspondence:

• The inner product between the states agree in the two theories

• The eigenvalues and eigenstates of $\widehat{H}_N$ in the first theory agree with the eigenvalues and eigenstates of $\widehat{H}$ in the second theory.

However there is one major difference.

In the first description, for each N we have a different quantum theory.

In the second description, a single quantum theory can describe all N.

$a^\dagger_n|0\rangle$ is a single particle state, $\quad |0\rangle$ is a zero particle state,

$a^\dagger_{n_1}a^\dagger_{n_2}|0\rangle$ is a two particle state etc.

In the second system, the information on N is encoded in the eigenvalue of a certain operator, called the number operator:

$$\widehat{N} = \sum_{n=1}^{\infty} a^\dagger_n a_n, \qquad \left[\widehat{N}, a^\dagger_p\right] = a^\dagger_p, \qquad [\widehat{N}, a_p] = -a_p$$

$$\widehat{N}\, a^\dagger_{n_1} \cdots a^\dagger_{n_N}|0\rangle = N\, a^\dagger_{n_1} \cdots a^\dagger_{n_N}|0\rangle$$

What about mapping of other operators between the two descriptions?

We begin by asking what kind of operators we have in the description 1.

Examples:

$$\widehat{x}_1, \quad \widehat{p}_{1y}\widehat{x}_3, \quad \vec{\nabla}_1^2, \quad \cdots$$

Numbers in the subscript denote particle labels.

However none of these are good operators.

Since the particles are indistinguishable, it does not make sense to consider the operator $\widehat{x}_1$.

Which particle is particle 1?

Good operators: Those which are invariant under permutations of $1, \cdots, N$, e.g.

$$\sum_{i=1}^{N} \widehat{x}_i, \quad \sum_{i,j=1}^{N} \widehat{p}_{iy}\widehat{x}_j, \quad \sum_{i=1}^{N} \vec{\nabla}_i^2, \quad \cdots$$

We can divide good operators into different classes.

- One body operators:

$$\widehat{B}_N = \sum_{i=1}^{N} \widehat{b}_i$$

$\widehat{b}_i$ is constructed from the position and momentum operators of the i-th particle, e.g.

$$\widehat{x}_i\widehat{p}_{iy}, \quad \vec{\nabla}_i^2, \quad \cdots$$

• 2-body operators:

$$\widehat{V}_N = \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \widehat{v}_{ij}$$

$\widehat{v}_{ij}$ is constructed from the position and momentum operators of the i-th particle and j-th particle, e.g.

$$\widehat{x}_i \widehat{p}_{jy}, \quad 1/|\vec{r}_i - \vec{r}_j|$$

Note 1: The $i = j$ term is removed from the sum since that sum can be regarded as a one body operators.

Note 2: The two body operator vanishes if $N = 1$.

Similarly one can define 3-body, 4-body, $\cdots$, k-body operators.

Question: What do these operators get mapped to in the second description?

Consider the one body operators:

$$\widehat{B}_N = \sum_{i=1}^{N} \widehat{b}_i$$

Claim: This is mapped to the operator:

$$\widehat{B} = \sum_{n,p=1}^{\infty} b_{np} a_n^\dagger a_p$$

$$b_{np} = \int d^3 r_1 \, u_n^*(\vec{r}_1) \, \widehat{b}_1 \; u_p(\vec{r}_1)$$

Note: Since $\vec{r}_1$ is an integration variable, we could replace $_1$ by $_i$ for any i

To verify this we need to check:

$$\int d^3 r_1 \cdots d^3 r_N u_{n_1,\cdots,n_N}(\vec{r}_1, \cdots, \vec{r}_N)^* \sum_{i=1}^{N} \widehat{b}_i \, u_{n'_1,\cdots,n'_N}(\vec{r}_1, \cdots, \vec{r}_N)$$
$$= \langle 0| a_{n_1} \cdots a_{n_N} \, \widehat{B} \, a^\dagger_{n'_1} \cdots a^\dagger_{n'_N} |0\rangle$$

We'll check this for $N = 1$ and leave the general case as exercise.

For $N = 1$,

$$\text{r.h.s.} = \langle 0| a_{n_1} \left(\sum_{n,p=1}^{\infty} b_{np} a_n^\dagger a_p \right) a^\dagger_{n'_1} |0\rangle = \sum_{n,p=1}^{\infty} b_{np} \, \delta_{n_1 n} \delta_{p n'_1} = b_{n_1 n'_1}$$

$$\text{l.h.s.} = \int d^3 r_1 u_{n_1}(\vec{r}_1)^* \, \widehat{b}_1 \, u_{n'_1}(\vec{r}_1) = b_{n_1 n'_1}$$

Therefore two sides agree!

For general case, first show that on both sides of the identity, the sets $\{n_1, \cdots n_N\}$ and $\{n'_1, \cdots n'_N\}$ can differ at most in one entry for non-zero result.

Then study this case.

Next consider 2-body operators:

$$\widehat{V}_N = \sum_{\substack{i,j=1\\i\neq j}}^{N} \widehat{v}_{ij}, \qquad \widehat{v}_{ij} = \widehat{v}_{ji}$$

Claim: In the second description, this is mapped to the operator:

$$\widehat{V} = \sum_{m,n,p,q=1}^{\infty} v_{m,n,p,q} a_m^\dagger a_n^\dagger a_p a_q$$

$$v_{m,n,p,q} = \int d^3r_1 d^3r_2 \, u_m(\vec{r}_1)^* u_n(\vec{r}_2)^* \, \widehat{v}_{12} \, u_p(\vec{r}_1) u_q(\vec{r}_2)$$

To verify this we need to check:

$$\begin{aligned} &\int d^3r_1 \cdots d^3r_N u_{n_1,\cdots,n_N}(\vec{r}_1,\cdots,\vec{r}_N)^* \left(\sum_{\substack{i,j=1\\i\neq j}}^{N} \widehat{v}_{ij} \right) u_{n'_1,\cdots,n'_N}(\vec{r}_1,\cdots,\vec{r}_N) \\ = \ & \langle 0| a_{n_1} \cdots a_{n_N} \, \widehat{V} \, a^\dagger_{n'_1} \cdots a^\dagger_{n'_N} |0\rangle \end{aligned}$$

We'll check this for $N = 2$ and leave the general case as exercise.

$$\begin{aligned} \text{r.h.s.} &= \langle 0| a_{n_1} a_{n_2} \sum_{m,n,p,q=1}^{\infty} v_{m,n,p,q} a_m^\dagger a_n^\dagger a_p a_q \, a^\dagger_{n'_1} a^\dagger_{n'_2} |0\rangle \\ &= v_{n_1,n_2,n'_1,n'_2} + v_{n_2,n_1,n'_1,n'_2} + v_{n_1,n_2,n'_2,n'_1} + v_{n_2,n_1,n'_2,n'_1} \end{aligned}$$

$$\begin{aligned} \text{l.h.s.} &= \frac{1}{2} \int d^3r_1 d^3r_2 \{ u_{n_1}(\vec{r}_1)^* u_{n_2}(\vec{r}_2)^* + u_{n_1}(\vec{r}_2)^* u_{n_2}(\vec{r}_1)^* \} \\ & \qquad (\widehat{v}_{12} + \widehat{v}_{21}) \{ u_{n'_1}(\vec{r}_1) u_{n'_2}(\vec{r}_2) + u_{n'_1}(\vec{r}_2) u_{n'_2}(\vec{r}_1) \} \\ &= v_{n_1,n_2,n'_1,n'_2} + v_{n_2,n_1,n'_1,n'_2} + v_{n_1,n_2,n'_2,n'_1} + v_{n_2,n_1,n'_2,n'_1} \end{aligned}$$

The two sides agree!

For the general case, first show that the result vanishes if the sets $\{n_1, \cdots, n_N\}$ and $\{n'_1, \cdots, n'_N\}$ differ in more than two entries, and then study the non-vanishing case.

Possible applications:

Suppose we have a system of particles, each moving under some potential U, and furthermore they interact pairwise via some interaction terms v_{ij}.

Hamiltonian in the first description:

$$\widehat{H}_N = \sum_{i=1}^{N} \widehat{h}_i + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \widehat{v}_{ij}$$

In the second description this can be translated to,

$$\widehat{H} = \sum_{n=1}^{\infty} e_n \, a_n^\dagger a_n + \frac{1}{2} \sum_{m,n,p,q=1}^{\infty} v_{m,n,p,q} a_m^\dagger a_n^\dagger a_p a_q$$

The second description does not have any explicit N.

N is the eigenvalue of the number operator $\widehat{N} = \sum_{n=1}^{\infty} a_n^\dagger a_n$

Using the second description we can study properties of the system for all N at one go

We can find similar maps for general k-body operator.

Note: In the second description, these operators commute with the number operator

$$\widehat{N} = \sum_{k=1}^{\infty} a_k^\dagger a_k$$

since there are equal number of a and $a^\dagger$ in $\widehat{B}$, $\widehat{V}$ etc.

$$\widehat{B} = \sum_{n,p=1}^{\infty} b_{np} a_n^\dagger a_p, \qquad \widehat{V} = \sum_{m,n,p,q=1}^{\infty} v_{m,n,p,q} a_m^\dagger a_n^\dagger a_p a_q$$

e.g.

$$[\widehat{N}, a_n^\dagger a_p] = [\widehat{N}, a_n^\dagger] a_p + a_n^\dagger [\widehat{N}, a_p] = a_n^\dagger a_p - a_n^\dagger a_p = 0$$

This is not surprising since in the first description all operators conserve particle number.

The converse is not true.

In the second description we can construct operators like:

$$\sum_{m,n,p=1}^{\infty} C_{mnp}\, a_m^\dagger a_n^\dagger a_p$$

which do not conserve particle number.

$a_m^\dagger a_n^\dagger a_p$ increases the number of particles by 1

– removes a particle in state u_p and creates a pair of particles in state u_m and u_n.

Such operators do not exist in the first description since we work with a fixed number of particles.

Due to this, the second description has certain advantages.

If experiments show that particle number is not conserved, we can try to explain this by adding operators, that do not conserve particle number, to the Hamiltonian.

3 Second quantization

Today we shall consider a third quantum system.

We go back to the single particle Schrodinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = \widehat{h}\psi$$

$$\widehat{h}\,\psi = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi + U(\vec{r})\,\psi$$

We regard this as the classical equation of a field $\psi(t,\vec{r})$ and quantize it

– second quantization.

How do we do this?

1. Find a classical Lagrangian that gives this equation of motion

2. Find the canonically conjugate momenta and the Hamiltonian

3. Quantize it using standard procedure

Classical Lagrangian:

$$L = \int d^3r\,\psi(t,\vec{r})^*\,\left[i\hbar\frac{\partial\psi(t,\vec{r})}{\partial t} - \widehat{h}\psi(t,\vec{r})\right]$$

In this equation think of $\widehat{h}$ as some differential operator and not an operator acting on the Hilbert space.

We can now write down the action and use variational principle to derive the equations of motion.

$$S = \int dt \int d^3r\, \psi(t,\vec{r})^* \left[i\hbar \frac{\partial \psi(t,\vec{r})}{\partial t} - \widehat{h}\psi(t,\vec{r}) \right]$$

Under arbitrary variation of ψ,

$$\delta S = \int dt \int d^3r \left[\delta\psi(t,\vec{r})^* \left\{ i\hbar \frac{\partial \psi(t,\vec{r})}{\partial t} - \widehat{h}\psi(t,\vec{r}) \right\} + \psi(t,\vec{r})^* \left\{ i\hbar \frac{\partial \delta\psi(t,\vec{r})}{\partial t} - \widehat{h}\delta\psi(t,\vec{r}) \right\} \right]$$

In the second { }, we carry out two manipulations:

• We integrate by parts in t and ignore the boundary terms since, as part of the action principle, $\delta\psi$ vanishes at initial and final time.

• We use the hermiticity of $\widehat{h}$

$$\delta S = \int dt \int d^3r \left[\delta\psi(t,\vec{r})^* \left\{ i\hbar \frac{\partial \psi(t,\vec{r})}{\partial t} - \widehat{h}\psi(t,\vec{r}) \right\} + \left\{ -i\hbar \frac{\partial \psi(t,\vec{r})^*}{\partial t} \delta\psi(t,\vec{r}) - \{\widehat{h}\psi(t,\vec{r})\}^* \delta\psi(t,\vec{r}) \right\} \right]$$

Even though the correct procedure will be to write $\psi = \psi_1 + i\psi_2$ and set the coefficients of $\delta\psi_1$ and $\delta\psi_2$ to zero, this is equivalent to setting the coefficients of $\delta\psi$ and $\delta\psi^*$ to zero

– as if ψ and ψ^* are independent variables

e.g.

$$\delta\psi\, F + \delta\psi^* G = (\delta\psi_1 + i\delta\psi_2)F + (\delta\psi_1 - i\delta\psi_2)G = \delta\psi_1(F+G) + i\,\delta\psi_2\,(F-G)$$

$$\Rightarrow F + G = 0, \quad F - G = 0 \quad \Rightarrow \quad F = G = 0$$

Therefore we get

$$i\hbar \frac{\partial \psi(t,\vec{r})}{\partial t} - \widehat{h}\psi(t,\vec{r}) = 0, \qquad -i\hbar \frac{\partial \psi(t,\vec{r})^*}{\partial t} - \{\widehat{h}\psi(t,\vec{r})\}^* = 0$$

– correct equations

$\Rightarrow$ we have the correct Lagrangian

$$L = \int d^3r \, \psi(t,\vec{r})^* \left[i\hbar \frac{\partial \psi(t,\vec{r})}{\partial t} - \widehat{h}\psi(t,\vec{r}) \right]$$

We shall now directly check that the Euler-Lagrange equations from L gives the equations of motion.

Since $\{u_n\}$ form a complete basis of functions, we can expand ψ as:

$$\psi(t,\vec{r}) = \sum_{n=1}^{\infty} a_n(t) \, u_n(\vec{r})$$

Knowing the a_n's we can determine ψ and vice versa.

Therefore we can regard the a_n's as our degrees of freedom.

Substitute this in the expression for L:

$$L = \sum_{m,n=1}^{\infty} a_n(t)^* \int d^3r \, u_n(\vec{r})^* \left[i\hbar \frac{da_m}{dt} \, u_m(\vec{r}) - a_m(t) \, \widehat{h} u_m(\vec{r}) \right]$$

Using $\widehat{h} u_m(\vec{r}) = e_m \, u_m(\vec{r})$ and the orthonormality of the u_m's we get

$$L = \sum_{m,n=1}^{\infty} a_n(t)^* \left[i\hbar \frac{da_m(t)}{dt} - e_m \, a_m(t) \right] \delta_{mn} = \sum_{n=1}^{\infty} a_n(t)^* \left[i\hbar \frac{da_n(t)}{dt} - e_n \, a_n(t) \right]$$

$$L = \sum_{n=1}^{\infty} a_n(t)^* \left[i\hbar \frac{da_n(t)}{dt} - e_n\, a_n(t) \right]$$

Treat a_n and a_n^* as independent degrees of freedom and write down the Euler-Lagrange equations.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{a}_n} - \frac{\partial L}{\partial a_n} = 0, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{a}_n^*} - \frac{\partial L}{\partial a_n^*} = 0$$

$$\Rightarrow \quad i\hbar \frac{d}{dt} a_n^* + e_n a_n^* = 0, \qquad i\hbar \frac{da_n(t)}{dt} - e_n\, a_n(t) = 0$$

Using $\psi = \sum_n a_n(t) u_n(\vec{r})$, we get

$$i\hbar \frac{\partial \psi(t, \vec{r})}{\partial t} - \widehat{h}\psi(t, \vec{r}) = 0, \qquad -i\hbar \frac{\partial \psi(t, \vec{r})^*}{\partial t} - \widehat{h}\psi(t, \vec{r})^* = 0$$

– correct equations!

If in doubt, write $a_n = b_n + i\, c_n$ with real b_n, c_n and write the equations for b_n and c_n

– gives the same result.

$$L = \sum_{n=1}^{\infty} a_n(t)^* \left[i\hbar \frac{da_n(t)}{dt} - e_n\, a_n(t) \right]$$

Find the canonically conjugate momenta p_n, $\tilde{p}_n$.

$$p_n = \frac{\partial L}{\partial \dot{a}_n} = i\hbar a_n^*, \quad \tilde{p}_n = \frac{\partial L}{\partial \dot{a}_n^*} = 0$$

$$H = \sum_{n=1}^{\infty} [p_n \dot{a}_n + \tilde{p}_n \dot{a}_n^* - L] = \sum_{n=1}^{\infty} e_n a_n^* a_n$$

Note: The p_n's and $\tilde{p}_n$'s are determined in terms of a_n and a_n^* and are not independent variables.

To proceed systematically, we should use Dirac's procedure for quantizing constrained systems.

However in this case the constraints are simple and we can simply regard $a_n(t)$ and $p_n(t) = i\hbar a_n(t)^*$ as canonically conjugate variables and proceed.

$$H = \frac{1}{i\hbar} \sum_{n=1}^{\infty} e_n\, p_n\, a_n$$

Hamilton's equations:

$$\frac{dp_n}{dt} = -\frac{\partial H}{\partial a_n}, \quad \frac{da_n}{dt} = \frac{\partial H}{\partial p_n}$$

gives

$$i\hbar \frac{da_n^*}{dt} = -\frac{1}{i\hbar} e_n p_n = -e_n a_n^*, \qquad \frac{da_n}{dt} = \frac{1}{i\hbar} e_n a_n$$

– reproduces the correct equations.

$$p_n(t) = i\hbar a_n(t)^*, \qquad H = \sum_{n=1}^{\infty} e_n a_n^* a_n$$

We can now proceed to quantize.

$$[a_n, p_m] = i\hbar\,\delta_{mn} \quad \Rightarrow \quad [a_n, i\hbar a_m^\dagger] = i\hbar\,\delta_{mn} \quad \Rightarrow \quad [a_n, a_m^\dagger] = \delta_{mn}$$

$$[a_m, a_n] = 0, \qquad [a_m^\dagger, a_n^\dagger] = 0$$

$$\widehat{H} = \sum_{n=1}^{\infty} e_n a_n^\dagger a_n$$

– precisely the second system that we studied.

Summary

Given a single particle Schrodinger equation, the dynamics of multi-particle states without any mutual interaction is described by the following steps:

- Regard the wave-function as a classical field and the Schrodinger equation as the equation of motion of the classical field
- Quantize this classical field theory

– second quantization.

We shall now try to rewrite the commutation relations and the Hamiltonian in terms of ψ, $\psi^\dagger$.

$$[a_m, a_n] = 0, \qquad [a_m^\dagger, a_n^\dagger] = 0, \qquad [a_n, a_m^\dagger] = \delta_{mn}$$

$$\widehat{H} = \sum_{n=1}^{\infty} e_n a_n^\dagger a_n$$

$$\psi(t, \vec{r}) = \sum_{n=1}^{\infty} a_n(t) u_n(\vec{r}), \qquad \psi(t, \vec{r})^\dagger = \sum_{n=1}^{\infty} a_n(t)^\dagger u_n(\vec{r})^*$$

Note: $\psi, \psi^\dagger, a_n, a_n^\dagger$ are operators

u_n and u_n^* are fixed functions (eigenfunctions of $\widehat{h}$)

$$[\psi(t, \vec{r}), \psi(t, \vec{r}\,')] = \sum_{m,n=1}^{\infty} u_m(\vec{r}) u_n(\vec{r}\,')[a_m(t), a_n(t')] = 0, \qquad [\psi(t, \vec{r})^\dagger, \psi(t, \vec{r}\,')^\dagger] = 0$$

$$[\psi(t, \vec{r}), \psi(t, \vec{r}\,')^\dagger] = \sum_{m,n=1}^{\infty} u_m(\vec{r}) u_n(\vec{r}\,')^*[a_m(t), a_n(t)^\dagger]$$

$$= \sum_{m,n=1}^{\infty} u_m(\vec{r}) u_n(\vec{r}\,')^* \, \delta_{mn} = \sum_{n=1}^{\infty} u_n(\vec{r}) u_n(\vec{r}\,')^* = \delta^{(3)}(\vec{r} - \vec{r}\,')$$

using completeness relation.

Next we shall study the representation of other operators in this description.

4 Second quantization

We shall repeat the previous analysis with real variables:

$$L = \int d^3r \, \left[\frac{1}{2} i\hbar \psi(t,\vec{r})^* \, \frac{\partial \psi(t,\vec{r})}{\partial t} - \frac{1}{2} i\hbar \, \frac{\partial \psi(t,\vec{r})^*}{\partial t} \, \psi(t,\vec{r}) - \psi^*(t,\vec{r}) \, \widehat{h} \psi(t,\vec{r}) \right]$$

Since $\{u_n\}$ form a complete basis of functions, we can expand ψ as:

$$\psi(t,\vec{r}) = \sum_{n=1}^{\infty} a_n(t) \, u_n(\vec{r})$$

This gives

$$L = \sum_{n=1}^{\infty} \left[\frac{1}{2} i\hbar \, a_n(t)^* \, \frac{da_n(t)}{dt} - \frac{1}{2} i\hbar \, \frac{da_n(t)^*}{dt} \, a_n(t) - e_n \, a_n(t)^* \, a_n(t) \right]$$

Write

$$a_n(t) = b_n(t) + i c_n(t)$$

This gives

$$L = \sum_{n=1}^{\infty} \left[\hbar \, c_n(t) \frac{db_n(t)}{dt} - \hbar \, b_n(t) \frac{dc_n(t)}{dt} - e_n \{ b_n(t)^2 + c_n(t)^2 \} \right]$$

$b_n(t)$ and $c_n(t)$ are the degrees of freedom

$$L = \sum_{n=1}^{\infty} \left[\hbar\, c_n(t) \frac{db_n(t)}{dt} - \hbar\, b_n(t) \frac{dc_n(t)}{dt} - e_n \{ b_n(t)^2 + c_n(t)^2 \} \right]$$

Equations of motion:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{b}_n} - \frac{\partial L}{\partial b_n} = 0, \qquad \frac{d}{dt} \frac{\partial L}{\partial \dot{c}_n} - \frac{\partial L}{\partial c_n} = 0$$

$$\Rightarrow \quad \hbar \frac{dc_n}{dt} + \hbar \frac{dc_n}{dt} + 2e_n b_n = 0, \qquad -\hbar \frac{db_n}{dt} - \hbar \frac{db_n}{dt} + 2e_n c_n = 0$$

Use first + i × second equation and $a_n = b_n + ic_n$

$$\Rightarrow \quad -i\hbar \frac{da_n}{dt} + e_n a_n = 0 \quad \Rightarrow \quad i\hbar \frac{\partial \psi(t, \vec{r})}{\partial t} - \widehat{h} \psi(t, \vec{r}) = 0$$

– correct equations

$$L = \sum_{n=1}^{\infty} \left[\hbar\, c_n(t) \frac{db_n(t)}{dt} - \hbar\, b_n(t) \frac{dc_n(t)}{dt} - e_n \{ b_n(t)^2 + c_n(t)^2 \} \right]$$

Define 'momenta' conjugate to the 'coordinates' b_n, c_n:

$$\widetilde{b}_n = \frac{\partial L}{\partial \dot{b}_n} = \hbar\, c_n, \qquad \widetilde{c}_n = \frac{\partial L}{\partial \dot{c}_n} = -\hbar\, b_n$$

$$H = \sum_{n=1}^{\infty} \left\{ \widetilde{b}_n\, \dot{b}_n + \widetilde{c}_n\, \dot{c}_n \right\} - L = \sum_{m=1}^{\infty} e_n \left\{ b_n(t)^2 + c_n(t)^2 \right\} = \sum_{m=1}^{\infty} e_n\, a_n(t)^* a_n(t)$$

This recovers the previous expression for the Hamiltonian.

Now we need to understand how to quantize the system in terms of the independent variables b_n and c_n, i.e. how to compute $[b_n, c_m]$ etc.

If we just substitute $c_n = \widetilde{b}_n/\hbar$ and use $[b_m, \widetilde{b}_n] = i\hbar\, \delta_{mn}$, we shall get

$$[b_m, c_n] = i\hbar\, \delta_{mn}/\hbar = i\, \delta_{mn}$$

This is wrong!

Both constraints need to be accounted for together.

We have to go through Dirac's procedure for quantizing constrained system.

For this we shall restrict to some particular n since variables for different n commute anyway.

Define:

$$\chi_1 = \widetilde{b}_n - \hbar\, c_n, \qquad \chi_2 = \widetilde{c}_n + \hbar\, b_n$$

The constraints are

$$\chi_1 = 0, \qquad \chi_2 = 0$$

We shall now review Dirac's procedure and apply it to our system.

Suppose we have a system with coordinates $q_1, \cdots, q_K$, conjugate momenta $p_1, \cdots, p_K$ and M constraints:

$$\chi_1(\vec{q}, \vec{p}) = 0, \quad \chi_2(\vec{q}, \vec{p}) = 0, \cdots, \quad \chi_M(\vec{q}, \vec{p}) = 0$$

Define:

$$M_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\}_{PB} = \sum_{i=1}^{K} \left[\frac{\partial \chi_\alpha}{\partial q_i} \frac{\partial \chi_\beta}{\partial p_i} - \frac{\partial \chi_\alpha}{\partial p_i} \frac{\partial \chi_\beta}{\partial q_i} \right], \quad 1 \leq \alpha, \beta \leq M$$

PB stands for Poisson bracket

If $\det M \neq 0$, the constraints are known as second class.

In that case, given any pair of functions F and G of q_i's and p_i's, we define the Dirac bracket as follows:

$$\{F, G\}_{DB} = \{F, G\}_{PB} - \sum_{\alpha,\beta=1}^{M} \{F, \chi_\alpha\}_{PB} \, (M^{-1})_{\alpha\beta} \, \{\chi_\beta, G\}_{PB}$$

M^{-1}: matrix inverse of M (defined if $\det M \neq 0$)

Note: Like PB, DB is anti-symmetric under $F \leftrightarrow G$

Now among q_i's and p_i's choose a set of $2K - M$ independent variables

Call them $Q_1, \cdots, Q_{2K-M}$

Compute $\{Q_a, Q_b\}_{DB}$ and call the result W_{ab}

Quantization: Regard Q_a's as operators with commutation relations:

$$[Q_a, Q_b] = i\,\hbar\, W_{ab} \qquad \text{for } 1 \le a, b \le 2\,K - M$$

Note: In the absence of constraints, $DB = PB$ and this gives the standard quantization procedure, i.e. in the Poisson bracket relations between the q's and p's, replace $\{\ ,\ \}_{PB}$ by $[\ ,\ ]/(i\hbar)$.

For full details of quantization of constrained systems, see the book

Paul Dirac, Lectures on Quantum Mechanics, Lecture 1

Now apply this to our system.

$$\chi_1 = \widetilde{b}_n - \hbar\, c_n, \qquad \chi_2 = \widetilde{c}_n + \hbar\, b_n$$

$$\{\chi_1, \chi_1\}_{PB} = 0, \quad \{\chi_2, \chi_2\}_{PB} = 0, \quad \{\chi_1, \chi_2\}_{PB} = -2\hbar, \quad \{\chi_2, \chi_1\}_{PB} = 2\hbar$$

$$M = \begin{pmatrix} 0 & -2\hbar \\ 2\hbar & 0 \end{pmatrix}, \quad M^{-1} = \frac{1}{2\hbar}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\{b_n, \chi_1\}_{PB} = 1, \quad \{\chi_1, b_n\}_{PB} = -1, \quad \{b_n, \chi_2\}_{PB} = 0, \quad \{\chi_2, b_n\}_{PB} = 0,$$

$$\{c_n, \chi_1\}_{PB} = 0, \quad \{\chi_1, c_n\}_{PB} = 0, \quad \{c_n, \chi_2\}_{PB} = 1, \quad \{\chi_2, c_n\}_{PB} = -1,$$

$$\{b_n, c_n\}_{DB} = \{b_n, c_n\}_{PB} - \{b_n, \chi_1\}_{PB}(M^{-1})_{12}\{\chi_2, c_n\}_{PB} = 0 - (1)\frac{1}{2\hbar}(-1) = \frac{1}{2\hbar}$$

Quantization:

$$\frac{1}{i\,\hbar}[b_n, c_n] = \frac{1}{2\hbar} \quad \Rightarrow \quad [b_n, c_n] = \frac{i}{2}$$

Full set of commutation relations:

$$[b_n, c_m] = \frac{i}{2}\,\delta_{mn}, \quad [b_m, b_n] = 0, \quad [c_m, c_n] = 0$$

$$[b_n, c_m] = \frac{i}{2}\,\delta_{mn}, \quad [b_m, b_n] = 0, \quad [c_m, c_n] = 0$$

Using $a_m = b_m + ic_m$, $a_n^\dagger = b_n - ic_n$ we get:

$$[a_m, a_n] = [b_m + ic_m, b_n + ic_n] = i[b_m, c_n] + i[c_m, b_n] = 0$$

Similarly we can compute other commutators, leading to:

$$[a_m, a_n^\dagger] = \delta_{mn}, \quad [a_m, a_n] = 0, \quad [a_m^\dagger, a_n^\dagger] = 0$$

$$\begin{aligned}
\widehat{H} &= \sum_{n=1}^{\infty} e_n\,(b_n^2 + c_n^2) = \frac{1}{4}\sum_{n=1}^{\infty} e_n\,\left[(a_n + a_n^\dagger)^2 - (a_n - a_n^\dagger)^2\right] \\
&= \frac{1}{4}\sum_{n=1}^{\infty} e_n\,\left[(a_n)^2 + (a_n^\dagger)^2 + a_n a_n^\dagger + a_n^\dagger a_n - (a_n)^2 - (a_n^\dagger)^2 + a_n a_n^\dagger + a_n^\dagger a_n\right] \\
&= \frac{1}{2}\sum_{n=1}^{\infty} e_n\,\left[a_n a_n^\dagger + a_n^\dagger a_n\right] = \sum_{n=1}^{\infty} e_n\, a_n^\dagger a_n + \text{constant}
\end{aligned}$$

– same as what we had found earlier.

$$\psi(t, \vec{r}) = \sum_{n=1}^{\infty} a_n(t)\, u_n(\vec{r})$$

$$[\psi(t,\vec{r}), \psi(t,\vec{r}\,')^\dagger] = \delta^{(3)}(\vec{r}-\vec{r}\,'), \qquad [\psi(t,\vec{r}), \psi(t,\vec{r}\,')] = 0, \qquad [\psi(t,\vec{r})^\dagger, \psi(t,\vec{r}\,')^\dagger] = 0$$

We shall now try to find representation of other operators in the third description.

One body operator $\widehat{B}_N = \sum_{i=1}^{N} \widehat{b}_i$:

$$\widehat{B} = \sum_{n,p=1}^{\infty} b_{np} a_n(t)^\dagger a_p(t)$$

$$b_{np} = \int d^3r_1 \, u_n^*(\vec{r}_1) \, \widehat{b}_1 \; u_p(\vec{r}_1)$$

$$\widehat{B} = \sum_{n,p=1}^{\infty} \int d^3r_1 \, u_n^*(\vec{r}_1) \, \widehat{b}_1 \, u_p(\vec{r}_1) a_n(t)^\dagger \, a_p(t) = \int d^3r_1 \psi(t,\vec{r}_1)^\dagger \, \widehat{b}_1 \, \psi(t,\vec{r}_1)$$

Note: ψ and $\psi^\dagger$ are the operators in the Hilbert space and $\widehat{b}$ is just a differential operator

Algorithm:

1. Take expectation value of $\widehat{b}_1$ in the single particle wave-function $\psi(t,\vec{r}_1)$

2. In the resulting expression, regard $\psi, \psi^\dagger$ as operators.

The number operator:

$$\widehat{N} = \sum_{n=1}^{\infty} a_n^\dagger a_n = \sum_{m,n=1}^{\infty} a_m^\dagger a_n \int d^3r \, u_m(\vec{r})^* u_n(\vec{r}) = \int d^3r \, \psi(t,\vec{r})^\dagger \psi(t,\vec{r})$$

The Hamiltonian:

$$\widehat{H} = \int d^3r \, \psi(t,\vec{r})^\dagger \, \widehat{h} \, \psi(t,\vec{r})$$

Two body operator:

$$\widehat{V}_N = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \widehat{v}_{ij}, \qquad \widehat{v}_{ij} = \widehat{v}_{ji}$$

In the second description, this is mapped to the operator:

$$\widehat{V} = \sum_{m,n,p,q=1}^{\infty} v_{m,n,p,q} a_m^\dagger a_n^\dagger a_p a_q$$

$$v_{m,n,p,q} = \int d^3 r_1 d^3 r_2 \, u_m(\vec{r}_1)^* u_n(\vec{r}_2)^* \, \widehat{v}_{12} \, u_p(\vec{r}_1) u_q(\vec{r}_2)$$

In the third description:

$$\begin{aligned} \widehat{V} &= \sum_{m,n,p,q=1}^{N} \int d^3 r_1 d^3 r_2 \, u_m(\vec{r}_1)^* u_n(\vec{r}_2)^* \, \widehat{v}_{12} \, u_p(\vec{r}_1) u_q(\vec{r}_2) a_m^\dagger a_n^\dagger a_p a_q \\ &= \int d^3 r_1 d^3 r_2 \psi(t,\vec{r}_1)^\dagger \psi(t,\vec{r}_2)^\dagger \, \widehat{v}_{12} \, \psi(t,\vec{r}_1) \, \psi(t,\vec{r}_2) \end{aligned}$$

$\widehat{v}_{12}$ is not an operator in the Hilbert space but should be regarded as a differential operator acting on ψ

e.g. if $\widehat{v}_{12} = \vec{p}_1.\vec{p}_2$ then

$$\widehat{v}_{12} \, \psi(t,\vec{r}_1) \, \psi(t,\vec{r}_2) = (-i\hbar)^2 \vec{\nabla}_1 \psi(t,\vec{r}_1).\vec{\nabla}_2 \psi(t,\vec{r}_2)$$

Algorithm:

1. Take expectation value of $\widehat{v}_{12}$ in the two particle wave-function $\psi(t,\vec{r}_1)\psi(t,\vec{r}_2)$

2. In the resulting expression, regard $\psi, \psi^\dagger$ as operators.

The same procedure works for k-body interaction.

5 Second quantization

Recall that we started with:

$$L = \int d^3r\, \psi^* \left[i\hbar \frac{\partial \psi}{\partial t} - \widehat{h}\, \psi \right]$$

and derived, after quantization:

$$\widehat{H} = \int d^3r\, \psi(t,\vec{r})^\dagger\, \widehat{h}\, \psi(t,\vec{r}) = \sum_{n=1}^{\infty} e_n\, a_n(t)^\dagger a_n(t)$$

$$\psi(t,\vec{r}) = \sum_{n=1}^{\infty} a_n(t)\, u_n(\vec{r}), \qquad \psi(t,\vec{r})^\dagger = \sum_{n=1}^{\infty} a_n(t)^\dagger\, u_n(\vec{r})^*$$

$$[a_m(t), a_n(t)^\dagger] = \delta_{mn}, \quad [a_m(t), a_n(t)] = 0, \quad [a_m(t)^\dagger, a_n(t)^\dagger] = 0$$

$$[\psi(t,\vec{r}), \psi(t,\vec{r}\,')^\dagger] = \delta^{(3)}(\vec{r}-\vec{r}\,'), \quad [\psi(t,\vec{r}), \psi(t,\vec{r}\,')] = 0, \quad [\psi(t,\vec{r})^\dagger, \psi(t,\vec{r}\,')^\dagger] = 0$$

We shall now try to give physical interpretation of $\psi(t,\vec{r})$ and $\psi(t,\vec{r})^\dagger$.

Recall the interpretation of $a_n, a_n^\dagger$:

At $t = 0$,

$a_n^\dagger|0\rangle$ describes a single particle state with wave-function $u_n(\vec{r})$

$a_n^\dagger\, a_m^\dagger|0\rangle$ describes a two particle state in states $u_n(\vec{r})$ and $u_m(\vec{r})$ etc.

Q. What state does $\psi(0,\vec{r}_0)^\dagger|0\rangle$ represent for some fixed vector $\vec{r}_0$?

$$\psi(0,\vec{r}_0)^\dagger|0\rangle = \sum_{n=1}^{\infty} u_n(\vec{r}_0)^* a_n(0)^\dagger|0\rangle$$

1. Since this is a linear combination of single particle states, it is a single particle state

2. The wave-function of the state:

$$\sum_{n=1}^{\infty} u_n(\vec{r}_0)^* u_n(\vec{r}) = \delta^{(3)}(\vec{r} - \vec{r}_0)$$

Therefore $\psi(0,\vec{r}_0)^\dagger|0\rangle$ represents a one particle state in position eigenstate at position $\vec{r}_0$.

$\Rightarrow$ $\psi, \psi^\dagger$ play the same role as $a_n, a_n^\dagger$, but in a basis of position eigenstates instead of basis of energy eigenstates.

Given this, one can ask: Is it possible to derive the commutation relations of $\psi, \psi^\dagger$ and the expression for the Hamiltonian in terms of $\psi, \psi^\dagger$ directly, instead of going through the $a_n, a_n^\dagger$'s?

The main issue: Dealing with a continuous label $\vec{r}$ instead of discrete label n

Why do we want to rederive an established result?

Recall the motivation from lecture 1:

Quantum field theory is useful for multi-particle system but essential for cases when the particle number is not conserved.

Eventually we want to apply it to the cases where the particle number is not conserved and state labels are continuous

– relativistic quantum field theory

But before applying a new method to a new system, we need to check that the new method works in systems where old methods also work and results are known.

The multi-particle quantum mechanics with discrete energy levels is such a system.

We want to test all the techniques of quantum field theory on this system before applying it to systems for which the answer is not known by any other method.

$$L = \int d^3r\, \psi^* \left[i\hbar \frac{\partial \psi}{\partial t} - \widehat{h}\, \psi \right]$$

Trick: Use the definition of an integral as the limit of a sum, but take the limit later.

Discretize space as a lattice of points:

$$\vec{r} = w\vec{i}, \qquad w \text{ small}, \qquad \vec{i} = (i_x, i_y, i_z), \qquad i_x, i_y, i_z\text{: integers}$$

Dynamical variables:

$$\psi(t,\vec{r}) \to \psi_{\vec{i}}(t), \qquad \psi(t,\vec{r})^* \to \psi_{\vec{i}}(t)^*, \qquad \int d^3r \to w^3 \sum_{\vec{i}}$$

$$\int d^3r\, \psi(t,\vec{r})^* \frac{\partial \psi(t,\vec{r})}{\partial t} \quad \Rightarrow \quad w^3 \sum_{\vec{i}} \psi_{\vec{i}}(t)^* \frac{\partial \psi_{\vec{i}}(t)}{\partial t}$$

Defining $\widehat{h}$ requires a little more effort since $\widehat{h}$ contains spatial derivatives

$$\partial_x \psi(t,\vec{r}) \to w^{-1} \left(\psi_{i_x+1, i_y. i_z} - \psi_{i_x, i_y, i_z} \right) \quad \text{etc.}$$

$$\begin{aligned}
\partial_x^2 \psi(t,\vec{r}) \;&\to\; w^{-1} \left(\partial_x \psi(t, \vec{r} + (w,0,0)) - \partial_x \psi(t,\vec{r}) \right) \\
&\to\; w^{-2} \left\{ \left(\psi_{i_x+2, i_y. i_z} - \psi_{i_x+1, i_y, i_z} \right) - \left(\psi_{i_x+1, i_y. i_z} - \psi_{i_x, i_y, i_z} \right) \right\} \\
&=\; w^{-2} \left\{ \psi_{i_x+2, i_y. i_z} - 2\, \psi_{i_x+1, i_y, i_z} + \psi_{i_x, i_y, i_z} \right\} \qquad \text{etc.}
\end{aligned}$$

Using this we can map $\widehat{h}\psi(t,\vec{r})$ to some quantity that we shall call $\widehat{h}\, \psi_{\vec{i}}(t)$

$$L = w^3 \sum_{\vec{i}} \psi_{\vec{i}}(t)^* \left[i\hbar \frac{\partial \psi_{\vec{i}}(t)}{\partial t} - \widehat{h} \psi_{\vec{i}}(t) \right]$$

$$L = w^3 \sum_{\vec{i}} \psi_{\vec{i}}(t)^* \left[i\hbar \frac{\partial \psi_{\vec{i}}(t)}{\partial t} - \widehat{h}\psi_{\vec{i}}(t) \right]$$

We can now regard $\psi_{\vec{i}}(t)^*$ and $\psi_{\vec{i}}(t)$ as independent dynamical variables and proceed.

e.g. conjugate momenta $\pi_{\vec{i}}$ to $\psi_{\vec{i}}$ and $\widetilde{\pi}_{\vec{i}}$ to $\psi^*_{\vec{i}}$ are:

$$\pi_{\vec{i}} = \frac{\partial L}{\partial \dot{\psi}_{\vec{i}}} = i\,\hbar\, w^3\, \psi^*_{\vec{i}}, \qquad \widetilde{\pi}_{\vec{i}} = \frac{\partial L}{\partial \dot{\psi}^*_{\vec{i}}} = 0$$

$$H = \sum_{\vec{i}} \{\pi_{\vec{i}}\dot{\psi}_{\vec{i}} + \widetilde{\pi}_{\vec{i}}\dot{\psi}^*_{\vec{i}}\} - L = w^3 \sum_{\vec{i}} \psi_{\vec{i}}(t)^*\, \widehat{h}\, \psi_{\vec{i}} \to \int d^3r \psi(t,\vec{r})^* \widehat{h} \psi(t,\vec{r})$$

Quantization: Make $\psi, \psi^\dagger$ as operators and use equal time commutators:

$$[\psi_{\vec{i}}, \psi_{\vec{j}}] = 0, \qquad [\pi_{\vec{i}}, \pi_{\vec{j}}] = 0, \qquad [\psi_{\vec{i}}, \pi_{\vec{j}}] = i\hbar\delta_{\vec{i},\vec{j}}, \qquad \delta_{\vec{i},\vec{j}} = \delta_{i_x j_x}\delta_{i_y j_y}\delta_{i_z j_z}$$

This translates to

$$[\psi_{\vec{i}}, \psi_{\vec{j}}] = 0, \qquad [\psi^\dagger_{\vec{i}}, \psi^\dagger_{\vec{j}}] = 0, \qquad [\psi_{\vec{i}}, \psi^\dagger_{\vec{j}}] = \frac{1}{w^3}\,\delta_{\vec{i},\vec{j}}$$

Let $D(\vec{r},\vec{r}\,')$ be the continuous function whose discrete version is $\delta_{\vec{i},\vec{j}}/w^3$ with $\vec{r} = w\,\vec{i}$, $\vec{r}\,' = w\,\vec{j}$

Then given any function $F(\vec{r}\,')$, we have

$$\int d^3r'\, D(\vec{r},\vec{r}\,')\, F(\vec{r}\,') \quad \to \quad w^3 \sum_{\vec{j}} \frac{1}{w^3}\,\delta_{\vec{i},\vec{j}}\, F_{\vec{j}} = F_{\vec{i}} \quad \to \quad F(\vec{r})$$

$$\Rightarrow D(\vec{r},\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

$$\Rightarrow [\psi(t,\vec{r}), \psi(t,\vec{r}\,')^\dagger] = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

This is not yet fully satisfactory.

1. We would like to get a more direct approach without going through discretization

2. The formula $\pi_{\vec{i}} = i\,\hbar\, w^3\, \psi^*_{\vec{i}}$ does not have a good continuum limit

In the $w \to 0$ limit the rhs vanishes

We shall now try to rectify both shortcomings using 'functional derivative'

Define:

$$\frac{\delta F}{\delta \psi(t,\vec{r})} = \frac{1}{w^3}\,\frac{\partial F}{\partial \psi_{\vec{i}}(t)}$$

for any functional F, as $w \to 0$.

$$\frac{\delta \psi(t,\vec{r}\,')}{\delta \psi(t,\vec{r})} = \frac{1}{w^3}\,\frac{\partial \psi_{\vec{j}}(t)}{\partial \psi_{\vec{i}}(t)} = \frac{1}{w^3}\delta_{\vec{i},\vec{j}} = \delta^{(3)}(\vec{r}-\vec{r}\,'), \quad \vec{r} = w\,\vec{i}, \quad \vec{r}\,' = w\,\vec{j}$$

Define:

$$\Pi(t,\vec{r}) = \frac{\delta L}{\delta \dot{\psi}(t,\vec{r})} \quad \to \quad \Pi_{\vec{i}} = \frac{1}{w^3}\,\frac{\partial L}{\partial \dot{\psi}_{\vec{i}}(t)} = \frac{1}{w^3}\pi_{\vec{i}} = i\hbar\psi^*_{\vec{i}} \quad \to \quad \Pi(t,\vec{r}) = i\hbar\psi(t,\vec{r})^*$$

$\Rightarrow$ there is a well-defined relation between Π and ψ^* as $w \to 0$.

$$H = \sum_{\vec{i}} \pi_{\vec{i}}\,\dot{\psi}_{\vec{i}} - L = w^3 \sum_{\vec{i}} \Pi_{\vec{i}}\,\dot{\psi}_{\vec{i}} - L \quad \to \quad \int d^3r\,\Pi(t,\vec{r})\,\dot{\psi}(t,\vec{r}) - L$$

$$[\psi(t,\vec{r}),\Pi(t,\vec{r}\,')] \quad \to \quad [\psi_{\vec{i}},\Pi_{\vec{j}}] = \frac{1}{w^3}[\psi_{\vec{i}},\pi_{\vec{j}}] = \frac{1}{w^3}\,i\hbar\,\delta_{\vec{i},\vec{j}} = i\,\hbar\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

Based on this analysis, we can now formulate the rules for quantization directly in the continuum.

Define conjugate momentum to $\psi(t,\vec{r})$ as

$$\Pi(t,\vec{r}) = \frac{\delta L}{\delta\dot{\psi}(t,\vec{r})}$$

$$H = \int d^3r\, \Pi(t,\vec{r})\dot{\psi}(t,\vec{r}) - L$$

Quantization:

$$[\psi(t,\vec{r}), \Pi(t,\vec{r}\,')] = i\hbar\delta^{(3)}(\vec{r}-\vec{r}\,')$$

For

$$L = \int d^3r\, \psi(t,\vec{r})^* \left[i\hbar \frac{\partial\psi(t,\vec{r})}{\partial t} - \widehat{h}\,\psi(t,\vec{r})\right]$$

this gives

$$\Pi(t,\vec{r}') = i\hbar \int d^3r\, \psi(t,\vec{r})^*\delta^{(3)}(\vec{r}-\vec{r}\,') = i\hbar\psi(t,r\,')^*$$

$$H = \int d^3r\, \psi(t,\vec{r})^*\widehat{h}\,\psi(t,\vec{r})$$

Quantization:

$$[\psi(t,\vec{r}), \psi(t,\vec{r}\,')^\dagger] = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

$$\widehat{H} = \int d^3r\, \psi(t,\vec{r})^\dagger\widehat{h}\,\psi(t,\vec{r})$$

– New derivation of earlier results

A useful relation:

For any quantity F that depends on ψ, under an arbitrary variation of ψ:

$$\delta F = \sum_{\vec{i}} \frac{\partial F}{\partial \psi_{\vec{i}}} \, \delta\psi_{\vec{i}} = w^3 \sum_{\vec{i}} \frac{1}{w^3} \frac{\partial F}{\partial \psi_{\vec{i}}} \, \delta\psi_{\vec{i}} \quad \to \quad \int d^3 r \, \frac{\delta F}{\delta \psi(t, \vec{r})} \, \delta\psi(t, \vec{r})$$

6 Second quantization of fermions

We shall now repeat the analysis (from lecture 1 onwards) for identical fermions (spinless).

Single particles satisfy the same Schrodinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \widehat{h}\psi$$

$$\widehat{h}\,\psi = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi + U(\vec{r})\,\psi$$

$\widehat{h}$: Single particle Hamiltonian

Eigenvalues and eigenstates of $\widehat{h}$.

$$\widehat{h}\,u_n(\vec{r}) = e_n u_n(\vec{r})$$

$\{u_n\}$: complete basis of states, normalized as

$$\int d^3r\, u_n^*(\vec{r})\, u_m(\vec{r}) = \delta_{mn}$$

General solution:

$$\psi(t, \vec{r}) = \sum_n a_n(t) u_n(\vec{r}), \qquad a_n(t) = a_n(0)\, e^{-ie_n t/\hbar}$$

No difference with the case of single boson.

Now consider a system of N identical fermions of same mass, each moving under the same potential, and with no mutual interaction

$\vec{r}_i$: position of the i'th particle

$\vec{\nabla}_i$: the gradient operators with respect to the i'th particle coordinate

The wave function $\psi(t, \vec{r}_1, \cdots, \vec{r}_N)$ satisfies Schrodinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \widehat{H}_N \, \psi, \qquad \widehat{H}_N = \sum_{i=1}^{N} \widehat{h}_i, \qquad \widehat{h}_i = -\frac{\hbar^2}{2m} \vec{\nabla}_i^2 + U(\vec{r}_i)$$

ψ is <u>anti-symmetric</u> under the exchange of any two $\vec{r}_i$'s.

Choose basis of states:

$$u_{n_1, \cdots, n_N}(\vec{r}_1, \cdots, \vec{r}_N) \equiv \frac{1}{\sqrt{N!}} \sum_{\text{Permutations of } \vec{r}_1, \cdots \vec{r}_N} u_{n_1}(\vec{r}_1) \cdots u_{n_N}(\vec{r}_N) (-1)^P$$

$(-1)^P$:

1 if we reach the permutation by even number of exchanges from $1, 2, \cdots, N$

-1 if we reach the permutation by odd number of exchanges from $1, 2, \cdots, N$

Example: For $N = 3$,

$(-1)^P = 1$ for 123, 231, 312

$(-1)^P = -1$ for 213, 132, 321

$$u_{n_1,\cdots,n_N}(\vec{r}_1,\cdots,\vec{r}_N) \equiv \frac{1}{\sqrt{N!}} \sum_{\text{Permutations of } \vec{r}_1,\cdots\vec{r}_N} u_{n_1}(\vec{r}_1)\cdots u_{n_N}(\vec{r}_N)(-1)^P$$

This basis is also automatically anti-symmetric under the exchange of the n_i's

e.g. $u_{1,2}$ and $u_{2,1}$ should not be counted as separate basis states

We label the subscripts in a fixed order e.g. in the order of increasing energy and/or other quantum numbers, as in the case of bosonic theory

e.g. $u_{1,2,4}$ but not $u_{2,1,4}$ or $u_{1,4,2}$

Furthermore, a given index cannot be repeated more than once since the result will vanish by anti-symmetry, *e.g.* $u_{1,1,2} = 0$

– different from bosons

In the occupancy number representation this means that occupancy number m_n of the n-th state can be either 0 or 1, but not > 1

– Pauli exclusion principle

$$u_{n_1,\cdots,n_N}(\vec{r}_1,\cdots,\vec{r}_N) \equiv \frac{1}{\sqrt{N!}} \sum_{\text{Permutations of } \vec{r}_1,\cdots\vec{r}_N} u_{n_1}(\vec{r}_1)\cdots u_{n_N}(\vec{r}_N)(-1)^P$$

Normalization of the basis states.

$$\int d^3r_1\cdots d^3r_N\, u_{n_1,\cdots,n_N}(\vec{r}_1,\cdots,\vec{r}_N)^*\, u_{l_1,\cdots,l_N}(\vec{r}_1,\cdots,\vec{r}_N) = \delta_{l_1n_1}\delta_{l_2n_2}\cdots\delta_{l_Nn_N}$$

All cross terms vanish as in the bosonic case, e.g.

$$\int d^3r_1 d^3r_2 u^*_{n_1}(\vec{r}_1)u^*_{n_2}(\vec{r}_2)u_{n_2}(\vec{r}_1)u_{n_1}(\vec{r}_2) = 0$$

if $n_1 \neq n_2$.

The extra complication that we had in the bosonic case for occupancy number > 1 is absent here since $m_n = 0$ or 1.

Action of N particle Hamiltonian on these basis states.

$$\widehat{H}_N = \sum_{i=1}^N \widehat{h}_i$$

$$\widehat{H}_N\, u_{n_1,\cdots,n_N}(\vec{r}_1,\cdots,\vec{r}_N) = \left(\sum_{i=1}^N e_{n_i}\right) u_{n_1,\cdots,n_N}(\vec{r}_1,\cdots,\vec{r}_N)$$

– same as for bosons.

We shall now consider a second quantum system.

A Hilbert space on which acts a set of operators $a_1, a_2, \cdots$ and their hermitian conjugates $a_1^\dagger, a_2^\dagger, \cdots$ satisfying:

$$\{a_m, a_n\} = 0, \qquad \{a_m^\dagger, a_n^\dagger\} = 0, \qquad \{a_m, a_n^\dagger\} = \delta_{mn}$$

Definition:

$$\{A, B\} = A\,B + B\,A$$

Hamiltonian:

$$\widehat{H} = \sum_{n=1}^{\infty} e_n\, a_n^\dagger a_n$$

This system is some time called the 'fermionic harmonic oscillator', but this is not a harmonic oscillator.

As in the case of harmonic oscillators, the ground state $|0\rangle$, also called the vacuum state, is defined as

$$a_n|0\rangle = 0 \quad \text{for every } n$$

Other states are created by applying arbitrary combinations of $a_n^\dagger$ on the vacuum state.

Note: Since $(a_n^\dagger)^2 = 0$, each $a_n^\dagger$ can be applied at most once.

Goal: We shall show that this quantum theory is related to the system considered earlier under certain identifications.

Identification of states:

$$u_{n_1,n_2,\cdots,n_N} \leftrightarrow a^\dagger_{n_1} \cdots a^\dagger_{n_N}|0\rangle$$

Note that like $u_{n_1,n_2,\cdots,n_N}$, the right hand side is automatically anti-symmetric under $n_i \leftrightarrow n_j$

Check that r.h.s have the same inner product as l.h.s

$$\langle 0|a_{n_N} a_{n_{N-1}} \cdots a_{n_1} a^\dagger_{l_1} \cdots a^\dagger_{l_N}|0\rangle = \delta_{l_1 n_1}\delta_{l_2 n_2} \cdots \delta_{l_N n_N}$$

Note: Since a_n's for different n anti-commute, the ordering is important for determining the sign.

Conclusion: $a^\dagger_{n_1} \cdots a^\dagger_{n_N}|0\rangle$ has the same norm as $u_{n_1,\cdots,n_N}$ in the first theory.

This gives a map between the Hilbert spaces of the two theories.

Let us compare the energy eigenvalues in the two quantum systems.

$$\widehat{H} a^\dagger_{n_1} \cdots a^\dagger_{n_N} |0\rangle \;\; = \;\; \left(\sum_{n=1}^{\infty} \hbar \omega_n a^\dagger_n a_n \right) a^\dagger_{n_1} \cdots a^\dagger_{n_N} |0\rangle$$

Use

$$[\widehat{H}, a^\dagger_p] = \left[\sum_{n=1}^{\infty} \hbar \omega_n a^\dagger_n a_n, a^\dagger_p \right] = \left[\sum_{n=1}^{\infty} \hbar \omega_n (a^\dagger_n a_n a^\dagger_p - a^\dagger_p a^\dagger_n a_n) \right]$$
$$= \left[\sum_{n=1}^{\infty} \hbar \omega_n (a^\dagger_n a_n a^\dagger_p + a^\dagger_n a^\dagger_p a_n - a^\dagger_n a^\dagger_p a_n - a^\dagger_p a^\dagger_n a_n) \right]$$
$$= \left[\sum_{n=1}^{\infty} \hbar \omega_n a^\dagger_n \delta_{np} \right] = \hbar \, \omega_p \, a^\dagger_p$$

$$\widehat{H} a^\dagger_{n_1} \cdots a^\dagger_{n_N} |0\rangle = (e_{n_1} + \cdots + e_{n_N}) \, a^\dagger_{n_1} \cdots a^\dagger_{n_N} |0\rangle$$

- same energy eigenvalue as the first system

Map of other operators:

Consider the one body operators:

$$\widehat{B}_N = \sum_{i=1}^{N} \widehat{b}_i$$

Claim: This is mapped to the operator:

$$\widehat{B} = \sum_{n,p=1}^{\infty} b_{np} a_n^\dagger a_p$$

$$b_{np} = \int d^3 r_1 \, u_n^*(\vec{r}_1) \, \widehat{b}_1 \; u_p(\vec{r}_1)$$

Next consider 2-body operators:

$$\widehat{V}_N = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \widehat{v}_{ij}, \qquad \widehat{v}_{ij} = \widehat{v}_{ji}$$

Claim: In the second description, this is mapped to the operator:

$$\widehat{V} = \sum_{m,n,p,q=1}^{\infty} v_{m,n,p,q} a_m^\dagger a_n^\dagger a_q a_p$$

$$v_{m,n,p,q} = \int d^3 r_1 d^3 r_2 \, u_m(\vec{r}_1)^* u_n(\vec{r}_2)^* \, \widehat{v}_{12} \, u_p(\vec{r}_1) u_q(\vec{r}_2)$$

Note: Order of $a_q \, a_p$

Proof of these will be left as exercise.

7 Second quantization of fermions, Klein-Gordon equation

We shall now study quantum field theory for describing a system of identical fermions.

Goal: Reproduce the relations:

$$\{a_m, a_n\} = 0, \qquad \{a_m^\dagger, a_n^\dagger\} = 0, \qquad \{a_m, a_n^\dagger\} = \delta_{mn}$$

$$\widehat{H} = \sum_{n=1}^{\infty} e_n \, a_n^\dagger a_n$$

We shall begin with the same classical system that we had for the system of identical bosons.

$$L = \int d^3r \, \psi^* \left(i\hbar \frac{\partial \psi}{\partial t} - \widehat{h} \, \psi \right)$$

$$\Pi(t, \vec{r}) = \frac{\delta L}{\delta \dot{\psi}(t, \vec{r})} = i\hbar \psi(t, r)^*$$

$$H = \int d^3r \, \Pi(t, \vec{r}) \dot{\psi}(t, \vec{r}) - L = \int d^3r \, \psi(t, \vec{r})^* \widehat{h} \, \psi(t, \vec{r})$$

$$\Pi(t,\vec{r}) = i\hbar\psi(t,r)^*$$

$$H = \int d^3r\, \psi(t,\vec{r})^* \widehat{h}\, \psi(t,\vec{r})$$

Quantization: Regard the fields as operators, and use:

$$\{\psi(t,\vec{r}), \Pi(t,\vec{r}\,')\} = i\hbar\, \delta^{(3)}(\vec{r}-\vec{r}\,'), \quad \{\psi(t,\vec{r}), \psi(t,\vec{r}\,')\} = 0, \quad \{\Pi(t,\vec{r}), \Pi(t,\vec{r}\,')\} = 0$$

$$\widehat{H} = \int d^3r\, \psi(t,\vec{r})^\dagger \widehat{h}\, \psi(t,\vec{r}), \qquad \Pi(t,\vec{r}) = i\hbar\psi(t,r)^\dagger$$

$$\{A, B\} = AB + BA$$

Note the difference: $[\ ,\ ] \Rightarrow \{\ ,\ \}$

This is an ad hoc rule, and does not follow from the standard rules for quantizing a classical system.

For this reason, there is really no classical limit of a fermionic field theory.

Correspondence principle does not hold.

For example, in the bosonic theory we can construct a quantum state that closely approximates a classical field configuration $\psi(0,\vec{r}) = f(\vec{r})$ as follows:

1. Decompose $f(\vec{r})$ in eigenfunctions of $\widehat{h}$:

$$f(\vec{r}) = \sum_{n=1}^{\infty} f_n \, u_n(\vec{r})$$

f_n: numbers

2. Compare this with the expansion of the quantum field:

$$\psi(0,\vec{r}) = \sum_{n=1}^{\infty} a_n \, u_n(\vec{r}), \qquad a_n = a_n(0)$$

An approximate classical field configuration will be described by a quantum state for which the expectation value of a_n is f_n and the variances in various quantities are small compared to their mean

– coherent state

– requires applying large number of $a_n^\dagger$ on the ground state.

In fermionic theory, we cannot apply $a_n^\dagger$ more than once!

– no classical limit

Nevertheless, as a quantum theory it makes perfect sense since the laws of quantum mechanics are not violated, e.g. the Heisenberg equations of motion remain the same:

$$i\hbar \frac{\partial \psi(t,\vec{r}\,')}{\partial t} = -[\widehat{H}, \psi(t,\vec{r}\,')]$$

$$\{\psi(t,\vec{r}),\Pi(t,\vec{r}\,')\} = i\hbar\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

$$\Pi(t,\vec{r}) = i\hbar\psi(t,r)^\dagger$$

This gives

$$\left\{\psi(t,\vec{r}),\psi(t,\vec{r}\,')^\dagger\right\} = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

Also

$$\{\psi(t,\vec{r}),\psi(t,\vec{r}\,')\} = 0, \quad \left\{\psi(t,\vec{r})^\dagger,\psi(t,\vec{r}\,')^\dagger\right\} = 0$$

$$\widehat{H} = \int d^3r\,\psi(t,\vec{r})^\dagger\widehat{h}\,\psi(t,\vec{r})$$

Let us check that the Heisenberg equations of motion give the correct evolution.

$$\begin{aligned}
&[\widehat{H},\psi(t,\vec{r}\,')] = \widehat{H}\psi(t,\vec{r}\,') - \psi(t,\vec{r}\,')\widehat{H} \\
= &\int d^3r\,\left(\psi(t,\vec{r})^\dagger\widehat{h}\,\psi(t,\vec{r})\psi(t,\vec{r}\,') - \psi(t,\vec{r}\,')\,\psi(t,\vec{r})^\dagger\widehat{h}\,\psi(t,\vec{r})\right) \\
= &\int d^3r\,\left(-\psi(t,\vec{r})^\dagger\psi(t,\vec{r}\,')\widehat{h}\,\psi(t,\vec{r}) - \psi(t,\vec{r}\,')\,\psi(t,\vec{r})^\dagger\widehat{h}\,\psi(t,\vec{r})\right) \\
= &\int d^3r\,\left(-\delta^{(3)}(\vec{r}-\vec{r}\,')\widehat{h}\,\psi(t,\vec{r})\right) = -\widehat{h}'\psi(t,\vec{r}\,')
\end{aligned}$$

$\widehat{h}'$: $\quad\widehat{h}$ with $\vec{r}$ replaced by $\vec{r}\,'$

$$i\hbar\frac{\partial\psi(t,\vec{r}\,')}{\partial t} = -[\widehat{H},\psi(t,\vec{r}\,')] = \widehat{h}'\psi(t,\vec{r}\,')$$

– correct equation

Ehrenfest theorem still holds!

Recovering the algebra of a_n, $a_n^\dagger$:

Define a_n, $a_n^\dagger$ via the expansion of ψ, $\psi^\dagger$:

$$\psi(t,\vec{r}) = \sum_{n=1}^{\infty} a_n(t)\, u_n(\vec{r}), \qquad \psi(t,\vec{r})^\dagger = \sum_{n=1}^{\infty} a_n(t)^\dagger\, u_n(\vec{r})^*$$

$$a_m(t) = \int d^3r\, u_m(\vec{r})^* \psi(t,\vec{r}), \qquad a_m(t)^\dagger = \int d^3r\, u_m(\vec{r})\, \psi(t,\vec{r})^\dagger$$

$$\{a_m(t), a_n(t)^\dagger\} = \int d^3r \int d^3r'\, u_m(\vec{r})^*\, u_n(\vec{r}\,')\, \{\psi(t,\vec{r}), \psi(t,\vec{r}\,')^\dagger\}$$

$$= \int d^3r \int d^3r'\, u_m(\vec{r})^*\, u_n(\vec{r}\,')\, \delta^{(3)}(\vec{r}-\vec{r}\,') = \int d^3r\, u_m(\vec{r})^*\, u_n(\vec{r}) = \delta_{mn}$$

Similarly

$$\{a_m(t), a_n(t)\} = 0, \qquad \{a_m(t)^\dagger, a_n(t)^\dagger\} = 0$$

$$\widehat{H} = \int d^3r\, \psi(t,\vec{r})^\dagger \widehat{h}\, \psi(t,\vec{r}) = \sum_{m,n=1}^{\infty} \int d^3r\, a_m(t)^\dagger a_n(t)\, u_m(\vec{r})^* \widehat{h} u_n(\vec{r})$$

$$= \sum_{m,n=1}^{\infty} a_m(t)^\dagger a_n(t)\, e_n\, \delta_{mn} = \sum_{n=1}^{\infty} e_n\, a_n(t)^\dagger a_n(t)$$

Therefore we have reproduced the relations needed for describing a system of identical fermions.

Number operator:

$$\widehat{N}(t) = \int d^3r\, \psi(t,\vec{r})^\dagger\, \psi(t,\vec{r}) = \sum_{n=1}^{\infty} a_n(t)^\dagger a_n(t)$$

Map of other operators:

One body operator

$$\widehat{B}_N = \sum_{i=1}^{N} \widehat{b}_i$$

is mapped to the operator:

$$\widehat{B} = \sum_{n,p=1}^{\infty} b_{np} a_n^{\dagger} a_p$$

$$b_{np} = \int d^3 r_1 \, u_n^*(\vec{r}_1) \, \widehat{b}_1 \; u_p(\vec{r}_1)$$

This gives

$$\widehat{B} = \sum_{n,p=1}^{\infty} \int d^3 r_1 \, u_n^*(\vec{r}_1) \, \widehat{b}_1 \; u_p(\vec{r}_1) \, a_n^{\dagger} a_p = \int d^3 r_1 \, \psi(t, \vec{r}_1)^{\dagger} \, \hat{b}_1 \, \psi(t, \vec{r}_1)$$

2-body operator

$$\widehat{V}_N = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \widehat{v}_{ij}, \qquad \widehat{v}_{ij} = \widehat{v}_{ji}$$

is mapped to the operator:

$$\widehat{V} = \sum_{m,n,p,q=1}^{\infty} v_{m,n,p,q} a_m^{\dagger} a_n^{\dagger} a_q a_p$$

$$v_{m,n,p,q} = \int d^3 r_1 d^3 r_2 \, u_m(\vec{r}_1)^* u_n(\vec{r}_2)^* \, \widehat{v}_{12} \, u_p(\vec{r}_1) u_q(\vec{r}_2)$$

This gives:

$$\widehat{V} = \int d^3 r_1 \, d^3 r_2 \, \psi(t, \vec{r}_1)^{\dagger} \psi(t, \vec{r}_2)^{\dagger} \, \widehat{v}_{12} \, \psi(t, \vec{r}_2) \, \psi(t, \vec{r}_1)$$

Now we shall discuss relativistic particles.

From now on, we shall work in units:

$$\hbar = 1, \qquad c\,(\text{speed of light}) = 1$$

In any formula, we can recover powers of $\hbar$ and c using dimensional analysis.

If we get an equation $A = B$, its actual form is

$$A = \hbar^a\, c^b\, B$$

Find a and b by comparing the dimensions of the two sides.

In these units, the relativistic relation $E = \sqrt{\vec{p}^2 c^2 + m^2 c^4}$ between energy E and momentum $\vec{p}$ takes the form

$$E = \sqrt{\vec{p}^2 + m^2}$$

What is the generalization of the free particle Schrodinger equation?

Recall the original motivation that led to Schrodinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\vec{\nabla}^2\psi$$

– plane waves of the form $e^{-i\,E\,t+i\vec{p}.\vec{r}}$ should satisfy Schrodinger equation when the non-relativistic relation between E and $\vec{p}$ holds:

$$E = \frac{\vec{p}^2}{2\,m}$$

Now we want:

$$E = \sqrt{\vec{p}^2 + m^2}$$

Problem: Any differential operator with finite number of derivatives will always produce polynomial in $\vec{p}$ acting on plane waves

– simple modification of the right hand side of Schrodinger equation will not work.

One route: Dirac equation – make the wave-function multi-component

Apparently simpler route: Square the relation between E and $\vec{p}$:

$$E^2 = \vec{p}^2 + m^2$$

– follows from the equation:

$$-\frac{\partial^2\phi}{\partial t^2} = -\vec{\nabla}^2\phi + m^2\phi$$

– Klein-Gordon equation

$$-\frac{\partial^2\phi}{\partial t^2} = -\vec{\nabla}^2\phi + m^2\phi$$

1. The equation has second order time derivative

– we need to specify $\phi(t_0, \vec{r})$ and $\dot{\phi}(t_0, \vec{r})$ initially to find $\phi(t, \vec{r})$ at later time

2. The equation is real

– if the initial ϕ and $\dot{\phi}$ are real, $\phi(t, \vec{r})$ will remain real.

Put another way, if we take $\phi = \phi_R + i\,\phi_I$, then the real and imaginary parts of K-G equation gives:

$$-\frac{\partial^2\phi_R}{\partial t^2} = -\vec{\nabla}^2\phi_R + m^2\phi_R, \qquad -\frac{\partial^2\phi_I}{\partial t^2} = -\vec{\nabla}^2\phi_I + m^2\phi_I$$

ϕ_R and ϕ_I evolve independently, and can be considered as two real fields.

For this reason we shall take ϕ to be real.

3. If we substitute the plane wave:

$$e^{-i\,E\,t + i\vec{p}.\vec{r}}$$

into the K-G equation, we get

$$E^2 = \vec{p}^2 + m^2$$

– has two solutions:

$$E = \pm\sqrt{\vec{p}^2 + m^2}$$

If we start with a generic initial condition, and then express the solution as superposition of plane waves, then both solutions will appear.

$$\phi(t, \vec{r}) = \int d^3p \left[A(\vec{p})e^{-it\sqrt{\vec{p}^2+m^2}+i\vec{p}.\vec{r}} + B(\vec{p})e^{it\sqrt{\vec{p}^2+m^2}+i\vec{p}.\vec{r}}\right], \quad B(\vec{p})^* = A(-\vec{p})$$

$$\phi(t,\vec{r}) = \int d^3p \left[A(\vec{p})e^{-it\sqrt{\vec{p}^2+m^2}+i\vec{p}.\vec{r}} + B(\vec{p})e^{it\sqrt{\vec{p}^2+m^2}+i\vec{p}.\vec{r}}\right], \quad B(\vec{p})^* = A(-\vec{p})$$

Alternatively, we could allow ϕ to be complex and set $B(\vec{p}) = 0$.

However the initial condition that leads to such solutions involve long range correlation between ϕ and $\dot{\phi}$:

$$\phi(t,\vec{r}) = \int d^3p\, A(\vec{p})\, e^{-it\sqrt{\vec{p}^2+m^2}+i\vec{p}.\vec{r}}$$

$$\dot{\phi}(t,\vec{r}) = \int d^3p\, A(\vec{p})\, \{-i\sqrt{\vec{p}^2+m^2}\}\, e^{-it\sqrt{\vec{p}^2+m^2}+i\vec{p}.\vec{r}}$$

At $t = t_0$, the initial conditions on ϕ and $\dot{\phi}$ are not independent.

But $\dot{\phi}(t,\vec{r})$ depends on $\phi(t,\vec{r}\,')$ for $\vec{r}\,'$ far away from $\vec{r}$, since

$$A(\vec{p}) = \frac{1}{(2\pi)^3}\int d^3r'\, e^{it_0\sqrt{\vec{p}^2+m^2}-i\vec{p}.\vec{r}\,'}\,\phi(t_0,\vec{r}\,')$$

– non-local initial condition.

For free particle one could live with it, but it is difficult to construct interacting theories out of this without introducing action at a distance.

Conclusion: We should proceed by taking ϕ to be real and keeping both solutions.

What is the interpretation of the negative energy solutions?

It is difficult to find an interpretation in the first quantized formulation in which we regard ϕ as the wave-function of a particle.

However we get a consistent interpretation in the second quantized formulation where we regard ϕ as a quantum field describing a system of bosons.

Quantization of the Klein-Gordon equation

Step 1: Write down a Lagrangian from which we can derive the K-G equations

$$L = \frac{1}{2}\int d^3r \left[(\partial_t\phi)^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right], \qquad \partial_t\phi = \frac{\partial\phi}{\partial t}$$

Action

$$S = \int dt\, L = \frac{1}{2}\int dt \int d^3r \left[(\partial_t\phi)^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right]$$

Under an arbitrary variation $\delta\phi$,

$$\delta S = \int dt \int d^3r \left[(\partial_t\delta\phi)\partial_t\phi - (\vec{\nabla}\delta\phi).\vec{\nabla}\phi - m^2(\delta\phi)\,\phi\right]$$

1. In the first term, integrate by parts in time and ignore boundary terms at initial and final time since $\delta\phi$ vanishes there

2. In the second term, integrate by parts in x, y, z and ignore boundary terms by requiring ϕ to vanish at spatial infinity.

$$\delta S = \int dt \int d^3r \left[-(\delta\phi)\partial_t^2\phi + (\delta\phi)\,\vec{\nabla}^2\phi - m^2(\delta\phi)\,\phi\right]$$

Requiring $\delta S = 0$ for arbitrary $\delta\phi$ gives:

$$-\partial_t^2\phi + \vec{\nabla}^2\phi - m^2\phi = 0$$

– Klein-Gordon equation

Can also be derived using Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{\phi}(t,\vec{r}\,')}\right) - \left(\frac{\delta L}{\delta\phi(t,\vec{r}\,')}\right) = 0$$

8 Klein-Gordon field

Last time we derived the Lagrangian for Klein-Gordon equation

$$L = \frac{1}{2} \int d^3r \left[(\dot{\phi})^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2 \right]$$

We now turn to step 2 for quantization

– define conjugate momentum and Hamiltonian

$$\Pi(t,\vec{r}\,') = \frac{\delta L}{\delta\dot{\phi}(t,\vec{r}\,')} = \int d^3r \, \dot{\phi}(t,\vec{r}) \frac{\delta\dot{\phi}(t,\vec{r})}{\delta\dot{\phi}(t,\vec{r}\,')} = \int d^3r \, \dot{\phi}(t,\vec{r}) \, \delta^{(3)}(\vec{r} - \vec{r}\,') = \dot{\phi}(t,\vec{r}\,')$$

$$H = \int d^3r \, \Pi(t,\vec{r})\dot{\phi}(t,\vec{r}) - L = \frac{1}{2} \int d^3r \left[\Pi(t,\vec{r})^2 + (\vec{\nabla}\phi(t,\vec{r}))^2 + m^2 \, \phi(t,\vec{r})^2 \right]$$

$$H = \frac{1}{2}\int d^3r \left[\Pi(t,\vec{r})^2 + (\vec{\nabla}\phi(t,\vec{r}))^2 + m^2\,\phi(t,\vec{r})^2\right]$$

It will be useful to use momentum space basis, and introduce new independent variables $\widetilde{\phi}(t,\vec{p}), \widetilde{\Pi}(t,\vec{p})$ via:

$$\phi(t,\vec{r}) = \frac{1}{(2\pi)^{3/2}}\int d^3p\,\widetilde{\phi}(t,\vec{p})\,e^{i\vec{p}.\vec{r}}, \qquad \Pi(t,\vec{r}) = \frac{1}{(2\pi)^{3/2}}\int d^3p\,\widetilde{\Pi}(t,\vec{p})\,e^{i\vec{p}.\vec{r}},$$

Inverse relations:

$$\widetilde{\phi}(t,\vec{p}) = \frac{1}{(2\pi)^{3/2}}\int d^3r\,\phi(t,\vec{r})\,e^{-i\vec{p}.\vec{r}}, \qquad \widetilde{\Pi}(t,\vec{p}) = \frac{1}{(2\pi)^{3/2}}\int d^3r\,\Pi(t,\vec{r})\,e^{-i\vec{p}.\vec{r}},$$

$$\widetilde{\phi}(t,\vec{p})^* = \widetilde{\phi}(t,-\vec{p}), \quad \widetilde{\Pi}(t,\vec{p})^* = \widetilde{\Pi}(t,-\vec{p})$$

Note different use of the word ‘momentum’

– $\vec{p}$ refers to arguments of functions after Fourier transformation

– conjugate momentum Π is canonically conjugate variable to ϕ

This gives

$$\int d^3r\,\phi(t,\vec{r})^2 = \int d^3r \int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3p'}{(2\pi)^{3/2}}\,\widetilde{\phi}(t,\vec{p})\,e^{i\vec{p}.\vec{r}}\,\widetilde{\phi}(t,\vec{p}\,')\,e^{i\vec{p}\,'.\vec{r}}$$

$$= \int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3p'}{(2\pi)^{3/2}}\,\widetilde{\phi}(t,\vec{p})\,\widetilde{\phi}(t,\vec{p}\,')\,(2\pi)^3\delta^{(3)}(\vec{p}+\vec{p}\,') = \int d^3p\,\widetilde{\phi}(t,\vec{p})\,\widetilde{\phi}(t,-\vec{p})$$

Similar analysis can be done for other terms, leading to

$$\begin{aligned} H &= \frac{1}{2}\int d^3p\,\left[\widetilde{\Pi}(t,-\vec{p})\widetilde{\Pi}(t,\vec{p}) + \vec{p}^{\,2}\,\widetilde{\phi}(t,-\vec{p})\widetilde{\phi}(t,\vec{p}) + m^2\,\widetilde{\phi}(t,-\vec{p})\widetilde{\phi}(t,\vec{p})\right] \\ &= \frac{1}{2}\int d^3p\,\left[\widetilde{\Pi}(t,\vec{p})^*\widetilde{\Pi}(t,\vec{p}) + \vec{p}^{\,2}\,\widetilde{\phi}(t,\vec{p})^*\widetilde{\phi}(t,\vec{p}) + m^2\,\widetilde{\phi}(t,\vec{p})^*\widetilde{\phi}(t,\vec{p})\right] \end{aligned}$$

Why did we expand in the momentum space basis?

General rule: Expand in the basis of eigenfunctions of the spatial part of the differential operator that appears in the equations of motion

– in this case the operator is $\vec{\nabla}^2 - m^2$

Its eigenfunctions are $e^{i\vec{p}.\vec{r}}$

This simplifies the part of H given by

$$\frac{1}{2}\int d^3r \left[(\vec{\nabla}\phi(t,\vec{r}))^2 + m^2\,\phi(t,\vec{r})^2\right] = \frac{1}{2}\int d^3r\,\phi(t,\vec{r})\,\left[-\vec{\nabla}^2 + m^2\right]\phi(t,\vec{r})$$

This is the analog of finding the normal modes in the small oscillation problem in classical mechanics.

$$H = \frac{1}{2}\int d^3p \,\left[\widetilde{\Pi}(t,\vec{p})^*\widetilde{\Pi}(t,\vec{p}) + \vec{p}^{\,2}\,\widetilde{\phi}(t,\vec{p})^*\widetilde{\phi}(t,\vec{p}) + m^2\,\widetilde{\phi}(t,\vec{p})^*\widetilde{\phi}(t,\vec{p})\right]$$

$$\widetilde{\phi}(t,\vec{p})^* = \widetilde{\phi}(t,-\vec{p}), \quad \widetilde{\Pi}(t,\vec{p})^* = \widetilde{\Pi}(t,-\vec{p})$$

Step 3: Quantization

Regard ϕ and Π as operators satisfying:

$$[\phi(t,\vec{r}),\Pi(t,\vec{r}\,')] = i\,\delta^{(3)}(\vec{r}-\vec{r}\,'), \quad [\phi(t,\vec{r}),\phi(t,\vec{r}\,')] = 0, \quad [\Pi(t,\vec{r}),\Pi(t,\vec{r}\,')] = 0$$

This gives

$$[\widetilde{\phi}(t,\vec{p}),\widetilde{\Pi}(t,\vec{p}\,')] = \left[\frac{1}{(2\pi)^{3/2}}\int d^3r\,\phi(t,\vec{r})\,e^{-i\vec{p}.\vec{r}}, \frac{1}{(2\pi)^{3/2}}\int d^3r'\,\Pi(t,\vec{r}\,')\,e^{-i\vec{p}\,'.\vec{r}\,'}\right]$$

$$= \frac{i}{(2\pi)^3}\int d^3r\,e^{-i\vec{p}.\vec{r}}\int d^3r'\,e^{-i\vec{p}\,'.\vec{r}\,'}\delta^{(3)}(\vec{r}-\vec{r}\,') = \frac{i}{(2\pi)^3}\int d^3r\,e^{-i(\vec{p}+\vec{p}\,').\vec{r}} = i\,\delta^{(3)}(\vec{p}+\vec{p}\,')$$

$$[\widetilde{\phi}(t,\vec{p}),\widetilde{\phi}(t,\vec{p}\,')] = 0, \qquad [\widetilde{\Pi}(t,\vec{p}),\widetilde{\Pi}(t,\vec{p}\,')] = 0$$

$$\widetilde{\phi}(t,\vec{p})^\dagger = \widetilde{\phi}(t,-\vec{p}), \quad \widetilde{\Pi}(t,\vec{p})^\dagger = \widetilde{\Pi}(t,-\vec{p})$$

$$\begin{aligned}
\widehat{H} &= \frac{1}{2}\int d^3p\,\left[\widetilde{\Pi}(t,\vec{p})^\dagger\widetilde{\Pi}(t,\vec{p}) + \vec{p}^{\,2}\,\widetilde{\phi}(t,\vec{p})^\dagger\widetilde{\phi}(t,\vec{p}) + m^2\,\widetilde{\phi}(t,\vec{p})^\dagger\widetilde{\phi}(t,\vec{p})\right] \\
&= \frac{1}{2}\int d^3p\,\left[\widetilde{\Pi}(t,-\vec{p})\widetilde{\Pi}(t,\vec{p}) + \vec{p}^{\,2}\,\widetilde{\phi}(t,-\vec{p})\widetilde{\phi}(t,\vec{p}) + m^2\,\widetilde{\phi}(t,-\vec{p})\widetilde{\phi}(t,\vec{p})\right]
\end{aligned}$$

$$[\widetilde{\phi}(t,\vec{p}),\widetilde{\Pi}(t,\vec{p}\,')] = i\,\delta^{(3)}(\vec{p}+\vec{p}\,'), \quad [\widetilde{\phi}(t,\vec{p}),\widetilde{\phi}(t,\vec{p}\,')] = 0, \quad [\widetilde{\Pi}(t,\vec{p}),\widetilde{\Pi}(t,\vec{p}\,')] = 0$$

$$\widetilde{\phi}(t,\vec{p})^\dagger = \widetilde{\phi}(t,-\vec{p}), \quad \widetilde{\Pi}(t,\vec{p})^\dagger = \widetilde{\Pi}(t,-\vec{p})$$

$$\widehat{H} = \frac{1}{2}\int d^3p \left[\widetilde{\Pi}(t,\vec{p})^\dagger\widetilde{\Pi}(t,\vec{p}) + (\vec{p}^{\,2}+m^2)\,\widetilde{\phi}(t,\vec{p})^\dagger\widetilde{\phi}(t,\vec{p})\right]$$

Except for the †'s, the system looks like independent harmonic oscillators for each $\vec{p}$, with angular frequency $\sqrt{\vec{p}^{\,2}+m^2}$

Define

$$E_{\vec{p}} = \sqrt{\vec{p}^{\,2}+m^2}$$

$$a(t,\vec{p}) = \frac{1}{\sqrt{2}}\left(E_{\vec{p}}^{1/2}\widetilde{\phi}(t,\vec{p}) + i\,E_{\vec{p}}^{-1/2}\widetilde{\Pi}(t,\vec{p})\right)$$

$$a(t,\vec{p})^\dagger = \frac{1}{\sqrt{2}}\left(E_{\vec{p}}^{1/2}\widetilde{\phi}(t,-\vec{p}) - i\,E_{\vec{p}}^{-1/2}\widetilde{\Pi}(t,-\vec{p})\right)$$

Note that $a(t,\vec{p})^\dagger$ is the hermitian conjugate of $a(t,\vec{p})$.

Ex. Check that

$$[a(t,\vec{p}),a(t,\vec{p}\,')] = 0, \qquad [a(t,\vec{p})^\dagger,a(t,\vec{p}\,')^\dagger] = 0, \qquad [a(t,\vec{p}),a(t,\vec{p}\,')^\dagger] = \delta^{(3)}(\vec{p}-\vec{p}\,')$$

$$\widehat{H} = \int d^3p\, E_{\vec{p}}\, a(t,\vec{p})^\dagger a(t,\vec{p}) + \text{constant}$$

$$[\widehat{H},a(t,\vec{p})] = -E_{\vec{p}}\,a(t,\vec{p}), \quad [\widehat{H},a(t,\vec{p})^\dagger] = E_{\vec{p}}\,a(t,\vec{p})^\dagger, \quad E_{\vec{p}} = \sqrt{\vec{p}^{\,2}+m^2}$$

$$[a(t,\vec{p}), a(t,\vec{p}\,')] = 0, \qquad [a(t,\vec{p})^\dagger, a(t,\vec{p}\,')^\dagger] = 0, \qquad [a(t,\vec{p}), a(t,\vec{p}\,')^\dagger] = \delta^{(3)}(\vec{p}-\vec{p}\,')$$

$$\widehat{H} = \int d^3p\, E_{\vec{p}}\, a(t,\vec{p})^\dagger a(t,\vec{p}) + \text{constant}$$

$$[\widehat{H}, a(t,\vec{p})] = -E_{\vec{p}}\, a(t,\vec{p}), \quad [\widehat{H}, a(t,\vec{p})^\dagger] = E_{\vec{p}}\, a(t,\vec{p})^\dagger$$

We introduce basis states at $t=0$:

Define vacuum state $|0\rangle$ to satisfy

$$a(\vec{p})|0\rangle = 0 \qquad \text{for all } \vec{p}$$

Excited states:

$$|\vec{p}_1, \cdots, \vec{p}_N\rangle = a(\vec{p}_1)^\dagger a(\vec{p}_2)^\dagger \cdots a(\vec{p}_N)^\dagger |0\rangle$$

$$\widehat{H}|\vec{p}_1, \cdots, \vec{p}_N\rangle = (E_{\vec{p}_1} + E_{\vec{p}_2} + \cdots + E_{\vec{p}_N})|\vec{p}_1, \cdots, \vec{p}_N\rangle$$

– same energy as N free particles with momentum $\vec{p}_1, \cdots, \vec{p}_N$

So far we have not introduced momentum operator.

Next time we shall use Noether's theorem to define the momentum operator in this quantum field theory.

Result:

$$\widehat{P}_i = \int d^3p \, p_i \, a(t,\vec{p})^\dagger a(t,\vec{p}), \qquad i = x, y, z$$

Using

$$|\vec{p}_1, \cdots, \vec{p}_N\rangle = a(\vec{p}_1)^\dagger a(\vec{p}_2)^\dagger \cdots a(\vec{p}_N)^\dagger |0\rangle$$

one can easily show that

$$\widehat{P}_i \, |\vec{p}_1, \cdots, \vec{p}_N\rangle = (p_{1i} + \cdots + p_{Ni})|\vec{p}_1, \cdots, \vec{p}_N\rangle$$

We also had

$$\widehat{H}|\vec{p}_1, \cdots, \vec{p}_N\rangle = (E_{\vec{p}_1} + E_{\vec{p}_2} + \cdots + E_{\vec{p}_N})|\vec{p}_1, \cdots, \vec{p}_N\rangle$$

We shall interpret $|\vec{p}_1, \cdots, \vec{p}_N\rangle$ as a state containing N free particles with momenta $\vec{p}_1, \cdots, \vec{p}_N$ and energy $E_{\vec{p}_1}, \cdots, E_{\vec{p}_N}$.

Noether's theorem in classical theory

For every continuous global symmetry, there is a conserved quantity

– valid in classical mechanics as well as in classical field theory

The proof will be given in the next lecture.

Today we shall introduce some notations.

Consider a general classical field theory with fields $\phi_1, \phi_2, \cdots, \phi_n$

– they could be multiple scalars, components of a vector (like the vector potential in electromagnetic theory) etc.

Action $S[\phi_1, \cdots, \phi_n]$ is functional of these fields

This means that given n functions of space-time coordinates, $\phi_1(t, \vec{x}), \cdots, \phi_n(t, \vec{r})$, S generates a number

e.g. for a single scalar field describing Klein-Gordon theory,

$$S[\phi] = \frac{1}{2} \int dt \int d^3r \left[(\partial_t \phi)^2 - (\vec{\nabla}\phi)^2 - m^2 \phi^2 \right]$$

A transformation: Given a set of functions $\phi_1, \cdots, \phi_n$, a transformation is a rule by which we generate new set of functions $\widetilde{\phi}_1, \cdots, \widetilde{\phi}_n$.

Examples:

1. $\widetilde{\phi}_i(t, \vec{r}) = -\phi_i(t, \vec{r})$

2. $\widetilde{\phi}_i(t, \vec{r}) = \phi_i(t, -\vec{r})$

3. $\widetilde{\phi}_1(t, \vec{r}) = \phi_2(t, \vec{r}), \quad \widetilde{\phi}_2(t, \vec{r}) = \phi_1(t, \vec{r}), \quad \widetilde{\phi}_i(t, \vec{r}) = \phi_i(t, \vec{r})$ for $i \geq 3$

Note: In all these cases the new functions are determined by old functions.

A transformation is a symmetry of the action if

$$S[\widetilde{\phi}_1, \cdots, \widetilde{\phi}_n] = S[\phi_1, \cdots, \phi_n]$$

Example: The transformation $\widetilde{\phi}(t, \vec{r}) = -\phi(t, \vec{r})$ is a symmetry of

$$S[\phi] = \frac{1}{2} \int dt \int d^3r \left[(\partial_t \phi)^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2 \right]$$

since

$$S[\widetilde{\phi}] = \frac{1}{2} \int dt \int d^3r \left[(\partial_t \widetilde{\phi})^2 - (\vec{\nabla}\widetilde{\phi})^2 - m^2\widetilde{\phi}^2 \right] = S[\phi]$$

9 Symmetries and conservation laws

Consider a general classical field theory with fields $\phi_1, \phi_2, \cdots, \phi_n$

– they could be multiple scalars, components of a vector (like the vector potential in electromagnetic theory) etc.

Action $S[\phi_1, \cdots, \phi_n]$ is functional of these fields

This means that given n functions of space-time coordinates, $\phi_1(t, \vec{r}), \cdots, \phi_n(t, \vec{r})$, S generates a number

e.g. for a single scalar field describing Klein-Gordon theory,

$$S[\phi] = \frac{1}{2} \int dt \int d^3r \left[(\partial_t \phi)^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2 \right]$$

A transformation: Given a set of functions $\phi_1, \cdots, \phi_n$, a transformation is a rule by which we generate new set of functions $\widetilde{\phi}_1, \cdots, \widetilde{\phi}_n$.

Examples:

1. $\widetilde{\phi}_i(t, \vec{r}) = -\phi_i(t, \vec{r})$

2. $\widetilde{\phi}_i(t, \vec{r}) = \phi_i(t, -\vec{r})$

3. $\widetilde{\phi}_1(t, \vec{r}) = \phi_2(t, \vec{r}), \quad \widetilde{\phi}_2(t, \vec{r}) = \phi_1(t, \vec{r}), \quad \widetilde{\phi}_i(t, \vec{r}) = \phi_i(t, \vec{r})$ for $i \geq 3$

Note: In all these cases the new functions are determined by old functions.

We shall consider transformations that are smooth and invertible.

Invertible:

For a given transformation,

the knowledge of ϕ_i determines $\widetilde{\phi}_i$ uniquely,

and

the knowledge of $\widetilde{\phi}_i$ determines ϕ_i uniquely.

Smooth:

Two field configurations, that are close to each other, gets mapped to field configurations that are close to each other.

If ϕ_i is mapped to $\widetilde{\phi}_i$ then $\phi_i + \delta\phi_i$ is mapped to $\widetilde{\phi}_i + \delta\widetilde{\phi}_i$

If $\delta\phi_i = \epsilon f_i$ than $\delta\widetilde{\phi}_i = \epsilon g_i + \mathcal{O}(\epsilon^2)$ for smooth functions f_i, g_i.

Smoothness + Invertibility

$\Rightarrow$ for given sets of functions $f_1, \cdots, f_n$ we can determine the functions $g_1, \cdots, g_n$ uniquely and vice versa.

A transformation is a symmetry of the action if

$$S[\widetilde{\phi}_1, \cdots, \widetilde{\phi}_n] = S[\phi_1, \cdots, \phi_n]$$

Example: The transformation $\widetilde{\phi}(t, \vec{r}) = -\phi(t, \vec{r})$ is a symmetry of

$$S[\phi] = \frac{1}{2} \int dt \int d^3r \left[(\partial_t \phi)^2 - (\vec{\nabla}\phi)^2 - m^2 \phi^2 \right]$$

since

$$S[\widetilde{\phi}] = \frac{1}{2} \int dt \int d^3r \left[(\partial_t \widetilde{\phi})^2 - (\vec{\nabla}\widetilde{\phi})^2 - m^2 \widetilde{\phi}^2 \right] = S[\phi]$$

Similarly $\widetilde{\phi}(t, \vec{r}) = \phi(t, -\vec{r})$ is also a symmetry.

$$S[\widetilde{\phi}] = \frac{1}{2} \int dt \int d^3r \left[(\partial_t \phi(t, -\vec{r}))^2 - (\vec{\nabla}\phi(t, -\vec{r}))^2 - m^2 \phi(t, -\vec{r})^2 \right]$$

Change variable of integration $\vec{r}\,' = -\vec{r}$

$\vec{\nabla} = -\vec{\nabla}'$, $d^3r = d^3r'$ after changing limits of integration

$$S[\widetilde{\phi}] = \frac{1}{2} \int dt \int d^3r' \left[(\partial_t \phi(t, \vec{r}\,'))^2 - (-\vec{\nabla}'\phi(t, \vec{r}\,'))^2 - m^2 \phi(t, \vec{r}\,')^2 \right] = S[\phi]$$

While checking if a transformation is a symmetry of the action, we shall ignore integrals of total derivatives.

If a transformation is a symmetry of the action then it maps solutions of the equations of motion to solutions of the equations of motion.

– if $\phi_i(t, \vec{r})$ satisfies equations of motion, so does $\widetilde{\phi}_i(t, \vec{r})$

Proof:

Suppose that under the symmetry transformation:

ϕ_i is mapped to $\widetilde{\phi}_i$

Smoothness: Given any set of functions $f_1, \cdots, f_n$,

$\phi_i + \epsilon f_i$ is mapped to $\widetilde{\phi}_i + \epsilon g_i + \mathcal{O}(\epsilon^2)$ for some set of functions $g_1, \cdots, g_n$

We have, by symmetry,

$$S[\{\widetilde{\phi}_i\}] = S[\{\phi_i\}], \qquad S[\{\widetilde{\phi}_i + \epsilon g_i\}] = S[\{\phi_i + \epsilon f_i\}] + \mathcal{O}(\epsilon^2)$$

Now suppose that $\phi_i(t, \vec{r})$ satisfies equations of motion.

$$S[\{\phi_i + \epsilon\, f_i\}] = S[\{\phi_i\}] + \mathcal{O}(\epsilon^2)$$

since equations of motion $\Rightarrow$ first order term in the variation of S vanishes

This gives, up to correction terms of order ϵ^2

$$S[\{\widetilde{\phi}_i + \epsilon\, g_i\}] = S[\{\phi_i + \epsilon f_i\}] = S[\{\phi_i\}] = S[\{\widetilde{\phi}_i\}]$$

g_i's are arbitrary, since by invertibility, for any set of g_i's we can find the corresponding f_i's

$\Rightarrow \widetilde{\phi}_i(t, \vec{r})$'s satisfy the equations of motion.

Special case: Continuous symmetry:

The transformation laws depend on a parameter that can be varied continuously, such that for every value of the parameter the transformation is a symmetry.

Example: $\widetilde{\phi}(t,\vec{r}) = \phi(t,\vec{r}+\vec{a})$ is a symmetry of K-G action for every real value of $\vec{a} = (a_x, a_y, a_z)$.

$$S[\widetilde{\phi}] = \frac{1}{2}\int dt \int d^3r \left[(\partial_t\phi(t,\vec{r}+\vec{a}))^2 - (\vec{\nabla}\phi(t,\vec{r}+\vec{a}))^2 - m^2\phi(t,\vec{r}+\vec{a})^2\right]$$

Change integration variable to $\vec{r}\,' = \vec{r}+\vec{a}, \quad \Rightarrow \quad d^3r' = d^3r, \quad \vec{\nabla}' = \vec{\nabla}$

$$S[\widetilde{\phi}] = \frac{1}{2}\int dt \int d^3r' \left[(\partial_t\phi(t,\vec{r}\,'))^2 - (\vec{\nabla}'\phi(t,\vec{r}\,')^2 - m^2\phi(t,\vec{r}\,')^2\right] = S[\phi]$$

In this case we have three continuous symmetries labelled by parameters a_x, a_y, a_z

Time translation is also a continuous symmetry.

Another example: Consider Schrodinger field theory

$$S = \int dt \int d^3r\, \psi^* \left[i\hbar\frac{\partial\psi}{\partial t} - \widehat{h}\psi\right]$$

This has a symmetry:

$$\widetilde{\psi}(t,\vec{r}) = e^{i\theta}\psi(t,\vec{r}), \qquad \widetilde{\psi}(t,\vec{r})^* = e^{-i\theta}\psi(t,\vec{r})^*$$

for any real parameter θ.

Identity transformation:

$$\widetilde{\phi}_i(t, \vec{r}) = \phi_i(t, \vec{r})$$

– always a symmetry of the action.

A continuous symmetry is called connected to identity if for some value of the continuous parameter(s) the transformation reduces to identity transformation.

Example: $\vec{a} = 0$ for translation, $\theta = 0$ for Schrodinger action

We shall focus on these symmetries although this is not strictly necessary.

We normally choose the parameter of transformation a in such a way that $a = 0$ corresponds to identity transformation.

Infinitesimal transformation: a infinitesimal

– transformations close to identity

Continuous <u>global</u> symmetry: The symmetry transformation parameter does not depend on space-time coordinates $t, \vec{r}$

e.g. $\widetilde{\phi}(t, \vec{r}) = \phi(t, \vec{r} + \vec{a})$ is not a symmetry of K-G action if $\vec{a}$ becomes an arbitrary function of $t, \vec{r}$.

We can still change integration variable to $\vec{r}\,' = \vec{r} + \vec{a}(t, \vec{r})$, but

$\Rightarrow \quad d^3r' \neq d^3r, \quad \vec{\nabla}' \neq \vec{\nabla}$

Noether's theorem: For every continuous global symmetry, the theory has a conserved quantity $Q(t)$:

$$\frac{d}{dt}Q = 0$$

Global symmetry: Transformation parameters do not depend on $t, \vec{r}$.

The conserved quantities associated with space translation are called momentum

– universal definition in all theories

– measures total physical momentum of a state

The conserved quantity associated with time translation is called energy

– coincides with the Hamiltonian

10 Symmetries and conservation laws

Noether's theorem: For every continuous global symmetry, the theory has a conserved quantity $Q(t)$:

$$\frac{d}{dt}Q = 0$$

Global symmetry: Transformation parameters do not depend on $t, \vec{r}$.

We'll now give a proof of Noether's theorem.

For this it will be useful (not necessary) to set up a relativistic notation

Define: $x^0 = t, \qquad (x^1, x^2, x^3) = \vec{r}$

Four vector $x = (x^0, x^1, x^2, x^3)$

$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$

The matrix $\eta^{\mu\nu} = \text{diag}\,(-1, 1, 1, 1), \qquad \eta_{\mu\nu} = \text{diag}\,(-1, 1, 1, 1)$

Indices appearing twice in a product are automatically summed over 0,1,2,3, e.g.

$$\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi = \sum_{\mu,\nu=0}^{3} \eta^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi = -(\partial_0\phi)^2 + (\partial_1\phi)^2 + (\partial_2\phi)^2 + (\partial_3\phi)^2 = -(\partial_t\phi)^2 + (\vec{\nabla}\phi)^2$$

In this notation, the K-G action may be written as:

$$S = \frac{1}{2}\int d^4x\,\left[-\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2\right]$$

Raising and lowering indices: Given any 4-vector A^μ, we define:

$$A_\mu = \eta_{\mu\nu}A^\nu, \qquad A^\mu = \eta^{\mu\nu}\,A_\nu$$

Proof of Noether theorem:

Consider an infinitesimal continuous global symmetry

$$\widetilde{\phi}_i(x) = \phi_i(x) + \epsilon\, \chi_i(x)$$

ϵ: an infinitesimal parameter

$\chi_i(x)$: Some known functions, constructed from the ϕ_i's

Example: Consider infinitesimal translation along x^1:

$\widetilde{\phi}_i(x^0, x^1, x^2, x^3) = \phi_i(x^0, x^1 + \epsilon, x^2, x^3) = \phi_i(x^0, x^1, x^2, x^3) + \epsilon\, \partial_1 \phi_i(x^0, x^1, x^2, x^3)$

In this case,

$$\chi_i(x) = \partial_1 \phi_i(x)$$

Symmetry:

$$S[\{\phi_i\}] = S[\{\phi_i + \epsilon\chi_i\}]$$

Now take an arbitrary function $f(x)$.

$S[\{\phi_i\}] = S[\{\phi_i + \epsilon f \chi_i\}]$ if $f(x)$ is a constant.

$\Rightarrow$ when $f(x)$ is not a constant:

$$S[\{\phi_i + \epsilon f \chi_i\}] = S[\{\phi_i\}] + \epsilon \int d^4x [K_1^\mu(x)\, \partial_\mu f + K_2^{\mu\nu}(x) \partial_\mu \partial_\nu f + \cdots]$$

$K_1^\mu, K_2^{\mu\nu}, \cdots$ are constructed from the ϕ_i's and their derivatives (but not f)

Taking f to vanish at ∞, we can integrate by parts and get

$$S[\{\phi_i + \epsilon f \chi_i\}] = S[\{\phi_i\}] + \epsilon \int d^4x\, f(x)\, \partial_\mu J^\mu(x), \quad J^\mu = -K_1^\mu(x) + \partial_\nu\, K_2^{\mu\nu}(x) + \cdots$$

We have proved that if

$$\widetilde{\phi}_i(x) = \phi_i(x) + \epsilon\,\chi_i(x)$$

is an infinitesimal symmetry transformation, then for any function $f(x)$ that vanishes at ∞, we have

$$S[\{\phi_i + \epsilon f \chi_i\}] = S[\{\phi_i\}] + \epsilon \int d^4x f(x)\, \partial_\mu J^\mu(x)$$

for some quantity $J^\mu(x)$ constructed from the fields $\phi_i(x)$.

Now suppose that ϕ_i's satisfy equations of motion.

In this case under any variation of the fields, the change in the action vanishes to first order.

$$S[\{\phi_i + \epsilon f \chi_i\}] = S[\{\phi_i\}] + \mathcal{O}(\epsilon^2)$$

Therefore

$$\int d^4x f(x)\, \partial_\mu J^\mu(x) = 0$$

Since $f(x)$ is an arbitrary function, this gives

$$\partial_\mu J^\mu(x) = 0$$

when equations of motion are satisfied

J^μ is called conserved current

$$\partial_0 J^0 = -\vec{\nabla}.\vec{J}, \qquad \vec{J} = (J^1, J^2, J^3)$$

$$\frac{d}{dt}\int d^3r\, J^0(t, \vec{r}) = -\int d^3r\, \vec{\nabla}.\vec{J} = 0$$

assuming that the boundary terms vanish

– no current flowing out to infinity

Conclusion: Given a continuous global symmetry, we can construct a quantity

$$Q = \int d^3r \, J^0(t, \vec{r})$$

which satisfies

$$\frac{dQ}{dt} = 0$$

when equations of motion are satisfied.

Construction of Q simplifies for a special class of actions:

$$S = \int d^4x \, \mathcal{L}(\{\phi_i\}, \{\partial_\mu \phi_i\})$$

$\mathcal{L}$: Lagrangian density – an ordinary function of the $n + 4n$ variables ϕ_i's and $\partial_\mu \phi_i$'s

e.g. for K-G field theory

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2$$

– an ordinary function of ϕ, $\partial_0\phi$, $\partial_1\phi$, $\partial_2\phi$, $\partial_3\phi$

Suppose

$$\widetilde{\phi}_i(x) = \phi_i(x) + \epsilon \, \chi_i(x), \qquad \partial_\mu \widetilde{\phi}_i(x) = \partial_\mu \phi_i(x) + \epsilon \, \partial_\mu \chi_i(x)$$

is a symmetry.

Then, in order that $\int d^4x \, \mathcal{L}$ remains invariant under this transformation,

$$\mathcal{L}(\{\widetilde{\phi}_i\}, \{\partial_\mu \widetilde{\phi}_i\}) = \mathcal{L}(\{\phi_i\}, \{\partial_\mu \phi_i\}) + \epsilon \, \partial_\mu K^\mu$$

for some K^μ.

$$\Rightarrow \qquad \sum_{i=1}^{n} \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \epsilon \chi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \epsilon \, \partial_\mu \chi_i \right] = \epsilon \, \partial_\mu K^\mu$$

This gives the identity:

$$\sum_{i=1}^{n} \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \chi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu \chi_i \right] = \partial_\mu K^\mu$$

$$\sum_{i=1}^{n}\left[\frac{\partial\mathcal{L}}{\partial\phi_i}\,\chi_i+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\partial_\mu\chi_i\right]=\partial_\mu K^\mu$$

Now define

$$\widehat{\phi}_i(x)=\phi_i(x)+\epsilon\,f(x)\,\chi_i(x),\qquad \partial_\mu\widehat{\phi}_i(x)=\partial_\mu\phi_i(x)+\epsilon\,\partial_\mu(f(x)\chi_i(x))$$

Then

$$\mathcal{L}(\{\widehat{\phi}_i(x)\},\{\partial_\mu\widehat{\phi}_i(x)\})=\mathcal{L}(\{\phi_i(x)\},\{\partial_\mu\phi_i(x)\})+\sum_{i=1}^{n}\left[\frac{\partial\mathcal{L}}{\partial\phi_i}\,\epsilon f\chi_i+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\epsilon\,\partial_\mu(f\,\chi_i)\right]$$

$$\begin{aligned}
S[\{\phi_i+\epsilon f\chi_i\}]-S[\{\phi_i\}] &= \int d^4x\sum_{i=1}^{n}\left[\frac{\partial\mathcal{L}}{\partial\phi_i}\,\epsilon f\chi_i+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\epsilon\,\partial_\mu(f\,\chi_i)\right]\\
&=\int d^4x\sum_{i=1}^{n}\left[\frac{\partial\mathcal{L}}{\partial\phi_i}\,\epsilon f\chi_i+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\epsilon\,f\,\partial_\mu\chi_i+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\epsilon\,\partial_\mu f\,\chi_i)\right]\\
&=\int d^4x\,\epsilon\,f\,\partial_\mu\left[K^\mu-\sum_{i=1}^{n}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\chi_i\right]
\end{aligned}$$

On the other hand, we defined J^μ via

$$S[\{\phi_i+\epsilon f\chi_i\}]-S[\{\phi_i\}]=\int d^4x\,\epsilon\,f\,\partial_\mu J^\mu$$

Comparison of the two equations gives:

$$J^\mu=-\sum_{i=1}^{n}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\,\chi_i+K^\mu$$

By our previous argument, $\partial_\mu J^\mu=0, \quad \Rightarrow \quad Q=\int d^3r J^0$ is conserved

11 Symmetries and conservation laws

Relevant result for

$$S = \int d^4x \, \mathcal{L}(\{\phi_i\}, \{\partial_\mu \phi_i\})$$

Suppose

$$\widetilde{\phi}_i(x) = \phi_i(x) + \epsilon \, \chi_i(x), \qquad \partial_\mu \widetilde{\phi}_i(x) = \partial_\mu \phi_i(x) + \epsilon \, \partial_\mu \chi_i(x)$$

is a symmetry.

Then, in order that $\int d^4x \, \mathcal{L}$ remains invariant under this transformation,

$$\mathcal{L}(\{\widetilde{\phi}_i\}, \{\partial_\mu \widetilde{\phi}_i\}) = \mathcal{L}(\{\phi_i\}, \{\partial_\mu \phi_i\}) + \epsilon \, \partial_\mu K^\mu$$

for some K^μ.

Then

$$J^\mu = -\sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \, \chi_i + K^\mu$$

satisfies

$$\partial_\mu \, J^\mu = 0$$

$Q = \int d^3r J^0$ is conserved

Example: Schrodinger field theory

$$\begin{aligned} S &= \int dt \int d^3r\, \psi^* \left[i\, \frac{\partial \psi}{\partial t} + \frac{1}{2m} \vec{\nabla}^2 \psi - U(\vec{r})\psi \right] \\ &= \int dt \int d^3r \left[i\psi^* \, \frac{\partial \psi}{\partial t} - \frac{1}{2m} \vec{\nabla}\psi^* . \vec{\nabla}\psi - U(\vec{r})\psi^*\psi \right] \end{aligned}$$

$$\mathcal{L}(\psi, \psi^*, \partial_\mu \psi, \partial_\mu \psi^*) = \left[i\psi^* \, \frac{\partial \psi}{\partial t} - \frac{1}{2m} \vec{\nabla}\psi^* . \vec{\nabla}\psi - U(\vec{r})\psi^*\psi \right]$$

Symmetry

$$\widetilde{\psi}(t, \vec{r}) = e^{i\theta} \psi(t, \vec{r}), \qquad \widetilde{\psi}(t, \vec{r})^* = e^{-i\theta} \psi(t, \vec{r})^*$$

$$\mathcal{L}(\widetilde{\psi}, \widetilde{\psi}^*, \{\partial_\mu \widetilde{\psi}\}, \{\partial_\mu \widetilde{\psi}^*\}) = \mathcal{L}(\psi, \psi^*, \{\partial_\mu \psi\}, \{\partial_\mu \psi^*\})$$

Infinitesimal version: $\theta = \epsilon$:

$$\widetilde{\psi}(t, \vec{r}) = \psi(t, \vec{r}) + i\epsilon\psi(t, \vec{r}), \qquad \widetilde{\psi}(t, \vec{r})^* = \psi(t, \vec{r})^* - i\epsilon\psi(t, \vec{r})^*$$

Recall our definition of χ_i and K^μ:

$$\widetilde{\phi}_i(x) = \phi_i(x) + \epsilon\, \chi_i(x), \quad \mathcal{L}(\{\widetilde{\phi}_i\}, \{\partial_\mu \widetilde{\phi}_i\}) = \mathcal{L}(\{\phi_i\}, \{\partial_\mu \phi_i\}) + \epsilon\, \partial_\mu K^\mu$$

under infinitesimal symmetry.

$$\Rightarrow \qquad \chi = i\, \psi(t, \vec{r}), \qquad \chi^* = -i\psi(t, \vec{r})^*, \qquad K^\mu = 0$$

$$\mathcal{L}(\psi, \psi^*, \{\partial_\mu \psi\}, \{\partial_\mu \psi^*\}) = \left[i\psi^* \frac{\partial \psi}{\partial t} - \frac{1}{2m} \vec{\nabla}\psi^* . \vec{\nabla}\psi - U(\vec{r})\psi^*\psi \right]$$

$$\Rightarrow \qquad \chi = i\,\psi(t, \vec{r}), \qquad \chi^* = -i\psi(t, \vec{r})^*, \qquad K^\mu = 0$$

$$J^\mu = -\sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\,\chi_i + K^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\,(i\,\psi) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)}\,(-i\,\psi^*)$$

$$J^0 = -\frac{\partial L}{\partial(\partial_t \psi)}\,(i\psi) = \psi^*\psi$$

$$J^i = \frac{1}{2m}\partial_i \psi^*(i\psi) + \frac{1}{2m}\partial_i \psi(-i\psi^*) = \frac{i}{2m}\{\psi \partial_i \psi^* - \psi^* \partial_i \psi\}$$

Conserved quantity

$$N = \int d^3 r\, \psi(t, \vec{r})^* \psi(t, \vec{r})$$

– in quantum theory this becomes the number operator

Next we find conserved quantities in K-G theory associated with space-time translation.

$$\mathcal{L} = \frac{1}{2}\left[-\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2\right]$$

Symmetry:

$$\widetilde{\phi}(x) = \phi(x+a)$$

$a = (a^0, a^1, a^2, a^3)$: An arbitrary constant four vector

We should have four conserved quantities and the associated currents $J^\mu_{(\rho)}$

Infinitesimal version: $a = (\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3)$

$$\widetilde{\phi}(x) = \phi(x+\epsilon) = \phi(x) + \epsilon^\rho\,\partial_\rho\phi$$

$$\begin{aligned}\mathcal{L}(\widetilde{\phi}(x), \{\partial_\nu\widetilde{\phi}(x)\}) &= \mathcal{L}(\phi(x+\epsilon), \{\partial_\nu\phi(x+\epsilon)\}) \\ &= \mathcal{L}(\phi(x), \{\partial_\nu\phi(x)\}) + \epsilon^\rho\,\partial_\rho\mathcal{L}(\phi(x), \{\partial_\nu\phi(x)\})\end{aligned}$$

Compare with our definition of χ_i and K^μ:

$$\widetilde{\phi}_i(x) = \phi_i(x) + \epsilon\,\chi_i(x), \quad \mathcal{L}(\{\widetilde{\phi}_i\}, \{\partial_\nu\widetilde{\phi}_i\}) = \mathcal{L}(\{\phi_i\}, \{\partial_\nu\phi_i\}) + \epsilon\,\partial_\mu K^\mu$$

Since we have four transformations, and one field, we write this as

$$\widetilde{\phi}(x) = \phi(x) + \epsilon^\rho\,\chi_{(\rho)}(x), \quad \mathcal{L}(\widetilde{\phi}, \{\partial_\nu\widetilde{\phi}\}) = \mathcal{L}(\phi, \{\partial_\nu\phi\}) + \epsilon^\rho\,\partial_\mu K^\mu_{(\rho)}$$

This gives

$$\chi_{(\rho)} = \partial_\rho\phi, \qquad \partial_\mu K^\mu_{(\rho)} = \partial_\rho\mathcal{L} \quad \Rightarrow \quad K^\mu_{(\rho)} = \mathcal{L}\,\delta^\mu_\rho, \qquad \delta^\mu_\rho : \text{Kronecker } \delta$$

$$\mathcal{L} = \frac{1}{2}\left[-\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2\right]$$

$$\chi_{(\rho)} = \partial_\rho\phi, \qquad K^\mu_{(\rho)} = \mathcal{L}\,\delta^\mu_\rho, \qquad \delta^\mu_\rho : \text{Kronecker } \delta$$

$$J^\mu_{(\rho)} = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\chi_{(\rho)} + K^\mu_{(\rho)} = \eta^{\mu\nu}\,\partial_\nu\phi\,\partial_\rho\phi + \mathcal{L}\,\delta^\mu_\rho$$

Conserved quantity:

$$P_\rho = \int d^3r\, J^0_{(\rho)} = \int d^3r\, \left[\eta^{0\nu}\,\partial_\nu\phi\,\partial_\rho\phi + \mathcal{L}\,\delta^0_\rho\right]$$

We define

$$P^\sigma = \eta^{\sigma\rho}\,P_\rho = \int d^3r\, \left[\eta^{0\nu}\eta^{\sigma\rho}\partial_\nu\phi\,\partial_\rho\phi + \mathcal{L}\,\eta^{0\sigma}\right]$$

$$P^0 = \int d^3r\, \left[(\partial_0\phi)^2 - \mathcal{L}\right] = \frac{1}{2}\left[(\partial_0\phi)^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right]$$

agrees with H (energy)

$$P^i = P_i = \int d^3r\,[-\partial_0\phi\,\partial_i\phi] = -\int d^3r\,\Pi\,\partial_i\phi \qquad \text{for } i = 1,2,3$$

– defined as i-th component of physical momentum

Recall Fourier transformation formulæ:

$$\phi(t,\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3p\, \widetilde{\phi}(t,\vec{p})\, e^{i\vec{p}.\vec{r}}, \qquad \Pi(t,\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3p\, \widetilde{\Pi}(t,\vec{p})\, e^{i\vec{p}.\vec{r}},$$

$$P^i = -\int d^3r\, \Pi(t,\vec{r})\, \partial_i\phi = -\int d^3p\, \widetilde{\Pi}(t,-\vec{p})\, (i\, p_i)\, \widetilde{\phi}(t,\vec{p}), \qquad \text{for } i = 1,2,3$$

In quantum theory, where ϕ and Π are operators, we have

$$a(t,\vec{p}) = \frac{1}{\sqrt{2}} \left(E_{\vec{p}}^{1/2} \widetilde{\phi}(t,\vec{p}) + i\, E_{\vec{p}}^{-1/2} \widetilde{\Pi}(t,\vec{p}) \right)$$

$$a(t,\vec{p})^\dagger = \frac{1}{\sqrt{2}} \left(E_{\vec{p}}^{1/2} \widetilde{\phi}(t,-\vec{p}) - i\, E_{\vec{p}}^{-1/2} \widetilde{\Pi}(t,-\vec{p}) \right)$$

$$E_{\vec{p}} = \sqrt{\vec{p}^{\,2} + m^2}$$

Using this we can rewrite P^i in the quantum theory as:

$$\widehat{P}_i = \int d^3p\, p_i\, a(t,\vec{p})^\dagger\, a(t,\vec{p}) \quad + \quad \text{constant}$$

This was used earlier to show that the state:

$$|\vec{p}_1, \cdots, \vec{p}_N\rangle = a(\vec{p}_1)^\dagger a(\vec{p}_2)^\dagger \cdots a(\vec{p}_N)^\dagger |0\rangle$$

satisfies

$$\widehat{P}_i\, |\vec{p}_1, \cdots, \vec{p}_N\rangle = (p_{1i} + \cdots + p_{Ni}) |\vec{p}_1, \cdots, \vec{p}_N\rangle$$

We also had

$$\widehat{H} |\vec{p}_1, \cdots, \vec{p}_N\rangle = (E_{\vec{p}_1} + E_{\vec{p}_2} + \cdots + E_{\vec{p}_N}) |\vec{p}_1, \cdots, \vec{p}_N\rangle$$

This allowed us to interpret $|\vec{p}_1, \cdots, \vec{p}_N\rangle$ as a state containing N free particles with momenta $\vec{p}_1, \cdots, \vec{p}_N$ and energy $E_{\vec{p}_1}, \cdots, E_{\vec{p}_N}$.

Why did we have to construct a conserved quantity for defining momentum?

For free particles we did not need to go through the exercise.

Momentum of each particle is individually conserved, i.e. the state $|\vec{p}_1, \cdots, \vec{p}_N\rangle$, if evolved in time via Schrodinger equation, will just pick up a phase and will still have the interpretation of an N particle states with momenta $\vec{p}_1, \cdots, \vec{p}_N$.

However, once we introduce interactions, this will no longer be the case.

The particles may be able to exchange momenta, produce new particles, annihilate themselves etc.

Nevertheless, the total momentum and energy should not change.

This will be the case if in the interacting theory we define the energy and momenta as conserved quantities associated with space-time translation, provided space and time translations are symmetries of the theory.

For Noether theorem to be useful in quantum field theory, it is important that a quantity conserved in the classical theory is also conserved in the quantum theory

– generally true since the quantum field operators satisfy the same equations of motion as the classical fields.

However in some cases this fails.

Such symmetries are known as anomalous

– will not be discussed any further in this course.

Now let us return briefly to the currents $J^{\mu}_{(\rho)}$ associated with space-time translation.

It is called the energy-momentum tensor and written as $T^{\mu}{}_{\rho}$.

Note that the two indices are on different footing.

μ labels the component of the conserved four current

ρ labels the particular direction of translation that leads to this conserved quantity.

$\rho = 0$ associated to time translation, $\rho = 1, 2, 3$ associated to space translation symmetry

We also define

$$T^{\mu\sigma} = \eta^{\sigma\rho} T^{\mu}{}_{\rho}$$

Then

$$\text{Energy} = \int d^3r\, T^{00}, \qquad \text{momentum} = \int d^3r\, T^{0i}$$

For the Klein-Gordon field:

$$T^{\mu\sigma} = \eta^{\sigma\rho}\, J^{\mu}_{(\rho)} = \eta^{\sigma\rho} \left[\eta^{\mu\nu}\, \partial_\nu\phi\, \partial_\rho\phi + \mathcal{L}\, \delta^{\mu}_{\rho}\right] = \eta^{\sigma\rho}\eta^{\mu\nu}\, \partial_\nu\phi\, \partial_\rho\phi + \mathcal{L}\, \eta^{\mu\sigma}$$

– symmetric under $\mu \leftrightarrow \sigma$

This is a general property of the energy momentum tensor

– can be made symmetric, if necessary, by adding a tensor $K^{\mu\sigma}$ with the properties:

1. $\partial_\mu K^{\mu\sigma} = 0$, 2. $\int d^3r\, K^{0\sigma} = 0$

12 Lorentz invariance and associated conserved quantities

Last time we analyzed the conserved quantities in K-G theory associated with translation symmetry.

$$\widetilde{\phi}(x) = \phi(x+\epsilon) = \phi(x) + \epsilon^\rho\, \partial_\rho \phi$$

This gives

$$\mathcal{L}(\widetilde{\phi}(x), \{\partial_\mu \widetilde{\phi}(x)\}) = \mathcal{L}(\phi(x+\epsilon), \{\partial_\mu \phi(x+\epsilon)\})$$

$$= \mathcal{L}(\phi(x), \{\partial_\mu \phi(x)\}) + \epsilon^\rho\, \partial_\rho \mathcal{L}(\phi(x), \{\partial_\mu \phi(x)\})$$

Using the general definitions:

$$\widetilde{\phi}(x) = \phi(x) + \epsilon^\rho\, \chi_{(\rho)}(x), \qquad \mathcal{L}(\widetilde{\phi}(x), \{\partial_\mu \widetilde{\phi}(x)\}) = \mathcal{L}(\phi(x), \{\partial_\mu \phi(x)\}) + \epsilon^\rho \partial_\mu K^\mu_{(\rho)}$$

we get, by comparison,

$$\chi_{(\rho)} = \partial_\rho \phi, \qquad K^\mu_{(\rho)} = \mathcal{L}\, \delta^\mu_\rho, \qquad \delta^\mu_\rho : \text{Kronecker } \delta$$

This gave conserved current associated with translation along ρ direction:

$$J^\mu_{(\rho)} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\, \chi_{(\rho)} + K^\mu_{(\rho)} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\, \partial_\rho \phi + \mathcal{L}\, \delta^\mu_\rho$$

We called this $T^\mu{}_\rho$.

Note: Up to this point we do not need explicit form of $\mathcal{L}$, but after this step we used the form of $\mathcal{L}$ in K-G theory.

Up to this point the analysis also holds for multiple fields if we just sum over all fields in the first term.

Lorentz invariance of K-G action:

$$\widetilde{\phi}(x) = \phi(\Lambda\, x)$$

Λ: Lorentz transformation matrix

$$(\Lambda\, x)^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad \Lambda\, \eta\, \Lambda^T = \eta$$

In components

$$\Lambda^\rho{}_\mu\, \eta^{\mu\nu}\, \Lambda^\sigma{}_\nu = \eta^{\rho\sigma}$$

We shall first check that this is a symmetry of K-G action.

$$\begin{aligned}\mathcal{L}(\widetilde{\phi}(x), \{\partial_\nu \widetilde{\phi}(x)\}) &= -\frac{1}{2}\eta^{\mu\nu}\partial_\mu \widetilde{\phi}(x)\partial_\nu \widetilde{\phi}(x) - \frac{1}{2}m^2 \widetilde{\phi}(x)^2 \\ &= -\frac{1}{2}\eta^{\mu\nu}\partial_\mu \phi(\Lambda x)\partial_\nu \phi(\Lambda x) - \frac{1}{2}m^2 \phi(\Lambda x)^2\end{aligned}$$

Define

$$x'^\rho = \Lambda^\rho{}_\sigma\, x^\sigma$$

Then

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\rho}{\partial x^\mu}\, \frac{\partial}{\partial x'^\rho} = \Lambda^\rho{}_\mu\, \partial'_\rho$$

This gives

$$\begin{aligned}\mathcal{L}(\widetilde{\phi}(x), \{\partial_\nu \widetilde{\phi}(x)\}) &= -\frac{1}{2}\eta^{\mu\nu}\Lambda^\rho{}_\mu\, \partial'_\rho \phi(x')\Lambda^\sigma{}_\nu\, \partial'_\sigma \phi(x') - \frac{1}{2}m^2 \phi(x')^2 \\ &= -\frac{1}{2}\eta^{\rho\sigma}\partial'_\rho \phi(x')\, \partial'_\sigma \phi(x') - \frac{1}{2}m^2 \phi(x')^2 = \mathcal{L}(\phi(x'), \{\partial'_\nu \phi(x')\})\end{aligned}$$

Since $|\det \Lambda| = 1$, we also have $d^4x' = d^4x$

$$\mathcal{L}(\widetilde{\phi}(x), \{\partial_\nu \widetilde{\phi}(x)\}) = \mathcal{L}(\phi(x'), \{\partial'_\nu \phi(x')\}), \qquad d^4x = d^4x'$$

This gives

$$S[\widetilde{\phi}] = \int d^4x\, \mathcal{L}(\widetilde{\phi}(x), \{\partial_\nu \widetilde{\phi}(x)\}) = \int d^4x'\, \mathcal{L}(\phi(x'), \{\partial'_\nu \phi(x')\}) = S[\phi]$$

Therefore Lorentz transformation is a symmetry of the action.

We shall now construct the associated conserved quantity.

For this we need to consider the infinitesimal version of Lorentz transformation.

$$\Lambda = I + \epsilon\,\omega, \qquad \Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \epsilon\,\omega^{\mu}{}_{\nu}$$

$$\Lambda\,\eta\,\Lambda^T = \eta \quad \Rightarrow \quad (I + \epsilon\,\omega)\,\eta\,(I + \epsilon\,\omega^T) = \eta \quad \Rightarrow \quad \epsilon(\omega\,\eta + \eta\,\omega^T) = 0$$

$$\omega^{\mu}{}_{\rho}\,\eta^{\rho\nu} + \eta^{\mu\rho}\omega^{\nu}{}_{\rho} = 0 \quad \Rightarrow \quad \omega^{\mu\nu} + \omega^{\nu\mu} = 0$$

where we have defined

$$\omega^{\mu\nu} = \omega^{\mu}{}_{\rho}\,\eta^{\rho\nu}$$

Therefore $\omega^{\mu\nu}$ is an arbitrary anti-symmetric matrix

$\omega_{\rho\sigma} = \eta_{\rho\mu}\,\eta_{\sigma\nu}\,\omega^{\mu\nu}$ is also an antisymmetric matrix

– 6 independent paramaters

3 boost and 3 rotations

Goal: Find the conserved charges associated with Lorentz transformations

– we need to construct the χ_i's and K^{μ}'s for each of these six independent infinitesimal transformations.

We'll do it all together just as for translation we found all the conserved charges together.

Recall that

$$\mathcal{L}(\widetilde{\phi}(x), \{\partial_\nu \widetilde{\phi}(x)\}) = \mathcal{L}(\phi(x'), \{\partial'_\nu \phi(x')\})$$

For infinitesimal trasformation:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu = (\delta^\mu_\nu + \epsilon\, \omega^\mu{}_\nu) x^\nu = x^\mu + \epsilon\, \omega^\mu{}_\nu x^\nu$$

Therefore

$$\mathcal{L}(\widetilde{\phi}(x), \{\partial_\nu \widetilde{\phi}(x)\}) = \mathcal{L}(\phi(x), \{\partial_\nu \phi(x)\}) + \epsilon\, \omega^\mu{}_\nu\, x^\nu\, \partial_\mu \mathcal{L}(\phi(x), \{\partial_\nu \phi(x)\})$$

This can be written as

$$\mathcal{L}(\widetilde{\phi}(x), \{\partial_\nu \widetilde{\phi}(x)\}) = \mathcal{L}(\phi(x), \{\partial_\nu \phi(x)\}) + \epsilon\, \partial_\mu \left[\omega^\mu{}_\nu\, x^\nu\, \mathcal{L}(\phi(x), \{\partial_\nu \phi(x)\})\right]$$

since

$$\omega^\mu{}_\nu\, \partial_\mu\, x^\nu = \omega^\mu{}_\nu\, \delta^\nu_\mu = \omega^{\mu\rho} \eta_{\rho\nu}\, \delta^\nu_\mu = \omega^{\mu\rho} \eta_{\rho\mu} = 0$$

by anti-symmetry of $\omega^{\mu\rho}$.

Recall our general definition of χ_i and K^μ:

$$\widetilde{\phi}_i(x) = \phi_i(x) + \epsilon\,\chi_i(x), \quad \mathcal{L}(\{\widetilde{\phi}_i\}, \{\partial_\mu \widetilde{\phi}_i\}) = \mathcal{L}(\{\phi_i\}, \{\partial_\mu \phi_i\}) + \epsilon\,\partial_\mu K^\mu$$

under infinitesimal symmetry.

Here we have

$$\mathcal{L}(\widetilde{\phi}(x), \{\partial_\nu \widetilde{\phi}(x)\}) = \mathcal{L}(\phi(x), \{\partial_\nu \phi(x)\}) + \epsilon\,\partial_\mu\left[\omega^\mu{}_\nu\, x^\nu\, \mathcal{L}(\phi(x), \{\partial_\nu \phi(x)\})\right]$$

This gives

$$K^\mu = \omega^\mu{}_\nu\, x^\nu\, \mathcal{L}(\phi(x), \{\partial_\nu \phi(x)\})$$

We also have

$$\widetilde{\phi}(x) = \phi(x') = \phi(x) + \epsilon\,\omega^\mu{}_\nu\, x^\nu\, \partial_\mu \phi$$

This gives

$$\chi(x) = \omega^\mu{}_\nu\, x^\nu\, \partial_\mu \phi$$

$$\chi(x) = \omega^\rho{}_\nu \, x^\nu \, \partial_\rho \phi, \qquad K^\mu = \omega^\mu{}_\nu \, x^\nu \, \mathcal{L}(\phi(x), \{\partial_\nu \phi(x)\})$$

Conserved current

$$J^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \chi + K^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \omega^\rho{}_\nu \, x^\nu \, \partial_\rho \phi + \omega^\mu{}_\nu \, x^\nu \, \mathcal{L}(\phi(x), \{\partial_\nu \phi(x)\})$$

Recall the construction of $T^{\mu\nu} = \eta^{\nu\rho} J^\mu_{(\rho)}$:

$$\chi_{(\rho)} = \partial_\rho \phi, \qquad K^\mu_{(\rho)} = \mathcal{L} \, \delta^\mu_\rho, \qquad \delta^\mu_\rho : \text{Kronecker } \delta$$

$$J^\mu_{(\rho)} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \chi_{(\rho)} + K^\mu_{(\rho)} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\rho \phi + \mathcal{L} \, \delta^\mu_\rho$$

This gives the conserved current for Lorentz transformation:

$$J^\mu = \omega^\rho{}_\nu \, x^\nu \, J^\mu_{(\rho)} = \eta^{\rho\tau} \, \omega_{\tau\nu} \, x^\nu \, J^\mu_{(\rho)} = \omega_{\tau\nu} \, x^\nu \, T^{\mu\tau} = \frac{1}{2} \omega_{\tau\nu} \left[x^\nu \, T^{\mu\tau} - x^\tau T^{\mu\nu} \right]$$

Since $\omega_{\tau\nu}$ is an arbitrary antisymmetric matrix,

$$M^{\mu\nu\tau} = x^\nu \, T^{\mu\tau} - x^\tau T^{\mu\nu}$$

is conserved, i.e.

$$\partial_\mu \, M^{\mu\nu\tau} = 0$$

This implies that $T^{\mu\nu}$ must be symmetric.

$$0 = \partial_\mu \, M^{\mu\nu\tau} = \partial_\mu \{ x^\nu \, T^{\mu\tau} - x^\tau T^{\mu\nu} \} = \delta^\nu_\mu \, T^{\mu\tau} - \delta^\tau_\mu \, T^{\mu\nu} = T^{\nu\tau} - T^{\tau\nu}$$

Note: In this derivation we did not need to use the explicit form of $\mathcal{L}$

As long as the action is Lorentz invariant, this derivation will work for general Lagrangian, even for multiple scalar fields.

$$M^{\mu\nu\tau} = x^\nu T^{\mu\tau} - x^\tau T^{\mu\nu}$$

Conserved quantity

$$\mathcal{J}^{\nu\tau} = \int d^3r \, M^{0\nu\tau} = \int d^3r \, \left[x^\nu T^{0\tau} - x^\tau T^{0\nu}\right]$$

$$\mathcal{J}^{ij} = \int d^3r \left[x^i T^{0j} - x^j T^{0i}\right]$$

SInce T^{0i} represents momentum density, $\left[x^i T^{0j} - x^j T^{0i}\right]$ represents angular momentum density.

$\mathcal{J}^{ij}$ represents the three independent components of angular momenta of the system

$$\mathcal{J}^{0i} = \int d^3r \, M^{00i} = \int d^3r \, \left[x^0 T^{0i} - x^i T^{00}\right]$$

These are conserved quantities associated with Lorentz boost

Note that they have explicit time dependence

Therefore they are conserved but do not commute with the Hamiltonian.

$$0 = \frac{d}{dt}\mathcal{J}^{0i} = i[\widehat{H}, \mathcal{J}^{0i}] + \frac{\partial}{\partial t}\mathcal{J}^{0i}$$

$\Rightarrow \quad [\widehat{H}, \mathcal{J}^{0i}] \neq 0$

This reflects the fact that under Lorentz boost, energy of a particle changes.

13 Spin angular momentum

We considered Lorentz invariance of K-G action, under which:

$$\widetilde{\phi}(x) = \phi(\Lambda\, x)$$

$$\mathcal{L}(\widetilde{\phi}(x), \{\partial_\nu \widetilde{\phi}(x)\}) = \mathcal{L}(\phi(x'), \{\partial'_\nu \phi(x')\})$$

where,

$$x' = \Lambda x, \qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Infinitesimal version:

$$\Lambda = I + \epsilon\, \omega, \quad \Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon\, \omega^\mu{}_\nu$$

We found that $\omega_{\rho\nu} = \eta_{\rho\mu}\omega^\mu{}_\nu$ is anti-symmetric matrix.

Conserved current associated with this symmetry:

$$J^\mu = \frac{1}{2}\omega_{\tau\nu}\, [x^\nu\, T^{\mu\tau} - x^\tau T^{\mu\nu}]$$

The same result holds when we have more fields and more general $\mathcal{L}$ if:

$$\widetilde{\phi}_r(x) = \phi_r(\Lambda\, x) \quad \text{for } i = 1, \cdots, n$$

$$\mathcal{L}(\{\widetilde{\phi}_r(x)\}, \{\partial_\nu \widetilde{\phi}_r(x)\}) = \mathcal{L}(\{\phi_r(x')\}, \{\partial'_\nu \phi_r(x')\})$$

where

$$x' = \Lambda x, \qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Now consider a more general class of theories.

Consider a field theory with multiple fields $\phi_1, \cdots, \phi_n$

Some of these may represent components of vector fields or higher tensor fields.

The general Lorentz transformation law takes the form:

$$\widetilde{\phi}_r(x) = \sum_{s=1}^{n} S_{rs}(\Lambda)\, \phi_s(\Lambda x)$$

e.g. if $\{\phi_r\}$ represented the components of a Lorentz four vector A_μ, then

$$\widetilde{A}_\mu(x) = \Lambda^\nu{}_\mu\, A_\nu(\Lambda x) \qquad \Rightarrow \qquad S_{\mu\nu}(\Lambda) = \Lambda^\nu{}_\mu \quad \Rightarrow \quad S = \Lambda^T$$

We shall check while studying Maxwell's theory that this is a symmetry of the action.

If $\{\phi_r\}$ includes a scalar field ϕ and a vector field A_μ, then S will be a block diagonal 5×5 matrix:

$$S = \begin{pmatrix} 1 & \\ & \Lambda^T \end{pmatrix}$$

We shall assume that $\mathcal{L}$ still satisfies:

$$\mathcal{L}(\{\widetilde{\phi}_r(x)\}, \{\partial_\nu \widetilde{\phi}_r(x)\}) = \mathcal{L}(\{\phi_r(x')\}, \{\partial'_\nu \phi_r(x')\})$$

where

$$x' = \Lambda x, \qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\widetilde{\phi}_r(x) = \sum_{s=1}^{n} S_{rs}(\Lambda)\,\phi_s(\Lambda x)$$

Now suppose that we make another Lorentz transformation with Lorentz matrix Λ'.

This will transform the field configuration $\{\widetilde{\phi}_r\}$ to new configuration $\{\widehat{\phi}_r\}$:

$$\begin{aligned}\widehat{\phi}_r(x) &= \sum_{s=1}^{n} S_{rs}(\Lambda')\,\widetilde{\phi}_s(\Lambda' x) = \sum_{s=1}^{n} S_{rs}(\Lambda') \sum_{t=1}^{n} S_{st}(\Lambda)\phi_t(\Lambda\Lambda' x) \\ &= \sum_{t=1}^{n}\{S(\Lambda')S(\Lambda)\}_{rt}\phi_t(\Lambda\Lambda' x) = \sum_{s=1}^{n}\{S(\Lambda')S(\Lambda)\}_{rs}\phi_s(\Lambda\Lambda' x)\end{aligned}$$

This must represent a single Lorentz transformation with matrix $\Lambda\Lambda'$.

$$\widehat{\phi}_r(x) = \sum_{s=1}^{n} S_{rs}(\Lambda\Lambda')\}\phi_s(\Lambda\Lambda' x)$$

Therefore we should have

$$S(\Lambda')S(\Lambda) = S(\Lambda\Lambda')$$

Taking transpose:

$$S^T(\Lambda)S^T(\Lambda') = S^T(\Lambda\Lambda')$$

Composition law of $S^T(\Lambda)$ is the same as that of Λ

We say S^T gives a representation of the Lorentz group.

$$S^T(\Lambda)S^T(\Lambda') = S^T(\Lambda\Lambda')$$

Let us verify this in known cases.

For scalar, $S(\Lambda) = I$ (identity matrix)

$$S^T(\Lambda)S^T(\Lambda') = I, \qquad S^T(\Lambda\Lambda') = I$$

For vector, $S(\Lambda) = \Lambda^T$ and $S(\Lambda\Lambda') = (\Lambda\Lambda')^T$.

$$S^T(\Lambda)S^T(\Lambda') = \Lambda\Lambda', \qquad S^T(\Lambda\Lambda') = \Lambda\Lambda'$$

Goal: Study how the Noether current $M^{\mu\nu\tau}$ is modified when $S(\Lambda)$ is not the identity matrix.

For this we need to study infinitesimal transformation.

$$S^T(\Lambda')S^T(\Lambda) = S^T(\Lambda'\Lambda)$$

If $\Lambda' = I$ (identity matrix), then

$$S^T(I)S^T(\Lambda) = S^T(I\Lambda) = S^T(\Lambda)$$

This gives

$$S^T(I) = I, \qquad \Rightarrow \qquad S(I) = I^T = I$$

If Λ is close to the identity then S should be close to identity.

Take $\Lambda = I + \epsilon\,\omega$, $\quad \omega_{\mu\nu} = \eta_{\mu\rho}\,\omega^\rho{}_\nu$ is anti-symmetric matrix

Then

$$S(I + \epsilon\,\omega) = I + \epsilon\,T(\omega)$$

for some $n \times n$ matrix T.

Note: T_{rs} is a function of 6 variables $\{\omega_{\mu\nu}\}$

$$S(\Lambda)S(\Lambda') = S(\Lambda'\Lambda) \quad \Rightarrow \quad S(I + \epsilon\,\omega)S(I + \epsilon\,\omega') = S\left((I + \epsilon\,\omega')(I + \epsilon\,\omega)\right)$$

$$\Rightarrow \quad (I + \epsilon\,T(\omega))(I + \epsilon\,T(\omega')) = S(I + \epsilon\,(\omega + \omega')) = I + \epsilon\,T(\omega + \omega')$$

$$\Rightarrow \quad I + \epsilon\,T(\omega) + \epsilon\,T(\omega') = I + \epsilon\,T(\omega + \omega')$$

$$\Rightarrow \quad T(\omega) + T(\omega') = T(\omega + \omega')$$

T_{rs} is a function of 6 variables $\{\omega_{\mu\nu}\}$

$$T(\omega) + T(\omega') = T(\omega + \omega')$$

This shows that T_{rs} must be linear function of $\{\omega_{\mu\nu}\}$

General form:

$$T_{rs}(\omega) = \omega_{\tau\nu}\,\Sigma_{rs}^{\tau\nu}$$

for some constants $\Sigma_{rs}^{\tau\nu}$.

Infinitesimal transformation of ϕ_r:

$$\widetilde{\phi}_r(x) = \sum_{s=1}^{n}(\delta_{rs} + \epsilon\, T_{rs}(\omega))\phi_s((I + \epsilon\,\omega)x)$$

Note: The term involving T_{rs} is the extra term that was not present for scalars.

Extra term in $\widetilde{\phi}_r(x)$:

$$\epsilon\sum_{s=1}^{n} T_{rs}(\omega)\phi_s(x) = \epsilon\sum_{s=1}^{n}\,\omega_{\tau\nu}\,\Sigma_{rs}^{\tau\nu}\,\phi_s(x)$$

Recall our definition of $\chi_r(x)$:

$$\widetilde{\phi}_r(x) = \phi_r(x) + \epsilon\,\chi_r(x)$$

This gives the extra term in $\chi_r(x)$:

$$\omega_{\tau\nu}\sum_{s=1}^{n}\,\Sigma_{rs}^{\tau\nu}\,\phi_s(x)$$

Extra term in $\chi_r(x)$:

$$\omega_{\tau\nu} \sum_{s=1}^{n} \Sigma_{rs}^{\tau\nu} \, \phi_s(x)$$

Now recall the expression for the conserved current:

$$J^{\mu} = -\sum_{r=1}^{n} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \, \chi_r + K^{\mu}$$

This gives the expression for the extra term in J^{μ}:

$$-\sum_{r=1}^{n} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \, \omega_{\tau\nu} \sum_{s=1}^{n} \Sigma_{rs}^{\tau\nu} \, \phi_s(x)$$

Earlier expression for J^{μ}:

$$\frac{1}{2}\omega_{\tau\nu} \left[x^{\nu} \, T^{\mu\tau} - x^{\tau} T^{\mu\nu}\right]$$

New expression:

$$\frac{1}{2}\omega_{\tau\nu} \left[x^{\nu} \, T^{\mu\tau} - x^{\tau} T^{\mu\nu} - 2 \sum_{r,s=1}^{n} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \Sigma_{rs}^{\tau\nu} \, \phi_s(x)\right]$$

$$M^{\mu\nu\tau} = x^{\nu} \, T^{\mu\tau} - x^{\tau} T^{\mu\nu} - 2 \sum_{r,s=1}^{n} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \Sigma_{rs}^{\tau\nu} \, \phi_s(x)$$

The extra term is the relativistic generalization of spin angular momentum

14 Conservation laws to symmetries

Noether procedure gives a way to construct the conserved charge for a given continuous symmetry.

Can we go the other way?

Conserved charge $\rightarrow$ symmetry transformation

Suppose in a classical field theory, with action of the form:

$$S = \int d^4x \, \mathcal{L}(\{\phi_r\}, \{\partial_\mu \phi_r\})$$

we have an infinitesimal symmetry

$$\widetilde{\phi}_r(x) = \phi_r(x) + \epsilon \, \chi_r(x), \qquad \partial_\mu \widetilde{\phi}_r(x) = \partial_\mu \phi_r(x) + \epsilon \, \partial_\mu \chi_r(x)$$

Let us suppose that $Q = \int d^3r J^0(x)$ is the conserved charge.

Question: Can we recover the symmetry transformation by knowing Q?

Answer:

$$\chi_r(t, \vec{r}) = \{Q(t), \phi_r(t, \vec{r})\}_{PB}$$

In quantum theory,

$$\chi_r(t, \vec{r}) = -i \, [Q(t), \phi_r(t, \vec{r})]$$

$$\chi_r(t,\vec{r}) = -i\,[Q(t), \phi_r(t,\vec{r})]$$

Even though the result holds in general, we shall prove this for a special class of symmetries for which

• χ_r does not depend on any conjugate momentum Π_s

• K^0 vanishes

Ex. Check that most symmetries we have studied so far are of this type, except,

• time translation

• Lorentz boost

$$J^\mu = -\sum_{r=1}^{n} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\, \chi_r + K^\mu$$

Since K^0 vanishes,

$$Q = \int d^3r\, J^0(t,\vec{r}) = -\int d^3r \sum_{r=1}^{n} \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_r)}\, \chi_r$$

Now

$$L = \int d^3r'\, \mathcal{L}(\phi_r(t,\vec{r}\,'), \{\dot{\phi}_r(t,\vec{r}\,')\}, \{\partial_i' \phi_r(t,\vec{r}\,')\})$$

This gives

$$\Pi_r(t,\vec{r}) = \frac{\delta L}{\delta \dot{\phi}_r(t,\vec{r})} = \int d^3r'\, \delta^{(3)}(\vec{r} - \vec{r}\,')\, \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r(t,\vec{r}\,')} = \frac{\partial \mathcal{L}(t,\vec{r})}{\partial \dot{\phi}_r(t,\vec{r})}$$

This gives

$$Q(t) = -\int d^3r \sum_{r=1}^{n} \Pi_r(t,\vec{r})\, \chi_r(t,\vec{r})$$

This gives

$$[Q(t), \phi_s(t,\vec{r}\,')] = -\left[\int d^3r \sum_{r=1}^{n} \Pi_r(t,\vec{r})\, \chi_r(t,\vec{r}),\ \ \phi_s(t,\vec{r}\,')\right]$$

Since we have assumed that χ_r does not depend on Π_s's, only the commutator

$$[\Pi_r(t,\vec{r}), \phi_s(t,\vec{r}\,')] = -i\, \delta_{rs}\, \delta^{(3)}(\vec{r} - \vec{r}\,')$$

contributes.

$$[Q(t), \phi_s(t,\vec{r}\,')] = i \int d^3r \sum_{r=1}^{n}\ \delta_{rs}\, \delta^{(3)}(\vec{r} - \vec{r}\,')\, \chi_r(t,\vec{r}) = i\, \chi_s(t,\vec{r}\,')$$

$$\Rightarrow \qquad \chi_s(t,\vec{r}\,') = -i\, [Q(t), \phi_s(t,\vec{r}\,')]$$

$$\chi_s(t, \vec{r}') = -i\,[Q(t), \phi_s(t, \vec{r}')]$$

With some effort one can also prove this in the more general case where χ_r depends on the Π_s's and K^0 does not vanish.

Ex. Check that in K-G theory,

$$\left[\int d^3r(x^0T^{0i} - x^iT^{00}), \phi(t, \vec{r}')\right] = i(x'^i\partial_t\phi(t, \vec{r}') + x^0\partial_i'\phi(t, \vec{r}'))$$

This corresponds to taking $\omega_{0i} = -1, \quad \omega_{i0} = 1$

We shall now show how this relation can be used in quantum theory.

We take K-G theory to illustrate the point, but the results are more general.

Quantities of interest: Matrix elements of operators between states

Since states are created by applying $a^\dagger$'s on $|0\rangle$, and $a, a^\dagger$ can be expressed in terms of ϕ and $\dot{\phi}$'s, the general quantities of interest will be:

$$\langle 0|\phi(t_1, \vec{r}_1) \cdots \phi(t_N, \vec{r}_N)|0\rangle$$

Now suppose that Q is a conserved charge.

Suppose further that

$$Q|0\rangle = 0 \qquad \Rightarrow \qquad \langle 0|Q = 0$$

If this is not true, we say that the symmetry is spontaneously broken

– will not be discussed in this course

We have

$$\langle 0|[Q, \phi(t_1, \vec{r}_1) \cdots \phi(t_N, \vec{r}_N)]|0\rangle = 0$$

$$\langle 0|[Q, \phi(t_1, \vec{r}_1) \cdots \phi(t_N, \vec{r}_N)]|0\rangle = 0$$

$$\Downarrow$$

$$\langle 0| \left([Q, \phi(t_1, \vec{r}_1)]\phi(t_2, \vec{r}_2) \cdots \phi(t_N, \vec{r}_N) + \phi(t_1, \vec{r}_1)[Q, \phi(t_2, \vec{r}_2)]\phi(t_3, \vec{r}_3) \cdots \phi(t_N, \vec{r}_N) + \cdots \right) |0\rangle = 0$$

Since Q is time independent, we can choose the argument of Q to match the time coordinate of the field with which we calculate the commutator

– can use equal time commutator

$$-i \langle 0| \left(\chi(t_1, \vec{r}_1)\phi(t_2, \vec{r}_2) \cdots \phi(t_N, \vec{r}_N) + \phi(t_1, \vec{r}_1)\chi(t_2, \vec{r}_2)\phi(t_3, \vec{r}_3) \cdots \phi(t_N, \vec{r}_N) + \cdots \right) |0\rangle = 0$$

Now recall that

$$\widetilde{\phi}(t, \vec{r}) = \phi(t, \vec{r}) + \epsilon\, \chi(t, \vec{r})$$

Therefore we have

$$\begin{aligned}
&\langle 0|\widetilde{\phi}(t_1, \vec{r}_1)\widetilde{\phi}(t_2, \vec{r}_2) \cdots \widetilde{\phi}(t_N, \vec{r}_N)|0\rangle \\
&= \langle 0|\phi(t_1, \vec{r}_1)\phi(t_2, \vec{r}_2) \cdots \phi(t_N, \vec{r}_N)|0\rangle \\
&+\epsilon\langle 0| \left(\chi(t_1, \vec{r}_1)\phi(t_2, \vec{r}_2) \cdots \phi(t_N, \vec{r}_N) + \phi(t_1, \vec{r}_1)\chi(t_2, \vec{r}_2)\phi(t_3, \vec{r}_3) \cdots \phi(t_N, \vec{r}_N) + \cdots \right) |0\rangle + \mathcal{O}(\epsilon^2) \\
&= \langle 0|\phi(t_1, \vec{r}_1)\phi(t_2, \vec{r}_2) \cdots \phi(t_N, \vec{r}_N)|0\rangle + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Conclusion:

$$\langle 0|\widetilde{\phi}(t_1, \vec{r}_1)\widetilde{\phi}(t_2, \vec{r}_2) \cdots \widetilde{\phi}(t_N, \vec{r}_N)|0\rangle = \langle 0|\phi(t_1, \vec{r}_1)\phi(t_2, \vec{r}_2) \cdots \phi(t_N, \vec{r}_N)|0\rangle + \mathcal{O}(\epsilon^2)$$

$$\langle 0|\widetilde{\phi}(t_1,\vec{r}_1)\widetilde{\phi}(t_2,\vec{r}_2)\cdots\widetilde{\phi}(t_N,\vec{r}_N)|0\rangle = \langle 0|\phi(t_1,\vec{r}_1)\phi(t_2,\vec{r}_2)\cdots\phi(t_N,\vec{r}_N)|0\rangle + \mathcal{O}(\epsilon^2)$$

This has been proved for infinitesimal transformation.

However finite transformations can be built in K steps of size ϵ with $\epsilon \sim 1/K$

At each step we make an error of at most of order ϵ^2

Net error $\sim \quad \epsilon^2 K \sim 1/K$

$\to 0$ as $K \to \infty$

Conclusion:

$$\langle 0|\widetilde{\phi}(t_1,\vec{r}_1)\widetilde{\phi}(t_2,\vec{r}_2)\cdots\widetilde{\phi}(t_N,\vec{r}_N)|0\rangle = \langle 0|\phi(t_1,\vec{r}_1)\phi(t_2,\vec{r}_2)\cdots\phi(t_N,\vec{r}_N)|0\rangle$$

for finite transformation.

Note: This is a consequence of symmetry and holds irrespective of the form of the action.

Example:

1. Translation symmetry: $\widetilde{\phi}(x) = \phi(x + a)$

$a = (a^0, a^1, a^2, a^3)$: arbitrary constant 4-vector

Therefore:

$$\langle 0|\phi(x_1 + a)\phi(x_2 + a)\cdots\phi(x_N + a)|0\rangle = \langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_N)|0\rangle$$

This means that $\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_N)|0\rangle$ is a function of only the differences

$$x_2 - x_1, \quad x_3 - x_1, \quad \cdots, \quad x_N - x_1$$

2. Lorentz invariance: $\widetilde{\phi}(x) = \phi(\Lambda\, x)$

$$\langle 0|\phi(\Lambda x_1)\phi(\Lambda x_2)\cdots\phi(\Lambda x_N)|0\rangle = \langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_N)|0\rangle$$

Special case: $N = 2$

Translation invariance:

$$\langle 0|\phi(x_1)\phi(x_2)|0\rangle = f(x_1 - x_2)$$

Lorentz invariance: $f(x_1 - x_2)$ is a function of

$$(x_1 - x_2)^2 = \eta_{\mu\nu}(x_1^\mu - x_2^\mu)(x_1^\nu - x_2^\nu) = -(t_1 - t_2)^2 + (\vec{r}_1 - \vec{r}_2)^2$$

Some definitions:

Two space-time points x_1 and x_2 are outside each others light-cone if

$(x_1 - x_2)^2 > 0 \quad \Leftrightarrow \quad -(t_1 - t_2)^2 + (\vec{r}_1 - \vec{r}_2)^2 > 0 \quad \Leftrightarrow \quad (\vec{r}_1 - \vec{r}_2)^2 > (t_1 - t_2)^2$

In this case it is possible to find a Lorentz transformation such that

$$x'_1 = \Lambda x_1, \quad x'_2 = \Lambda x_2, \quad t'_1 = t'_2$$

Note: t'_1, t'_2 are the time components of x'_1, x'_2

A consequence: Commutator of fields vanish outside the light-cone

$$\langle 0|[\phi(x_1), \phi(x_2)]\phi(x_3) \cdots \phi(x_N)|0\rangle = 0 \quad \text{if} \quad (x_1 - x_2)^2 > 0$$

Proof:

There is a Lorentz transformation Λ such that

$$x'_1 = \Lambda x_1, \quad x'_2 = \Lambda x_2, \quad t'_1 = t'_2$$

On the other hand

$$\langle 0|[\phi(x_1), \phi(x_2)]\phi(x_3) \cdots \phi(x_N)|0\rangle = \langle 0|[\phi(\Lambda x_1), \phi(\Lambda x_2)]\phi(\Lambda x_3) \cdots \phi(\Lambda x_N)|0\rangle$$

$$= \langle 0|[\phi(x'_1), \phi(x'_2)]\phi(x'_3) \cdots \phi(x'_N)|0\rangle$$

$$[\phi(x'_1), \phi(x'_2)] = [\phi(t'_1, \vec{r}\,'_1), \phi(t'_2, \vec{r}\,'_2)] = [\phi(t'_1, \vec{r}\,'_1), \phi(t'_1, \vec{r}\,'_2)]$$

since $t'_1 = t'_2$

But the equal time commutator $[\phi(t'_1, \vec{r}\,'_1), \phi(t'_1, \vec{r}\,'_2)]$ vanishes

$$\Rightarrow \langle 0|[\phi(x_1), \phi(x_2)]\phi(x_3) \cdots \phi(x_N)|0\rangle = 0 \quad \text{if} \quad (x_1 - x_2)^2 > 0$$

This is true even if one or both of the ϕ's were Π since $\vec{r}\,'_1 \neq \vec{r}\,'_2$

15 Discrete symmetries

Discrete symmetries:

In this case there is no Noether theorem.

Nevertheless we still have consequences in the quantum theory.

We shall work in the Hamiltonian formalism from the beginning.

Suppose the discrete symmetry takes the form

$$\phi_r(x) \to \widetilde{\phi}_r(x), \qquad \Pi_r(x) \to \widetilde{\Pi}_r(x), \qquad H[\{\widetilde{\phi}_r\}, \{\widetilde{\Pi}_r\}] = H[\{\phi_r\}, \{\Pi_r\}]\,.$$

Let us further suppose that

$$\{\widetilde{\phi}_r(t, \vec{r}), \widetilde{\Pi}_s(t, \vec{r}\,')\}_{PB} = \delta_{rs}\, \delta^{(3)}(\vec{r} - \vec{r}\,')$$

Then in the quantum theory, we shall have

$$[\widetilde{\phi}_r(t, \vec{r}), \widetilde{\Pi}_s(t, \vec{r}\,')] = i\, \delta_{rs}\, \delta^{(3)}(\vec{r} - \vec{r}\,')$$

Let us work at $t = 0$.

In this case we can define a unitary operator U in the quantum theory that satisfies:

$$U\, \phi_r(x)\, U^{-1} = \widetilde{\phi}_r(x), \qquad U\, \Pi_r(x)\, U^{-1} = \widetilde{\Pi}_r(x)$$

Since all the states are created by combinations of $\{\phi_r\}$ and $\{\Pi_r\}$ acting on $|0\rangle$, this defines U in terms of $U|0\rangle$

We now need to check internal consistency

– e.g. by applying U on $|0\rangle$ and $[a, a^\dagger]|0\rangle$ we should get the same result.

Since all commutation relations of this type arise from ϕ-Π commutation relations, we test the consistency of

$$U\,\phi_r(x)\,U^{-1} = \widetilde{\phi}_r(x), \qquad U\,\Pi_r(x)\,U^{-1} = \widetilde{\Pi}_r(x)$$

with the ϕ-Π commutation relation.

We use:

$$U\,A_1 \cdots A_N\,U^{-1} = U\,A_1 U^{-1}\,\cdots U\,A_N\,U^{-1}$$

$$[\widetilde{\phi}_r(t,\vec{r}), \widetilde{\Pi}_s(t,\vec{r}\,')] = [U\phi_r(t,\vec{r})U^{-1}, U\,\Pi_s(t,\vec{r}\,')U^{-1}] = U[\phi_r(t,\vec{r}), \Pi_s(t,\vec{r}\,')]U^{-1}$$

$$l.h.s. = i\,\delta_{rs}\,\delta^{(3)}(\vec{r}-\vec{r}\,'), \qquad r.h.s. = i\,\delta_{rs}\,\delta^{(3)}(\vec{r}-\vec{r}\,')\,U\,U^{-1} = i\,\delta_{rs}\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

Therefore

$$U\,\phi_r(x)\,U^{-1} = \widetilde{\phi}_r(x), \qquad U\,\Pi_r(x)\,U^{-1} = \widetilde{\Pi}_r(x)$$

is consistent with the ϕ-Π commutation relation.

Since H is given as a functional of $\{\phi_r\}$ and $\{\Pi_r\}$, we have:

$$U\,H[\{\phi_r\},\{\Pi_r\}]\,U^{-1} = H[[\{U\,\phi_r U^{-1}\},\{U\,\Pi_r U^{-1}\}] = H[\{\widetilde{\phi}_r\},\{\widetilde{\Pi}_r\}] = H[\{\phi_r\},\{\Pi_r\}]$$

Note: For simplifying notation, we have dropped the 'hat' from H even though this is a quantum operator.

Therefore

$$U\,H = H\,U$$

$\Rightarrow U$ is time independent.

This implies that

$$U\,\phi_r(x)\,U^{-1} = \widetilde{\phi}_r(x), \qquad U\,\Pi_r(x)\,U^{-1} = \widetilde{\Pi}_r(x)$$

holds for all t, not just at $t = 0$.

Now suppose that the ground state is an eigenstate of U and $U^{-1} = U^\dagger$.

$$U^\dagger|0\rangle = e^{i\alpha}|0\rangle, \qquad \langle 0|U = e^{-i\alpha}\langle 0|$$

Then

$$\langle 0|\widetilde{\phi}(t_1,\vec{r}_1)\widetilde{\phi}(t_2,\vec{r}_2)\cdots\widetilde{\phi}(t_N,\vec{r}_N)|0\rangle = \langle 0|U\,\phi(t_1,\vec{r}_1)U^{-1}U\phi(t_2,\vec{r}_2)U^{-1}\cdots|0\rangle$$

$$= \langle 0|U\,\phi(t_1,\vec{r}_1)\phi(t_2,\vec{r}_2)\cdots\phi(t_N,\vec{r}_N)U^\dagger|0\rangle = \langle 0|\phi(t_1,\vec{r}_1)\phi(t_2,\vec{r}_2)\cdots\phi(t_N,\vec{r}_N)|0\rangle$$

– similar to what we had in the case of continuous symmetry.

If the ground state is not an eigenstate of U and $U^\dagger$, then there is no such relation

– we say that the symmetry is spontaneously broken.

Examples of discrete symmetry:

Parity: $\widetilde{\phi}(t,\vec{r}) = \phi(t,-\vec{r})$

Internal symmetry: $\widetilde{\phi}(t,\vec{r}) = -\phi(t,\vec{r})$

– does not change the argument of ϕ.

An exception: Time reversal symmetry

$$\widetilde{\phi}_r(t,\vec{r}) = \phi_r(-t,\vec{r}), \qquad \mathcal{L}(\{\widetilde{\phi}_r\},\{\partial_\mu\widetilde{\phi}_r\}) = \mathcal{L}(\{\phi_r(t',\vec{r})\},\{\partial'_\mu\phi_r(t',\vec{r})\}), \quad t'=-t$$

More general transformation involving some matrix acting on ϕ_r is possible, but the results we shall describe will not change.

$$\partial_t\widetilde{\phi}_r(t,\vec{r}) = \partial_t\phi_r(-t,\vec{r}) = -\partial'_t\phi_r(t',\vec{r})$$

$$\widetilde{\Pi}_r(t,\vec{r}) = \frac{\partial\mathcal{L}}{\partial(\partial_t\widetilde{\phi}_r(t,\vec{r}))} = -\frac{\partial\mathcal{L}}{\partial(\partial'_t\phi_r(t',\vec{r}))} = -\Pi_r(t',\vec{r}) = -\Pi_r(-t,\vec{r})$$

Ex. Using

$$H = \int d^3r \left[\sum_{r=1}^{n} \partial_t\phi_r(t,\vec{r})\,\Pi_r(t,\vec{r}) - \mathcal{L}\right]$$

show that

$$H[\{\widetilde{\phi}_r\},\{\widetilde{\Pi}_r\}] = H[\{\phi_r\},\{\Pi_r\}]$$

Now we have a problem.

$$\{\widetilde{\phi}_r(t,\vec{r}),\widetilde{\Pi}_s(t,\vec{r}\,')\}_{PB} = \{\phi_r(-t,\vec{r}),-\Pi_s(-t,\vec{r}\,')\}_{PB} = -\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

In quantum theory:

$$[\phi_r(t,\vec{r}),\Pi_s(t,\vec{r}\,')] = i\,\delta^{(3)}(\vec{r}-\vec{r}\,'), \qquad [\widetilde{\phi}_r(t,\vec{r}),\widetilde{\Pi}_s(t,\vec{r}\,')] = -i\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

This means that there cannot be a unitary operator U such that:

$$U\,\phi_r(x)\,U^{-1} = \widetilde{\phi}_r(x), \qquad U\,\Pi_r(x)\,U^{-1} = \widetilde{\Pi}_r(x)$$

since this will give

$$\begin{aligned} & [\widetilde{\phi}_r(t,\vec{r}),\widetilde{\Pi}_s(t,\vec{r}\,')] = [U\phi_r(t,\vec{r})U^{-1},U\Pi_s(t,\vec{r}\,')U^{-1}] \\ = \ & U[\phi_r(t,\vec{r}),\Pi_s(t,\vec{r}\,')]U^{-1} = U\,i\,\delta_{rs}\,\delta^{(3)}(\vec{r}-\vec{r}\,')\,U^{-1} = i\,\delta_{rs}\,\delta^{(3)}(\vec{r}-\vec{r}\,') \end{aligned}$$

Solution: Take U to be an anti-unitary operator.

If in a given basis $\{|\alpha\rangle\}$, we have a state:

$$|A\rangle = \sum_\alpha A_\alpha |\alpha\rangle$$

then,

$$U|A\rangle = \sum_{\alpha,\beta} A_\alpha^* \, W_{\alpha\beta} |\beta\rangle$$

W: A unitary matrix

Crucial new ingredient: $A_\alpha \to A_\alpha^*$

For a complex number c, this gives

$$Uc|A\rangle = U \sum_\alpha c\, A_\alpha |\alpha\rangle = \sum_{\alpha,\beta} c^* \, A_\alpha^* \, W_{\alpha\beta} |\beta\rangle = c^* \, U|A\rangle$$

Since this is true for any state $|A\rangle$, we have

$$U\, c = c^* \, U \qquad \Rightarrow \qquad U\, c\, U^{-1} = c^*$$

Therefore it is consistent to introduce an anti-unitary operator such that

$$U\, \phi_r(x)\, U^{-1} = \widetilde{\phi}_r(x), \qquad U\, \Pi_r(x)\, U^{-1} = \widetilde{\Pi}_r(x)$$

since this will give

$$\begin{aligned} & [\widetilde{\phi}_r(t,\vec{r}), \widetilde{\Pi}_s(t,\vec{r}\,')] = [U\phi_r(t,\vec{r})\, U^{-1}, U\, \Pi_s(t,\vec{r}\,')\, U^{-1}] \\ = \; & U[\phi_r(t,\vec{r}), \Pi_s(t,\vec{r}\,')]U^{-1} = U\, i\, \delta_{rs}\, \delta^{(3)}(\vec{r}-\vec{r}\,')\, U^{-1} = -i\, \delta_{rs}\, \delta^{(3)}(\vec{r}-\vec{r}\,') \end{aligned}$$

reproducing the correct commutator.

From this we can work out the consequence of time reversal symmetry as before.

16 Green's functions in Klein-Gordon theory

Goal: Develop tools for calculating matrix elements in K-G theory

We define

$$\Delta_+(x_1, x_2) = \langle 0|\phi(x_1)\phi(x_2)|0\rangle$$

We can calculate this by expressing $\phi(x)$ in terms of $a(t,\vec{p})$ and $a(t,\vec{p})^\dagger$.

For this we recall some relations derived earlier:

$$\phi(t,\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3p\, \widetilde{\phi}(t,\vec{p})\, e^{i\vec{p}.\vec{r}}, \qquad \Pi(t,\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3p\, \widetilde{\Pi}(t,\vec{p})\, e^{i\vec{p}.\vec{r}},$$

$$E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$$

$$a(t,\vec{p}) = \frac{1}{\sqrt{2}} \left(E_{\vec{p}}^{1/2} \widetilde{\phi}(t,\vec{p}) + i\, E_{\vec{p}}^{-1/2} \widetilde{\Pi}(t,\vec{p}) \right)$$

$$a(t,\vec{p})^\dagger = \frac{1}{\sqrt{2}} \left(E_{\vec{p}}^{1/2} \widetilde{\phi}(t,-\vec{p}) - i\, E_{\vec{p}}^{-1/2} \widetilde{\Pi}(t,-\vec{p}) \right)$$

$$[a(t,\vec{p}), a(t,\vec{p}\,')] = 0, \qquad [a(t,\vec{p})^\dagger, a(t,\vec{p}\,')^\dagger] = 0, \qquad [a(t,\vec{p}), a(t,\vec{p}\,')^\dagger] = \delta^{(3)}(\vec{p}-\vec{p}\,')$$

$$[H, a(t,\vec{p})] = -E_{\vec{p}}\, a(t,\vec{p}), \quad [H, a(t,\vec{p})^\dagger] = E_{\vec{p}}\, a(t,\vec{p})^\dagger$$

$$\frac{da(t,\vec{p})}{dt} = i[H, a(t,\vec{p})] = -i\, E_{\vec{p}}\, a(t,\vec{p}) \quad \Rightarrow \quad a(t,\vec{p}) = e^{-i\, E_{\vec{p}} t}\, a(0,\vec{p})$$

$$a(t,\vec{p})^\dagger = e^{i\, E_{\vec{p}} t} a(0,\vec{p})^\dagger$$

$$\Rightarrow \quad \widetilde{\phi}(t,\vec{p}) = \frac{1}{\sqrt{2\, E_{\vec{p}}}} (a(t,\vec{p}) + a(t,-\vec{p})^\dagger) = \frac{1}{\sqrt{2\, E_{\vec{p}}}} (e^{-i\, E_{\vec{p}} t}\, a(0,\vec{p}) + e^{i\, E_{\vec{p}} t} a(0,-\vec{p})^\dagger)$$

$$\widetilde{\phi}(t,\vec{p}) = \frac{1}{\sqrt{2\,E_{\vec{p}}}}\,(e^{-i\,E_{\vec{p}}t}\,a(0,\vec{p}) + e^{i\,E_{\vec{p}}t}a(0,-\vec{p})^\dagger)$$

$$\Rightarrow \quad \phi(t,\vec{r}) = \frac{1}{(2\pi)^{3/2}}\int d^3p\,\widetilde{\phi}(t,\vec{p})\,e^{i\vec{p}.\vec{r}} = \frac{1}{(2\pi)^{3/2}}\int d^3p\,e^{i\vec{p}.\vec{r}}\,\frac{1}{\sqrt{2\,E_{\vec{p}}}}\,(e^{-i\,E_{\vec{p}}t}\,a(0,\vec{p})+e^{i\,E_{\vec{p}}t}a(0,-\vec{p})^\dagger)$$

We substitute this into the expression for $\Delta_+(x_1,x_2)$

$$\Delta_+(x_1,x_2) = \frac{1}{(2\pi)^3}\int d^3p_1\,d^3p_2\,e^{i\vec{p}_1.\vec{r}_1+i\vec{p}_2.\vec{r}_2}\,\frac{1}{2\sqrt{E_{\vec{p}_1}\,E_{\vec{p}_2}}}$$
$$\langle 0|(e^{-i\,E_{\vec{p}_1}t_1}\,a(0,\vec{p}_1) + e^{i\,E_{\vec{p}_1}t_1}a(0,-\vec{p}_1)^\dagger)(e^{-i\,E_{\vec{p}_2}t_2}\,a(0,\vec{p}_2) + e^{i\,E_{\vec{p}_2}t_2}a(0,-\vec{p}_2)^\dagger)|0\rangle$$

Only the $a(0,\vec{p}_1)a(0,-\vec{p}_2)^\dagger$ term contributes, producing $\delta^{(3)}(\vec{p}_1+\vec{p}_2)$

Result (calling $\vec{p}_1$ as $\vec{p}$):

$$\Delta_+(x_1,x_2) = \frac{1}{(2\pi)^3}\int d^3p\,\frac{1}{2E_{\vec{p}}}\,e^{i\vec{p}.(\vec{r}_1-\vec{r}_2)}\,e^{-i\,E_{\vec{p}}\,(t_1-t_2)}$$

$$\Delta_+(x_1, x_2) = \frac{1}{(2\pi)^3} \int d^3p \, \frac{1}{2E_{\vec{p}}} \, e^{i\vec{p}.(\vec{r}_1 - \vec{r}_2)} \, e^{-i\,E_{\vec{p}}\,(t_1 - t_2)}$$

Claim: This can be written as

$$\Delta_+(x_1, x_2) = \frac{1}{(2\pi)^3} \int d^3p \, dp^0 \, \delta((p^0)^2 - \vec{p}^2 - m^2) \, H(p^0) \, e^{i\vec{p}.(\vec{r}_1 - \vec{r}_2) - i\,p^0(t_1 - t_2)}$$

Do the integration over p^0 using the delta function

– gives contribution from $p^0 = \pm\sqrt{\vec{p}^2 + m^2} = \pm E_{\vec{p}}$

The Heaviside function picks the contribution only from $p^0 = E_{\vec{p}}$

Using $\delta(f(x)) = \delta(x - x_0)/|f'(x_0)|$ if $f(x_0) = 0$, we get the $1/(2E_{\vec{p}})$ factor.

$$\Delta_+(x_1, x_2) = \frac{1}{(2\pi)^4} \int d^4p \, \widetilde{\Delta}(p) \, e^{ip.(x_1 - x_2)}, \qquad p = (p^0, \vec{p}),$$

$$\widetilde{\Delta}(p) = 2\pi \, \delta(-p^2 - m^2) \, H(p^0)$$

$$a.b = \eta_{\mu\nu} \, a^\mu \, b^\nu = -a^0 b^0 + \vec{a}.\vec{b}$$

Now we shall check Lorentz invariance of $\Delta_+(x_1, x_2)$

$$\Delta_+(\Lambda x_1, \Lambda x_2) = \frac{1}{(2\pi)^4} \int d^4p \, \widetilde{\Delta}(p) \, e^{ip.\Lambda(x_1 - x_2)}$$

Define $p' = \Lambda^{-1} p$ so that $p = \Lambda p'$

We have $d^4p = d^4p'$ and

$$p.\Lambda(x_1 - x_2) = \Lambda p'.\Lambda(x_1 - x_2) = p'.(x_1 - x_2)$$

since

$$\Lambda p'.\Lambda(x_1 - x_2) = \eta_{\mu\nu} \, \Lambda^\mu{}_\rho \, p'^\rho \, \Lambda^\nu{}_\sigma \, (x_1 - x_2)^\sigma = \eta_{\rho\sigma} \, p'^\rho \, (x_1 - x_2)^\sigma = p'.(x_1 - x_2)$$

This gives

$$\Delta_+(\Lambda x_1, \Lambda x_2) = \frac{1}{(2\pi)^4} \int d^4p' \, \widetilde{\Delta}(p) \, e^{i\Lambda p'.\Lambda(x_1 - x_2)} = \frac{1}{(2\pi)^4} \int d^4p' \, \widetilde{\Delta}(p) \, e^{ip'.(x_1 - x_2)}$$

We now use:

$$\widetilde{\Delta}(p) = 2\,\pi\,\delta(-p^2 - m^2)\,H(p^0) = 2\,\pi\,\delta(-p'^2 - m^2)\,H(p'^0) = \widetilde{\Delta}(p')$$

Note: In general $H(p^0) \neq H(p'^0)$, but when $p^2 + m^2 = 0$, i.e. $p^0 = \pm\sqrt{\vec{p}^2 + m^2}$, a Lorentz transformation connected to identity cannot change the sign of p^0.

$H(p^0) = H(p'^0)$ for time-like vector p.

This gives

$$\Delta_+(\Lambda x_1, \Lambda x_2) = \frac{1}{(2\pi)^4} \int d^4p' \, \widetilde{\Delta}(p') \, e^{ip'.(x_1 - x_2)} = \frac{1}{(2\pi)^4} \int d^4p \, \widetilde{\Delta}(p) \, e^{ip.(x_1 - x_2)} = \Delta_+(x_1, x_2)$$

Some other definitions:

$$\Delta(x_1, x_2) = \langle 0|[\phi(x_1), \phi(x_2)]|0\rangle = \Delta_+(x_1, x_2) - \Delta_+(x_2, x_1)$$

Using the expression for $\Delta(x_1, x_2)$, we get

$$\Delta(x_1, x_2) = \frac{1}{(2\pi)^3} \int d^4p\, \delta(-p^2 - m^2)\, H(p^0) \left\{ e^{ip.(x_1 - x_2)} - e^{-ip.(x_1 - x_2)} \right\} .$$

Ex. Show that this vanishes for $(x_1 - x_2)^2 > 0$

Definition: Time ordered product

$$T(\phi(x_1)\phi(x_2)\cdots\phi(x_n)) = \phi(x_{i_1})\,\phi(x_{i_2})\cdots\phi(x_{i_n})$$

$(i_1, \cdots, i_n)$: A specific permutation of $1, 2, \cdots, n$ such that

$$x^0_{i_1} \geq x^0_{i_2} \geq x^0_{i_3} \geq \cdots \geq x^0_{i_n}$$

We arrange the ϕ's so that their time arguments increase as we move from right to left.

Example:

$$T(\phi(x_1)\phi(x_2)) = H(x^0_1 - x^0_2)\,\phi(x_1)\phi(x_2) + H(x^0_2 - x^0_1)\,\phi(x_2)\phi(x_1)$$

We now define:

$$\Delta_F(x_1, x_2) = \langle 0|T(\phi(x_1)\phi(x_2))|0\rangle = H(x^0_1 - x^0_2)\,\Delta_+(x_1, x_2) + H(x^0_2 - x^0_1)\,\Delta_+(x_2, x_1)$$

Note:

$$\Delta_F(x_1, x_2) = \Delta_F(x_2, x_1)$$

Ex. Show that:

$$\Delta_F(x_1, x_2) = \frac{1}{(2\pi)^4} \int d^4p\, \widetilde{\Delta}_F(p)\, e^{ip.(x_1-x_2)}, \qquad p = (p^0, \vec{p}),$$

$$\widetilde{\Delta}_F(p) = \frac{i}{-p^2 - m^2 + i\epsilon}$$

ϵ: a small positive number, to be taken to 0 at the end

Hint: Do the p^0 integral

$$\int_{-\infty}^{\infty} dp^0 e^{-ip^0\,(x_1^0 - x_2^0)} \frac{i}{(p^0)^2 - \vec{p}^2 - m^2 + i\epsilon}$$

using residue formula.

Note: Depending on the sign of $x_1^0 - x_2^0$ we need to close the contour in upper or lower half plane so that the contribution to the integral from part of the contour at ∞ vanishes.

Differential equations:

Define

$$\Box = \eta^{\mu\nu}\,\partial_\mu\,\partial_\nu = -\partial_0^2 + \vec{\nabla}^2$$

Then K-G equation takes the form:

$$(\Box - m^2)\phi = 0$$

This gives:

$$(\Box_1 - m^2)\,\Delta_+(x_1, x_2) = (\Box_1 - m^2)\,\langle 0|\phi(x_1)\phi(x_2)|0\rangle = \langle 0|(\Box_1 - m^2)\,\phi(x_1)\phi(x_2)|0\rangle = 0$$

Ex. Using the expression for $\Delta_+(x_1, x_2)$, verify that $(\Box_1 - m^2)\,\Delta_+(x_1, x_2) = 0$.

Similarly $(\Box_2 - m^2)\,\Delta_+(x_1, x_2) = 0$.

Next consider:

$$(\Box_1 - m^2)\,\Delta_F(x_1,x_2)$$

$$= \left\{-\frac{\partial^2}{\partial(x_1^0)^2} + \vec{\nabla}_1^2 - m^2\right\}\left\{H(x_1^0-x_2^0)\,\Delta_+(x_1,x_2) + H(x_2^0-x_1^0)\,\Delta_+(x_2,x_1)\right\}$$

The $\vec{\nabla}_1^2 - m^2$ acts only on $\Delta_+(x_1,x_2)$ or $\Delta_+(x_2,x_1)$

Use:

$$\frac{\partial}{\partial x_1^0}H(x_1^0-x_2^0) = \delta(x_1^0-x_2^0), \quad \frac{\partial}{\partial x_1^0}H(x_2^0-x_1^0) = -\delta(x_1^0-x_2^0)$$

This gives

$$\begin{aligned}
&-\frac{\partial^2}{\partial(x_1^0)^2}\left\{H(x_1^0-x_2^0)\,\Delta_+(x_1,x_2) + H(x_2^0-x_1^0)\,\Delta_+(x_2,x_1)\right\} \\
&= -\frac{\partial}{\partial x_1^0}\Big[\delta(x_1^0-x_2^0)\ \left\{\Delta_+(x_1,x_2) - \Delta_+(x_2,x_1)\right\} \\
&\quad + H(x_1^0-x_2^0)\,\frac{\partial}{\partial x_1^0}\Delta_+(x_1,x_2) + H(x_2^0-x_1^0)\,\frac{\partial}{\partial x_1^0}\Delta_+(x_2,x_1)\Big]
\end{aligned}$$

The red term vanishes since for $x_1^0 = x_2^0$, $\phi(x_1)\phi(x_2) = \phi(x_2)\phi(x_1)$

We get,

$$\begin{aligned}
&-H(x_1^0-x_2^0)\,\frac{\partial^2}{\partial(x_1^0)^2}\Delta_+(x_1,x_2) + H(x_2^0-x_1^0)\,\frac{\partial^2}{\partial(x_1^0)^2}\Delta_+(x_2,x_1) \\
&\quad -\delta(x_1^0-x_2^0)\,\frac{\partial}{\partial x_1^0}\Delta_+(x_1,x_2) + \delta(x_2^0-x_1^0)\,\frac{\partial}{\partial x_1^0}\Delta_+(x_2,x_1)
\end{aligned}$$

The blue term combines with the $\vec{\nabla}_1^2 - m^2$ terms to give $\Box_1^2 - m^2$ and vanishes on $\Delta_+(x_1,x_2)$ or $\Delta_+(x_2,x_1)$.

The left-over equation:

$$\begin{aligned}
(\Box_1 - m^2)\,\Delta_F(x_1, x_2) &= -\delta(x_1^0 - x_2^0)\,\frac{\partial}{\partial x_1^0}\Delta_+(x_1, x_2) + \delta(x_2^0 - x_1^0)\,\frac{\partial}{\partial x_1^0}\Delta_+(x_2, x_1) \\
&= \delta(x_1^0 - x_2^0)\,\langle 0|\{-\dot{\phi}(x_1)\,\phi(x_2) + \phi(x_2)\,\dot{\phi}(x_1)\}|0\rangle \\
&= \delta(x_1^0 - x_2^0)\,\langle 0|[\phi(x_2), \Pi(x_1)]|0\rangle = \delta(x_1^0 - x_2^0)\,\delta^{(3)}(\vec{r}_1 - \vec{r}_2) \\
&= \delta^{(4)}(x_1 - x_2)
\end{aligned}$$

Ex. Verify this using the explicit form of $\Delta_F(x_1, x_2)$.

17 Anti-unitary operators, Feynman diagrams, composite operators

Some comments on anti-unitary operator:

If U is a unitary operator, then given two states $|C\rangle$, $|B\rangle$, we have

$$\langle C|U|B\rangle = \langle U^\dagger C|B\rangle = \langle U^{-1}C|B\rangle$$

Note: $\langle U^{-1}C|$ is by definition the conjugate of the state $U^{-1}|C\rangle$.

Claim: If U is a anti-unitary operator, then

$$\langle C|U|B\rangle = \langle U^{-1}C|B\rangle^*$$

To prove this we need to study some properties of U^{-1}.

Let us expand the states in some orthonormal basis $\{|\alpha\rangle\}$:

$$|B\rangle = \sum_\alpha B_\alpha\,|\alpha\rangle, \quad |C\rangle = \sum_\alpha C_\alpha\,|\alpha\rangle, \quad U^{-1}|C\rangle = |A\rangle = \sum_\alpha A_\alpha|\alpha\rangle$$

Then

$$\sum_\beta C_\beta\,|\beta\rangle = |C\rangle = U|A\rangle = \sum_{\alpha,\beta} A_\alpha^*\,W_{\alpha\beta}\,|\beta\rangle, \qquad WW^\dagger = I$$

$$\Rightarrow \quad C_\beta = \sum_\alpha A_\alpha^*\,W_{\alpha\beta} \quad \Rightarrow \quad A_\alpha = \sum_\beta (C_\beta W_{\beta\alpha}^{-1})^* = \sum_\beta C_\beta^*(W_{\beta\alpha}^\dagger)^* = \sum_\beta C_\beta^*\,W_{\alpha\beta}$$

Then

$$lhs = \langle C|U|B\rangle = \sum_\gamma C_\gamma^*\langle\gamma|\sum_{\alpha,\beta} B_\alpha^*\,W_{\alpha\beta}|\beta\rangle = \sum_\gamma\sum_{\alpha,\beta} C_\gamma^* B_\alpha^*\,W_{\alpha\beta}\,\delta_{\beta\gamma} = \sum_{\alpha,\beta} C_\beta^* B_\alpha^*\,W_{\alpha\beta}$$

$$rhs = \langle A|B\rangle^* = \sum_\alpha A_\alpha B_\alpha^* = \sum_{\alpha,\beta} C_\beta^*\,W_{\alpha\beta}B_\alpha^* = lhs$$

Consequences for time reversal invariance:

Let U be the time reversal operator:

$$U\,\phi(x)U^{-1} = \widetilde{\phi}(x), \qquad \widetilde{\phi}(t,\vec{r}) = \phi(-t,\vec{r})$$

Then

$$\begin{aligned} & \langle 0|\widetilde{\phi}(x_1)\cdots\widetilde{\phi}(x_n)|0\rangle = \langle 0|U\phi(x_1)U^{-1}\cdots U\phi(x_n)U^{-1}|0\rangle \\ = \; & \langle 0|U\phi(x_1)\cdots\phi(x_n)U^{-1}|0\rangle = \langle U^{-1}0|\phi(x_1)\cdots\phi(x_n)U^{-1}|0\rangle^* \end{aligned}$$

Suppose $U^{-1}|0\rangle = e^{i\alpha}|0\rangle \Rightarrow \langle U^{-1}0| = e^{-i\alpha}\langle 0|$

This gives:

$$\langle 0|\widetilde{\phi}(x_1)\cdots\widetilde{\phi}(x_n)|0\rangle = \langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle^*$$

For $\Delta_+(x_1,x_2) = \langle 0|\phi(x_1)\phi(x_2)|0\rangle$, this gives

$$\Delta_+(-t_1,\vec{r}_1,-t_2,\vec{r}_2) = \Delta_+(t_1,\vec{r}_1,t_2,\vec{r}_2)^*$$

We shall now verify this using the explicit form of Δ_+, but first we shall try to deal with the possible divergences.

Divergences:

$$\Delta_+(x_1, x_2) = \frac{1}{(2\pi)^3} \int d^3p \, \frac{1}{2E_{\vec{p}}} \, e^{i\vec{p}.(\vec{r}_1 - \vec{r}_2)} \, e^{-i\,E_{\vec{p}}\,(t_1 - t_2)}$$

The integral over $\vec{p}$ is not absolutely convergent.

If we replace the integrand by its absolute value, we get

$$\frac{1}{(2\pi)^3} \int d^3p \, \frac{1}{2E_{\vec{p}}}$$

For large $|\vec{p}|$ it takes the form

$$\frac{1}{(2\pi)^3} \int_0^\infty 4\pi \, p^2 \, dp \, \frac{1}{2p} = (2\pi)^{-2} \int_0^\infty p \, dp$$

– diverges from large p region.

We try to define the integral via suitable analytic continuation.

Consider the integral

$$\frac{1}{(2\pi)^3} \int d^3p \, \frac{1}{2E_{\vec{p}}} \, e^{i\vec{p}.(\vec{r}_1 - \vec{r}_2)} \, e^{-i\,E_{\vec{p}}\,(t_1 - t_2 - i\,\epsilon)}$$

The ϵ dependent factor gives a damping $e^{-\epsilon E_{\vec{p}}}$ and makes the integral converge.

Take $\epsilon \to 0^+$ limit after evaluating the integral

We need to check whether we get a finite result in this limit.

$$\frac{1}{(2\pi)^3} \int d^3p \, \frac{1}{2E_{\vec{p}}} \, e^{i\vec{p}.(\vec{r}_1-\vec{r}_2)} \, e^{-i\,E_{\vec{p}}\,(t_1-t_2-i\,\epsilon)}$$

Let θ be the angle between $\vec{p}$ and $(\vec{r}_1 - \vec{r}_2)$ and define $p = |\vec{p}|$.

Then the integral may be written as

$$\frac{1}{(2\pi)^3} \int p^2 \, dp \, \sin\theta \, d\theta \, d\phi \, \frac{1}{2E_{\vec{p}}} \, e^{ip\,|\vec{r}_1-\vec{r}_2|\,\cos\theta - i\,E_{\vec{p}}\,(t_1-t_2-i\,\epsilon)}$$

In the large p region where divergences may appear, we can replace $E_{\vec{p}}$ by p.

$$\frac{1}{(2\pi)^3} \, 2\pi \int_0^\pi d\theta \, \sin\theta \int_0^\infty p^2 \, dp \, \frac{1}{2\,p} \, e^{ipa-\epsilon p}, \qquad a = |\vec{r}_1 - \vec{r}_2| \, \cos\theta - (t_1 - t_2)$$

Now we have

$$\int_0^\infty dp \, p \, e^{i\,a\,p-\epsilon p} = i^{-1} \frac{d}{da} \int_0^\infty dp \, e^{i\,a\,p-\epsilon p} = i^{-1} \frac{d}{da} \left(\frac{1}{ia-\epsilon} e^{i\,a\,p-\epsilon p} \Big|_{p=0}^{p=\infty} \right)$$

$$= \; i^{-1} \frac{d}{da} \frac{1}{\epsilon - ia} = -\frac{1}{(a+i\epsilon)^2}$$

Defining $u = \cos\theta$, we can write the original expression as

$$\frac{1}{(2\pi)^2} \int_{-1}^1 du \, \frac{1}{2} \left(-\frac{1}{(|\vec{r}_1 - \vec{r}_2| \, u - (t_1 - t_2) + i\epsilon)^2} \right)$$

$$= \; \frac{1}{8\pi^2 |\vec{r}_1 - \vec{r}_2|} \left[\frac{1}{|\vec{r}_1 - \vec{r}_2| - (t_1 - t_2) + i\epsilon} + \frac{1}{|\vec{r}_1 - \vec{r}_2| + (t_1 - t_2) - i\epsilon} \right]$$

– this is finite as $\epsilon \to 0$ except for $t_1 - t_2 = \pm |\vec{r}_1 - \vec{r}_2|$, i.e. for $(x_1 - x_2)^2 \neq 0$

Since Δ_F is given in terms of Δ_+, $\Delta_F(x_1, x_2)$ also diverges for $(x_1 - x_2)^2 = 0$, but is otherwise finite.

Time reversal invariance:

The properly defined expression for $\Delta_+(x_1, x_2)$ is:

$$\Delta_+(x_1, x_2) = \frac{1}{(2\pi)^3} \int d^3p \, \frac{1}{2E_{\vec{p}}} \, e^{i\vec{p}.(\vec{r}_1-\vec{r}_2)} \, e^{-i\,E_{\vec{p}}\,(t_1-t_2-i\,\epsilon)}$$

Manifest Lorentz invariant form

$$\Delta_+(x_1, x_2) = \frac{1}{(2\pi)^3} \int d^3p \, dp^0 \, \delta((p^0)^2 - \vec{p}^2 - m^2) \, H(p^0) \, e^{i\vec{p}.(\vec{r}_1-\vec{r}_2) - i\,p^0(t_1-t_2-i\epsilon)}$$

We now need to check if

$$\Delta_+(t_1, \vec{r}_1, t_2, \vec{r}_2) = \Delta_+(-t_1, \vec{r}_1, -t_2, \vec{r}_2)^*$$

We get

$$\begin{aligned} & \Delta_+(-t_1, \vec{r}_1, -t_2, \vec{r}_2)^* \\ = \ & \frac{1}{(2\pi)^3} \int d^3p \, dp^0 \, \delta((p^0)^2 - \vec{p}^2 - m^2) \, H(p^0) \, e^{-i\vec{p}.(\vec{r}_1-\vec{r}_2) + i\,p^0(-t_1+t_2+i\epsilon)} \end{aligned}$$

Note that we have replaced all the i's by $-i$ and replaced t_1, t_2 by $-t_1, -t_2$.

We can now redefine $\vec{p}$ as $-\vec{p}$, and do some minor algebra in the exponent to write this as:

$$\begin{aligned} & \Delta_+(-t_1, \vec{r}_1, -t_2, \vec{r}_2)^* \\ = \ & \frac{1}{(2\pi)^3} \int d^3p \, dp^0 \, \delta((p^0)^2 - \vec{p}^2 - m^2) \, H(p^0) \, e^{i\vec{p}.(\vec{r}_1-\vec{r}_2) - i\,p^0(t_1-t_2-i\epsilon)} \\ = \ & \Delta_+(t_1, \vec{r}_1, t_2, \vec{r}_2) \end{aligned}$$

This establishes the desired relation.

Next we shall compute:

$$G(x_1, x_2, \cdots, x_{2n}) = \langle 0|T(\phi(x_1)\phi(x_2)\cdots\phi(x_{2n}))|0\rangle$$

Recall that the result vanishes for odd number of ϕ's due to the $\phi \to -\phi$ symmetry.

It follows from the definition of time ordering that:

$$G(x_1, x_2, \cdots, x_{2n}) = \langle 0|\phi(x_{i_1})\phi(x_{i_2})\cdots\phi(x_{i_{2n}})|0\rangle$$

$(i_1, \cdots, i_{2n})$: A specific permutation of $1, 2, \cdots, 2n$ such that

$$x^0_{i_1} \geq x^0_{i_2} \geq x^0_{i_3} \geq \cdots \geq x^0_{i_{2n}}$$

To compute this we write each $\phi(x)$ in terms of $\widetilde{\phi}(p)$ and express this as sum of a and $a^\dagger$.

We then move the a's to the right, picking up commutators with the $a^\dagger$'s.

The results can be expressed in terms of Δ_+'s.

Consider first $G(x_1, x_2, x_3, x_4)$

If we pick the commutator between the a of $\phi(x_{i_1})$ and $a^\dagger$ of $\phi(x_{i_2})$, and the a of $\phi(x_{i_3})$ and $a^\dagger$ of $\phi(x_{i_4})$, we get a factor of $\Delta_+(x_{i_1}, x_{i_2})\Delta_+(x_{i_3}, x_{i_4})$.

Net result:

$$\Delta_+(x_{i_1}, x_{i_2})\Delta_+(x_{i_3}, x_{i_4}) + \Delta_+(x_{i_1}, x_{i_3})\Delta_+(x_{i_2}, x_{i_4}) + \Delta_+(x_{i_1}, x_{i_4})\Delta_+(x_{i_2}, x_{i_3})$$

Ex. Check this by doing this calculation carefully.

$$
\begin{aligned}
& \langle 0|T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))|0\rangle \\
= \; & \Delta_+(x_{i_1}, x_{i_2})\Delta_+(x_{i_3}, x_{i_4}) + \Delta_+(x_{i_1}, x_{i_3})\Delta_+(x_{i_2}, x_{i_4}) + \Delta_+(x_{i_1}, x_{i_4})\Delta_+(x_{i_2}, x_{i_3})
\end{aligned}
$$

In this form, in order to evaluate the r.h.s. we need to know the ordering of $x_1^0, x_2^0, x_3^0, x_4^0$.

However, using the fact:

$$
x_{i_1}^0 \geq x_{i_2}^0 \geq x_{i_3}^0 \geq \cdots \geq x_{i_{2n}}^0
$$

we can write the equation as

$$
\begin{aligned}
& \langle 0|T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)))|0\rangle \\
= \; & \Delta_F(x_{i_1}, x_{i_2})\,\Delta_F(x_{i_3}, x_{i_4}) + \Delta_F(x_{i_1}, x_{i_3})\,\Delta_F(x_{i_2}, x_{i_4}) + \Delta_F(x_{i_1}, x_{i_4})\,\Delta_F(x_{i_2}, x_{i_3}) \\
= \; & \Delta_F(x_1, x_2)\,\Delta_F(x_3, x_4) + \Delta_F(x_1, x_3)\,\Delta_F(x_2, x_4) + \Delta_F(x_1, x_4)\,\Delta_F(x_2, x_3)
\end{aligned}
$$

– valid for any ordering of $x_1^0, x_2^0, x_3^0, x_4^0$.

General result: Wick's theorem

$$
\begin{aligned}
& \langle 0|T(\phi(x_1)\phi(x_2)\cdots\phi(x_{2n}))|0\rangle \\
= \; & \Delta_F(x_1, x_2)\,\Delta_F(x_3, x_4)\cdots\Delta_F(x_{2n-1}, x_{2n}) + \text{all inequivalent pairings}
\end{aligned}
$$

Ex. Try to convince yourself of this result.

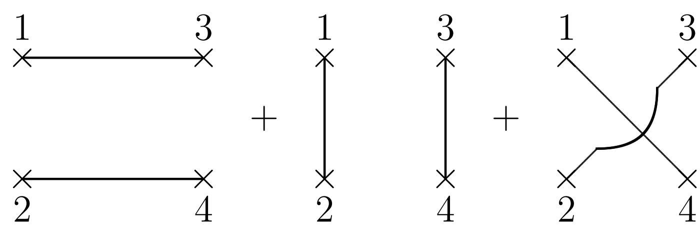
(can use method of induction)

Diagrammatic representation: Label every $\phi(x)$ by a $\times$ with a short line coming out.

Join the $2n$ such $\times$'s in pairs by lines.

A line connecting x and y represents $\Delta_F(x, y)$.

Final answer: Sum of the contribution from all the diagrams.



represents $G(x_1, x_2, x_3, x_4)$.

The lines, representing factors of $\Delta_F(x, y)$ are called propagators

Number of diagrams for $G(x_1, \cdots, x_{2n})$:

$$(2n-1)(2n-3)\cdots 1 = \frac{(2n)!}{(2n)(2n-2)\cdots 2} = \frac{(2n)!}{2^n n!}$$

Each diagram has n propagators.

These are called Feynman diagrams.

Composite operators:

$$\phi(x)^2, \quad \phi(x)^3, \quad \phi(x)\partial_\mu\phi(x)\,\partial_\nu\phi(x)$$

etc.

However, matrix elements involving these operators are divergent, *e.g.*,

$$\begin{aligned} & \langle 0|T(\phi(x_1)\phi(x_2)\phi(x)^2)|0\rangle \\ = \; & \Delta_F(x_1,x_2)\,\Delta_F(x,x) + \Delta_F(x_1,x)\,\Delta_F(x_2,x) + \Delta_F(x_1,x)\,\Delta_F(x_2,x) \end{aligned}$$

– diverges since $\Delta_F(x,x)=\infty$

Remedy: Define 'normal ordered operator'

$$:\phi(x)^2 := \lim_{y\to x}\{T(\phi(x)\phi(y)) - \Delta_F(x,y)\}$$

Then

$$\begin{aligned} & \langle 0|T(\phi(x_1)\phi(x_2)\; :\phi(x)^2:)|0\rangle \\ = \; & \lim_{y\to x}\langle 0|\{T(\phi(x_1)\phi(x_2)\phi(x)\phi(y)) - T(\phi(x_1)\phi(x_2))\Delta_F(x,y)\}|0\rangle \\ = \; & \lim_{y\to x}\{\Delta_F(x_1,x_2)\Delta_F(x,y) + \Delta_F(x_1,x)\Delta_F(x_2,y) + \Delta_F(x_1,y)\Delta_F(x_2,x) \\ & \qquad -\Delta_F(x_1,x_2)\Delta_F(x,y)\} \\ = \; & 2\,\Delta_F(x_1,x)\Delta_F(x_2,x)\,. \end{aligned}$$

Effectively in the definition of $:\phi(x)^2:$ we move all the a's to the right of all the $a^\dagger$'s.

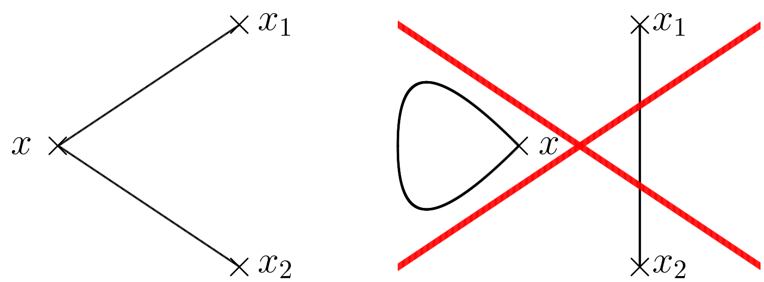
For more complicated operators, the terms we subtract get more complicated.

Feynman diagram: Represent $:\phi(x)^2:$ by a $\times$ with two short lines coming out.

In a given Green's function join pairs of short lines by propagators till no short line is left exposed.

Normal ordering: Never connect short lines coming out of the same $\times$ to each other by a propagator

e.g. the Feynman diagram for $\langle 0|T(\phi(x_1)\phi(x_2) :\phi(x)^2:)|0\rangle$:



Result: $2\,\Delta_F(x,x_1)\,\Delta_F(x,x_2)$

The factor of 2 comes from the fact that x_1 can be connected to any of the two lines coming out of x, and then x_2 is connected to the other line

– known as combinatoric factor

Once we use the rule of 'no connection between lines out of the same point', we do not need to keep track of all the terms that we need to subtract in the definition of the normal ordering.

Ex. Draw Feynman diagrams and compute

$$\langle 0|T(:\phi(x_1)^2::\phi(x_2)^2::\phi(x_3)^2:)|0\rangle, \qquad \langle 0|T(:\phi(x_1)^4::\phi(x_2)^4:)|0\rangle$$
$$\langle 0|T(:\phi(x_1)^2::\phi(x_2)^2:\phi(x_3)\,\phi(x_4))|0\rangle$$

18 Interacting scalar field theory

So far we have been dealing with Klein-Gordon theory

– the action is quadratic in ϕ

This class of theories are known as free theories

– no mutual interaction between the particles

Now we shall consider an action:

$$S = \int dt\, L = \int dt \int d^3r \left[\frac{1}{2}\,(\partial_t \phi)^2 - \frac{1}{2}\,(\vec{\nabla}\phi)^2 - \frac{1}{2}\,m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right]$$

λ is a constant,   4! is just a convention

Ex. The extra term is Lorentz invariant

We have

$$\Pi(t,\vec{r}) = \frac{\delta L}{\delta \dot{\phi}(t,\vec{r})} = \partial_t \phi(t,\vec{r})$$

$$H = \int d^3r \Pi(t,\vec{r})\dot{\phi}(t,\vec{r}) - L = \int d^3r \left[\frac{1}{2}\Pi(t,\vec{r})^2 + \frac{1}{2}(\vec{\nabla}\phi(t,\vec{r}))^2 + \frac{1}{2}m^2\,\phi(t,\vec{r})^2 + \frac{\lambda}{4!}\phi(t,\vec{r})^4 \right]$$

Observables:

$$\langle \Omega | \phi(x_1) \cdots \phi(x_n) | \Omega \rangle$$

$|\Omega\rangle$: The ground state of H which will be called the vacuum

Relation of these to physically measurable quantities depends on the problem where we apply this, *e.g.* we have different ways to use this in

1. Particle physics

2. Cosmology

3. Condensed matter physics

We shall focus on particle physics viewpoint, where we mostly need

$$G(x_1, \cdots, x_n) = \langle \Omega | T(\phi(x_1) \cdots \phi(x_n)) | \Omega \rangle$$

This is what we shall focus on.

$$H = \int d^3r \left[\frac{1}{2}\Pi(t,\vec{r})^2 + \frac{1}{2}(\vec{\nabla}\phi(t,\vec{r}))^2 + \frac{1}{2}m^2\,\phi(t,\vec{r})^2 + \frac{\lambda}{4!}\phi(t,\vec{r})^4 \right]$$

We cannot solve the theory exactly.

Instead we shall analyze it by assuming that λ is small and using perturbation theory.

Write H as

$$H_{free} + H_{int}$$

$$H_{free} = \int d^3r \left[\frac{1}{2}\Pi(t,\vec{r})^2 + \frac{1}{2}(\vec{\nabla}\phi(t,\vec{r}))^2 + \frac{1}{2}m^2\,\phi(t,\vec{r})^2 \right]$$

$$H_{int} = \frac{\lambda}{4!} \int d^3r \phi(t,\vec{r})^4$$

Note: H is conserved, but neither H_{free} nor H_{int} is separately conserved.

Choose some fixed time t_0 and define

$$H_0 = H_{free}(t_0)$$

Define interaction picture field operators:

$$\phi_I(t,\vec{r}) = e^{iH_0(t-t_0)}\,\phi(t_0,\vec{r})e^{-iH_0(t-t_0)}, \qquad \Pi_I(t,\vec{r}) = e^{iH_0(t-t_0)}\,\Pi(t_0,\vec{r})e^{-iH_0(t-t_0)}$$

$$\phi_I(t,\vec{r}) = e^{iH_0(t-t_0)}\,\phi(t_0,\vec{r})e^{-iH_0(t-t_0)}, \qquad \Pi_I(t,\vec{r}) = e^{iH_0(t-t_0)}\,\Pi(t_0,\vec{r})e^{-iH_0(t-t_0)},$$

Now,

$$H_0 = H_{free}(t_0) = \int d^3r \left[\frac{1}{2}\Pi(t_0,\vec{r})^2 + \frac{1}{2}(\vec{\nabla}\phi(t_0,\vec{r}))^2 + \frac{1}{2}m^2\,\phi(t_0,\vec{r})^2\right]$$

We can write

$$H_0 = e^{iH_0(t-t_0)}\,H_0\,e^{-iH_0(t-t_0)} = \int d^3r \left[\frac{1}{2}\Pi_I(t,\vec{r})^2 + \frac{1}{2}(\vec{\nabla}\phi_I(t,\vec{r}))^2 + \frac{1}{2}m^2\,\phi_I(t,\vec{r})^2\right]$$

Ex. Check that

$$[\phi_I(t,\vec{r}),\Pi_I(t,\vec{r}\,')] = i\,\delta^{(3)}(\vec{r}-\vec{r}\,'), \quad [\phi_I(t,\vec{r}),\phi_I(t,\vec{r}\,')] = 0, \quad [\Pi_I(t,\vec{r}),\Pi_I(t,\vec{r}\,')] = 0$$

$$\frac{\partial\phi_I(t,\vec{r})}{\partial t} = i[H_0,\phi_I(t,\vec{r})], \qquad \frac{\partial\Pi_I(t,\vec{r})}{\partial t} = i[H_0,\Pi_I(t,\vec{r})]$$

Therefore, if $|0\rangle$ is the ground state of H_0, then

$$\langle 0|T(\phi_I(x_1)\cdots\phi_I(x_{2n}))|0\rangle$$

are given by the free field results we have learned to compute

$$\Delta_F(x_1,x_2)\,\Delta_F(x_3,x_4)\cdots\Delta_F(x_{2n-1},x_{2n}) + \text{all inequivalent pairings}$$

Goal: Express $\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle$ in terms of $\langle 0|T(\phi_I(x_1)\cdots\phi_I(x_{2m}))|0\rangle$

– need to express $\phi(x)$ and $|\Omega\rangle$ in terms of $\phi_I(x)$ and $|0\rangle$.

$$\phi_I(t,\vec{r}) = e^{iH_0(t-t_0)}\,\phi(t_0,\vec{r})e^{-iH_0(t-t_0)}, \qquad \Pi_I(t,\vec{r}) = e^{iH_0(t-t_0)}\,\Pi(t_0,\vec{r})e^{-iH_0(t-t_0)},$$

$$\phi(t,\vec{r}) = e^{iH(t-t_0)}\,\phi(t_0,\vec{r})e^{-iH(t-t_0)}, \qquad \Pi(t,\vec{r}) = e^{iH(t-t_0)}\,\Pi(t_0,\vec{r})e^{-iH(t-t_0)},$$

– follows from Heisenberg equations of motion:

$$\frac{\partial\phi(t,\vec{r})}{\partial t} = i[H,\phi(t,\vec{r})], \qquad \frac{\partial\Pi(t,\vec{r})}{\partial t} = i[H,\Pi(t,\vec{r})]$$

Define:

$$U(t) = e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)} \qquad \Rightarrow \qquad U(t)^{-1} = e^{iH(t-t_0)}\,e^{-iH_0(t-t_0)}$$

Then

$$U(t)^{-1}\phi_I(t,\vec{r})U(t) = \phi(t,\vec{r}), \qquad U(t)^{-1}\Pi_I(t,\vec{r})U(t) = \Pi(t,\vec{r})$$

Define

$$U(t,t') \;=\; U(t)U(t')^{-1} = e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}\,e^{iH(t'-t_0)}\,e^{-iH_0(t'-t_0)}$$

Let us suppose

$$x^0_{i_1} \geq x^0_{i_2} \geq \cdots \geq x^0_{i_n}$$

Then

$$\begin{aligned}
&\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle = \langle\Omega|\phi(x_{i_1})\cdots\phi(x_{i_n})|\Omega\rangle \\
=\;& \langle\Omega|U(t_{i_1})^{-1}\phi_I(x_{i_1})U(t_{i_1})U(t_{i_2})^{-1}\phi_I(x_{i_2})U(t_{i_2})^{-1}\cdots U(t_{i_n})^{-1}\phi_I(x_{i_n})U(t_{i_n}|\Omega\rangle \\
=\;& \langle\Omega|U(t_{i_1})^{-1}\phi_I(x_{i_1})U(t_{i_1},t_{i_2})\phi_I(x_{i_2})U(t_{i_2},t_{i_3})\cdots U(t_{i_{n-1}},t_{i_n})\phi_I(x_{i_n})U(t_{i_n})|\Omega\rangle
\end{aligned}$$

We shall now try to relate $|\Omega\rangle$ and $|0\rangle$

Goal: Relate $|\Omega\rangle$ to $|0\rangle$

Let $|n\rangle'$ denote the basis of eigenstates of H with eigenvalue E_n.

Then

$$|0\rangle = \sum_n |n\rangle' \, '\langle n|0\rangle$$

$$e^{-iH\,(\mathcal{T}(1-i\epsilon)+t_0)}|0\rangle = \sum_n e^{-iH\,(\mathcal{T}(1-i\epsilon)+t_0)}|n\rangle' \, '\langle n|0\rangle = \sum_n e^{-iE_n\,(\mathcal{T}(1-i\epsilon)+t_0)}|n\rangle' \, '\langle n|0\rangle$$

$$e^{-iE_n\,(\mathcal{T}(1-i\epsilon)+t_0)} = e^{-iE_n\,(\mathcal{T}+t_0)-\epsilon\,E_n\,\mathcal{T}}$$

In the $\mathcal{T}\to\infty$ limit the contribution from the ground state $|\Omega\rangle$ dominates all other contributions.

Therefore, as $\mathcal{T}\to\infty$,

$$e^{-iH\,(\mathcal{T}(1-i\epsilon)+t_0)}|0\rangle = e^{-iE_\Omega\,(\mathcal{T}+t_0)-\epsilon\,E_\Omega\,\mathcal{T}}\;|\Omega\rangle\langle\Omega|0\rangle$$

Recall

$$U(t)^{-1} = e^{iH(t-t_0)}\,e^{-iH_0(t-t_0)}$$

$$\begin{aligned} U(-\mathcal{T}(1-i\epsilon))^{-1}|0\rangle &= e^{-iH\,(\mathcal{T}(1-i\epsilon)+t_0)}e^{iH_0(\mathcal{T}(1-i\epsilon)+t_0)}|0\rangle = e^{-iH\,(\mathcal{T}(1-i\epsilon)+t_0)}|0\rangle \\ &= e^{-iE_\Omega\,(\mathcal{T}+t_0)-\epsilon\,E_\Omega\,\mathcal{T}}\;|\Omega\rangle\langle\Omega|0\rangle = N_1\,|\Omega\rangle \end{aligned}$$

$$N_1 = e^{-iE_\Omega\,(\mathcal{T}+t_0)-\epsilon\,E_\Omega\,\mathcal{T}}\;\langle\Omega|0\rangle$$

Ex. Similarly, using

$$\langle 0| = \sum_n \langle 0|n\rangle' \, '\langle n|$$

we get

$$\langle 0|U(\mathcal{T}(1-i\epsilon)) = N_2\,\langle\Omega|, \qquad N_2 = e^{-iE_\Omega\,(\mathcal{T}-t_0)-\epsilon\,E_\Omega\,\mathcal{T}}\;\langle 0|\Omega\rangle$$

$$
\begin{aligned}
&\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle \\
= \ &\langle\Omega|U(t_{i_1})^{-1}\phi_I(x_{i_1})U(t_{i_1},t_{i_2})\phi_I(x_{i_2})U(t_{i_2},t_{i_3})\cdots U(t_{i_{n-1}},t_{i_n})\phi_I(x_{i_n})U(t_{i_n})|\Omega\rangle
\end{aligned}
$$

$$
U(-\mathcal{T}(1-i\epsilon))^{-1}|0\rangle = N_1\,|\Omega\rangle, \qquad \langle 0|U(\mathcal{T}(1-i\epsilon)) = N_2\,\langle\Omega|
$$

This gives

$$
\begin{aligned}
&\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle \\
= \ &(N_1N_2)^{-1}\langle 0|U(\mathcal{T}(1-i\epsilon))U(t_{i_1})^{-1}\phi_I(x_{i_1})U(t_{i_1},t_{i_2})\phi_I(x_{i_2})U(t_{i_2},t_{i_3})\cdots \\
&\qquad\cdots\ U(t_{i_{n-1}},t_{i_n})\phi_I(x_{i_n})U(t_{i_n})U(-\mathcal{T}(1-i\epsilon))^{-1}|0\rangle \\
= \ &(N_1N_2)^{-1}\langle 0|U(\mathcal{T}(1-i\epsilon),t_{i_1})\phi_I(x_{i_1})U(t_{i_1},t_{i_2})\phi_I(x_{i_2})U(t_{i_2},t_{i_3})\cdots \\
&\qquad\cdots U(t_{i_{n-1}},t_{i_n})\phi_I(x_{i_n})U(t_{i_n},-\mathcal{T}(1-i\epsilon))|0\rangle
\end{aligned}
$$

Remember that we need to first take $\mathcal{T}\to\infty$ limit and then $\epsilon\to 0^+$ limit.

Our next task will be to manipulate $U(t,t')$ to express this in terms of ϕ_I.

19 Perturbation theory for interacting scalar field

We were considering interacting scalar field theory with Hamiltonian:

$$H = H_{free} + H_{int}$$

$$H_{free} = \int d^3r \left[\frac{1}{2}\Pi(t,\vec{r})^2 + \frac{1}{2}(\vec{\nabla}\phi(t,\vec{r}))^2 + \frac{1}{2}m^2\,\phi(t,\vec{r})^2 \right]$$

$$H_{int} = \frac{\lambda}{4!}\int d^3r\,\phi(t,\vec{r})^4$$

We defined:

$$H_0 = H_{free}(t_0)$$

$$\phi_I(t,\vec{r}) = e^{iH_0(t-t_0)}\,\phi(t_0,\vec{r})\,e^{-iH_0(t-t_0)}, \qquad \Pi_I(t,\vec{r}) = e^{iH_0(t-t_0)}\,\Pi(t_0,\vec{r})\,e^{-iH_0(t-t_0)},$$

$|0\rangle$: Ground state of H_0

$|\Omega\rangle$ ground state of H

We showed that the quantities

$$\langle 0|T(\phi_I(x_1)\cdots\phi_I(x_{2n}))|0\rangle$$

are given by the free field results we have learned to compute

Our goal: Express $\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle$ in terms of $\langle 0|T(\phi_I(x_1)\cdots\phi_I(x_{2m}))|0\rangle$

We defined

$$U(t) = e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}, \qquad U(t,t') = U(t)U(t')^{-1}$$

and found that:

$$\begin{aligned} &\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle \\ = \;& (N_1N_2)^{-1}\langle 0|U(\mathcal{T}(1-i\epsilon), t_{i_1})\,\phi_I(x_{i_1})\,U(t_{i_1}, t_{i_2})\,\phi_I(x_{i_2})\,U(t_{i_2}, t_{i_3})\cdots \\ & \qquad\qquad \cdots U(t_{i_{n-1}}, t_{i_n})\,\phi_I(x_{i_n})\,U(t_{i_n}, -\mathcal{T}(1-i\epsilon))|0\rangle \\ & \qquad\qquad\qquad\qquad x^0_{i_1} \ge x^0_{i_2} \ge \cdots \ge x^0_{i_n} \end{aligned}$$

N_1, N_2: constants that will be determined later.

We need to take $\mathcal{T}\to\infty$ limit before taking $\epsilon\to 0^+$ limit.

If we can express $U(t,t')$ in terms of the ϕ_I's then we would have achieved our goal.

Strategy: Instead of calculating $U(t,t')$ directly, we shall regard this as a solution to a differential equation and then solve the differential equation.

Recall:
$$U(t,t') = U(t)U(t')^{-1}, \qquad U(t) = e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}$$
From this we get,
$$\frac{dU(t)}{dt} = e^{iH_0(t-t_0)}\,(i\,H_0 - i\,H)\,e^{-iH(t-t_0)} = -i\,e^{iH_0(t-t_0)}\,H_{int}(t_0)e^{-iH(t-t_0)}$$
since
$$H(t_0) = H_{free}(t_0) + H_{int}(t_0) = H_0 + H_{int}(t_0)$$
This gives:
$$\begin{aligned}
\frac{dU(t)}{dt} &= -i\,e^{iH_0(t-t_0)}\,H_{int}(\phi(t_0,\vec{r}),\Pi(t_0,\vec{r}))e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)} \\
&= -i\,H_{int}\left(e^{iH_0(t-t_0)}\,\phi(t_0,\vec{r})\,e^{-iH_0(t-t_0)}, e^{iH_0(t-t_0)}\,\Pi(t_0,\vec{r})\,e^{-iH_0(t-t_0)}\right)\,U(t) \\
&= -i\,H_{int}(\phi_I(t,\vec{r}),\Pi_I(t,\vec{r}))\,U(t)
\end{aligned}$$
We shall use the shorthand notation
$$H_I(t) = H_{int}[\phi_I(t,\vec{r}),\Pi_I(t,\vec{r})]$$
Then we have
$$\frac{dU(t)}{dt} = -i\,H_I(t)U(t)$$
Since $U(t,t') = U(t)U(t')^{-1}$, we get
$$\frac{\partial U(t,t')}{\partial t} = -i\,H_I(t)U(t)U(t')^{-1} = -i\,H_I(t)\,U(t,t'), \qquad U(t,t') = I \quad \text{at } t = t'$$

This first order differential equation and the boundary condition determines $U(t,t')$ uniquely.

$U(t,t')$ is determined from the equation:

$$\frac{\partial U(t,t')}{\partial t} = -i\, H_I(t)\, U(t,t'), \qquad U(t,t') = I \quad \text{at } t = t'$$

Claim;

$$U(t,t') = T\left(\exp\left[-i\int_{t'}^{t} H_I(t'')dt''\right]\right), \qquad H_I(t) = H_{int}(\phi_I(t,\vec{r}), \Pi_I(t,\vec{r}))$$

Meaning of rhs:

We can expand the exponential and write:

$$\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \cdots \int_{t'}^{t} dt_m\, T\left(H_I(t_1)H_I(t_2)\cdots H_I(t_m)\right)$$

Call this $V(t,t')$.

Out goal is to prove $U(t,t') = V(t,t')$.

For this we need to show that $V(t,t')$ satisfies the same differential equation and boundary condition as $U(t,t')$.

At $t = t'$, only the $m = 0$ term contribute to $V(t,t')$.

This gives $V(t',t') = I$ $\surd$

$$\begin{aligned}\frac{\partial V(t,t')}{\partial t} &= \sum_{m=1}^{\infty} \frac{(-i)^m}{m!}\, m \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \cdots \int_{t'}^{t} dt_{m-1}\, T\left(H_I(t_1)H_I(t_2)\cdots H_I(t_{m-1})H_I(t)\right) \\ &= -i\, H_I(t) \sum_{m=1}^{\infty} \frac{(-i)^{m-1}}{(m-1)!} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \cdots \int_{t'}^{t} dt_{m-1}\, T\left(H_I(t_1)H_I(t_2)\cdots H_I(t_{m-1})\right) \\ &= -i\, H_I(t)\, V(t,t') \qquad \surd\end{aligned}$$

Therefore $U(t,t') = V(t,t')$

Results so far:

$$\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle$$

$$= (N_1N_2)^{-1}\langle 0|U(\mathcal{T}(1-i\epsilon),t_{i_1})\phi_I(x_{i_1})U(t_{i_1},t_{i_2})\phi_I(x_{i_2})U(t_{i_2},t_{i_3})\cdots$$

$$\cdots U(t_{i_{n-1}},t_{i_n})\phi_I(x_{i_n})U(t_{i_n},-\mathcal{T}(1-i\epsilon))|0\rangle$$

$$U(t,t') = T\left(\exp\left[-i\int_{t'}^{t} H_I(t'')dt''\right]\right), \qquad H_I(t) = H_{int}(\phi_I(t,\vec{r}),\Pi_I(t,\vec{r}))$$

This gives

$$\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle$$

$$= (N_1N_2)^{-1}\left\langle 0\left|T\left(\exp\left[-i\int_{t_{i_1}}^{\mathcal{T}(1-i\epsilon)} H_I(t'')dt''\right]\right)\phi_I(x_{i_1})T\left(\exp\left[-i\int_{t_{i_2}}^{t_{i_1}} H_I(t'')dt''\right]\right)\phi_I(x_{i_2})\right.\right.$$
$$\left.\left.\cdots T\left(\exp\left[-i\int_{t_{i_{n-1}}}^{t_{i_n}} H_I(t'')dt''\right]\right)\phi_I(x_{i_n})T\left(\exp\left[-i\int_{-\mathcal{T}(1-i\epsilon)}^{t_{i_n}} H_I(t'')dt''\right]\right)\right|0\right\rangle$$

Note that all the terms are time ordered.

Therefore we can replace multiple time ordering by a single time ordering and rearrange the terms inside the time ordering arbitrarily.

$$\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle$$

$$= (N_1N_2)^{-1}\left\langle 0\left|T\left(\exp\left[-i\int_{t_{i_1}}^{\mathcal{T}(1-i\epsilon)} H_I(t'')dt''\right]\ \phi_I(x_{i_1})\ \exp\left[-i\int_{t_{i_2}}^{t_{i_1}} H_I(t'')dt''\right]\phi_I(x_{i_2})\right.\right.\right.$$
$$\left.\left.\left.\cdots\exp\left[-i\int_{t_{i_{n-1}}}^{t_{i_n}} H_I(t'')dt''\right]\phi_I(x_{i_n})\exp\left[-i\int_{-\mathcal{T}(1-i\epsilon)}^{t_{i_n}} H_I(t'')dt''\right]\right)\right|0\right\rangle$$
$$= (N_1N_2)^{-1}\left\langle 0\left|T\left(\phi_I(x_1)\cdots\phi_I(x_n)\ \exp\left[-i\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)} H_I(t'')dt''\right]\right)\right|0\right\rangle$$

$$
\begin{aligned}
&\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle \\
&= (N_1N_2)^{-1}\left\langle 0\left|T\left(\phi_I(x_1)\cdots\phi_I(x_n)\ \exp\left[-i\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)} H_I(t'')dt''\right]\right)\right|0\right\rangle \\
&= (N_1N_2)^{-1}\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)} dt'_1\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)} dt'_2\cdots\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)} dt'_m\,\langle 0|T(\phi_I(x_1)\cdots\phi_I(x_n) \\
&\qquad\qquad H_I(t'_1)\cdots H_I(t'_m))|0\rangle
\end{aligned}
$$

To simplify this we can also replace

$$
t_i \to t_i(1-i\epsilon), \qquad 1\le i\le n
$$

since for finite t_i this will have no effect when we take $\epsilon\to 0$ at the end.

This allows us to replace all the time arguments by $t\to u(1-i\epsilon)$ with u real.

For the current problem:

$$
\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)} H_I(t')dt' = \frac{\lambda}{4!}\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)} dt'\int d^3r'\,\phi_I(t',\vec{r}\,')^4
$$

We can evaluate individual terms in the sum using the result:

$$
\begin{aligned}
&\langle 0|T(\phi_I(x_1)\cdots\phi_I(x_{2n}))|0\rangle \\
&= \Delta_F(x_1,x_2)\,\Delta_F(x_3,x_4)\cdots\Delta_F(x_{2n-1},x_{2n}) + \text{all inequivalent pairings}
\end{aligned}
$$

– given by sum over Feynman diagrams.

Note: Due to presence of composite operator $\phi_I(x')^4$, there will be divergent terms like $\Delta_F(x',x')$

– can be normal ordered, but we shall proceed by ignoring this problem for now.

We now need to calculate $N_1 N_2$.

$$\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle$$

$$= (N_1N_2)^{-1}\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)}dt'_1\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)}dt'_2\cdots\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)}dt'_m\,\langle 0|T(\phi_I(x_1)\cdots\phi_I(x_n)$$
$$H_I(t'_1)\cdots H_I(t'_m))|0\rangle$$

Set $n = 0$

Then lhs $= \langle\Omega|\Omega\rangle = 1$

This gives

$$N_1N_2 = \sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)}dt'_1\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)}dt'_2\cdots\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)}dt'_m\,\langle 0|T(H_I(t'_1)\cdots H_I(t'_m))|0\rangle$$

– can also be evaluated with Feynman diagrams

$$\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle = \frac{\text{Numerator}}{\text{Denominator}}$$

Numerator: Given by sum of Feynman diagrams whose vertices include:

1. n 'external points', each of which has one line coming out

2. m 'internal points' each of which has 4 lines coming out (4-point vertex)

Draw all Feynman diagrams

Associated expression is given by

$$\frac{1}{m!}\left(-\frac{i\lambda}{4!}\right)^m \times \text{propagators} \times \text{combinatoric factors}$$

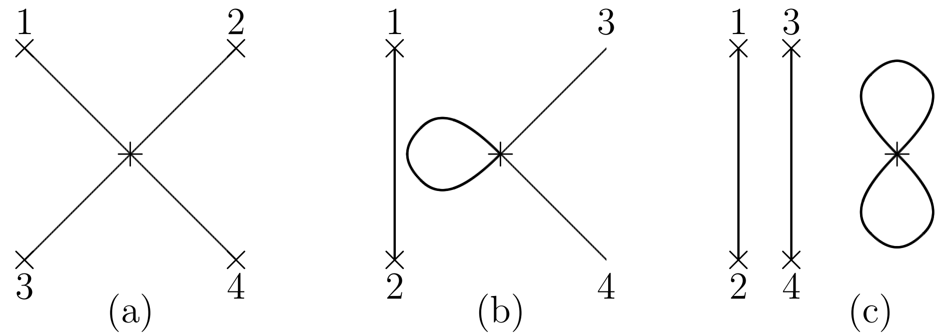
We need to integrate over the space-time coordinates of the internal points.

Denominator: Same rule except that there are no external points

Note: At order λ^m, there are m internal points

$\Rightarrow$ finite number of Feynman diagrams.

Example: $n = 4$, $m = 1$



Contribution from (a):

$$-i\frac{\lambda}{4!}\int d^4x\,\Delta_F(x,x_1)\,\Delta_F(x,x_2)\,\Delta_F(x,x_3)\,\Delta_F(x,x_4)\times 4\times 3\times 2$$

Contribution from (b):

$$-i\frac{\lambda}{4!}\int d^4x\,\Delta_F(x_1,x_2)\,\Delta_F(x,x_3)\,\Delta_F(x,x_4)\,\Delta_F(x,x)\ \times 4\times 3$$

Total $\binom{4}{2} = 6$ diagrams obtained by inequivalent permutations of 1,2,3,4

Contribution from (c):

$$-i\frac{\lambda}{4!}\int d^4x\,\Delta_F(x_1,x_2)\,\Delta_F(x_3,x_4)\,\Delta_F(x,x)\,\Delta_F(x,x)\ \times 3$$

Total 3 diagrams obtained by inequivalent permutations of 1,2,3,4

20 Feynman diagrams of scalar field theory

$$\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle = \frac{\text{Numerator}}{\text{Denominator}}$$

Numerator: Given by Feynman diagrams whose vertices include:

1. n 'external points', each of which has one line coming out

2. m 'internal points' each of which has 4 lines coming out (4-point vertex)

Draw all Feynman diagrams

Associated expression is given by

$$\frac{1}{m!}\left(-\frac{i\lambda}{4!}\right)^m \times \text{propagators} \times \text{combinatoric factors}$$

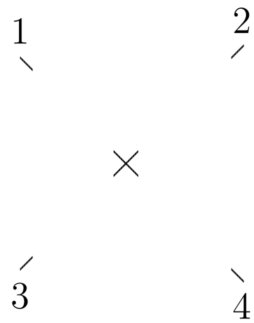
We need to integrate over the space-time coordinates of the internal points.

Denominator: Same rule except that there are no external points

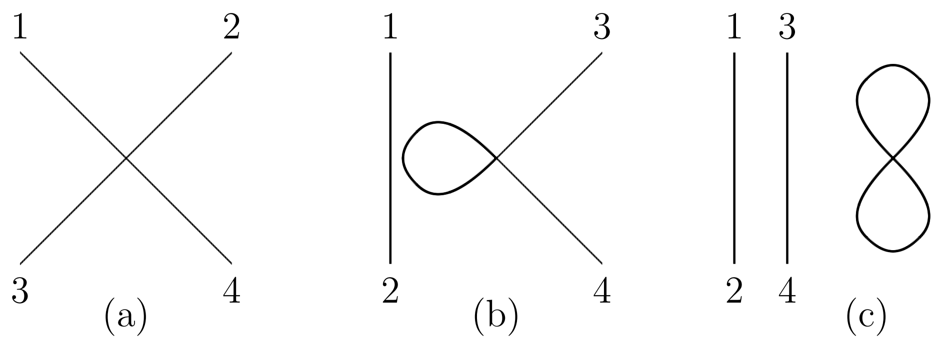
Note: At order λ^m, there are m internal points

$\Rightarrow$ finite number of Feynman diagrams.

Example: $n = 4$, $m = 1$



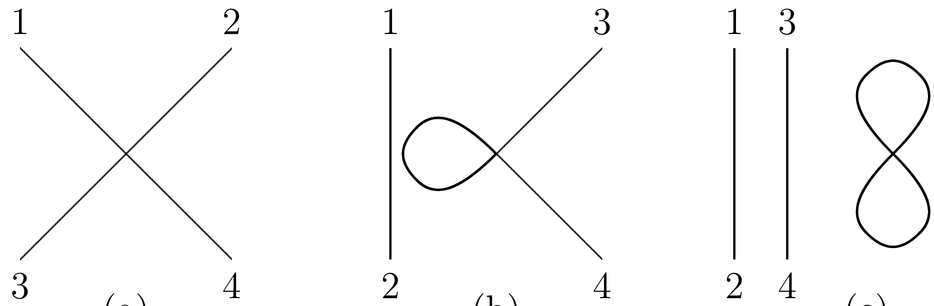
Connect the lines pairwise in all possible ways



Total $\binom{4}{2} = 6$ diagrams of type (b) obtained by inequivalent permutations of 1,2,3,4

Total 3 diagrams of type (c) obtained by inequivalent permutations of 1,2,3,4

Example: $n = 4$, $m = 1$



(a) (b) (c)

Contribution from (a):

$$-i\frac{\lambda}{4!}\int d^4x\,\Delta_F(x,x_1)\,\Delta_F(x,x_2)\,\Delta_F(x,x_3)\,\Delta_F(x,x_4)\times 4\times 3\times 2$$

Contribution from (b):

$$-i\frac{\lambda}{4!}\int d^4x\,\Delta_F(x_1,x_2)\,\Delta_F(x,x_3)\,\Delta_F(x,x_4)\,\Delta_F(x,x)\;\times 4\times 3$$

Total $\binom{4}{2} = 6$ diagrams obtained by inequivalent permutations of 1,2,3,4

Contribution from (c):

$$-i\frac{\lambda}{4!}\int d^4x\,\Delta_F(x_1,x_2)\,\Delta_F(x_3,x_4)\,\Delta_F(x,x)\,\Delta_F(x,x)\;\times 3$$

Total 3 diagrams obtained by inequivalent permutations of 1,2,3,4

Consistency check: Total combinatoric factor

$$24 + 12\times 6 + 3\times 3 = 105$$

Compare with $(2k)!/(2^k k!)$ with $2k = 8$.

$$\frac{8\times 7\times 6\times 5\times 4\times 3\times 2}{16\times 4\times 3\times 2} = 7\times 3\times 5 = 105$$

General consistency check:

Consider the collection of Feynman diagrams with n external points and m internal points.

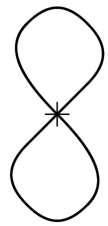
Sum of the combinatoric factors of all the Feynman diagrams will be given by:

$$\frac{(2k)!}{2^k k!}, \qquad 2k = n + 4\,m$$

Similar analysis can be carried out for the denominator $(N_1 N_2)$

The leading term is 1.

The order λ term is given by a single Feynman diagram



Its expression is

$$-i\frac{\lambda}{4!}\int d^4x\ \Delta_F(x,x)\,\Delta_F(x,x)\ \times 3$$

We shall call diagrams without external lines as bubble diagrams.

Denominator = 1 + sum over bubbles

Recall

$$N_1 N_2 = \sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)} dt_1' \int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)} dt_2' \cdots \int_{-\mathcal{T}(1-i\epsilon)}^{\mathcal{T}(1-i\epsilon)} dt_m' \, \langle 0|T(H_I(t_1')\cdots H_I(t_m'))|0\rangle$$

Claim:

Numerator = sum of diagrams without bubbles × (1 + sum over bubbles)

In that case,

$$\frac{\text{numerator}}{\text{denominator}} = \text{sum of diagrams without bubbles}$$

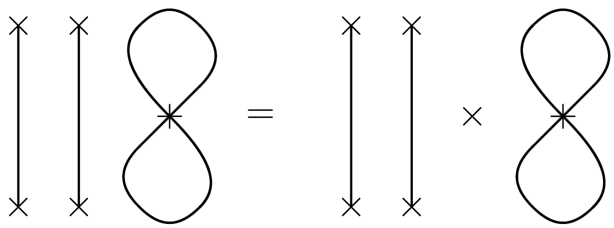
We shall now prove the claim.

Every diagram in the numerator is

either diagram without a bubble

or a diagram without bubble × a bubble diagram

e.g.



Therefore at the level of diagrams we have

numerator = sum of diagram without bubble × (1+ sum over bubbles)

We need to check various factors.

Suppose the diagram without bubble has m insertions of four point vertex.

Gives a factor of $(-i\lambda/4!)^m/m!$

Suppose the bubble diagram has k insertions of four point vertex.

Gives a factor of $(-i\lambda/4!)^k/k!$

Corresponding numerator diagram has $m+k$ insertion of four point vertex

Net factor: $(-i\lambda/4!)^{m+k}/(m+k)!$ = product on rhs $\times\, m!k!/(m+k)!$

We shall now show that the extra factor is cancelled by the combinatoric factor.

The numerator diagram has $m + k$ four point vertex

Of these we have to choose k that becomes part of the bubble

The rest becomes part of the diagram without bubble

This can be done in $\binom{m+k}{k} = (m+k)!/(m!k!)$ ways.

This cancels the extra $m!k!/(m+k)!$ factor that we had earlier.

The rest of the combinatoric factors are the same on the rhs and the lhs

– measures the number of ways a diagram without bubble can be formed $\times$ the number of ways a bubble diagram can be formed

numerator = diagram without bubble $\times$ (1+ sum over bubbles)

denominator = 1+ sum over bubbles

This gives

$$\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle = \text{sum of Feynman diagrams without bubbles}$$

21 Momentum space Feynman diagrams

Some comments on the combinatoric factor

We have seen that in a Green's function with a total of $2k$ ϕ's, the sum of the combinatoric factors of all the diagrams is given by:

$$\frac{(2k)!}{2^k k!}$$

– gives a way to check if we have got the combinatoric factors right and also if we have included all diagrams.

But we also saw that we need to exclude the bubble diagrams.

Therefore we need to subtract from $(2k)!/(2^k k!)$ the contribution from the bubble diagrams.

This can be done by noting that if we have a diagram that is the product of:

1. Diagram without bubbles with $2k_1$ ϕ's

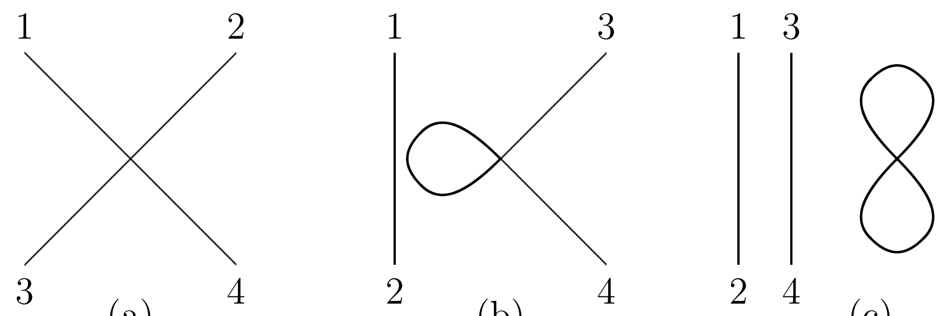
2. Bubble diagram with $2k_2$ ϕ's

then the total number of terms that it gives is:

$$\frac{(2k_1)!}{2^{k_1} k_1!} \times \frac{(2k_2)!}{2^{k_2} k_2!}$$

At some given order in perturbation expansion, we have to sum over possible choices of k_1, k_2 and then subtract this number from $(2k)!/(2^k k!)$ to get the number of terms without bubbles.

Example:



(a) (b) (c)

The last diagram is not be included.

For this we have $2k_1 = 4$, $2k_2 = 4$

The number of terms in such diagrams:

$$\frac{4!}{2^2\,2!} \times \frac{4!}{2^2\,2!} = 9$$

For the full diagram, we have $2k = 8$.

Therefore the total number of terms in diagrams without bubble is:

$$\frac{8!}{2^4 4!} - 9 = 105 - 9 = 96$$

A different perspective on the $i\epsilon$ prescription

We had a relation of the form:

$$e^{-iH\,(\mathcal{T}+t_0)}|0\rangle = \sum_n e^{-iH\,(\mathcal{T}+t_0)}|n\rangle'\,{}'\langle n|0\rangle = \sum_n e^{-iE_n\,(\mathcal{T}-t_0)}|n\rangle'\,{}'\langle n|0\rangle$$

$|n\rangle'$: eigenstates of H with eigenvalue E_n.

We wanted to make sure that only the ground state $|\Omega\rangle$ of H contributes in the $\mathcal{T} \to \infty$ limit

– achieved by replacing $\mathcal{T}$ by $\mathcal{T}(1 - i\epsilon)$

However the same goal will be achieved if we can make $E_n - E_\Omega$ acquire a small negative imaginary part i.e. if

$$E_n - E_\Omega = a_n - i\,\epsilon\, b_n, \qquad a_n, b_n > 0$$

In that case

$$e^{-iE_n\,(\mathcal{T}-t_0)} = e^{-iE_\Omega\,(\mathcal{T}-t_0)}\; e^{-ia_n(\mathcal{T}-t_0)-\epsilon\, b_n\,(\mathcal{T}-t_0)}$$

The second factor goes to 0 as $\mathcal{T} \to \infty$, showing that only the state $|\Omega\rangle$ contributes to the sum over n

– achieves the goal.

Similar result holds for

$$\langle 0|e^{-iH\,(\mathcal{T}+t_0)} = \sum_n \langle 0|n\rangle'\,{}'\langle n|\, e^{-iH\,(\mathcal{T}+t_0)}$$

Q. How can we add $-i\epsilon b_n$ to $E_n - E_\Omega$?

Q. How can we add $-i\epsilon b_n$ to $E_n - E_\Omega$?

Earlier approach $\mathcal{T} \to \mathcal{T}(1 - i\epsilon)$ is equivalent to $H \to H(1 - i\epsilon)$

This gives $E_n - E_\Omega \to (1 - i\epsilon)(E_n - E_\Omega)$

– generates a negative imaginary term in $E_n - E_\Omega$.

An equivalent procedure: $m^2 \to m^2 - i\epsilon$.

Consider the unperturbed theory first.

The difference between the energies of single particle state and vacuum:

$$\sqrt{\vec{p}^2 + m^2} \to \sqrt{\vec{p}^2 + m^2 - i\epsilon} \simeq \sqrt{\vec{p}^2 + m^2} - \frac{i\epsilon}{2\sqrt{\vec{p}^2 + m^2}}$$

– adds a negative imaginary contribution to the energy as desired.

For multi-particle states these negative imaginary contributions add, producing a net negative imaginary contribution.

When we include the effect of interactions, there may be additional contributions proportional to $i\epsilon\lambda$ and terms containing more powers of λ.

Since λ is taken to be a small parameter in which we do expansion, these higher order terms cannot change the sign of the leading λ independent term.

Conclusion: Replacing m^2 by $m^2 - i\epsilon$ achieves the same goal as $\mathcal{T} \to \mathcal{T}(1 - i\epsilon)$ or $H \to H(1 - i\epsilon)$.

From now on we shall follow this procedure and set all time arguments real.

Recall that replacing m^2 by $m^2 - i\epsilon$ has already been encountered before:

$$\Delta_F(x_1, x_2) = \frac{1}{(2\pi)^4} \int d^4p \, \widetilde{\Delta}_F(p) \, e^{ip.(x_1 - x_2)}, \qquad p = (p^0, \vec{p}),$$

$$\widetilde{\Delta}_F(p) = \frac{i}{-p^2 - m^2 + i\epsilon}$$

We shall define:

$$\widetilde{\Delta}_F(q_1, q_2) = (2\pi)^4 \, \delta^{(4)}(q_1 + q_2) \, \widetilde{\Delta}_F(q_1) = \frac{i}{-q_1^2 - m^2 + i\epsilon} \, (2\pi)^4 \, \delta^{(4)}(q_1 + q_2)$$

and write

$$\Delta_F(x_1, x_2) = \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \, \widetilde{\Delta}_F(q_1, q_2) \, e^{i(q_1.x_1 + q_2.x_2)}$$

Using this representation we shall now derive the Feynman rules for momentum space Green's function.

Given

$$G(x_1, \cdots, x_n) = \langle \Omega | T(\phi(x_1) \cdots \phi(x_n)) | \Omega \rangle$$

we define

$$\widetilde{G}(p_1, \cdots, p_n) = \int d^4x_1 \cdots d^4x_n \, e^{-i(p_1.x_1 + p_2.x_2 + \cdots + p_n.x_n)} \, G(x_1, \cdots, x_n)$$

Reverse relation:

$$G(x_1, \cdots, x_n) = \int \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_n}{(2\pi)^4} \, e^{i(p_1.x_1 + p_2.x_2 + \cdots + p_n.x_n)} \, \widetilde{G}(p_1, \cdots, p_n)$$

Since we have derived the Feynman rules for computing $G(x_1, \cdots, x_n)$, we can use this to compute $\widetilde{G}(p_1, \cdots, p_n)$.

We shall now try to manipulate these Feynman rules to find the Feynman rules for computing $\widetilde{G}(p_1, \cdots, p_n)$ directly, without having to first compute $G(x_1, \cdots, x_n)$.

$$\widetilde{G}(p_1, \cdots, p_n) = \int d^4x_1 \cdots d^4x_n \, e^{-i(p_1.x_1 + p_2.x_2 + \cdots + p_n.x_n)} \, G(x_1, \cdots, x_n)$$

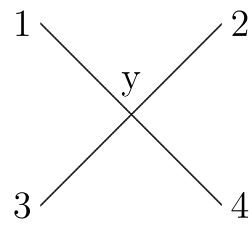
Strategy:

1. Write down the contribution to $G(x_1, \cdots, x_n)$ as sum of Feynman diagrams.

2. Use
$$\Delta_F(x_1, x_2) = \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \widetilde{\Delta}_F(q_1, q_2) \, e^{i(q_1.x_1 + q_2.x_2)}$$

3. Carry out the integrals over the external points x_i and the internal points y_i explicitly.

We shall illustrate this first by an example and then describe the general rules.



Contribution to $\widetilde{G}(p_1, p_2, p_3, p_4)$:

$$\int d^4x_1 \cdots d^4x_4 \, e^{-i(p_1.x_1+p_2.x_2+\cdots+p_4.x_4)} \int d^4y \, \Delta_F(x_1,y)\, \Delta_F(x_2,y)\, \Delta_F(x_3,y)\, \Delta_F(x_4,y) \times \left(-\frac{i\lambda}{4!}\right) \times \text{combinatoric factors}$$

$$= \int d^4x_1 \cdots d^4x_4 \int d^4y \, e^{-i(p_1.x_1+p_2.x_2+\cdots+p_4.x_4)} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \widetilde{\Delta}_F(q_1,q_2)\, e^{i(q_1.x_1+q_2.y)}$$
$$\int \frac{d^4q_3}{(2\pi)^4} \frac{d^4q_4}{(2\pi)^4} \widetilde{\Delta}_F(q_3,q_4)\, e^{i(q_3.x_2+q_4.y)} \int \frac{d^4q_5}{(2\pi)^4} \frac{d^4q_6}{(2\pi)^4} \widetilde{\Delta}_F(q_5,q_6)\, e^{i(q_5.x_3+q_6.y)}$$
$$\int \frac{d^4q_7}{(2\pi)^4} \frac{d^4q_8}{(2\pi)^4} \widetilde{\Delta}_F(q_7,q_8)\, e^{i(q_7.x_4+q_8.y)} \times \left(-\frac{i\lambda}{4!}\right) \times \text{combinatoric factors}$$

Now note that the integration over each x_i and y give simple delta functions, e.g.

$$\int d^4x_1 e^{-ip_1.x_1+iq_1.x_1} = (2\pi)^4\, \delta^{(4)}(p_1-q_1)$$

$$\int d^4y e^{i(q_2.y+q_4.y+q_6.y+q_8.y)} = (2\pi)^4\, \delta^{(4)}(q_2+q_4+q_6+q_8)$$

From this we get the rules for momentum space Green's function:

Now a propagator will carry momenta q_1, q_2 at the two ends instead of coordinates x_1, x_2 and will give factor of $\widetilde{\Delta}_F(q_1,q_2)$

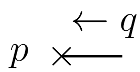
External points will be labelled by momenta $p_1, \cdots, p_n$ instead of coordinates $x_1, \cdots, x_n$ and will give factors of $(2\pi)^4\, \delta^{(4)}(p_i - q_j)$ for some j.

An internal vertex will carry four momenta $q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}$ and will represent a factor of

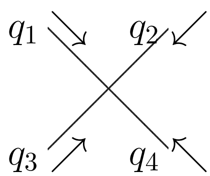
$$-\frac{i\lambda}{4!}\,(2\pi)^4\, \delta^{(4)}(q_{i_1}+q_{i_2}+q_{i_3}+q_{i_4})$$

Diagrammatic representation:

$$: \qquad \widetilde{\Delta}_F(q_1, q_2) = (2\pi)^4\, \delta^{(4)}(q_1 + q_2)\, \widetilde{\Delta}_F(q_1)$$



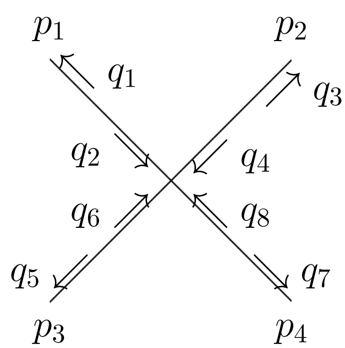
$$: \qquad (2\pi)^4\, \delta^{(4)}(p - q)$$



$$: \qquad -\frac{i\lambda}{4!}\, (2\pi)^4\, \delta^{(4)}(q_1 + q_2 + q_3 + q_4)$$

All momentum arguments of $\widetilde{\Delta}_F$ must be integrated over.

We can now evaluate the momentum space Green's function:



$$\int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} \frac{d^4q_4}{(2\pi)^4} \frac{d^4q_5}{(2\pi)^4} \frac{d^4q_6}{(2\pi)^4} \frac{d^4q_7}{(2\pi)^4} \frac{d^4q_8}{(2\pi)^4}$$

$$\widetilde{\Delta}_F(q_1, q_2)\, \widetilde{\Delta}_F(q_3, q_4)\, \widetilde{\Delta}_F(q_5, q_6)\, \widetilde{\Delta}_F(q_7, q_8)\, (2\pi)^4\delta^{(4)}(q_1 - p_1)\, (2\pi)^4\delta^{(4)}(q_3 - p_2)\, (2\pi)^4\delta^{(4)}(q_5 - p_3)$$

$$(2\pi)^4\delta^{(4)}(q_7 - p_4)\, \left(-\frac{i\lambda}{4!}\right)\, (2\pi)^4\delta^{(4)}(q_2 + q_4 + q_6 + q_8)\, \times \text{combinatoric factors}$$

Notice that this way of representing Feynman diagram is somewhat inefficient.

There are many factors of $d^4 q_i/(2\pi)^4$ and many delta functions $\delta^{(4)}(\sum_i q_i)$.

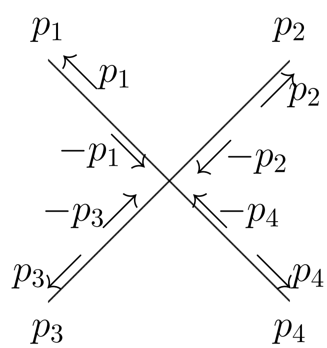
We can give a better representation of the Feynman diagrams by making use of the delta functions to do as many integrals as possible.

Rules:

1. Momentum is conserved at the vertices

2. Sum of momenta at the two ends of a propagator vanishes

Use these to label the momenta flowing in a diagram as we would do in labelling the currents in an electrical circuit.



At the end if at some vertex momentum is not manifestly conserved, we include the delta function for that vertex explicitly in our final expression

e.g. in the above diagram we have $(2\pi)^4\, \delta^{(4)}(-p_1 - p_2 - p_3 - p_4)$

If some momentum is not fixed in terms of external momenta, we explicitly integrate over it

– no such momentum in the diagram above.

22 Momentum space Feynman rules

Number of terms in a diagram without bubbles:

In ϕ^4 theory, suppose $S(2n, m)$ is the number of terms in a diagram without bubble with $2n$ external points and m internal points.

The total number of terms including bubbles is

$$\frac{(2n+4m)!}{2^{n+2m}(n+2m)!}$$

Number of terms in a bubble with k internal points is

$$\frac{(4k)!}{2^{2k}(2k)!}$$

Then

$$S(2n, m) = \frac{(2n+4m)!}{2^{n+2m}(n+2m)!} - \sum_{k=1}^{m} \frac{(4k)!}{2^{2k}(2k)!} \times S(2n, m-k) \times \binom{m}{k}$$

The k-th term gives

(the number of terms in bubble diagrams with k internal points)

$\times$

(the number of terms in diagrams without bubble with $m-k$ internal points and $2n$ external points)

$\times$

(the number of ways k internal points may be chosen out of m internal points)

Symmetry factor of a Feynman diagram:

In ϕ^4 theory, for a diagram with m internal points, it is the inverse of

$$\frac{1}{m!} \times \left(\frac{1}{4!}\right)^m \times \text{combinatoric factor}$$

Momentum space Feynman rules

Goal: Calculate

$$\widetilde{G}(p_1, \cdots, p_n) = \int d^4x_1 \cdots d^4x_n \, e^{-i(p_1.x_1+p_2.x_2+\cdots+p_n.x_n)} \, G(x_1, \cdots, x_n)$$

Draw Feynman diagram with external points carrying momenta $p_1, \cdots, p_n$ flowing outward.

Each propagator carries momentum.

Rules:

1. Momentum is conserved at the vertices

– sum of momenta entering a vertex vanishes

2. Momentum is conserved along the propagators

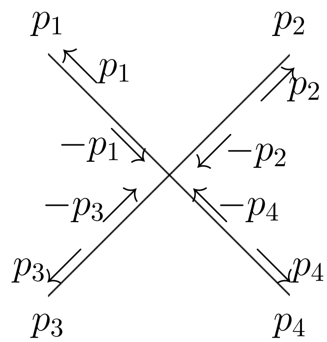
– sum of (oppositely directed) momenta at the two ends of a propagator vanishes.

Use these to label the momenta flowing in a diagram as we would do in labelling the currents in an electrical circuit.

At the end if at some vertex or propagator momentum is not manifestly conserved, we include the delta function for that vertex explicitly in our final expression

If some momentum is not fixed in terms of external momenta, we explicitly integrate over it.

A Feynman diagram representing a contribution to $\widetilde{G}(p_1, p_2, p_3, p_4)$



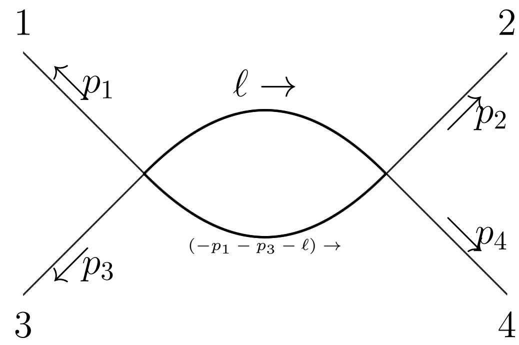
At the end if at some vertex or propagator momentum is not manifestly conserved, we include the delta function for that vertex explicitly in our final expression

e.g. in the above diagram we have $(2\pi)^4\, \delta^{(4)}(-p_1 - p_2 - p_3 - p_4)$

No unfixed momentum in the diagram above.

Net result:

$$-\frac{i\lambda}{4!} \left\{ \prod_{i=1}^{4} \frac{i}{-p_i^2 - m^2 + i\epsilon} \right\} (2\pi)^4\, \delta^{(4)}(-p_1 - p_2 - p_3 - p_4) \times 4 \times 3 \times 2$$



Expression:

$$\frac{1}{2!}\left(-\frac{i\lambda}{4!}\right)^2 \left\{\prod_{i=1}^{4} \widetilde{\Delta}_F(p_i)\right\} (2\pi)^4\, \delta^{(4)}(p_1+p_2+p_3+p_4) \int \frac{d^4\ell}{(2\pi)^4}\, \widetilde{\Delta}_F(\ell)\, \widetilde{\Delta}_F(-p_1-p_3-\ell)$$
$$\times \text{ combinatoric factor}$$

$$\widetilde{\Delta}_F(q) = \frac{i}{-q^2-m^2+i\epsilon}, \qquad \widetilde{\Delta}_F(q_1,q_2) = (2\pi)^4\, \delta^{(4)}(q_1+q_2)\, \widetilde{\Delta}_F(q_1)$$

ℓ is called loop momentum.

For every independent loop in the diagram, we have such a 4-momentum that we need to integrate over.

Momentum space Green's functions are directly related to observable quantities in particle physics.

In momentum space representation, divergences come from integration over loop momentum, e.g. in the above expression, the ℓ integral for large ℓ has the factor:

$$\int \frac{d^4\ell}{(-\ell^2 - m^2 + i\epsilon)(-(p_1 + p_3 + \ell)^2 - m^2 + i\epsilon)} \sim \int \frac{d^4\ell}{(\ell^2)^2} \sim \int \ell^3 d\ell\, \ell^{-4} = \int d\ell\, \ell^{-1}$$

– logarithmically divergent at large ℓ.

These are called ultra-violet (UV) divergence and can be removed by renormalization in a special class of theories called renormalizable theories.

$\Delta_F(x,x)$ divergences are also of this kind.

$$\Delta_F(x,y) = \int d^4\ell\, e^{i\ell.(x-y)} \frac{i}{-\ell^2 - m^2 + i\epsilon}$$

$$\Delta_F(x,x) = \int d^4\ell\, \frac{i}{-\ell^2 - m^2 + i\epsilon}$$

– diverges from large ℓ region.

There can also be divergences from vanishing denominator.

These are infra-red (IR) divergences and have physical origin

– indicates that we are trying to compute something that is not physically observable.

These divergences go away once we frame the question correctly.

Dealing with UV divergences:

All UV divergences come from integration over loop momenta from the region where the loop momenta are large.

Step 1. First find a regularization to make the integral finite.

e.g. we could just put a large cut-off Λ on the loop momentum: $|\ell^\mu| < \Lambda$

or multiply the integrand by some damping factor like $e^{-\sum_\mu |\ell^\mu|^2/\Lambda^2}$

The result depends on the cut-off Λ and diverges as $\Lambda \to \infty$.

2. Compute physical quantities $A_1, \cdots, A_n$ as functions of m, λ, Λ

$$A_i = f_i(m, \lambda, \Lambda) \qquad \text{for } i = 1, \cdots, n$$

– diverge as $\Lambda \to \infty$

3. Pick any two, say A_1, A_2 and solve for m, λ in terms of A_1, A_2, Λ

$$m = g_1(A_1, A_2, \Lambda), \qquad \lambda = g_2(A_1, A_2, \Lambda)$$

4. Use this to calculate $A_3, \cdots, A_n$ in terms of A_1, A_2, λ

$$A_i = f_i(g_1(A_1, A_2, \Lambda), g_2(A_1, A_2, \Lambda), \Lambda) \qquad \text{for } i = 3, \cdots, n$$

5. Now take the $\Lambda \to \infty$ limit.

In a renormalizable theory, this gives finite result:

$$A_i = h_i(A_1, A_2) \qquad \text{for } i = 3, \cdots, n$$

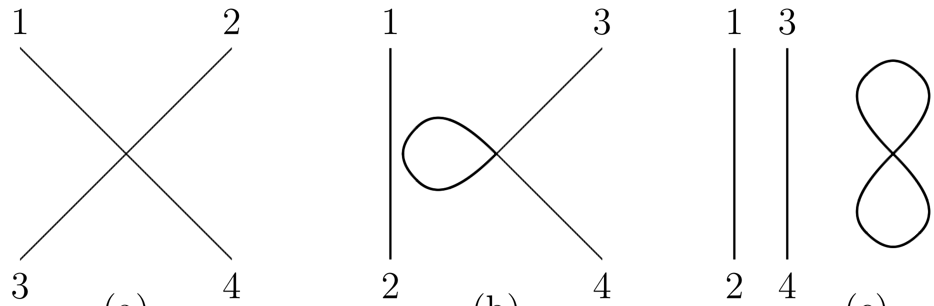
– predictions of the theory that can be tested experimentally.

In the rest of the course we shall avoid this problem by working with tree diagrams

– diagrams without loops

Typically the lowest order contribution to an amplitude comes from tree diagram.

Connected vs. disconnected diagram



(a) (b) (c)

A connected diagram is where every point is connected to another point via a collection of propagators and vertices, e.g. (a).

A disconnected diagram is where every point is not connected to another point via a collection of propagators and vertices, e.g. (b), (c).

Now, since every vertex and propagator has momentum conservation, every Feynman diagram will have overall momentum conservation, i.e. will be proportional to

$$(2\pi)^4 \, \delta^{(4)}(p_1 + \cdots + p_n)$$

Disconnected diagram has separate momentum conservation for each component, *e.g.* $\delta^{(4)}(p_1 + p_2) \, \delta^{(4)}(p_3 + p_4)$ in (b), (c).

Therefore, for generic external momenta, satisfying overall momentum conservation, only the connected diagrams contribute.

23 Computation of physical mass

We have learned how to calculate

$$G(x_1, \cdots, x_n) = \langle \Omega | T(\phi(x_1) \cdots \phi(x_n)) | \Omega \rangle$$

in an interacting scalar field theory.

Our goal now will be to study how to relate this to quantities that we can measure

1. Masses of particles

2. Scattering cross-section

– probability of finding a particular outcome when we take a set of particles and collide them.

We shall begin by studying some general properties of the spectrum of states in the theory.

Recall:

$$P_\rho = \int d^3x \, T^0{}_\rho$$

are the conserved quantities associated with infinitesimal space-time translation:

$$\widetilde{\phi}(x) = \phi(x+\epsilon) = \phi(x) + \epsilon^\rho \, \partial_\rho \phi(x)$$

$P_i = P^i$ are the spatial components of momenta

$P^0 = -P_0$ is the Hamiltonian

Recall the general relation:

If Q is the conserved quantity associated with the transformation $\phi \to \widetilde{\phi}$ with $\widetilde{\phi} = \phi + \epsilon \, \chi$, then

$$[Q, \phi(x)] = i \, \chi(x)$$

This gives

$$[P_\rho, \phi(x)] = i \, \partial_\rho \phi(x)$$

Claim:

$$[P_\mu, P_\nu] = 0$$

This can be checked using the known expressions for P_μ, but we shall describe a general procedure that works for calculating $[Q_i, Q_j]$ for any pair of Noether charges.

$$[P_\mu, [P_\nu, \phi(x)]] = i\,[P_\mu, \partial_\nu \phi(x)] = i\partial_\nu [P_\mu, \phi(x)] = i^2 \partial_\nu \partial_\mu \phi(x)$$
$$[P_\nu, [P_\mu, \phi(x)]] = i^2 \partial_\mu \partial_\nu \phi(x)$$

Now consider the Jacobi identity:

$$[[P_\mu, P_\nu], \phi(x)] + [[\phi(x), P_\mu], P_\nu] + [[P_\nu, \phi(x)], P_\mu] = 0$$

This gives, using $[A, B] = -[B, A]$,

$$\begin{aligned}
&[[P_\mu, P_\nu], \phi(x)] = -[[\phi(x), P_\mu], P_\nu] - [[P_\nu, \phi(x)], P_\mu] \\
&= -[P_\nu, [P_\mu, \phi(x)]] + [P_\mu, [P_\nu, \phi(x)]] = -i^2 \partial_\mu \partial_\nu \phi(x) + i^2 \partial_\nu \partial_\mu \phi(x) = 0
\end{aligned}$$

This proves

$$[[P_\mu, P_\nu], \phi(x)] = 0$$

Since P_μ and P_ν are time independent, similar relation holds for ϕ replaced by $\Pi = \partial_0 \phi$.

Since all operators are made from ϕ and Π we conclude that $[P_\mu, P_\nu]$ must commute with all the operators of the theory.

Therefore $[P_\mu, P_\nu]$ must be proportional to the identity operator.

We shall proceed by assuming that this is zero

$$[P_\mu, P_\nu] = 0$$

(In rare cases identity operator does appear in the commutator of conserved charges)

Since P_μ, P_ν commute, we can choose a complete set of orthonormal basis states $|n\rangle$ which are simultaneous eigenstates of all the P_μ's:

$$P_\mu |n\rangle = p_{(n)\mu} |n\rangle, \qquad \langle n| P_\mu = p_{(n)\mu} \langle n|$$

Recall that $\frac{1}{2}\omega_{\nu\tau}\mathcal{J}^{\tau\nu}$ is the conserved quantity associated with the infinitesimal Lorentz transformation:

$$\phi \to \widetilde{\phi} = \phi + \epsilon\, \omega^{\mu}{}_{\tau} x^{\tau} \partial_{\mu}\phi = \phi + \epsilon\, \omega_{\nu\tau} x^{\tau} \partial^{\nu}\phi, \quad \partial^{\nu}\phi = \eta^{\nu\mu}\partial_{\mu}\phi$$

This gives

$$\left[\frac{1}{2}\omega_{\nu\tau}\mathcal{J}^{\tau\nu}, \phi(x)\right] = i\, \omega_{\nu\tau} x^{\tau}\partial^{\nu}\phi = \frac{i}{2}\omega_{\nu\tau}(x^{\tau}\partial^{\nu}\phi - x^{\nu}\partial^{\tau}\phi)$$

Since $\omega_{\nu\tau}$ is an independent anti-symmetric matrix, this gives

$$[\mathcal{J}^{\tau\nu}, \phi(x)] = i(x^{\tau}\partial^{\nu}\phi - x^{\nu}\partial^{\tau}\phi)$$

Now recall

$$[P_{\rho}, \phi(x)] = i\, \partial_{\rho}\, \phi(x)$$

Ex. Using Jacobi identity as before, show that

$$[\mathcal{J}^{\tau\nu}, P_{\rho}] = i(P^{\nu}\delta^{\tau}_{\rho} - P^{\tau}\delta^{\nu}_{\rho})$$

ignoring terms proportional to the identity matrix.

We need to show that both sides have the same commutator with $\phi(x)$ and hence also $\Pi(x)$.

This can also be checked using the explicit form of P_{ρ} and $\mathcal{J}^{\tau\nu}$ in terms of ϕ and Π.

Ex. Repeat the analysis for $[\mathcal{J}^{\nu\tau}, \mathcal{J}^{\rho\sigma}]$.

$$[P_\mu, P_\nu] = 0$$

Since P_μ, P_ν commute, we can choose a complete set of orthonormal basis states $|n\rangle$ which are simultaneous eigenstates of all the P_μ's:

$$P_\mu|n\rangle = p_{(n)\mu}|n\rangle, \qquad \langle n|P_\mu = p_{(n)\mu}\,\langle n|$$

Now consider a different basis

$$|n\rangle' = \left(1 + \frac{1}{2}\,i\epsilon\,\omega_{\nu\tau}\,\mathcal{J}^{\nu\tau}\right)|n\rangle$$

We can use hermiticity of $\mathcal{J}^{\tau\nu}$ and,

$$[\mathcal{J}^{\tau\nu}, P_\rho] = i(P^\tau\delta^\nu_\rho - P^\nu\delta^\tau_\rho)$$

to show that

1. $'\langle m|n\rangle' = \langle m|n\rangle$.

2. **Ex.**

$$P_\mu|n\rangle' = p'_{(n)\mu}|n\rangle' + \mathcal{O}(\epsilon^2), \qquad p'_{(n)\mu} = (p_{(n)\mu} + \epsilon\,\omega_{\mu\tau}p^\tau_{(n)})$$

Note: $p'_{(n)}$ is the Lorentz transform of $p_{(n)}$ by the Lorentz matrix $(1+\epsilon\,\omega)$.

By applying infinitesimal transformations infinite number of times we can generate any finite Lorentz transformation that is connected to the identity

– can be used to construct a state carrying any momentum that is related to $p_{(n)}$ by Lorentz transformation.

Now let us suppose that the theory has a particle of mass m_p

$\Leftrightarrow$ there are states for which energy p^0 is fixed in terms of momenta $\vec{p}$ via the relation

$$p^0 = E_{\vec{p}} = \sqrt{\vec{p}^2 + m_p^2}$$

We shall denote these states by $|\vec{p}\rangle$.

It follows from our previous discussion that $|\vec{p}\rangle' \propto |\vec{p}\,'\rangle$, where

$$\begin{pmatrix} E_{\vec{p}\,'} \\ \vec{p}\,' \end{pmatrix} = \Lambda \begin{pmatrix} E_{\vec{p}} \\ \vec{p} \end{pmatrix}$$

We would like to normalize the states such that

$$|\vec{p}\rangle' = |\vec{p}\,'\rangle$$

without any additional normalization factor.

Since we have seen that $'\langle\vec{p}_1|\vec{p}_2\rangle' = \langle\vec{p}_1|\vec{p}_2\rangle$, we get

$$\langle\vec{p}_1'|\vec{p}_2'\rangle = \langle\vec{p}_1|\vec{p}_2\rangle$$

Problem: $\langle\vec{p}_1|\vec{p}_2\rangle = \delta^{(3)}(\vec{p}_1 - \vec{p}_2)$ is not compatible with this, since

$$\delta^{(3)}(\vec{p}_1 - \vec{p}_2) \neq \delta^{(3)}(\vec{p}_1' - \vec{p}_2')$$

Claim:

$$2E_{\vec{p}_1}\,\delta^{(3)}(\vec{p}_1 - \vec{p}_2) = 2E_{\vec{p}_1'}\,\delta^{(3)}(\vec{p}_1' - \vec{p}_2')$$

Therefore, we shall normalize the basis states for single particles of mass m_p as:

$$\langle\vec{p}_1|\vec{p}_2\rangle = 2E_{\vec{p}_1}\,\delta^{(3)}(\vec{p}_1 - \vec{p}_2)$$

Proof: Consider

$$
\begin{aligned}
& 2E_{\vec{p}_1}\,\delta^{(3)}(\vec{p}_1-\vec{p}_2)\,\delta(-p_1^2-m_p^2)\,H(p_1^0)\,\delta(-p_2^2-m_p^2)\,H(p_2^0) \\
= \; & 2E_{\vec{p}_1}\,\delta^{(3)}(\vec{p}_1-\vec{p}_2)\,\delta((p_1^0)^2-\vec{p}_1^{\,2}-m_p^2)\,H(p_1^0)\,\delta(-p_2^2-m_p^2)H(p_2^0)
\end{aligned}
$$

We can write this as

$$
\begin{aligned}
& 2E_{\vec{p}_1}\,\delta^{(3)}(\vec{p}_1-\vec{p}_2)\,\frac{1}{2E_{\vec{p}_1}}\,\delta(p_1^0-E_{\vec{p}_1})\,\delta(-p_2^2-m_p^2)H(p_2^0) \\
= \; & \delta^{(3)}(\vec{p}_1-\vec{p}_2)\,\delta(p_1^0-p_2^0)\delta(-p_2^2-m_p^2)H(p_2^0) \\
= \; & \delta^{(4)}(p_1-p_2)\,\delta(-p_2^2-m_p^2)\,H(p_2^0)
\end{aligned}
$$

Since rhs is Lorentz invariant, the starting expression is Lorentz invariant.

Therefore $2E_{\vec{p}_1}\,\delta^{(3)}(\vec{p}_1-\vec{p}_2)$ is Lorentz invariant.

From now on we shall normalize momentum eigenstates of single particles as

$$
\langle\vec{p}_1|\vec{p}_2\rangle = 2E_{\vec{p}_1}\,\delta^{(3)}(\vec{p}_1-\vec{p}_2)
$$

We now need to understand the meaning of completeness

$$\sum_n |n\rangle\langle n| = I$$

in this new normalization.

If n had taken discrete values then there is no confusion.

But we have continuous momentum label $\vec{p}$ of single particle states of mass m_p.

With the new normalization $\int d^3p\, |\vec{p}\rangle\langle\vec{p}|$ does not act as identity on the single particle states.

$$\int d^3p\, |\vec{p}\rangle\langle\vec{p}|\vec{p}_1\rangle = \int d^3p\, |\vec{p}\rangle\ (2E_{\vec{p}_1})\delta^{(3)}(\vec{p}-\vec{p}_1) = 2E_{\vec{p}_1}|\vec{p}_1\rangle$$

Therefore we use

$$\int \frac{d^3p}{2\, E_{\vec{p}}}\, |\vec{p}\rangle\langle\vec{p}|$$

This acts as identity on single particle states.

We shall proceed by assuming that the vacuum is Poincare invariant, i.e.

$$P_\mu|\Omega\rangle = 0 = \mathcal{J}^{\nu\rho}|\Omega\rangle, \qquad \langle\Omega|P_\mu = 0 = \langle\Omega|\mathcal{J}^{\nu\rho}$$

Note: This would not be possible if any of $[P_\mu, P_\nu]$, $[\mathcal{J}^{\nu\tau}, P_\mu]$ or $[\mathcal{J}^{\nu\tau}, \mathcal{J}^{\rho\sigma}]$ had terms proportional to identity.

We now consider the two point function

$$G(x_1,x_2) = \langle\Omega|T(\phi(x_1)\phi(x_2))|\Omega\rangle = H(x_1^0 - x_2^0)\, G_+(x_1,x_2) + H(x_2^0 - x_1^0)\, G_+(x_2,x_1)$$

where

$$G_+(x_1,x_2) = \langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle$$

We write

$$G_+(x_1,x_2) = \sum_n \langle\Omega|\phi(x_1)|n\rangle\langle n|\phi(x_2)|\Omega\rangle$$

We shall examine how single particles states of mass m_p contribute to this sum and use it to compute m_p, but for now we proceed by including all states.

Now

$$\partial_\mu\langle\Omega|\phi(x)|n\rangle = -i\,\langle\Omega|[P_\mu,\phi(x)]|n\rangle = -i\,\langle\Omega|(P_\mu\phi(x) - \phi(x)P_\mu)|n\rangle = i\,p_{(n)\mu}\langle\Omega|\phi(x)|n\rangle$$

This gives

$$\langle\Omega|\phi(x)|n\rangle = e^{ip_{(n)}.x}\,\langle\Omega|\phi(0)|n\rangle, \qquad \langle n|\phi(x)|\Omega\rangle = e^{-ip_{(n)}.x}\,\langle n|\phi(0)|\Omega\rangle$$

Therefore

$$G_+(x_1,x_2) = \sum_n e^{ip_{(n)}.(x_1-x_2)}\langle\Omega|\phi(0)|n\rangle\langle n|\phi(0)|\Omega\rangle$$

Use $\int d^4q\,\delta^{(4)}(p_{(n)} - q) = 1$ to write

$$\begin{aligned}
G_+(x_1,x_2) &= \sum_n e^{ip_{(n)}.(x_1-x_2)} \int d^4q\,\delta^{(4)}(p_{(n)} - q)\,\langle\Omega|\phi(0)|n\rangle\langle n|\phi(0)|\Omega\rangle \\
&= \int d^4q \sum_n e^{iq.(x_1-x_2)}\,\delta^{(4)}(p_{(n)} - q)\,\langle\Omega|\phi(0)|n\rangle\langle n|\phi(0)|\Omega\rangle \\
&= \int \frac{d^4q}{(2\pi)^4}\,e^{iq.(x_1-x_2)}\,F(q) \\
F(q) &= (2\pi)^4 \sum_n \delta^{(4)}(p_{(n)} - q)\,\langle\Omega|\phi(0)|n\rangle\langle n|\phi(0)|\Omega\rangle
\end{aligned}$$

$$\begin{aligned} G_+(x_1,x_2) &= \int \frac{d^4q}{(2\pi)^4}\, e^{iq.(x_1-x_2)}\, F(q) \\ F(q) &= (2\pi)^4 \sum_n \delta^{(4)}(p_{(n)}-q)\, \langle\Omega|\phi(0)|n\rangle\langle n|\phi(0)|\Omega\rangle \end{aligned}$$

Note that $G_+(x_1,x_2)$ depends only on the difference $x_1 - x_2$

– consequence of translation invariance $G_+(x_1,x_2) = G(x_1+a, x_2+a)$

Using this we can write:

$$F(q) = \int d^4x\, e^{-iq.x} G_+(x,0)$$

We shall now prove that

1. $F(q)$ is Lorentz invariant, i.e. $F(\Lambda q) = F(q)$

2. $F(q)$ vanishes for $q^2 > 0$ (space-like) and also for $q^0 < 0$.

24 Scalar mass computation in the interacting field theory

We defined

$$G_+(x_1, x_2) = \langle \Omega | \phi(x_1) \phi(x_2) | \Omega \rangle$$

From this we derived:

$$\begin{aligned} G_+(x_1, x_2) &= \int \frac{d^4 q}{(2\pi)^4} \, e^{iq.(x_1 - x_2)} \, F(q) \\ F(q) &= (2\pi)^4 \sum_n \delta^{(4)}(p_{(n)} - q) \, \langle \Omega | \phi(0) | n \rangle \langle n | \phi(0) | \Omega \rangle \end{aligned}$$

Inverting the first relation, we can write:

$$F(q) = \int d^4 x \, e^{-iq.x} G_+(x, 0)$$

We shall now prove that

1. $F(q)$ is Lorentz invariant, i.e. $F(\Lambda q) = F(q)$

2. $F(q)$ vanishes for $q^2 > 0$ (space-like) and also for $q^0 < 0$.

$$F(q) = (2\pi)^4 \sum_n \delta^{(4)}(p_{(n)} - q) \langle \Omega|\phi(0)|n\rangle\langle n|\phi(0)|\Omega\rangle$$

$$F(q) = \int d^4x \, e^{-iq.x} G_+(x, 0)$$

1. Proof of $F(\Lambda q) = F(q)$:

Recall Lorentz invariance of G_+:

$$\langle\Omega|\phi(\Lambda x_1)\phi(\Lambda x_2)|\Omega\rangle = \langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle \quad \Rightarrow \quad G_+(\Lambda x_1, \Lambda x_2) = G_+(x_1, x_2)$$

Now we have:

$$F(\Lambda q) = \int d^4x \, e^{-i\Lambda q.x} G_+(x, 0)$$

Change variable $x = \Lambda x'$, use $d^4x = d^4x'$ and the Lorentz invariance of G_+:

$$F(\Lambda q) = \int d^4x' \, e^{-i\Lambda q.\Lambda x'} G_+(\Lambda x', 0) = \int d^4x' \, e^{-iq.x'} G_+(x', 0) = F(q)$$

2. Proof that $F(q)$ vanishes for $q^2 > 0$ (space-like) and also for $q^0 < 0$:

For $F(q)$ to not vanish, there must be a state $|n\rangle$ with $p_{(n)} = q$.

Since the vacuum has zero energy, there is no state with $p_n^0 < 0$.

Therefore $F(q)$ vanishes for $q^0 < 0$.

If $q^2 > 0$, i.e. space-like, we can find a Lorentz transformation that makes $q^0 < 0$.

Therefore by the previous argument, and Lorentz invariance, $F(q)$ must vanish for $q^2 > 0$.

$$G_+(x_1, x_2) = \int \frac{d^4q}{(2\pi)^4}\, e^{iq.(x_1-x_2)}\, F(q)$$

1. $F(q)$ is Lorentz invariant, i.e. $F(\Lambda q) = F(q)$

2. $F(q)$ vanishes for $q^2 > 0$ (space-like) and also for $q^0 < 0$.

Therefore we can write:

$$F(q) = 2\pi\, f(-q^2)\, H(q^0)$$

$f(u)$: a function that vanshes for $u < 0$.

Substituting this into the expression for G_+ we get

$$G_+(x_1, x_2) = \int \frac{d^4q}{(2\pi)^4}\, e^{iq.(x_1-x_2)}\, 2\pi\, f(-q^2)\, H(q^0)$$

This can be written as:

$$G_+(x_1, x_2) = \int_0^\infty du \int \frac{d^4q}{(2\pi)^4}\, e^{iq.(x_1-x_2)}\, 2\pi\, f(u)\delta(u+q^2)\, H(q^0)$$

Now recall that for a free particle of mass m, $\langle 0|\phi(x_1)\phi(x_2)|0\rangle$ is given by:

$$\Delta_+(x_1, x_2; m) = \int \frac{d^4q}{(2\pi)^4}\, e^{iq.(x_1-x_2)}\, 2\pi\, \delta(q^2+m^2)\, H(q^0)$$

Therefore we can write:

$$G_+(x_1, x_2) = \int_0^\infty du\, f(u)\, \Delta_+(x_1, x_2; \sqrt{u})$$

– as if G_+ is s superposition for free particle results for different masses $\sqrt{u}$ with weight $f(u)$.

$f(u)$ is called the spectral density.

$$G_+(x_1,x_2) = \int_0^\infty du\, f(u)\, \Delta_+(x_1,x_2;\sqrt{u})$$

– as if G_+ is a superposition of free particle results for different masses $\sqrt{u}$ with weight $f(u)$.

This also gives

$$\begin{aligned} G(x_1,x_2) &= H(x_1^0-x_2^0)\, G_+(x_1,x_2) + H(x_2^0-x_1^0)\, G_+(x_2,x_1) \\ &= \int_0^\infty du\, f(u) \left(H(x_1^0-x_2^0)\Delta_+(x_1,x_2;\sqrt{u}) + H(x_2^0-x_1^0)\Delta_+(x_2,x_1;\sqrt{u}) \right) \\ &= \int_0^\infty du\, f(u)\, \Delta_F(x_1,x_2;\sqrt{u}) \end{aligned}$$

$G(x_1,x_2)$ can be calculated using perturbation theory, from which we can compute $f(u)$.

Typically $f(u)$ is non-zero for all u above some value.

This does not mean that the theory has particles with all possible masses.

Instead this implies that the G_+ receives contribution from multiparticle intermediate states for which $p^2 = -(p^0)^2 + \vec{p}^2$ can take continuous values.

e.g. if m_p is the physical mass of the particle, then for a two particle state

$$p^2 = (p_1+p_2)^2 = p_1^2+p_2^2+2p_1.p_2 = -m_p^2-m_p^2+2\left\{-\sqrt{\vec{p}_1^2+m_p^2}\sqrt{\vec{p}_2^2+m_p^2}+\vec{p}_1.\vec{p}_2\right\}$$

Note: For free K-G theory

$$G_+(x_1,x_2) = \sum_n \langle 0|\phi(x_1)|n\rangle\langle n|\phi(x_2)|0\rangle$$

gets contribution from single particle intermediate state only.

Q. How can we extract the physical mass m_p of a particle from the knowledge of $G(x_1, x_2)$?

Strategy:

1. Assume that there are single particles states with particle mass m_p.

2. Study their contribution to $G(x_1, x_2)$.

3. Then look for such contributions in $G(x_1, x_2)$.

So we begin by assuming that the P_μ eigenstates contain single particles states of mass m_p

– gives states with $p_{(n)}.p_{(n)} = -m_p^2$.

We now analyze the contribution to G_+ from these single particle states

$$\begin{aligned} G'_+(x_1, x_2) &= {\sum_n}' e^{ip_{(n)}.(x_1 - x_2)} \langle\Omega|\phi(0)|n\rangle\langle n|\phi(0)|\Omega\rangle \\ &= \int \frac{d^3p}{2\,E_{\vec{p}}}\, e^{ip.(x_1-x_2)} \langle\Omega|\phi(0)|\vec{p}\rangle\langle\vec{p}|\phi(0)|\Omega\rangle \end{aligned}$$

$|\vec{p}\rangle$: single particle states of momentum $\vec{p}$, mass m_p and energy $E_{\vec{p}} = \sqrt{\vec{p}^2 + m_p^2}$

Claim: $\langle\Omega|\phi(0)|\vec{p}\rangle$ is independent of $\vec{p}$, i.e. $\langle\Omega|\phi(0)|\vec{p}\rangle = \langle\Omega|\phi(0)|\vec{p}\,'\rangle$

Proof: Given a single particle state $|\vec{p}\rangle$, a state $|\vec{p}\,'\rangle$ with a nearby momentum $\vec{p}\,'$ may be related to $|\vec{p}\rangle$ by Lorentz transformation:

$$|\vec{p}\,'\rangle = |\vec{p}\rangle' = \left(1 + \frac{1}{2}\, i\epsilon\, \omega_{\nu\tau}\, \mathcal{J}^{\nu\tau}\right) |\vec{p}\rangle$$

for some $\omega_{\nu\tau}$ and some small ϵ.

We now use

$$[\mathcal{J}^{\nu\tau}, \phi(x)] = i(x^\nu \partial^\tau \phi - x^\tau \partial^\nu \phi) \quad \Rightarrow \quad [\mathcal{J}^{\nu\tau}, \phi(0)] = 0$$

and $\langle\Omega|\mathcal{J}^{\mu\nu} = 0$, to write

$$\langle\Omega|\phi(0)\left(1 + \frac{1}{2}\, i\epsilon\, \omega_{\nu\tau}\, \mathcal{J}^{\nu\tau}\right)|\vec{p}\rangle = \langle\Omega|\left(1 + \frac{1}{2}\, i\epsilon\, \omega_{\nu\tau}\, \mathcal{J}^{\nu\tau}\right)\phi(0)|\vec{p}\rangle = \langle\Omega|\phi(0)|\vec{p}\rangle$$

Once we have proven $\langle\Omega|\phi(0)|\vec{p}\rangle = \langle\Omega|\phi(0)|\vec{p}\,'\rangle$ for infinitesimal Lorentz transformation, it also holds for finite Lorentz transformation.

This gives

$$\langle\Omega|\phi(0)|\vec{p}\rangle\langle\vec{p}|\phi(0)|\Omega\rangle = \frac{1}{(2\pi)^3}\, Z, \qquad (Z = \text{some constant})$$

We have

$$G_+(x_1,x_2) = \int \frac{d^3p}{2\,E_{\vec{p}}}\, e^{ip.(x_1-x_2)}\, \langle\Omega|\phi(0)|\vec{p}\rangle\langle\vec{p}|\phi(0)|\Omega\rangle$$

$$\langle\Omega|\phi(0)|\vec{p}\rangle\langle\vec{p}|\phi(0)|\Omega\rangle = \frac{1}{(2\pi)^3}\, Z$$

This gives

$$\begin{aligned} G'_+(x_1,x_2) &= \frac{Z}{(2\pi)^3} \int \frac{d^3p}{2E_{\vec{p}}}\, e^{ip.(x_1-x_2)} = \frac{Z}{(2\pi)^3} \int d^4p\, \delta(-p^2 - m_p^2) H(p^0)\, e^{ip.(x_1-x_2)} \\ &= Z\, \Delta_+(x_1,x_2;m_p) \end{aligned}$$

Compare this with the full expression for G_+:

$$G_+(x_1,x_2) = \int_0^\infty du\, f(u)\, \Delta_+(x_1,x_2;\sqrt{u})$$

From this we see that the contribution of single particle states of mass m_p to $f(u)$ is given by:

$$f_s(u) = Z\,\delta(u - m_p^2)$$

From this we can find the contribution to $G(x_1, x_2)$ due to the single particle intermediate states.

Recall:

$$G(x_1, x_2) = \int_0^\infty du\, f(u)\, \Delta_F(x_1, x_2; \sqrt{u})$$

Since single particle contribution to $f(u)$ is $Z\,\delta(u - m_p^2)$, its contribution to $G(x_1, x_2)$ is:

$$G(x_1, x_2) = \int_0^\infty du\, Z\, \delta(u - m_p^2)\, \Delta_F(x_1, x_2; \sqrt{u}) + \cdots = Z\, \Delta_F(x_1, x_2; m_p) + \cdots$$

$\cdots$: contributions from other states.

We now translate this to the momentum space Green's function:

$$\begin{aligned} \widetilde{G}(p_1, p_2) &= \int d^4x_1 d^4x_2\, e^{-i(p_1.x_1 + p_2.x_2)}\, G(x_1, x_2) \\ &= Z \int d^4x_1 d^4x_2\, e^{-i(p_1.x_1 + p_2.x_2)}\, \Delta_F(x_1, x_2; m_p) + \cdots \\ &= Z\, (2\pi)^4\, \delta^{(4)}(p_1 + p_2)\, \frac{i}{-p_1^2 - m_p^2 + i\epsilon} + \cdots \end{aligned}$$

This shows that a single particle state of mass m_p produces a pole in $\widetilde{G}(p_1, p_2)$ at $p_1^2 = -m_p^2$.

Conversely, by finding the pole locations of $\widetilde{G}(p_1, p_2)$ in the variable p_1^2, we can determine the mass of the particle.

We shall now describe how to do this in perturbation expansion.

Note: This analysis goes through for two point function of any pair of scalar operators, e.g.

$$\langle \Omega | T(\phi(x_1)^2 \phi(x_2)^4) | \Omega \rangle$$

It also goes through with few modification for tensor operators like $\phi(x)\partial_\mu\partial_\nu\phi(x)$.

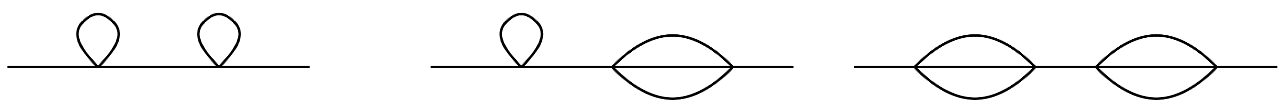
We introduce the notion of truncated, one particle irreducible (1PI) Green's functions.

Truncated: Remove the external propagator factors:

$$\prod_{i=1}^{n} \frac{i}{-p_i^2 - m^2 + i\epsilon}$$

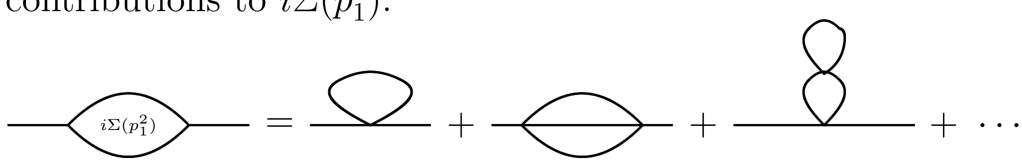
1PI: Remove all Feynman diagrams that fall apart into two disjoint parts by cutting one internal propagator.

Example of diagrams to be discarded in the evaluation of 1PI 2-point function:

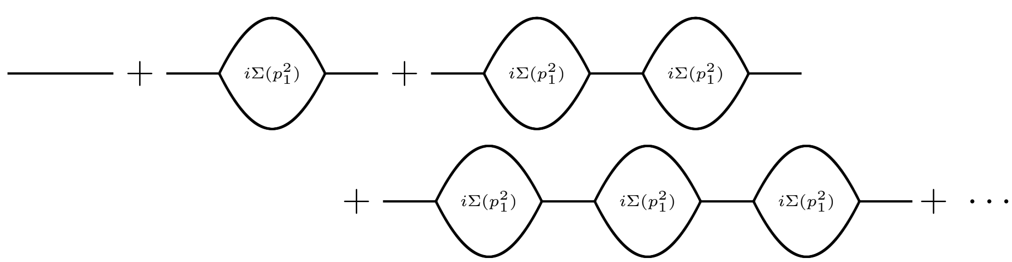


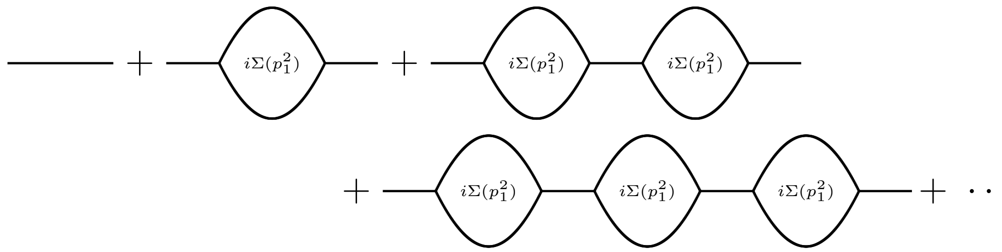
We shall denote by $i\Sigma(p_1^2)$ the sum of all the truncated 1PI 2-point Green's function without the $(2\pi)^4\delta^{(4)}(p_1 + p_2)$ factor.

Example of contributions to $i\Sigma(p_1^2)$:



We can now express $\widetilde{G}(p_1, p_2)$ as:





This gives

$$\begin{aligned}\widetilde{G}(p_1,p_2) &= (2\pi)^4\,\delta^{(4)}(p_1+p_2)\left[\frac{i}{-p_1^2-m^2+i\epsilon}+\left(\frac{i}{-p_1^2-m^2+i\epsilon}\right)^2 i\Sigma(p_1^2)\right.\\ &+ \left.\left(\frac{i}{-p_1^2-m^2+i\epsilon}\right)^3\left(i\Sigma(p_1^2)\right)^2+\cdots\right]\\ &= (2\pi)^4\,\delta^{(4)}(p_1+p_2)\,\frac{i}{-p_1^2-m^2+\Sigma(p_1^2)+i\epsilon}\end{aligned}$$

Since m_p^2 is the location of the pole in $-p_1^2$, we can write

$$m_p^2-m^2+\Sigma(-m_p^2)=0 \quad\Rightarrow\quad m_p^2=m^2-\Sigma(-m_p^2)$$

Since $\Sigma(-p^2)$ has an expansion starting at order λ, we can treat it as small and solve the equation iteratively.

Leading order

$$m_p^2=m^2$$

Next order

$$m_p^2=m^2-\Sigma(-m^2)$$

Next order

$$m_p^2=m^2-\Sigma(-m^2+\Sigma(-m^2))$$

etc.

This determines m_p in terms of m and λ.

25 S-matrix

We have seen how to extract the masses of physical particles from the analysis of the 2-point Green's function.

We shall now how to calculate scattering amplitudes from the analysis of n-point Green's functions.

Goal: Compute the probability of observing a certain outcome, like m particles going out along some specific directions, when we collide n particles with each other.

For this we need to first go back to free field theory.

$$\phi(t,\vec{r}) = \int \frac{d^3p}{(2\pi)^{3/2}}\, e^{i\vec{p}.\vec{r}}\, \frac{1}{\sqrt{2E_{\vec{p}}}}\, (a(t,\vec{p}) + a(t,-\vec{p})^\dagger)$$

Work at $t=0$ and define

$$|f\rangle = \int d^3r\, f(\vec{r})\, \phi(0,\vec{r})|0\rangle = \int d^3r\, f(\vec{r}) \int \frac{d^3p}{(2\pi)^{3/2}}\, e^{-i\vec{p}.\vec{r}}\, \frac{1}{\sqrt{2E_{\vec{p}}}}\, a(\vec{p})^\dagger|0\rangle$$

$f(\vec{r})$: some function whose choice will be specified later.

Define the state

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\, a(\vec{p})^\dagger|0\rangle$$

and the function

$$\widetilde{f}(\vec{p}) = \int \frac{d^3r}{(2\pi)^{3/2}}\, e^{-i\vec{p}.\vec{r}}\, f(\vec{r})$$

Then

$$|f\rangle = \int \frac{d^3p}{\sqrt{2E_{\vec{p}}}}\, \widetilde{f}(\vec{p})\, a(\vec{p})^\dagger|0\rangle = \int \frac{d^3p}{2E_{\vec{p}}}\, \widetilde{f}(\vec{p})\, |\vec{p}\rangle$$

We shall choose $\widetilde{f}(\vec{p})$ to be a gaussian centered around some momentum $\vec{k}$.

$$\widetilde{f}(\vec{p}) = \sigma^{-3/4}\, e^{-(\vec{p}-\vec{k})^2/(2\,\sigma)} \qquad \Rightarrow \qquad f(\vec{r}) = \sigma^{3/4}\, e^{i\vec{k}.\vec{r}}\, e^{-\sigma\vec{r}^2/2}$$

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\, a(\vec{p})^\dagger |0\rangle$$

$$|f\rangle = \int d^3r\, f(\vec{r})\, \phi(0,\vec{r})|0\rangle = \int \frac{d^3p}{2E_{\vec{p}}}\, \widetilde{f}(\vec{p})\, |\vec{p}\rangle$$

$$\widetilde{f}(\vec{p}) = \int \frac{d^3r}{(2\pi)^{3/2}}\, e^{-i\vec{p}.\vec{r}}\, f(\vec{r})$$

$$\widetilde{f}(\vec{p}) = \sigma^{-3/4}\, e^{-(\vec{p}-\vec{k})^2/(2\,\sigma)} \qquad \Rightarrow \qquad f(\vec{r}) = \sigma^{3/4}\, e^{i\vec{k}.\vec{r}}\, e^{-\sigma\vec{r}^2/2}$$

Interpretation in the language of quantum mechanics:

Since $|\vec{p}\rangle$ is interpreted as the momentum eigenstate of a single particle of eigenvalue $\vec{p}$, $\widetilde{f}(\vec{p})$ is the momentum space wave-function of the particle

$\Rightarrow$ $f(\vec{r})$ should be interpreted as position space wave-function

This shows that the particles states whose position space wave-functions are localized around $\vec{r}=0$ are obtained by acting on $|0\rangle$ by $\phi(0,\vec{r})$ around $\vec{r}=0$

– by translation invariance, this statement holds for any point in space.

We shall choose $\sqrt{\sigma}$ to be small compared to $|\vec{k}|$ so that the width of the distribution in the momentum space is small compared to the central value of the momentum

In that case the $E_{\vec{p}}$ factors in the integrand remain almost constant

– can be approximated by $E_{\vec{k}}$ and taken out of the integral.

We have been using Heisenberg picture.

We shall continue to do so.

However let us ask, how the state $|f\rangle$ would have looked if we used Schrodinger picture to evolve it is time.

Non-relativistic QM: In position space, the peak will move according to classical equations of motion and the width will increase with time as $\sqrt{|t|}$

We shall now see that the same is true in this case, except that the peak will move along the trajectory of a relativistic particle.

$|t, f\rangle$: The state $|f\rangle$ at time t if it evolves according to the Schrodinger picture

$$i\frac{\partial}{\partial t}|t, f\rangle = H|t, f\rangle, \qquad |0, f\rangle = |f\rangle$$

Trial solution:

$$\int \frac{d^3p}{2E_{\vec{p}}}\, \widetilde{f}(t, \vec{p})\, |\vec{p}\rangle$$

for some function $\widetilde{f}(t, \vec{p})$

Then, using $H|\vec{p}\rangle = E_{\vec{p}}|\vec{p}\rangle$, we get

$$i\,\frac{\partial \widetilde{f}(t, \vec{p})}{\partial t} = E_{\vec{p}}\, \widetilde{f}(t, \vec{p}), \quad \widetilde{f}(0, \vec{p}) = \widetilde{f}(\vec{p}) \qquad \Rightarrow \qquad \widetilde{f}(t, \vec{p}) = e^{-i\, E_{\vec{p}}\, t}\, \widetilde{f}(\vec{p})$$

Therefore in all our earlier formulae we simply have to replace $\widetilde{f}(\vec{p})$ by $e^{-i\, E_{\vec{p}}\, t}\, \widetilde{f}(\vec{p})$.

This gives

$$|t, f\rangle = \int d^3r\, f(t, \vec{r})\, \phi(0, \vec{r})|0\rangle, \qquad f(t, \vec{r}) = \int \frac{d^3p}{(2\pi)^{3/2}}\, e^{i\vec{p}.\vec{r}}\, e^{-i\, E_{\vec{p}}\, t}\, \widetilde{f}(\vec{p})$$

$$|t, f\rangle = \int d^3r\, f(t,\vec{r})\, \phi(0,\vec{r})|0\rangle, \qquad f(t,\vec{r}) = \int \frac{d^3p}{(2\pi)^{3/2}}\, e^{i\vec{p}.\vec{r}}\, e^{-i\,E_{\vec{p}}\,t}\, \widetilde{f}(\vec{p})$$

$$f(t,\vec{r}) = \int \frac{d^3p}{(2\pi)^{3/2}}\, e^{i\vec{p}.\vec{r}}\, e^{-i\,E_{\vec{p}}\,t}\, \sigma^{-3/4}\, e^{-(\vec{p}-\vec{k})^2/(2\,\sigma)}$$

Recall that we have taken σ to be small so that the $\vec{p}$ integral is limited to within about $\sqrt{\sigma}$ of $\vec{k}$.

Therefore, we can evaluate the $\vec{p}$ integral by expanding the rest of the terms in Taylor series expansion around $\vec{k}$.

Let us define $\vec{q} = \vec{p} - \vec{k}$. Then

$$f(t,\vec{r}) \simeq \int \frac{d^3q}{(2\pi)^{3/2}}\, e^{i(\vec{q}+\vec{k}).\vec{r}}\, e^{-i\,(E_{\vec{k}}+\vec{v}_{\vec{k}}.\vec{q})t}\, \sigma^{-3/4}\, e^{-\vec{q}^2/(2\,\sigma)}$$

where,

$$(\vec{v}_{\vec{k}})_i = \frac{\partial E_{\vec{k}}}{\partial k_i} = \frac{k_i}{E_{\vec{k}}}, \qquad \text{using } E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$$

$\vec{v}_{\vec{k}}$ has the interpretation of the velocity of a particle carrying momentum $\vec{k}$.

This gives

$$\begin{aligned} f(t,\vec{r}) &\simeq \int \frac{d^3q}{(2\pi)^{3/2}}\, e^{i\vec{k}.\vec{r}}\, e^{-i\,E_{\vec{k}}\,t}\;\; \sigma^{-3/4}\, e^{-\vec{q}^2/(2\,\sigma)+i\vec{q}.\vec{r}-i\vec{q}.\vec{v}_{\vec{k}}\,t} \\ &= \int \frac{d^3q}{(2\pi)^{3/2}}\, e^{i\vec{k}.\vec{r}}\, e^{-i\,E_{\vec{k}}\,t}\;\; \sigma^{-3/4}\, e^{-(\vec{q}-i\sigma\,\vec{r}+i\sigma\,\vec{k})^2/(2\,\sigma)-\sigma(\vec{r}-\vec{v}_{\vec{k}}\,t)^2/2} \\ &= \sigma^{3/4}\, e^{i\vec{k}.\vec{r}}\, e^{-i\,E_{\vec{k}}\,t}\, e^{-\sigma(\vec{r}-\vec{v}_{\vec{k}}\,t)^2/2} \end{aligned}$$

– a gaussian peaked around $\vec{r} = \vec{v}_{\vec{k}}\,t$.

Note: In the approximation we have make, we do not see the increase in the width, but this can be seen by keeping terms of order $\vec{q}^2$ in the expansion of $E_{\vec{p}}$.

Now consider multi-particle states:

$$|\vec{p}_1, \cdots, \vec{p}_n\rangle = \prod_{i=1}^{n} \left\{ \sqrt{2E_{\vec{p}_i}}\, a(\vec{p}_i)^\dagger \right\} |0\rangle$$

$$|f_1, \cdots, f_n\rangle = \int \prod_{i=1}^{n} \left\{ d^3 r_i\, f_i(\vec{r}_i)\, \phi(0, \vec{r}_i) \right\} |0\rangle = \int \prod_{i=1}^{n} \left\{ \frac{d^3 p_i}{2E_{\vec{p}_i}}\, \widetilde{f}_i(\vec{p}_i) \right\} |\vec{p}_1, \cdots, \vec{p}_n\rangle$$

We shall choose

$$\widetilde{f}_i(\vec{p}_i) = \sigma^{-3/4}\, e^{-(\vec{p}_i - \vec{k}_i)^2/(2\,\sigma)}, \qquad \vec{k}_i \neq \vec{k}_j$$

$$\Downarrow$$

$$f_i(\vec{r}_i) = \int \frac{d^3 p_i}{(2\pi)^{3/2}}\, e^{i\vec{p}_i . \vec{r}_i}\, \widetilde{f}_i(\vec{p}_i) = \sigma^{3/4}\, e^{i\vec{k}_i . \vec{r}_i}\, e^{-\sigma \vec{r}_i^{\,2}/2}$$

Note: The wave-functions $f_i(\vec{r}_i)$ all peak around $\vec{r}_i = 0$.

Denote by $|t, f_1, \cdots, f_n\rangle$ the same state at time t if we use the Schrodinger representation.

Since

$$H|\vec{p}_1, \cdots, \vec{p}_n\rangle = \left(\sum_{i=1}^{n} E_{\vec{p}_i} \right) |\vec{p}_1, \cdots, \vec{p}_n\rangle$$

we can find $|t, f_1, \cdots, f_n\rangle$ using the same procedure as before.

Ex. Result: $\widetilde{f}_i(\vec{p}_i) \to \widetilde{f}_i(t, \vec{p}_i) = e^{-iE_{\vec{p}_i} t}\, \widetilde{f}_i(\vec{p}_i)$

$$|t, f_1, \cdots, f_n\rangle = \int \prod_{i=1}^{n} \left\{ d^3 r_i\, f_i(t, \vec{r}_i)\, \phi(0, \vec{r}_i) \right\} |0\rangle$$

$$f_i(t, \vec{r}_i) = \sigma^{3/4}\, e^{i\vec{k}_i . \vec{r}_i}\, e^{-i\, E_{\vec{k}_i}\, t}\, e^{-\sigma(\vec{r}_i - \vec{v}_{\vec{k}_i}\, t)^2/2}$$

For large $|t|$, the peaks get well separated

The width increases as $\sqrt{|t|}$ but since the separation increases as $|t|$, for large $|t|$ we can approximate the states as a set of well separated particles.

Now we turn to the interacting theory.

We assume the existence of single particles states of mass m_p.

$|\vec{p}\rangle$: single particle states of mass m_p, momentum $\vec{p}$ and energy $E_{\vec{p}} = \sqrt{\vec{p}^2 + m_p^2}$, normalized as:

$$\langle \vec{p} | \vec{p}\,' \rangle = 2\, E_{\vec{p}}\, \delta^{(3)}(\vec{p} - \vec{p}\,')$$

Define the state:

$$|f\rangle = \int \frac{d^3 p}{2E_{\vec{p}}}\, \widetilde{f}(\vec{p})\, |\vec{p}\rangle, \qquad f(\vec{p}) = \sigma^{-3/4}\, e^{-(\vec{p}-\vec{k})^2/(2\,\sigma)}$$

In the Schrodinger picture, the state at time t would have the form:

$$\int \frac{d^3 p}{2E_{\vec{p}}}\, \widetilde{f}(t, \vec{p})\, |\vec{p}\rangle, \qquad \widetilde{f}(t, \vec{p}) = e^{-i\, E_{\vec{p}}\, t}\, \widetilde{f}(\vec{p})$$

Interpretation: $\widetilde{f}(t, \vec{p})$: momentum space wave-function at time t

Its Fourier transform $f(t, \vec{r})$ will have the interpretation of position space wave-function at time t:

$$f(t, \vec{r}) \simeq \sigma^{3/4}\, e^{i\vec{k}.\vec{r}}\, e^{-i\, E_{\vec{k}}\, t}\, e^{-\sigma(\vec{r}-\vec{v}_{\vec{k}}\, t)^2/2}$$

– peaks along the classical trajectory $\vec{r} = \vec{v}_{\vec{k}}\, t$.

Note: Now $\vec{v}_{\vec{k}}$ denotes the velocity of a particle of momentum $\vec{k}$ and mass m_p.

Consider now multi-particle states.

From physical consideration it is clear that at any time t, there should be a state in which n particles are moving along widely separated trajectories which will eventually intersect at some space-time point in the future.

We shall take this space-time point to be at the origin $(t = 0, \vec{r} = 0)$.

There is a state which in the far past appears to have n particles moving along the trajectories

$$\vec{r}_i = \vec{v}_{\vec{k}_i}\, t, \qquad i = 1, 2, \cdots, n$$

with wave-function:

$$\widetilde{f}_i(t, \vec{p}) = e^{-i\,E_{\vec{p}}\,t}\,\widetilde{f}_i(\vec{p}) = e^{-i\,E_{\vec{p}} t}\,\sigma^{-3/4}\,e^{-(\vec{p}-\vec{k}_i)^2/(2\,\sigma)}, \qquad f_i(t, \vec{r}_i) \simeq \sigma^{3/4}\,e^{i\vec{k}_i.\vec{r}_i}\,e^{-i\,E_{\vec{k}_i}\,t}\,e^{-\sigma(\vec{r}_i - \vec{v}_{\vec{k}_i}\,t)^2/2}$$

If we do any local experiment around $\vec{r}_i = \vec{v}_{\vec{k}_i}\,t$, the state will be indistinguishable from the single particle state

$$\int \frac{d^3p}{2E_{\vec{p}}}\,\widetilde{f}_i(t, \vec{p})\,|\vec{p}\rangle$$

Postulate: At $t = 0$, there is a basis of states labelled as $|\vec{p}_1, \cdots, \vec{p}_n\rangle_{in}$ such that the state:

$$\int \prod_{i=1}^{n} \left\{ \frac{d^3p_i}{2E_{\vec{p}_i}}\,\widetilde{f}_i(t, \vec{p}_i) \right\}\,|\vec{p}_1, \cdots, \vec{p}_n\rangle_{in}$$

gives the state described above for large negative t.

We can repeat the analysis for future trajectories.

At any time, there should be a state in which there are n particles moving along widely separated trajectories, which intersect at some space-time point in the past.

We shall take this space-time point to be at the origin $(t = 0, \vec{r} = 0)$.

There is a state which in the far future appears to have n particles moving along the trajectories

$$\vec{r}_i = \vec{v}_{\vec{k}_i}\, t, \qquad i = 1, 2, \cdots, n$$

with wave-function:

$$\widetilde{f}_i(t, \vec{p}) = e^{-i\, E_{\vec{p}}\, t}\, \widetilde{f}_i(\vec{p}) = e^{-i\, E_{\vec{p}} t}\, \sigma^{-3/4}\, e^{-(\vec{p}-\vec{k}_i)^2/(2\,\sigma)}, \qquad f_i(t, \vec{r}_i) \simeq \sigma^{3/4}\, e^{i\vec{k}_i.\vec{r}_i}\, e^{-i\, E_{\vec{k}_i}\, t}\, e^{-\sigma(\vec{r}_i - \vec{v}_{\vec{k}_i}\, t)^2/2}$$

If we do any local experiment around $\vec{r}_i = \vec{v}_{\vec{k}_i}\, t$, the state will be indistinguishable from the single particle state

$$\int \frac{d^3p}{2E_{\vec{p}}}\, \widetilde{f}_i(t, \vec{p})\, |\vec{p}\rangle$$

Postulate: At $t = 0$, there is a basis of states labelled as $|\vec{p}_1, \cdots, \vec{p}_n\rangle_{out}$ such that the state:

$$\int \prod_{i=1}^{n} \left\{ \frac{d^3p_i}{2E_{\vec{p}_i}}\, \widetilde{f}_i(t, \vec{p}_i) \right\} |\vec{p}_1, \cdots, \vec{p}_n\rangle_{out}$$

gives the state described above for large positive t.

Summary of postulates:

1. At $t = 0$, there is a basis of states labelled as $|\vec{p}_1, \cdots, \vec{p}_n\rangle_{in}$ such that the state:

$$\int \prod_{i=1}^{n} \left\{ \frac{d^3 p_i}{2E_{\vec{p}_i}} \, \widetilde{f}_i(t, \vec{p}_i) \right\} |\vec{p}_1, \cdots, \vec{p}_n\rangle_{in}$$

describes n widely separated particles moving along the trajectories $\vec{r}_i = \vec{v}_{\vec{k}_i}\, t$, with momentum space wave function $\widetilde{f}_i(t, \vec{p}_i) = \widetilde{f}_i(\vec{p}_i)\, e^{-iE_{\vec{p}_i} t}$, for large negative t.

2. At $t = 0$, there is a basis of states labelled as $|\vec{p}_1, \cdots, \vec{p}_n\rangle_{out}$ such that the state:

$$\int \prod_{i=1}^{n} \left\{ \frac{d^3 p_i}{2E_{\vec{p}_i}} \, \widetilde{f}_i(t, \vec{p}_i) \right\} |\vec{p}_1, \cdots, \vec{p}_n\rangle_{out}$$

describes n widely separated particles moving along the trajectories $\vec{r}_i = \vec{v}_{\vec{k}_i}\, t$, with momentum space wave function $\widetilde{f}_i(t, \vec{p}_i) = \widetilde{f}_i(\vec{p}_i)\, e^{-iE_{\vec{p}_i} t}$, for large positive t.

If

$$|\vec{p}_1, \cdots, \vec{p}_n\rangle_{in} = |\vec{p}_1, \cdots, \vec{p}_n\rangle_{out}$$

then incoming particles in the far past along the trajectories $\vec{r}_i = \vec{v}_{\vec{k}_i}\, t$ with momentum space wave function $\widetilde{f}_i(\vec{p}_i)\, e^{-iE_{\vec{p}_i} t}$ will evolve to outgoing particles in the far future along the trajectories $\vec{r}_i = \vec{v}_{\vec{k}_i}\, t$ with momentum space wave function $\widetilde{f}_i(\vec{p}_i)\, e^{-iE_{\vec{p}_i} t}$.

This is what happens in the free K-G theory.

But we do not expect this in an interacting theory like ϕ^4 theory.

We define the S-matrix:

$$S(\vec{q}_1, \cdots, \vec{q}_m | \vec{p}_1, \cdots, \vec{p}_n) = {}_{out}\langle \vec{q}_1, \cdots, \vec{q}_m | \vec{p}_1, \cdots, \vec{p}_n\rangle_{in}$$

$$S(\vec{q}_1, \cdots, \vec{q}_m | \vec{p}_1, \cdots, \vec{p}_n) = \ _{out}\langle \vec{q}_1, \cdots, \vec{q}_m | \vec{p}_1, \cdots, \vec{p}_n \rangle_{in}$$

– should reflect the probability amplitude for scattering of a set of incoming states of momenta $p_1, \cdots, p_n$ to a set of outgoing particles of momenta $q_1, \cdots, q_m$.

1. How do we calculate this?

– LSZ (Lehmann, Symanzik, Zimmermann) formalism

2. How do we relate this to experimental measurement?

– has different answers in different applications, e.g. in particle physics, condensed matter physics and cosmology.

We shall focus on the particle physics viewpoint.

Energy and momentum eigenvalues of $|p_1, \cdots, p_n\rangle_{in}$ and $|p_1, \cdots, p_n\rangle_{out}$:

Since energy and momenta are conserved, we can calculate them by evolving the state backward or forward in time.

Recall that

$$\int \prod_{i=1}^{n} \left\{ \frac{d^3 p_i}{2E_{\vec{p}_i}} \, \widetilde{f}_i(\vec{p}_i) \right\} |\vec{p}_1, \cdots, \vec{p}_n\rangle_{in}$$

when evolved backward using the Schrodinger equation to large negative t, describes n widely separated particles moving along the trajectories $\vec{r}_i = \vec{v}_{\vec{k}_i}\, t$ with momentum space wave function $\widetilde{f}_i(\vec{p}_i)\, e^{-iE_{\vec{p}_i} t}$.

For small σ, $\widetilde{f}_i(\vec{p}_i)$ is sharply localized around $\vec{p}_i = \vec{k}_i$, and this state has momentum $\sum_{i=1}^{n} \vec{k}_i$ and energy $\sum_{i=1}^{n} E_{\vec{k}_i}$

Therefore we conclude:

$$\vec{P}|\vec{k}_1, \cdots, \vec{k}_n\rangle_{in} = \left(\sum_{i=1}^{n} \vec{k}_i \right) |\vec{k}_1, \cdots, \vec{k}_n\rangle_{in}, \qquad H|\vec{k}_1, \cdots, \vec{k}_n\rangle_{in} = \left(\sum_{i=1}^{n} E_{\vec{k}_i} \right) |\vec{k}_1, \cdots, \vec{k}_n\rangle_{in}$$

The same result can be proved for the out states by evolving them forward in time.

Similar result holds for other conserved charges.

We also have the results:

$$|\vec{k}_1\rangle_{in} = |\vec{k}_1\rangle_{out}, \qquad |\Omega\rangle_{in} = |\Omega\rangle_{out}$$

26 S-matrix

We have introduced in and out states:

$$|\vec{k}_1, \cdots, \vec{k}_n\rangle_{in} \qquad \text{and} \qquad |\vec{k}_1, \cdots, \vec{k}_n\rangle_{out}$$

Intuitive understanding:

$|\vec{k}_1, \cdots, \vec{k}_n\rangle_{in}$ represents a state at $t = 0$, which, when evolved to large negative t by Schroginger equation, describes n well separated particles of momentum $\vec{k}_1, \cdots, \vec{k}_n$.

This cannot be strictly true, since single particle momentum eigenstates have infinite spread in position space, and the wave-functions of the n particles will continue to overlap even in the far past.

Solution: Spread out the momentum a bit around $\vec{k}_i$ by smearing function $\widetilde{f}_i(\vec{p})$ so that the position space wave-functions are localized around classical trajectories.

In that case, by going sufficiently far back in the past, we can make the wave-functions non-overlapping and the earlier interpretation holds.

In a similar spirit, $|\vec{k}_1, \cdots, \vec{k}_n\rangle_{out}$ represents a state at $t = 0$, which, when evolved to large positive t by Schroginger equation, describes n well separated particles of momentum $\vec{k}_1, \cdots, \vec{k}_n$.

For many applications, we can just use this simplistic view, keeping the idea of the smearing function at the back of our mind.

We derived the relations:

$$\vec{P}|\vec{k}_1, \cdots, \vec{k}_n\rangle_{in} = \left(\sum_{i=1}^{n} \vec{k}_i\right) |\vec{k}_1, \cdots, \vec{k}_n\rangle_{in}, \qquad H|\vec{k}_1, \cdots, \vec{k}_n\rangle_{in} = \left(\sum_{i=1}^{n} E_{\vec{k}_i}\right) |\vec{k}_1, \cdots, \vec{k}_n\rangle_{in}$$

The same result can be proved for the out states by evolving them forward in time.

Similar result holds for other conserved charges.

We also have the results:

$$|\vec{k}_1\rangle_{in} = |\vec{k}_1\rangle_{out}, \qquad |\Omega\rangle_{in} = |\Omega\rangle_{out}$$

How do we calculate

$$S(\vec{q}_1, \cdots, \vec{q}_m | \vec{p}_1, \cdots, \vec{p}_n) = {}_{out}\langle \vec{q}_1, \cdots, \vec{q}_m | \vec{p}_1, \cdots, \vec{p}_n \rangle_{in}$$

in an interacting scalar field theory?

Go back to some results in the free K-G theory:

$$\phi(t,\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3p \, \widetilde{\phi}(t,\vec{p}) \, e^{i\vec{p}.\vec{r}} = \frac{1}{(2\pi)^{3/2}} \int d^3p \, e^{i\vec{p}.\vec{r}} \frac{1}{\sqrt{2\,E_{\vec{p}}}} \left(a(t,\vec{p}) + a(t,-\vec{p})^\dagger\right)$$

$$= \frac{1}{(2\pi)^{3/2}} \int d^3p \, e^{i\vec{p}.\vec{r}} \frac{1}{\sqrt{2\,E_{\vec{p}}}} \left(a(\vec{p}) \, e^{-iE_{\vec{p}}t} + a(-\vec{p})^\dagger \, e^{iE_{\vec{p}}t}\right)$$

$$a(\vec{p}) = a(0,\vec{p}), \qquad a(\vec{p})^\dagger = a(0,\vec{p})^\dagger$$

Define

$$g_{\vec{p}}(t,\vec{r}) = \frac{1}{(2\pi)^{3/2}} \, e^{i\vec{p}.\vec{r} - iE_{\vec{p}}t} \, \frac{1}{\sqrt{2\,E_{\vec{p}}}}$$

$$A \overleftrightarrow{\partial_t} B = A\partial_t B - (\partial_t A) B$$

Ex. Check that

$$i\,a(\vec{p})^\dagger = \int d^3r \, g_{\vec{p}}(t,\vec{r}) \, \overleftrightarrow{\partial_t} \, \phi(t,\vec{r}), \qquad -i\,a(\vec{p}) = \int d^3r \, g_{\vec{p}}(t,\vec{r})^* \, \overleftrightarrow{\partial_t} \, \phi(t,\vec{r})$$

Note: Even though rhs seems to have time dependence, it is actually time independent.

In the interacting theory we define $g_{\vec{p}}$ as before with m replaced by m_p, and define

$$i\,a(t,\vec{p})^\dagger = \int d^3r \, g_{\vec{p}}(t,\vec{r}) \, \overleftrightarrow{\partial_t} \, \phi(t,\vec{r}), \qquad -i\,a(t,\vec{p}) = \int d^3r \, g_{\vec{p}}(t,\vec{r})^* \, \overleftrightarrow{\partial_t} \, \phi(t,\vec{r})$$

– no longer time independent.

Ex. Check that (no need for equation of motion of ϕ)

$$\partial_t \, a(t,\vec{p})^\dagger = -i \int d^3r \, g_{\vec{p}}(t,\vec{r}) \, (-\Box + m_p^2)\phi(t,\vec{r}),$$

$$\partial_t \, a(t,\vec{p}) = i \int d^3r \, g_{\vec{p}}(t,\vec{r})^* \, (-\Box + m_p^2)\phi(t,\vec{r}), \qquad \Box = -\partial_t^2 + \vec{\nabla}^2$$

Define

$$a_{in}(\vec{p})^\dagger = \frac{1}{\sqrt{Z}} \lim_{T\to\infty} a(-T(1-i\epsilon), \vec{p})^\dagger$$

Z: Constant fixed earlier via

$$\langle\Omega|\phi(0)|\vec{p}\rangle = \sqrt{Z}/(2\pi)^{3/2}$$

Similarly, we define:

$$a_{in}(\vec{p}) = \frac{1}{\sqrt{Z}} \lim_{T\to\infty} a(-T(1-i\epsilon), \vec{p})$$

$$a_{out}(\vec{p})^\dagger = \frac{1}{\sqrt{Z}} \lim_{T\to\infty} a(T(1-i\epsilon), \vec{p})^\dagger, \qquad a_{out}(\vec{p}) = \frac{1}{\sqrt{Z}} \lim_{T\to\infty} a(T(1-i\epsilon), \vec{p})$$

Note $\dagger$ does not act on $i\epsilon$.

Claim (to be proved later):

$$a_{in}(\vec{p}_1)^\dagger|\vec{p}_2, \cdots, \vec{p}_n\rangle_{in} = \frac{1}{\sqrt{2E_{\vec{p}_1}}} |\vec{p}_1, \cdots, \vec{p}_n\rangle_{in}$$

$$_{out}\langle\vec{p}_2, \cdots, \vec{p}_n|a_{out}(\vec{p}_1) = \frac{1}{\sqrt{2E_{\vec{p}_1}}} \ _{out}\langle\vec{p}_1, \cdots, \vec{p}_n|$$

$$a_{in}(\vec{p})|\vec{p}_1, \cdots, \vec{p}_n\rangle_{in} = \sum_{i=1}^{n} \sqrt{2E_{\vec{p}_i}}\, \delta^{(3)}(\vec{p}-\vec{p}_i)\, |\vec{p}_1, \cdots \vec{p}_{i-1}, \vec{p}_{i+1}, \cdots, \vec{p}_n\rangle_{in}$$

$$_{out}\langle\vec{p}_1, \cdots, \vec{p}_n|a_{out}(\vec{p})^\dagger = \sum_{i=1}^{n} \sqrt{2E_{\vec{p}_i}}\, \delta^{(3)}(\vec{p}-\vec{p}_i)\ \ _{out}\langle\vec{p}_1, \cdots \vec{p}_{i-1}, \vec{p}_{i+1}, \cdots, \vec{p}_n|$$

Therefore $a_{in}, a_{in}^\dagger$ and $a_{out}, a_{out}^\dagger$ act as creation / annihilation operators on the in and the out states.

We shall first assume this and see what it gives.

$$S(\vec{q}_1,\cdots,\vec{q}_m|\vec{p}_1,\cdots,\vec{p}_n) = {}_{out}\langle\vec{q}_1,\cdots,\vec{q}_m|\vec{p}_1,\cdots,\vec{p}_n\rangle_{in}$$

$$= \sqrt{2E_{\vec{p}_1}}\ {}_{out}\langle\vec{q}_1,\cdots,\vec{q}_m|a_{in}(\vec{p}_1)^\dagger|\vec{p}_2,\cdots,\vec{p}_n\rangle_{in}$$

We shall proceed by assuming that none of the q_i's are equal to any of the p_i's.

Otherwise we shall get extra 'forward scattering terms'.

$$S(\vec{q}_1,\cdots,\vec{q}_m|\vec{p}_1,\cdots,\vec{p}_n) = \sqrt{2E_{\vec{p}_1}}\ {}_{out}\langle\vec{q}_1,\cdots,\vec{q}_m|(a_{in}(\vec{p}_1)^\dagger - a_{out}(\vec{p}_1)^\dagger)|\vec{p}_2,\cdots,\vec{p}_n\rangle_{in}$$

Using

$$a_{in}(\vec{p})^\dagger = \frac{1}{\sqrt{Z}}\lim_{T\to\infty} a(-T(1-i\epsilon),\vec{p})^\dagger, \qquad a_{out}(\vec{p})^\dagger = \frac{1}{\sqrt{Z}}\lim_{T\to\infty} a(T(1-i\epsilon),\vec{p})^\dagger$$

we can express the S-matrix as:

$$-Z^{-1/2}\sqrt{2E_{\vec{p}_1}}\ {}_{out}\langle\vec{q}_1,\cdots,\vec{q}_m|\int_{-T(1-i\epsilon)}^{T(1-i\epsilon)} dt\,\partial_t a(t,\vec{p}_1)^\dagger|\vec{p}_2,\cdots,\vec{p}_n\rangle_{in}$$

Using

$$\partial_t\, a(t,\vec{p})^\dagger = -i\int d^3r\, g_{\vec{p}}(t,\vec{r})\,(-\Box + m_p^2)\phi(t,\vec{r})$$

we get,

$$i\,Z^{-1/2}\sqrt{2E_{\vec{p}_1}}\int d^4x\ g_{\vec{p}_1}(x)\,(-\Box + m_p^2)\ {}_{out}\langle\vec{q}_1,\cdots,\vec{q}_m|\phi(x)|\vec{p}_2,\cdots,\vec{p}_n\rangle_{in}$$

$$\int d^4x = \int_{-T(1-i\epsilon)}^{T(1-i\epsilon)} dt\int d^3r, \qquad x = (t,\vec{r})$$

We rename x as x_1 to write this as:

$$i\,Z^{-1/2}\sqrt{2E_{\vec{p}_1}}\int d^4x_1\ g_{\vec{p}_1}(x_1)\,(-\Box_{x_1} + m_p^2)\ {}_{out}\langle\vec{q}_1,\cdots,\vec{q}_m|\phi(x_1)|\vec{p}_2,\cdots,\vec{p}_n\rangle_{in}$$

$$i\,Z^{-1/2}\,\sqrt{2E_{\vec p_1}}\,\int d^4x_1\;g_{\vec p_1}(x_1)\,(-\Box_{x_1}+m_p^2)\;{}_{out}\langle\vec q_1,\cdots,\vec q_m|\phi(x_1)|\vec p_2,\cdots,\vec p_n\rangle_{in}$$

$$= i\,Z^{-1/2}\,\sqrt{2E_{\vec p_1}}\,\sqrt{2E_{\vec p_2}}\,\int d^4x_1\;g_{\vec p_1}(x_1)\,(-\Box_{x_1}+m_p^2)\;{}_{out}\langle\vec q_1,\cdots,\vec q_m|\phi(x_1)a_{in}(\vec p_2)^\dagger|\vec p_3,\cdots,\vec p_n\rangle_{in}$$

$$= i\,Z^{-1/2}\,\sqrt{2E_{\vec p_1}}\,\sqrt{2E_{\vec p_2}}\,\int d^4x_1\;g_{\vec p_1}(x_1)$$
$$(-\Box_{x_1}+m_p^2)\;{}_{out}\langle\vec q_1,\cdots,\vec q_m|\{\phi(x_1)a_{in}(\vec p_2)^\dagger - a_{out}(\vec p_2)^\dagger\phi(x_1)\}|\vec p_3,\cdots,\vec p_n\rangle_{in}$$

$$= -i\,Z^{-1}\,\sqrt{2E_{\vec p_1}}\,\sqrt{2E_{\vec p_2}}\,\int d^4x_1\,\int_{-T(1-i\epsilon)}^{T(1-i\epsilon)} dt_2\,g_{\vec p_1}(x_1)$$
$$(-\Box_{x_1}+m_p^2)\,\partial_{t_2}\;{}_{out}\langle\vec q_1,\cdots,\vec q_m|T(\phi(x_1)a(t_2,\vec p_2)^\dagger)|\vec p_3,\cdots,\vec p_n\rangle_{in}$$

Now use

$$a(t_2,\vec p_2)^\dagger = -i\,\int d^3r_2\,g_{\vec p_2}(t_2,\vec r_2)\,\overleftrightarrow{\partial_{t_2}}\,\phi(t_2,\vec r_2)$$

to express the S-matrix as:

$$= (i)^2\,Z^{-1}\,\sqrt{2E_{\vec p_1}}\,\sqrt{2E_{\vec p_2}}\,\int d^4x_1\,\int d^4x_2\,g_{\vec p_1}(x_1)$$
$$(-\Box_{x_1}+m_p^2)\,\partial_{t_2}\Big[g_{\vec p_2}(t_2,\vec r_2)\;{}_{out}\langle\vec q_1,\cdots,\vec q_m|T(\phi(x_1)\partial_{t_2}\phi(t_2,\vec p_2))|\vec p_3,\cdots,\vec p_n\rangle_{in}$$
$$-\partial_{t_2}g_{\vec p_2}(t_2,\vec r_2)\;{}_{out}\langle\vec q_1,\cdots,\vec q_m|T(\phi(x_1)\phi(t_2,\vec p_2))|\vec p_3,\cdots,\vec p_n\rangle_{in}\Big]$$

Use

$$\partial_{t_2}T(\phi(x_1)\phi(x_2)) = T(\phi(x_1)\partial_{t_2}\phi(x_2)) + \delta(t_1-t_2)(\phi(x_2)\phi(x_1)-\phi(x_1)\phi(x_2)) = T(\phi(x_1)\partial_{t_2}\phi(x_2))$$

to express the S-matrix as

$$= (i)^2\,Z^{-1}\,\sqrt{2E_{\vec p_1}}\,\sqrt{2E_{\vec p_2}}\,\int d^4x_1\,\int d^4x_2\;g_{\vec p_1}(x_1)$$
$$(-\Box_{x_1}+m_p^2)\,\partial_{t_2}\Big[g_{\vec p_2}(t_2,\vec r_2)\partial_{t_2}\;{}_{out}\langle\vec q_1,\cdots,\vec q_m|T(\phi(x_1)\phi(t_2,\vec p_2))|\vec p_3,\cdots,\vec p_n\rangle_{in}$$
$$-\partial_{t_2}g_{\vec p_2}(t_2,\vec r_2)\;{}_{out}\langle\vec q_1,\cdots,\vec q_m|T(\phi(x_1)\phi(t_2,\vec p_2))|\vec p_3,\cdots,\vec p_n\rangle_{in}\Big]$$

S-matrix is given by

$$= (i)^2\, Z^{-1}\, \sqrt{2E_{\vec{p}_1}}\, \sqrt{2E_{\vec{p}_2}} \int d^4x_1 \int d^4x_2\ g_{\vec{p}_1}(x_1)$$

$$(-\Box_{x_1} + m_p^2)\, \partial_{t_2} \Big[g_{\vec{p}_2}(t_2, \vec{r}_2) \partial_{t_2}\ {}_{out}\langle \vec{q}_1, \cdots, \vec{q}_m | T(\phi(x_1)\phi(t_2, \vec{p}_2)) | \vec{p}_3, \cdots, \vec{p}_n \rangle_{in}$$

$$-\partial_{t_2} g_{\vec{p}_2}(t_2, \vec{r}_2)\ {}_{out}\langle q_1, \cdots, q_m | T(\phi(x_1)\phi(t_2, \vec{p}_2)) | p_3, \cdots, p_n \rangle_{in} \Big]$$

$$= (i)^2\, Z^{-1}\, \sqrt{2E_{\vec{p}_1}}\, \sqrt{2E_{\vec{p}_2}} \int d^4x_1 \int d^4x_2\ g_{\vec{p}_1}(x_1)$$

$$(-\Box_{x_1} + m_p^2) \Big[g_{\vec{p}_2}(t_2, \vec{r}_2) \partial_{t_2}^2\ {}_{out}\langle \vec{q}_1, \cdots, \vec{q}_m | T(\phi(x_1)\phi(t_2, \vec{p}_2)) | \vec{p}_3, \cdots, \vec{p}_n \rangle_{in}$$

$$-\partial_{t_2}^2 g_{\vec{p}_2}(t_2, \vec{r}_2)\ {}_{out}\langle q_1, \cdots, q_m | T(\phi(x_1)\phi(t_2, \vec{p}_2)) | p_3, \cdots, p_n \rangle_{in} \Big]$$

$$g_{\vec{p}}(t, \vec{r}) = \frac{1}{(2\pi)^{3/2}}\, e^{i\vec{p}.\vec{r} - iE_{\vec{p}}t}\, \frac{1}{\sqrt{2\, E_{\vec{p}}}}$$

$$\Rightarrow \qquad \partial_{t_2}^2 g_{\vec{p}_2}(t_2, \vec{r}_2) = -E_{\vec{p}_2}^2 g_{\vec{p}_2}(t_2, \vec{r}_2) = -(\vec{p}_2^{\,2} + m_p^2) g_{\vec{p}_2}(t_2, \vec{r}_2) = (\vec{\nabla}_2^2 - m_p^2) g_{\vec{p}_2}(t_2, \vec{r}_2)$$

We substitute this into the expression for the S-matrix and integrate by parts in $\vec{r}_2$ to have $\vec{\nabla}_2^2$ act on the matrix element.

This expresses the S-matrix as

$$= (i)^2\, Z^{-1}\, \sqrt{2E_{\vec{p}_1}}\, \sqrt{2E_{\vec{p}_2}} \int d^4x_1 \int d^4x_2\ g_{\vec{p}_1}(x_1)\, g_{\vec{p}_2}(x_2)$$

$$(-\Box_{x_1} + m_p^2)\, (-\Box_{x_2} + m_p^2)\ {}_{out}\langle \vec{q}_1, \cdots, \vec{q}_m | T(\phi(x_1)\phi(x_2)) | \vec{p}_3, \cdots, \vec{p}_n \rangle_{in}$$

We can continue this procedure the express the S-matrix as

$$= (i)^n\, Z^{-n/2} \left\{ \prod_{i=1}^n \sqrt{2E_{\vec{p}_i}} \right\} \int d^4x_1 \cdots d^4x_n\ \left\{ \prod_{i=1}^n g_{\vec{p}_i}(t_i, \vec{r}_i) \right\}$$

$$(-\Box_{x_1} + m_p^2) \cdots (-\Box_{x_n} + m_p^2)\ {}_{out}\langle \vec{q}_1, \cdots, \vec{q}_m | T(\phi(x_1) \cdots \phi(x_n)) | \Omega \rangle$$

We now manipulate this as:

$${}_{out}\langle q_1, \cdots, q_m | T(\phi(x_1) \cdots \phi(x_n)) | \Omega \rangle = \sqrt{2E_{\vec{q}_1}}\ {}_{out}\langle \vec{q}_2, \cdots, \vec{q}_m | a_{out}(\vec{q}_1) T(\phi(x_1) \cdots \phi(x_n)) | \Omega \rangle$$

$$= \sqrt{2E_{\vec{q}_1}}\ {}_{out}\langle \vec{q}_2, \cdots, \vec{q}_m | \left\{ a_{out}(\vec{q}_1) T(\phi(x_1) \cdots \phi(x_n)) - T(\phi(x_1) \cdots \phi(x_n)) a_{in}(\vec{q}_1) \right\} | \Omega \rangle$$

and express this as time ordered product involving time integral of $\partial_t a(t, \vec{q}_1)$.

Ex. Final result:

$$S(\vec{q}_1,\cdots,\vec{q}_m|\vec{p}_1,\cdots,\vec{p}_n)$$
$$= (i)^{n+m}\, Z^{-(m+n)/2} \left\{\prod_{i=1}^n \sqrt{2E_{\vec{p}_i}}\right\}\left\{\prod_{j=1}^m \sqrt{2E_{\vec{q}_j}}\right\}\int d^4x_1\cdots d^4x_n\; d^4y_1\cdots d^4y_m$$
$$\left\{\prod_{i=1}^n g_{\vec{p}_i}(x_i)\right\}\left\{\prod_{j=1}^m g_{\vec{q}_j}(y_j)^*\right\}(-\Box_{x_1}+m_p^2)\cdots(-\Box_{x_n}+m_p^2)(-\Box_{y_1}+m_p^2)\cdots(-\Box_{y_m}+m_p^2)$$
$$\langle\Omega|T(\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m))|\Omega\rangle$$

Use momentum space representation of Green's function:

$$\langle\Omega|T(\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m))|\Omega\rangle$$
$$= \int \left\{\prod_{i=1}^n \frac{d^4k_i}{(2\pi)^4}\right\}\left\{\prod_{j=1}^m \frac{d^4\ell_i}{(2\pi)^4}\right\}\left\{\prod_{i=1}^n e^{ik_i.x_i}\right\}\left\{\prod_{j=1}^m e^{i\ell_j.y_j}\right\}\widetilde{G}(k_1,\cdots,k_n,\ell_1,\cdots,\ell_m)$$

we get

$$S(\vec{q}_1,\cdots,\vec{q}_m|\vec{p}_1,\cdots,\vec{p}_n)$$
$$= (i)^{n+m}\, Z^{-(m+n)/2} \left\{\prod_{i=1}^n \sqrt{2E_{\vec{p}_i}}\right\}\left\{\prod_{j=1}^m \sqrt{2E_{\vec{q}_j}}\right\}\int d^4x_1\cdots d^4x_n\; d^4y_1\cdots d^4y_m$$
$$\int \left\{\prod_{i=1}^n \frac{d^4k_i}{(2\pi)^4}\right\}\left\{\prod_{j=1}^m \frac{d^4\ell_i}{(2\pi)^4}\right\}\left\{\prod_{i=1}^n g_{\vec{p}_i}(x_i)\right\}\left\{\prod_{j=1}^m g_{\vec{q}_j}(y_j)^*\right\}\left\{\prod_{i=1}^n e^{ik_i.x_i}\right\}\left\{\prod_{j=1}^m e^{i\ell_j.y_j}\right\}$$
$$(k_1^2+m_p^2)\cdots(k_n^2+m_p^2)(\ell_1^2+m_p^2)\cdots(\ell_m^2+m_p^2)\,\widetilde{G}(k_1,\cdots,k_n,\ell_1,\cdots,\ell_m)$$

Now use

$$g_{\vec{p}}(x) = \frac{1}{(2\pi)^{3/2}}\, e^{ip.x}\, \frac{1}{\sqrt{2\,E_{\vec{p}}}}, \qquad g_{\vec{q}}(y)^* = \frac{1}{(2\pi)^{3/2}}\, e^{-iq.y}\, \frac{1}{\sqrt{2\,E_{\vec{q}}}}$$

The x_i and y_j integrals generate

$$\left\{\prod_{i=1}^n (2\pi)^4\, \delta^{(4)}(p_i+k_i)\right\}\left\{\prod_{j=1}^m (2\pi)^4\, \delta^{(4)}(q_j-\ell_j)\right\}, \qquad p_i^0 = E_{\vec{p}_i},\; q_i^0 = E_{\vec{q}_i}$$

This sets $k_i = -p_i$, $\ell_j = q_j$ and gives

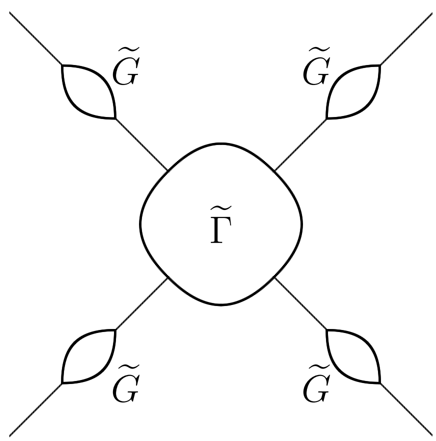
$$
\begin{aligned}
& S(\vec{q}_1, \cdots, \vec{q}_m | \vec{p}_1, \cdots, \vec{p}_n) \\
& = (i)^{n+m} \, Z^{-(m+n)/2} \, (2\pi)^{-3(m+n)/2} \, \left\{ \prod_{i=1}^{n} (p_i^2 + m_p^2) \right\} \left\{ \prod_{j=1}^{m} (q_j^2 + m_p^2) \right\} \\
& \qquad \qquad \widetilde{G}(-p_1, \cdots, -p_n, q_1, \cdots, q_m)
\end{aligned}
$$

Note: We have $p_i^2 + m_p^2 = 0$ and $q_j^2 + m_p^2 = 0$.

Therefore only the part of $\widetilde{G}$ that has poles at $p_i^2 + m_p^2 = 0$ and $q_j^2 + m_p^2 = 0$ will contribute.

How do we know if $\widetilde{G}$ has such poles?

Feynman diagram representation:



$\widetilde{\Gamma}$: amputated Green's function

Recall: Near $p_1^2 + m_p^2 = 0$,

$$
\widetilde{G}(p_1, p_2) \simeq (2\pi)^4 \, \delta^{(4)}(p_1 + p_2) \, Z \, \frac{i}{-p_1^2 - m_p^2 + i\epsilon}
$$

$$
\Rightarrow \quad S(\vec{q}_1, \cdots, \vec{q}_m | \vec{p}_1, \cdots, \vec{p}_n) = Z^{(m+n)/2} \, (2\pi)^{-3(m+n)/2} \, \widetilde{\Gamma}(-p_1, \cdots, -p_n, q_1, \cdots, q_m)
$$

27 LSZ formalism

In the last lecture we found the expression for the S-matrix in terms of the momentum space Green's function, assuming certain relations:

$$a_{in}(\vec{p}_1)^\dagger|\vec{p}_2,\cdots,\vec{p}_n\rangle_{in} = \frac{1}{\sqrt{2E_{\vec{p}_1}}}\,|\vec{p}_1,\cdots,\vec{p}_n\rangle_{in}$$

$$_{out}\langle\vec{p}_2,\cdots,\vec{p}_n|a_{out}(\vec{p}_1) = \frac{1}{\sqrt{2E_{\vec{p}_1}}}\;_{out}\langle\vec{p}_1,\cdots,\vec{p}_n|$$

$$a_{in}(\vec{p})|\vec{p}_1,\cdots,\vec{p}_n\rangle_{in} = \sum_{i=1}^{n}\sqrt{2E_{\vec{p}_i}}\,\delta^{(3)}(\vec{p}-\vec{p}_i)\,|\vec{p}_1,\cdots\vec{p}_{i-1},\vec{p}_{i+1},\cdots,\vec{p}_n\rangle_{in}$$

$$_{out}\langle\vec{p}_1,\cdots,\vec{p}_n|a_{out}(\vec{p})^\dagger = \sum_{i=1}^{n}\sqrt{2E_{\vec{p}_i}}\,\delta^{(3)}(\vec{p}-\vec{p}_i)\;\;_{out}\langle\vec{p}_1,\cdots\vec{p}_{i-1},\vec{p}_{i+1},\cdots,\vec{p}_n|$$

We shall now try to justify these relations.

Recall some definitions:

$$i\, a(t,\vec{p})^\dagger = \int d^3r\, g_{\vec{p}}(t,\vec{r})\, \overleftrightarrow{\partial_t}\, \phi(t,\vec{r})$$

$$a_{in}(\vec{p})^\dagger = \frac{1}{\sqrt{Z}} \lim_{T\to\infty} a(-T(1-i\epsilon),\vec{p})^\dagger$$

We shall first try to prove that

$$a_{in}(\vec{p})^\dagger|\Omega\rangle = \frac{1}{\sqrt{2E_{\vec{p}}}}\,|\vec{p}\rangle_{in} = \frac{1}{\sqrt{2E_{\vec{p}}}}\,|\vec{p}\rangle$$

Write

$$\begin{aligned} a(t,\vec{p})^\dagger|\Omega\rangle &= \sum_\alpha |\alpha\rangle\langle\alpha|a(t,\vec{p})^\dagger|\Omega\rangle \\ &= -i \int d^3r\, g_{\vec{p}}(t,\vec{r})\, \overleftrightarrow{\partial_t} \sum_\alpha |\alpha\rangle\langle\alpha|\phi(t,\vec{r})|\Omega\rangle \end{aligned}$$

Recall

$$\langle\alpha|\phi(t,\vec{r})|\Omega\rangle = e^{-i\vec{p}_{(\alpha)}.\vec{r}+iE_{(\alpha)}t}\langle\alpha|\phi(0)|\Omega\rangle, \qquad g_{\vec{p}}(t,\vec{r}) = \frac{1}{(2\pi)^{3/2}}\, e^{i\vec{p}.\vec{r}-iE_{\vec{p}}t}\, \frac{1}{\sqrt{2\,E_{\vec{p}}}}$$

Integration over $\vec{r}$ now gives $(2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}_{(\alpha)})$.

$\overleftrightarrow{\partial_t}$ bring down a factor of $i(E_{(\alpha)}+E_{\vec{p}})$

$$a(t,\vec{p})^\dagger|\Omega\rangle = -i\, \frac{1}{(2\pi)^{3/2}}\, \frac{1}{\sqrt{2\,E_{\vec{p}}}}\, (2\pi)^3 \sum_\alpha \delta^{(3)}(\vec{p}-\vec{p}_{(\alpha)})\, i(E_{(\alpha)}+E_{\vec{p}})\, e^{i(E_{(\alpha)}-E_{\vec{p}})t} \\ |\alpha\rangle\langle\alpha|\phi(0)|\Omega\rangle$$

$$a(t,\vec{p})^\dagger|\Omega\rangle = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\,E_{\vec{p}}}}\,(2\pi)^3\sum_\alpha \delta^{(3)}(\vec{p}-\vec{p}_{(\alpha)})\,(E_{(\alpha)}+E_{\vec{p}})\,e^{i(E_{(\alpha)}-E_{\vec{p}})t}\;|\alpha\rangle\langle\alpha|\phi(0)|\Omega\rangle$$

We need to set $t = -T(1-i\epsilon)$, take $T\to\infty$ limit and multiply by $1/\sqrt{Z}$ to get $a_{in}(\vec{p})^\dagger$

$$e^{i(E_{(\alpha)}-E_{\vec{p}})t} \quad\to\quad e^{-i(E_{(\alpha)}-E_{\vec{p}})\,T-(E_{(\alpha)}-E_{\vec{p}})\,T\,\epsilon}$$

As $T\to\infty$, the lowest $E_{(\alpha)}$ state has the dominant contribution.

$|\Omega\rangle$ cannot contribute since $\vec{p}_{(\alpha)}=0$ and the $\delta^{(3)}(\vec{p}-\vec{p}_{(\alpha)})$ make it vanish.

Next best choice: Single particle state of momentum $\vec{p}$

In the $T\to\infty$ limit we can ignore all other contributions.

Recall that for single particle states of momentum $\vec{q}$, $\sum_\alpha \to \int d^3q/(2E_{\vec{q}})$

Also $\langle\vec{q}|\phi(0)|\Omega\rangle = \sqrt{Z}/(2\pi)^{3/2}$.

This gives

$$\begin{aligned} a_{in}(\vec{p})^\dagger|\Omega\rangle &= \frac{1}{\sqrt{Z}}\,\frac{1}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{2\,E_{\vec{p}}}}\,(2\pi)^3\int\frac{d^3q}{2E_{\vec{q}}}\,\delta^{(3)}(\vec{p}-\vec{q})\,e^{-iT(1-i\epsilon)(E_{\vec{q}}-E_{\vec{p}})}\,(E_{\vec{q}}+E_{\vec{p}})\,|\vec{q}\rangle\,\frac{\sqrt{Z}}{(2\pi)^{3/2}} \\ &= \frac{1}{\sqrt{2\,E_{\vec{p}}}}\,|\vec{p}\rangle \end{aligned}$$

This is the desired relation.

Next we shall try to prove

$$a_{in}(\vec{p})|\vec{p}_1\rangle_{in} = \sqrt{2E_{\vec{p}_1}}\,\delta^{(3)}(\vec{p}-\vec{p}_1)\,|\Omega\rangle_{in}$$

Recall

$$-i\,a(t,\vec{p}) = \int d^3r\, g_{\vec{p}}(t,\vec{r})^*\,\overleftrightarrow{\partial_t}\,\phi(t,\vec{r}), \qquad a_{in}(\vec{p}) = \frac{1}{\sqrt{Z}}\lim_{T\to\infty} a(-T(1-i\epsilon),\vec{p})$$

Write

$$\begin{aligned} a(t,\vec{p})|\vec{p}_1\rangle_{in} &= \sum_\alpha |\alpha\rangle\langle\alpha|a(t,\vec{p})|\vec{p}_1\rangle_{in} \\ &= i\int d^3r\, g_{\vec{p}}(t,\vec{r})^*\,\overleftrightarrow{\partial_t}\,\sum_\alpha |\alpha\rangle\langle\alpha|\phi(t,\vec{r})|\vec{p}_1\rangle_{in} \end{aligned}$$

Using $[P_\mu,\phi(x)] = i\partial_\mu\phi(x)$, we get

$$i\partial_\mu\langle\alpha|\phi(t,\vec{r})|\vec{p}_1\rangle_{in} = \langle\alpha|[P_\mu,\phi(t,\vec{r})]|\vec{p}_1\rangle_{in} = (p_{(\alpha)}-p_1)_\mu\langle\alpha|\phi(t,\vec{r})|\vec{p}_1\rangle_{in}, \qquad p_1^0 = E_{\vec{p}_1}$$

This gives

$$\langle\alpha|\phi(t,\vec{r})|\vec{p}_1\rangle_{in} = e^{-i(p_{(\alpha)}-p_1).x}\langle\alpha|\phi(0)|\vec{p}_1\rangle_{in} = e^{-i(\vec{p}_{(\alpha)}-\vec{p}_1).\vec{r}+i(E_{(\alpha)}-E_{\vec{p}_1})t}\langle\alpha|\phi(0)|\vec{p}_1\rangle_{in}$$

Also recall

$$g_{\vec{p}}(t,\vec{r})^* = \frac{1}{(2\pi)^{3/2}}\,e^{-i\vec{p}.\vec{r}+iE_{\vec{p}}t}\,\frac{1}{\sqrt{2\,E_{\vec{p}}}}$$

$$a(t,\vec{p})|\vec{p}_1\rangle_{in} = i\int d^3r\,\frac{1}{(2\pi)^{3/2}}\,e^{-i\vec{p}.\vec{r}+iE_{\vec{p}}t}\,\frac{1}{\sqrt{2\,E_{\vec{p}}}}\,\overleftrightarrow{\partial_t}\,\sum_\alpha|\alpha\rangle e^{-i(\vec{p}_{(\alpha)}-\vec{p}_1).\vec{r}+i(E_{(\alpha)}-E_{\vec{p}_1})t}\langle\alpha|\phi(0)|\vec{p}_1\rangle_{in}$$

Integration over $\vec{r}$ now gives $(2\pi)^3\delta^{(3)}(\vec{p}+\vec{p}_{(\alpha)}-\vec{p}_1)$.

$\overleftrightarrow{\partial_t}$ bring down a factor of $i(E_{(\alpha)}-E_{\vec{p}_1}-E_{\vec{p}})$

$$\begin{aligned} a(t,\vec{p})|\vec{p}_1\rangle_{in} &= i\,\frac{1}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{2\,E_{\vec{p}}}}\,(2\pi)^3\sum_\alpha\delta^{(3)}(\vec{p}+\vec{p}_{(\alpha)}-\vec{p}_1)\,i(E_{(\alpha)}-E_{\vec{p}_1}-E_{\vec{p}}) \\ &\qquad e^{i(E_{(\alpha)}-E_{\vec{p}_1}+E_{\vec{p}})t}\,|\alpha\rangle\langle\alpha|\phi(0)|\vec{p}_1\rangle_{in} \end{aligned}$$

$$
\begin{aligned}
a(t,\vec{p})|\vec{p}_1\rangle_{in} &= i\,\frac{1}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{2\,E_{\vec{p}}}}\,(2\pi)^3\sum_\alpha \delta^{(3)}(\vec{p}+\vec{p}_{(\alpha)}-\vec{p}_1)\,i(E_{(\alpha)}-E_{\vec{p}_1}-E_{\vec{p}}) \\
&\qquad e^{i(E_{(\alpha)}-E_{\vec{p}_1}+E_{\vec{p}})t}\,|\alpha\rangle\langle\alpha|\phi(0)|\vec{p}_1\rangle_{in}
\end{aligned}
$$

We need to set $t=-T(1-i\epsilon)$, take $T\to\infty$ limit and multiply by $1/\sqrt{Z}$ to get $a_{in}(\vec{p})$

$$
e^{i(E_{(\alpha)}-E_{\vec{p}_1}+E_{\vec{p}})t} \qquad \to \qquad e^{-i(E_{(\alpha)}-E_{\vec{p}_1}+E_{\vec{p}})\,T-(E_{(\alpha)}-E_{\vec{p}_1}+E_{\vec{p}})\,T\,\epsilon}
$$

As $T\to\infty$, the lowest $E_{(\alpha)}$ state has the dominant contribution.

$|\Omega\rangle$ gives the leading contribute with $E_{(\alpha)}=0$, $\vec{p}_{(\alpha)}=0$.

Presence of $\delta^{(3)}(\vec{p}-\vec{p}_1)$ factor now allows us to set $E_{\vec{p}}=E_{\vec{p}_1}$.

Also $\langle\Omega|\phi(0)|\vec{p}_1\rangle=\sqrt{Z}/(2\pi)^{3/2}$.

This gives

$$
\begin{aligned}
a_{in}(\vec{p})|\vec{p}_1\rangle_{in} &= \frac{1}{\sqrt{Z}}\,i\,\frac{1}{(2\pi)^{3/2}}\,\frac{1}{\sqrt{2\,E_{\vec{p}}}}\,(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{p}_1)(-i)\,(E_{\vec{p}_1}+E_{\vec{p}})\,\frac{\sqrt{Z}}{(2\pi)^{3/2}}|\Omega\rangle \\
&= \sqrt{2\,E_{\vec{p}_1}}\,\delta^{(3)}(\vec{p}-\vec{p}_1)\,|\Omega\rangle
\end{aligned}
$$

This is the desired relation.

We shall now prove the general case.

$$\left\{\prod_{i=1}^{n}\sqrt{2E_{\vec{p}_i}}\right\}\, a_{in}(\vec{p}_1)^\dagger\cdots a_{in}(p_n)^\dagger|\Omega\rangle = |\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n\rangle_{in}$$

Consider

$$\prod_{i=1}^{n}\left\{\int\frac{d^3p_i}{2E_{\vec{p}_i}}\,\widetilde{f}_i(\vec{p}_i)\right\}\ \times\ \text{above equation},\qquad \widetilde{f}_i(\vec{p}_i) = \sigma^{-3/4}\,e^{-(\vec{p}_i-\vec{k}_i)^2/(2\,\sigma)}$$

If our claim is correct, then when we evolve this state to large negative t using the Schrodinger equation, the lhs should have the interpretation that it describes n particles with momenta $\vec{k}_1,\cdots,\vec{k}_n$, moving along the trajectories $\vec{r}_i=\vec{k}_i\,t$, with momentum space wave-function $\widetilde{f}_i(t,\vec{p}_i)$.

We already know that this is true for $n=1$.

Consider the i-th term in the product:

$$\int\frac{d^3p}{\sqrt{2E_{\vec{p}}}}\,\widetilde{f}_i(\vec{p})a_{in}(\vec{p})^\dagger = \frac{1}{\sqrt{Z}}\int\frac{d^3p}{\sqrt{2E_{\vec{p}}}}\,\widetilde{f}_i(\vec{p})a(t',\vec{p})^\dagger \qquad \text{at } t'=-T(1-i\epsilon)$$

$$= -\frac{i}{\sqrt{Z}}\int\frac{d^3p}{\sqrt{2E_{\vec{p}}}}\,\widetilde{f}_i(\vec{p})\int d^3r\, g_{\vec{p}}(t',\vec{r})\,\overleftrightarrow{\partial_{t'}}\,\phi(t',\vec{r})$$

$$= -\frac{i}{\sqrt{Z}}\int\frac{d^3p}{\sqrt{2E_{\vec{p}}}}\,\widetilde{f}_i(\vec{p})\int d^3r\,\frac{1}{(2\pi)^{3/2}}\,e^{i\vec{p}.\vec{r}-iE_{\vec{p}}t'}\,\frac{1}{\sqrt{2\,E_{\vec{p}}}}\,\overleftrightarrow{\partial_{t'}}\,\phi(t',\vec{r})$$

Do the integration over $\vec{p}$ using the fact that $\widetilde{f}_i$ is sharpely peaked around $E_{\vec{k}_i}$:

$$\int\frac{d^3p}{2E_{\vec{p}}}\,\frac{1}{(2\pi)^{3/2}}\,\widetilde{f}_i(\vec{p})\,e^{i\vec{p}.\vec{r}-iE_{\vec{p}}t'} \simeq \frac{1}{2E_{\vec{k}_i}}\int d^3p\,\frac{1}{(2\pi)^{3/2}}\,\widetilde{f}_i(\vec{p})\,e^{i\vec{p}.\vec{r}-iE_{\vec{p}}t'} = \frac{1}{2E_{\vec{k}_i}}\,f_i(t',\vec{r})$$

$$f_i(t,\vec{r}) \simeq \sigma^{3/4}\,e^{i\vec{k}_i.\vec{r}}\,e^{-i\,E_{\vec{k}_i}\,t}\,e^{-\sigma(\vec{r}-\vec{v}_{\vec{k}_i}\,t)^2/2}$$

This gives

$$\int\frac{d^3p}{\sqrt{2E_{\vec{p}}}}\,\widetilde{f}_i(\vec{p})a_{in}(\vec{p})^\dagger \simeq -\frac{i}{\sqrt{Z}}\,\frac{1}{2E_{\vec{k}_i}}\int d^3r\, f_i(t',\vec{r})\overleftrightarrow{\partial_{t'}}\,\phi(t',\vec{r})$$

$$\int \frac{d^3p}{\sqrt{2E_{\vec{p}}}}\, \widetilde{f}_i(\vec{p}) a_{in}(\vec{p})^\dagger \simeq -\frac{i}{\sqrt{Z}}\, \frac{1}{2E_{\vec{k}_i}} \int d^3r\, f_i(t',\vec{r}) \overleftrightarrow{\partial_{t'}}\, \phi(t',\vec{r}), \qquad t' = -T(1-i\epsilon)$$

$$f_i(t',\vec{r}) \simeq \sigma^{3/4}\, e^{i\vec{k}_i.\vec{r}}\, e^{-i\, E_{\vec{k}_i}\, t'}\, e^{-\sigma(\vec{r}-\vec{v}_{\vec{k}_i}\, t')^2/2}$$

$f_i(t',\vec{r})$ is a wave-packet localized around $\vec{r} = \vec{v}_{\vec{k}_i} t'$.

As long as $\vec{k}_i \neq \vec{k}_j$, the wave-packets for different i's are non-overlapping for large T.

$\Rightarrow$ each factor acts independently, as if it is acting on the vacuum.

We have already seen that each factor, acting on the vacuum, creates the desired single particle state with the desired wave-function.

Therefore the product of all the factors create the desired multi-particle state with the correct wave-function.

Similar argument can be used to prove the relation:

$$a_{in}(\vec{p})|\vec{p}_1,\cdots,\vec{p}_n\rangle_{in} = \sum_{i=1}^{n} \sqrt{2E_{\vec{p}_i}}\, \delta^{(3)}(\vec{p}-\vec{p}_i)\, |\vec{p}_1,\cdots \vec{p}_{i-1},\vec{p}_{i+1},\cdots,\vec{p}_n\rangle_{in}$$

Since after taking the convolution with $\widetilde{f}_i(\vec{p}_i)$, different $a_{in}(\vec{p}_i)^\dagger$ factors act independently, the action of $a_{in}(\vec{p})$ on the state can be studied independently for each $a_{in}(\vec{p}_i)^\dagger$.

This in turn is determined by the result for $a_{in}(\vec{p})|\vec{p}_1\rangle_{in}$ derived earlier.

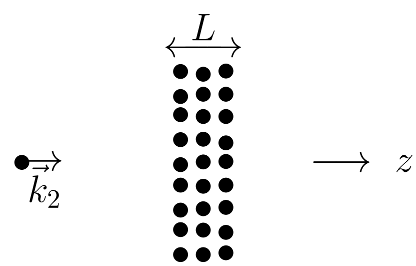
28 Scattering cross section

We have learned how to compute S-matrix in perturbation theory.

How do we relate the S-matrix to physically measurable quantities?

Direct relation: Via scattering experiments.

Fixed target:



Consider a stationary slab of ϕ particles of width L and large area, kept parallel to the x-y plane, filling the region $-L/2 < z < L/2$.

Take a ϕ particle at $z = -\widetilde{L}$, $x = y = 0$, at $t = t_0$ and throw it towards the slab with momentum $\vec{k}_2 = (0, 0, k_{2z})$.

Note: These numbers will have to be understood as peaks of the wave-function of the particle in position / momentum space.

The particle will eventually hit the slab at some time, taken to be $t = 0$.

Question: After the collision, what is the probability of finding m ϕ particles with momenta $\vec{q}_1, \cdots, \vec{q}_m$ lying inside some range R?

Collider: A ϕ particle collides with a beam of ϕ particle moving in the opposite direction

– equivalent to a moving slab in the z direction, with each particle carrying momentum $\vec{k}_1 = (0, 0, k_{1z})$.

Goal: Express the result in terms of the S-matrix which we have learned to calculate.

General structure of the S-matrix

$$S(\vec{q}_1,\cdots,\vec{q}_m|\vec{p}_1,\cdots,\vec{p}_n) = \ _{out}\langle\vec{q}_1,\cdots,\vec{q}_m|\vec{p}_1,\cdots,\vec{p}_n\rangle_{in}$$
$$= \prod_{i=1}^{m}(2E_{\vec{q}_i})\prod_{i=1}^{n}(2E_{\vec{p}_i})\ \left[\delta_{mn}\,\delta^{(3)}(\vec{p}_1-\vec{q}_1)\cdots\delta^{(3)}(\vec{p}_2-\vec{q}_2)\cdots\delta^{(3)}(\vec{p}_n-\vec{q}_n) + \text{permutations of } \vec{p}_1,\cdots,\vec{p}_n\right]$$
$$+\,i\,T(\vec{q}_1,\cdots,\vec{q}_m|\vec{p}_1,\cdots,\vec{p}_n)$$

– defines T- matrix.

T-matrix has the structure

$$T(\vec{q}_1,\cdots,\vec{q}_m|\vec{p}_1,\cdots,\vec{p}_n) = (2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{m}q_i - \sum_{j=1}^{n}p_j\right)\,(2\pi)^{-3(m+n)/2}\,\mathcal{M}(\vec{q}_1,\cdots,\vec{q}_m|\vec{p}_1,\cdots,\vec{p}_n)$$

We shall ignore the forward scattering term and express the result in terms of $\mathcal{M}$.

In the current problem, even though there are many target particles, scattering from them can be taken to be independent

– need to consider $n=2$.

Take the 'initial state', described in terms of basis of in states, to be

$$|f_1,f_2\rangle_{in} = \prod_{i=1}^{2}\left\{\int\frac{d^3p_i}{2E_{\vec{p}_i}}\,\widetilde{f}_i(\vec{p}_i)\right\}|\vec{p}_1,\vec{p}_2\rangle_{in}$$

$$f_i(t,\vec{r}_i) = \int\frac{d^3p}{(2\pi)^{3/2}}\,e^{i\vec{p}.\vec{r}_i - iE_{\vec{p}}t}\,\widetilde{f}_i(\vec{p}),\quad f_i(0,\vec{r}_i) = \int\frac{d^3p}{(2\pi)^{3/2}}\,e^{i\vec{p}.\vec{r}_i}\,\widetilde{f}_i(\vec{p})$$

Our choice of f_1, f_2:

$f_1(t,\vec{r})$, describing the target, should be peaked around $\vec{r}=\vec{b}$ at $t=0$, with $\vec{b}$ lying in the slab:

$$-L/2 \le b_z \le L/2, \qquad b_x, b_y \quad \text{arbitrary}$$

$\widetilde{f}_1(\vec{p})$ should be peaked around $\vec{p}=\vec{k}_1=(0,0,k_{1z})$.

$f_2(t,\vec{r})$, describing the incident particle, should peak around $\vec{r}=\vec{0}$ at $t=0$.

$\widetilde{f}_2(\vec{p})$ should peak around $\vec{k}_2=(0,0,k_{2z})$

Normalization: We would like to normalize both the incoming particle and the target particle state to have norm=1.

$${}_{in}\left\langle \vec{p} \left| \left\{ \int \frac{d^3p}{2E_{\vec{p}}}\, \widetilde{f}_i(\vec{p})^* \right\} \left\{ \int \frac{d^3p'}{2E_{\vec{p}'}}\, \widetilde{f}_i(\vec{p}') \right\} \right| \vec{p}' \right\rangle_{in} = 1 \quad \text{for } i=1,2$$

Using ${}_{in}\langle \vec{p}|\vec{p}'\rangle_{in} = 2\,E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{p}')$, we get

$$\int \frac{d^3p}{2E_{\vec{p}}} |\widetilde{f}_i(\vec{p})|^2 = 1$$

This is different from the Gaussian that we were using before, but can be incorporated by multiplying earlier expression for $\widetilde{f}_i(\vec{p})$ by $\sqrt{2E_{\vec{p}}}$.

The $\sqrt{2E_{\vec{p}}}$ factor can be taken out of all integrals since $\widetilde{f}(\vec{p})$ is sharply peaked.

Probability of finding m particles with momenta $\vec{q}_1, \cdots \vec{q}_m$ in some range R:

$$P = \int_R \frac{d^3q_1}{2E_{\vec{q}_1}} \cdots \frac{d^3q_m}{2E_{\vec{q}_m}} \left| {}_{out}\langle \vec{q}_1, \cdots, \vec{q}_m | f_1, f_2 \rangle_{in} \right|^2$$

Note: This formula is valid when the ranges of $\vec{q}_i$, $\vec{q}_j$ are non-overlapping for all pairs i, j, since otherwise we have to divide by a symmetry factor.

Probability of finding m particles with momenta $\vec{q}_1, \cdots \vec{q}_m$ in some range R:

$$P(\vec{b}, \vec{k}_1, \vec{k}_2, R) = \int_R \frac{d^3q_1}{2E_{\vec{q}_1}} \cdots \frac{d^3q_m}{2E_{\vec{q}_m}} \left| \, _{out}\langle \vec{q}_1, \cdots, \vec{q}_m | f_1, f_2 \rangle_{in} \right|^2$$

This result is for the collision with a single particle in the slab.

By taking into account the possibility of collision with all the particles in the slab we get the total probability:

$$\mathcal{P} = \rho \int d^3b \, P(\vec{b}, \vec{k}_1, \vec{k}_2, R)$$

ρ: no of ϕ particles per unit volume of the slab.

We shall assume the width L of the slab to be small so that the b_z integral gives L.

$$\mathcal{P} = \rho \, L \int d^2b_\perp \, P(\vec{b}, \vec{k}_1, \vec{k}_2, R), \qquad \vec{b}_\perp = (b_x, b_y), \quad \vec{b} = (\vec{b}_\perp, 0)$$

$$\mathcal{P} = \rho \, L \int d^2b_\perp \int_R \frac{d^3q_1}{2E_{\vec{q}_1}} \cdots \frac{d^3q_m}{2E_{\vec{q}_m}} \left| \prod_{i=1}^{2} \left\{ \int \frac{d^3p_i}{2E_{\vec{p}_i}} \, \widetilde{f}_i(\vec{p}_i) \right\} \, _{out}\langle \vec{q}_1, \cdots, \vec{q}_m | \vec{p}_1, \vec{p}_2 \rangle_{in} \right|^2$$

$$= \rho \, L \int d^2b_\perp \int_R \frac{d^3q_1}{2E_{\vec{q}_1}} \cdots \frac{d^3q_m}{2E_{\vec{q}_m}} \prod_{i=1}^{2} \left\{ \int \frac{d^3p_i}{2E_{\vec{p}_i}} \, \widetilde{f}_i(\vec{p}_i) \right\} \, _{out}\langle \vec{q}_1, \cdots, \vec{q}_m | \vec{p}_1, \vec{p}_2 \rangle_{in}$$

$$\prod_{i=1}^{2} \left\{ \int \frac{d^3p'_i}{2E_{\vec{p}'_i}} \, \widetilde{f}_i(\vec{p}'_i)^* \right\} \, _{out}\langle \vec{q}_1, \cdots, \vec{q}_m | \vec{p}'_1, \vec{p}'_2 \rangle^*_{in}$$

$$= \rho \, L \int d^2b_\perp \int_R \frac{d^3q_1}{2E_{\vec{q}_1}} \cdots \frac{d^3q_m}{2E_{\vec{q}_m}} \prod_{i=1}^{2} \left\{ \int \frac{d^3p_i}{2E_{\vec{p}_i}} \right\} \prod_{i=1}^{2} \left\{ \int \frac{d^3p'_i}{2E_{\vec{p}'_i}} \right\}$$

$$\prod_{i=1}^{2} \left\{ \widetilde{f}_i(\vec{p}_i) \, \widetilde{f}_i(\vec{p}'_i)^* \right\} (2\pi)^8 \delta^{(4)}(q_1 + \cdots + q_m - p_1 - p_2) \delta^{(4)}(q_1 + \cdots + q_m - p'_1 - p'_2)$$

$$(2\pi)^{-3(m+2)} \, \mathcal{M}(\vec{q}_1, \cdots, \vec{q}_m | \vec{p}_1, \vec{p}_2) \, \mathcal{M}(\vec{q}_1, \cdots, \vec{q}_m | \vec{p}'_1, \vec{p}'_2)^*$$

$$
\begin{aligned}
\mathcal{P} \;=\; & \rho\, L \int d^2 b_\perp \int_R \frac{d^3 q_1}{2E_{\vec q_1}} \cdots \frac{d^3 q_m}{2E_{\vec q_m}} \prod_{i=1}^{2} \left\{ \int \frac{d^3 p_i}{2E_{\vec p_i}} \right\} \prod_{i=1}^{2} \left\{ \int \frac{d^3 p_i'}{2E_{\vec p_i'}} \right\} \\
& \prod_{i=1}^{2} \left\{ \widetilde{f}_i(\vec p_i)\, \widetilde{f}_i(\vec p_i')^* \right\} (2\pi)^8 \delta^{(4)}(q_1 + \cdots + q_m - p_1 - p_2) \delta^{(4)}(q_1 + \cdots + q_m - p_1' - p_2') \\
& (2\pi)^{-3(m+2)}\; \mathcal{M}(\vec q_1, \cdots, \vec q_m | \vec p_1, \vec p_2)\; \mathcal{M}(\vec q_1, \cdots, \vec q_m | \vec p_1', \vec p_2')^*
\end{aligned}
$$

$\vec b$ dependence of the integrand comes from $\widetilde{f}_1(\vec p_1)$ and $\widetilde{f}_1(\vec p_1')^*$.

$$
f_1^{\vec b}(\vec r) = f_1^{\vec 0}(\vec r - \vec b) \quad \Rightarrow \quad \widetilde{f}_1^{\vec b}(\vec p_1) = e^{-i\vec p_1 . \vec b}\, \widetilde{f}_1^{\vec 0}(\vec p_1)
$$

b_z integral has already been performed giving the factor of L

The $\vec b_\perp$ integral gives

$$
\int d^2 b\, e^{-i(\vec p_{1\perp} - \vec p_{1\perp}').\vec b_\perp} = (2\pi)^2\, \delta^{(2)}(\vec p_{1\perp} - \vec p_{1\perp}')
$$

Also, using the first $\delta^{(4)}$ in the expression for $\mathcal{P}$, the second $\delta^{(4)}$ may be replaced by $\delta^{(4)}(p_1 + p_2 - p_1' - p_2')$.

We shall show that

$$
\delta^{(4)}(p_1 + p_2 - p_1' - p_2')\, \delta^{(2)}(\vec p_{1\perp} - \vec p_{1\perp}') = J\, \delta^{(3)}(\vec p_1 - \vec p_1')\, \delta^{(3)}(\vec p_2 - \vec p_2')
$$

J: A jacobian which, upon setting $\vec p_i = \vec k_i$, takes the form

$$
J = |\vec v_{\vec k_1} - \vec v_{\vec k_2}|^{-1}
$$

Using these δ-functions we get factors of

$$
\prod_{i=1}^{2} \int \frac{d^3 p_i}{2E_{\vec p_i}}\, f_i(\vec p_i) f_i(\vec p_i)^* = 1
$$

All other $\vec p_i$ dependent term in the integrand can be taken outside the $\vec p_i$ integration by setting $\vec p_i = \vec k_i$ for $i = 1, 2$

Final result:

$$\mathcal{P} = \rho\, L \int_R \frac{d^3q_1}{2E_{\vec{q}_1}} \cdots \frac{d^3q_m}{2E_{\vec{q}_m}}\ (2\pi)^{-3m}\,(2\pi)^4\delta^{(4)}(q_1+\cdots+q_m-k_1-k_2)$$
$$\frac{1}{2E_{\vec{k}_1}}\,\frac{1}{2E_{\vec{k}_2}}\,\mathcal{M}(\vec{q}_1,\cdots,\vec{q}_m|\vec{k}_1,\vec{k}_2)\ \mathcal{M}(\vec{q}_1,\cdots,\vec{q}_m|\vec{k}_1,\vec{k}_2)^*\,|\vec{v}_{\vec{k}_1}-\vec{v}_{\vec{k}_2}|^{-1}$$

The integrand is called the differential cross section.

The integral over the full range of $\vec{q}_1, \cdots \vec{q}_m$ is called the total cross section.

Note: If we integrate over the full range of each $\vec{q}_i$, we need to divide the final result by $m!$ since the particles are identical.

Full integration over each q_i will count every final state configuration $m!$ times, which differ from each other by the exchange of the q_i's

– related to the fact that

$$_{out}\langle\vec{q}_1,\cdots,\vec{q}_m|\vec{p}_1,\cdots,\vec{p}_m\rangle_{out}$$

has $m!$ terms, each involving products of m delta functions.

Proof of

$$\delta^{(4)}(p_1+p_2-p_1'-p_2')\,\delta^{(2)}(\vec{p}_{1\perp}-\vec{p}'_{1\perp}) = J\,\delta^{(3)}(\vec{p}_1-\vec{p}'_1)\,\delta^{(3)}(\vec{p}_2-\vec{p}'_2)$$

$$\begin{aligned}
&\delta^{(4)}(p_1+p_2-p_1'-p_2')\,\delta^{(2)}(\vec{p}_{1\perp}-\vec{p}'_{1\perp}) \\
&= \delta(E_{\vec{p}_1}+E_{\vec{p}_2}-E_{\vec{p}'_1}-E_{\vec{p}'_2})\,\delta(p_{1z}+p_{2z}-p'_{1z}-p'_{2z})\delta^{(2)}(\vec{p}_{1\perp}+\vec{p}_{2\perp}-\vec{p}'_{1\perp}-\vec{p}'_{2\perp})\,\delta^{(2)}(\vec{p}_{1\perp}-\vec{p}'_{1\perp}) \\
&= \delta(E_{\vec{p}_1}+E_{\vec{p}_2}-E_{\vec{p}'_1}-E_{\vec{p}'_2})\,\delta(p_{1z}+p_{2z}-p'_{1z}-p'_{2z})\delta^{(2)}(\vec{p}_{2\perp}-\vec{p}'_{2\perp})\,\delta^{(2)}(\vec{p}_{1\perp}-\vec{p}'_{1\perp})
\end{aligned}$$

Using the second delta function, we replace p'_{1z} by $p_{1z}+p_{2z}-p'_{2z}$ in the argument of the first delta function.

Also since the wave-functions are peaked around $\vec{p}_{1\perp}=0$, $\vec{p}_{2\perp}=0$, we set them to 0 in the argument of the first delta function.

Ex. Show that with these replacements, the expression reduces to

$$J\,\delta(p_{2z}-p'_{2z})\,\delta(p_{1z}+p_{2z}-p'_{1z}-p'_{2z})\,\delta^{(2)}(\vec{p}_{2\perp}-\vec{p}'_{2\perp})\,\delta^{(2)}(\vec{p}_{1\perp}-\vec{p}'_{1\perp}) = J\,\delta^{(3)}(\vec{p}_1-\vec{p}'_1)\,\delta^{(3)}(\vec{p}_2-\vec{p}'_2)$$

J: A jacobian which, upon setting $\vec{p}_i=\vec{k}_i$, takes the form

$$J = |\vec{v}_{\vec{k}_1}-\vec{v}_{\vec{k}_2}|^{-1}$$

For $m = 2$, we have only two particles in the final state.

$\vec{q}_1, \vec{q}_2$ give 6 variables.

$\delta^{(4)}(q_1 + q_2 - k_1 - k_2)$ fixes four of them.

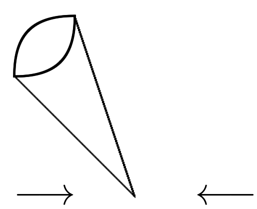
Therefore we have two independent variables which we can take to be the angle (θ_1, ϕ_1) giving the direction of momentum of $\vec{q}_1$.

After making appropriate change of variables, we can write:

$$\mathcal{P} = \int_R d^3q_1\, d^3q_2\, \delta^{(4)}(q_1 + \cdots + q_m - k_1 - k_2)\, \cdots = \int_\Omega \sin\theta_1\, d\theta_1\, d\phi_1\, \mathcal{J}\ \cdots$$

for appropriate jacobian $\mathcal{J}$.

The integrand on the right hand side gives the probability of detecting a ϕ particle within a solid angle Ω around the center of scattering.



29 Decay of unstable particles

Consider a field theory of two scalar fields ϕ and χ with action:

$$S = \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{2}\eta^{\mu\nu}\,\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}m_\chi^2\chi^2 - \frac{\lambda}{2!}\chi\,\phi^2 \right]$$

At the leading order in perturbation theory, we have two particles of masses m_χ and m_ϕ.

If $m_\chi > 2m_\phi$ then the χ particle can decay into two ϕ particles

In the center of mass of the χ particle, the two ϕ particles will carry momenta $\pm\vec{q}$ and energy conservation will give

$$m_\chi = 2\sqrt{m_\phi^2 + \vec{q}^2}$$

In a general Lorentz frame the χ particle will have momentum $\vec{p}$ and ϕ particles will have different momenta $\vec{q}_1, \vec{q}_2$.

The decay should be described by ${}_{out}\langle\phi, \vec{q}_1; \phi, \vec{q}_2|\chi, \vec{p}\rangle_{in}$.

Note: $|\chi, \vec{p}\rangle_{in}$ is not strictly an in state since it does not live forever, but we shall proceed by ignoring this problem.

${}_{out}\langle\phi, \vec{q}_1; \phi, \vec{q}_2|\chi, \vec{p}\rangle_{in}$ can be calculated in terms of three point Green's function as before.

Question: How to translate this into something that we can measure physically, e.g. half-life of the χ particle?

We shall consider a general case in which there are m decay products carrying momentum $\vec{q}_1, \cdots \vec{q}_m$

– possible if $m_\chi > m\, m_\phi$.

As before, we describe the in state as

$$|\chi, f\rangle_{in} = \int \frac{d^3p}{2E_{\vec{p}}}\, \widetilde{f}(\vec{p})|\chi,\vec{p}\rangle_{in}, \qquad \int \frac{d^3p}{2E_{\vec{p}}}\, \widetilde{f}(\vec{p})^*\widetilde{f}(\vec{p}) = 1$$

Peak of $\widetilde{f}(\vec{p})$ is taken to be at some value $\vec{k}$.

Probability of finding m ϕ-particles with momenta $\vec{q}_1, \cdots \vec{q}_m$ in some range R:

$$P = \int_R \frac{d^3q_1}{2E_{\vec{q}_1}} \cdots \frac{d^3q_m}{2E_{\vec{q}_m}} \left|\ _{out}\langle\phi,\vec{q}_1,\cdots,\phi,\vec{q}_m|\chi, f\rangle_{in}\right|^2$$

Use

$$_{out}\langle\phi,\vec{q}_1,\cdots,\phi,\vec{q}_m|\chi,\vec{p}\rangle_{in} = i\,(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^m q_i - p\right)\,(2\pi)^{-3(m+1)/2}\,\mathcal{M}(\vec{q}_1,\cdots,\vec{q}_m|\vec{p})$$

This gives

$$\begin{aligned} P \;=\; & \int_R \frac{d^3q_1}{2E_{\vec{q}_1}} \cdots \frac{d^3q_m}{2E_{\vec{q}_m}} \int \frac{d^3p}{2E_{\vec{p}}} \int \frac{d^3p'}{2E_{\vec{p}'}}\, \widetilde{f}(\vec{p})\, \widetilde{f}(\vec{p}')^*(2\pi)^8\delta^{(4)}\left(\sum_{i=1}^m q_i - p\right) \\ & \delta^{(4)}\left(\sum_{i=1}^m q_i - p'\right)\,(2\pi)^{-3(m+1)}\,\mathcal{M}(\vec{q}_1,\cdots,\vec{q}_m|\vec{p})\,\mathcal{M}(\vec{q}_1,\cdots,\vec{q}_m|\vec{p}')^* \end{aligned}$$

Using the first δ function, we can replace the second δ function by

$$\delta^{(4)}(p-p') = \delta^{(3)}(\vec{p}-\vec{p}')\,\delta(E_{\vec{p}} - E_{\vec{p}'})$$

Now we have a problem, since using $\delta^{(3)}(\vec{p}-\vec{p}')$ the second delta function becomes $\delta(0)$.

Solution: Write

$$\delta(p^0 - p'^0) = \frac{1}{2\pi}\int dt\, e^{it(p^0-p'^0)} = \frac{1}{2\pi}\int dt \quad \text{for } p^0 = p'^0$$

Therefore transition probability per unit time is finite.

$$\begin{aligned} \dot{P} \;=\; & \int_R \frac{d^3q_1}{2E_{\vec{q}_1}} \cdots \frac{d^3q_m}{2E_{\vec{q}_m}} \int \frac{d^3p}{2E_{\vec{p}}} \int \frac{d^3p'}{2E_{\vec{p}'}}\, \widetilde{f}(\vec{p})\, \widetilde{f}(\vec{p}')^*(2\pi)^7\delta^{(4)}\left(\sum_{i=1}^m q_i - p\right) \\ & \delta^{(3)}\left(\vec{p}-\vec{p}'\right)\,(2\pi)^{-3(m+1)}\,\mathcal{M}(\vec{q}_1,\cdots,\vec{q}_m|\vec{p})\,\mathcal{M}(\vec{q}_1,\cdots,\vec{q}_m|\vec{p}')^* \end{aligned}$$

$$
\begin{aligned}
\dot{P} &= \int_R \frac{d^3q_1}{2E_{\vec{q}_1}} \cdots \frac{d^3q_m}{2E_{\vec{q}_m}} \int \frac{d^3p}{2E_{\vec{p}}} \int \frac{d^3p'}{2E_{\vec{p}'}} \widetilde{f}(\vec{p})\, \widetilde{f}(\vec{p}')^* (2\pi)^7 \delta^{(4)}\left(\sum_{i=1}^m q_i - p\right) \\
&\quad \delta^{(3)}(\vec{p} - \vec{p}')\ (2\pi)^{-3(m+1)}\, \mathcal{M}(\vec{q}_1, \cdots, \vec{q}_m|\vec{p}) \mathcal{M}(\vec{q}_1, \cdots, \vec{q}_m|\vec{p}')^* \\
&= \int_R \frac{d^3q_1}{2E_{\vec{q}_1}} \cdots \frac{d^3q_m}{2E_{\vec{q}_m}} \int \frac{d^3p}{2E_{\vec{p}}} \frac{1}{2E_{\vec{p}}} \widetilde{f}(\vec{p})\, \widetilde{f}(\vec{p})^* (2\pi)^7 \delta^{(4)}\left(\sum_{i=1}^m q_i - p\right) \\
&\quad (2\pi)^{-3(m+1)}\, \mathcal{M}(\vec{q}_1, \cdots, \vec{q}_m|\vec{p})\, \mathcal{M}(\vec{q}_1, \cdots, \vec{q}_m|\vec{p})^*
\end{aligned}
$$

Now use

$$
\int \frac{d^3p}{2E_{\vec{p}}} \widetilde{f}(\vec{p})\, \widetilde{f}(\vec{p})^* = 1
$$

and set $\vec{p} = \vec{k}$ in the rest of the terms in the integrand.

$$
\begin{aligned}
\dot{P} &= \frac{1}{2E_{\vec{k}}} \int_R \frac{d^3q_1}{2E_{\vec{q}_1}} \cdots \frac{d^3q_m}{2E_{\vec{q}_m}} (2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^m q_i - k\right) \\
&\quad (2\pi)^{-3m}\, \mathcal{M}(\vec{q}_1, \cdots, \vec{q}_m|\vec{k})\, \mathcal{M}(\vec{q}_1, \cdots, \vec{q}_m|\vec{k})^*
\end{aligned}
$$

For computing the total decay probability we need to divide by $m!$ to account for the symmetry under permutation of the m final state particles.

If we start with N_0 χ particles each moving with momentum $\vec{k}$, then the number of particles at time t is determined from the equation:

$$
\frac{dN}{dt} = -\dot{P} N \quad \Rightarrow \quad N = N_0\, e^{-\dot{P} t}
$$

Half life τ is determined by

$$
e^{-\tau \dot{P}} = \frac{1}{2} \quad \Rightarrow \quad \tau = \frac{1}{\dot{P}} \ln 2
$$

Note: $1/(2E_{\vec{k}})$ factor in $\dot{P}$ shows that $\tau \propto E_{\vec{k}}$

Energetic particles have long life $\quad \Leftrightarrow \quad$ time dilation

Quantization of electromagnetic field

Strategy:

1. First manipulate the classical theory to bring it to a form suitable for quantization.

2. Then quantize it.

Action:

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma}$$

A_μ: Vector potential

Under an arbitrary variation of A_μ,

$$\delta S = -\frac{1}{2}\int d^4x F^{\mu\nu}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) = -\int d^4x F^{\mu\nu}(\partial_\mu \delta A_\nu) = \int d^4x \partial_\mu F^{\mu\nu}\,\delta A_\nu$$

after integration by parts,

Since δA_ν is arbitrary, we get equations of motion

$$\partial_\mu F^{\mu\nu} = 0$$

– covariant form of Maxwell's equation.

Lagrangian:

$$\begin{aligned} L &= -\frac{1}{4}\int d^3r F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\int d^3r \left[F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}\right] \\ &= \int d^3r \left[\frac{1}{2}(\partial_0 A_i - \partial_i A_0)(\partial_0 A_i - \partial_i A_0) - \frac{1}{4}F_{ij}F_{ij}\right], \qquad 1 \le i, j \le 3 \end{aligned}$$

Note: Repeated i, j indices are summed from 1 to 3.

$$L=\int d^3r\left[\frac{1}{2}(\partial_0 A_i-\partial_i A_0)(\partial_0 A_i-\partial_i A_0)-\frac{1}{4}F_{ij}F_{ij}\right]$$

Naively, to go to the Hamiltonian formalism, we would define

$$\Pi_i=\frac{\delta L}{\delta \dot{A}_i},\qquad \Pi_0=\frac{\delta L}{\delta \dot{A}_0}$$

Since L does not depend on $\dot{A}_0$, we get $\Pi_0=0$

– a constrained system.

We could use Dirac's formalism for quantizing constrained system.

We shall follow a slightly different approach based on the Routhian.

Suppose we have a dynamical system with coordinates $\{q_a\}$ with $a=1,\cdots,m$ and $\{q_\alpha\}$ with $\alpha=m+1,\cdots,m+n$.

How we divide the coordinate variables into the two sets is up to us.

L is a function of $\{q_a\}$, $\{\dot{q}_a\}$, $\{q_\alpha\}$, $\{\dot{q}_\alpha\}$.

Define

$$p_a=\frac{\partial L}{\partial \dot{q}_a},\qquad R(\{q_a\},\{p_a\},\{q_\alpha\},\{\dot{q}_\alpha\})=\sum_{a=1}^{m}p_a\dot{q}_a-L$$

Ex. Check that the original Euler-Lagrange equations for all the variables may be rewritten as

$$\frac{dq_a}{dt}=\frac{\partial R}{\partial p_a},\quad \frac{dp_a}{dt}=-\frac{\partial R}{\partial q_a},\quad 1\le a\le m,\quad \Rightarrow\quad \text{like Hamiltonian}$$

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_\alpha}-\frac{\partial R}{\partial q_\alpha}=0,\quad m+1\le\alpha\le m+n,\quad \Rightarrow\quad \text{like Lagrangian}$$

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad R(\{q_a\}, \{p_a\}, \{q_\alpha\}, \{\dot{q}_\alpha\}) = \sum_{a=1}^{m} p_a \dot{q}_a - L$$

$$\frac{dq_a}{dt} = \frac{\partial R}{\partial p_a}, \quad \frac{dp_a}{dt} = -\frac{\partial R}{\partial q_a}, \quad 1 \le a \le m, \quad \Rightarrow \quad \text{like Hamiltonian}$$

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_\alpha} - \frac{\partial R}{\partial q_\alpha} = 0, \quad m+1 \le \alpha \le m+n, \quad \Rightarrow \quad \text{like Lagrangian}$$

In Maxwell's theory we shall treat A_i's as the q_a's and A_0 as the q_α's.

$$L = \int d^3r \left[\frac{1}{2}(\partial_0 A_i - \partial_i A_0)(\partial_0 A_i - \partial_i A_0) - \frac{1}{4} F_{ij} F_{ij} \right]$$

$$\Pi_i(t, \vec{r}) = \frac{\delta L}{\delta \dot{A}_i(t, \vec{r})} = \partial_0 A_i - \partial_i A_0 = -E_i(t, \vec{r})$$

$$R = \int d^3r \left[\Pi_i(t,\vec{r}) \dot{A}_i(t,\vec{r}) - L \right] = \int d^3r \, \left[\frac{1}{2}\Pi_i(t,\vec{r})\Pi_i(t,\vec{r}) + \frac{1}{2} B_i(t,\vec{r}) B_i(t,\vec{r}) - A_0 \partial_i \Pi_i(t,\vec{r}) \right]$$

$$B_i = \epsilon_{ijk}\partial_j A_k, \quad \epsilon_{123} = 1, \quad \epsilon_{ijk} \text{ is antisymmetric in } i,j,k, \quad \Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

Equation of motion for A_0 gives

$$\frac{\partial}{\partial t}\frac{\delta R}{\delta \dot{A}_0} - \frac{\delta R}{\delta A_0} = 0 \quad \Rightarrow \quad \partial_i \Pi_i(t, \vec{r}) = 0$$

Equations of motion for A_i and Π_i give:

$$\frac{\partial A_i}{\partial t} = \frac{\delta R}{\delta \Pi_i} = \Pi_i + \partial_i A_0$$

$$\frac{\partial \Pi_i}{\partial t} = -\frac{\delta R}{\delta A_i} = -\epsilon_{ijk} \partial_j B_k$$

Note 1: The second equation is compatible with the constraint $\partial_i \Pi_i = 0$ in the sense that $\partial_t(\partial_i \Pi_i)$ vanishes as a consequence of the second equation

– once we set $\partial_i \Pi_i = 0$ at some time, it remains zero as a consequence of the equations of motion.

Note 2: There is no equation that gives the time evolution of A_0.

The inability to determine A_0 arises from the gauge invariance of the theory.

If A_μ solves classical equations of motion, so does $A_\mu + \partial_\mu \Lambda$ for any function $\Lambda(t, \vec{r})$.

Therefore starting from a given configuration, the time evolution is not unique

– can always add $\partial_\mu \Lambda$ to A_μ for a function $\Lambda(t, \vec{r})$ for which $\partial_0 \Lambda$ vanishes at the initial time but can be chosen arbitrarily later.

$\mathbf{F}_{\mu\nu}$ and hence $\vec{E}$, $\vec{B}$, L etc. are all invariant under this gauge transformation.

We declare that A_μ and $A_\mu + \partial_\mu \Lambda$ describe equivalent field configurations

– the time evolution of A_μ is not determined but the undetermined part is not physically relevant.

We shall solve this ambiguity by using gauge fixing

– pick one representative configuration among the gauge equivalent ones.

We shall use Coulomb gauge $\vec{\nabla}.\vec{A} = 0$.

Is it always possible?

Given a configuration for which $\vec{\nabla}.\vec{A} \neq 0$, can we find a Λ such that $\vec{\nabla}.(\vec{A} + \vec{\nabla}\Lambda) = 0$?

This gives $\vec{\nabla}^2\Lambda = -\vec{\nabla}.\vec{A}$

– can be solved by takine

$$\Lambda = -\int d^3r' \mathcal{G}(\vec{r}, \vec{r}\,')\, \vec{\nabla}'.\vec{A}(t, \vec{r}\,')\, d^3r', \qquad \mathcal{G}(\vec{r}, \vec{r}\,') = -\frac{1}{4\pi|\vec{r} - \vec{r}\,'|}$$

since $\vec{\nabla}^2\mathcal{G}(\vec{r}, \vec{r}\,') = \delta^{(3)}(\vec{r} - \vec{r}\,')$

What does it give for time evolution of A_0?

Use $\partial_t A_i = \Pi_i + \partial_i A_0$ to get

$$0 = \partial_t(\vec{\nabla}.\vec{A}) = \partial_i\Pi_i + \vec{\nabla}^2 A_0 = \vec{\nabla}^2 A_0$$

Therefore in the Coulomb gauge $\vec{\nabla}^2 A_0 = 0$.

If we use the boundary condition $A_0(t, \vec{r}) \to 0$ for large $|\vec{r}|$, then this gives $A_0 = 0$.

In this gauge the non-trivial equations of motion may be written as

$$\frac{\partial A_i}{\partial t} = \frac{\delta R}{\delta \Pi_i} = \Pi_i, \quad \frac{\partial \Pi_i}{\partial t} = -\frac{\delta R}{\delta A_i} = -\epsilon_{ijk}\partial_j B_k$$

$$R = \int d^3r \, \left[\frac{1}{2}\Pi_i(t, \vec{r})\Pi_i(t, \vec{r}) + \frac{1}{2}B_i(t, \vec{r})B_i(t, \vec{r})\right]$$

$$\partial_i A_i = 0, \quad \partial_i \Pi_i = 0 \quad \Rightarrow \quad \text{external constraint}$$

In the presence of source J^μ, the Maxwell's equation is modified to

$$\partial_\mu F^{\mu\nu} = -J^\nu, \qquad \partial_\nu J^\nu = 0$$

Ex. This can be achieved by adding to L:

$$\int d^3r\, A_\mu J^\mu$$

R gets a term

$$-\int d^3r\, A_\mu J^\mu$$

The A_0 equation of motion becomes

$$\partial_i \Pi_i + J^0 = 0$$

We can still fix Coulomb gauge, but A_0 is now determined from

$$\vec{\nabla}^2 A_0 = -\partial_i \Pi_i = J^0 \quad \Rightarrow \quad A_0(t,\vec{r}) = \int d^3r'\, \mathcal{G}(\vec{r},\vec{r}\,')\, J^0(t,\vec{r}\,')$$

Using this the extra term in R may be written as

$$-\int d^3r \int d^3r'\, \mathcal{G}(\vec{r},\vec{r}\,')\, J^0(t,\vec{r}\,')\, J^0(t,\vec{r})$$

External constraints

$$\partial_i \Pi_i = -J^0, \qquad \partial_i A_i = 0$$

30 Quantization of the Maxwell field

Classical Maxwell's theory can be described by a Routhian, which, in the Coulomb gauge, takes the form:

$$R = \int d^3r \left[\frac{1}{2}\Pi_i(t,\vec{r})\Pi_i(t,\vec{r}) + \frac{1}{2}B_i(t,\vec{r})B_i(t,\vec{r})\right], \qquad B_i = \epsilon_{ijk}\partial_j A_k, \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

Since we have eliminated the A_0 variable, the equations of motion derived from the Routhian take the form of Hamiltonian equations:

$$\frac{\partial A_i}{\partial t} = \frac{\delta R}{\delta \Pi_i} = \Pi_i, \quad \frac{\partial \Pi_i}{\partial t} = -\frac{\delta R}{\delta A_i} = -\epsilon_{ijk}\partial_j B_k$$

We also have some additional constraints:

$$\partial_i A_i = 0, \quad \partial_i \Pi_i = 0 \quad \Rightarrow \quad \text{external constraint}$$

Constraints are preserved by the equations of motion, i.e. once we impose them at $t = t_0$, they remain valid at all times due to equations of motion.

We can introduce Poisson brackets:

$$\{A_i(t,\vec{r}), \Pi_j(t,\vec{r}\,')\}_{PB} = \delta_{ij}\delta^{(3)}(\vec{r} - \vec{r}\,')$$

$$\{A_i(t,\vec{r}), A_j(t,\vec{r}\,')\}_{PB} = 0, \quad \{\Pi_i(t,\vec{r}), \Pi_j(t,\vec{r}\,')\}_{PB} = 0$$

and write the equations of motion as:

$$\frac{\partial A_i(t,\vec{r})}{\partial t} = \{A_i(t,\vec{r}), R(t)\}_{PB}, \quad \frac{\partial \Pi_i(t,\vec{r})}{\partial t} = \{\Pi_i(t,\vec{r}), R(t)\}_{PB}$$

The constraints are second class and can be easily eliminated.

We shall see this explicitly in the Fourier transformed variables.

We now introduce Fourier transformed variables:

$$A_i(t,\vec{r}) = \int \frac{d^3p}{(2\pi)^{3/2}}\, e^{i\vec{p}.\vec{r}}\, \widetilde{A}_i(t,\vec{p}), \qquad \Pi_i(t,\vec{r}) = \int \frac{d^3p}{(2\pi)^{3/2}}\, e^{i\vec{p}.\vec{r}}\, \widetilde{\Pi}_i(t,\vec{p})$$

Introduce vectors $\vec{\epsilon}^{(a)}(\vec{p})$ for $a=1,2$ such that

$$\vec{p}.\vec{\epsilon}^{(a)}(\vec{p}) = 0, \qquad \vec{\epsilon}^{(a)}(\vec{p}).\vec{\epsilon}^{(b)}(\vec{p}) = \delta_{ab}, \qquad \vec{\epsilon}^{(a)}(-\vec{p}) = \vec{\epsilon}^{(a)}(\vec{p})$$

Define:

$$\widetilde{A}^{(a)}(t,\vec{p}) = \epsilon^{(a)}_j(\vec{p})\, \widetilde{A}_j(t,\vec{p}), \qquad \widetilde{\Pi}^{(a)}(t,\vec{p}) = \epsilon^{(a)}_j(\vec{p})\, \widetilde{\Pi}_j(t,\vec{p})$$

$$\widetilde{A}_{\|}(t,\vec{p}) = i\, \frac{p_j \widetilde{A}_j(t,\vec{p})}{|\vec{p}|}, \qquad \widetilde{\Pi}_{\|}(t,\vec{p}) = i\, \frac{p_j \widetilde{\Pi}_j(t,\vec{p})}{|\vec{p}|}$$

This gives

$$\widetilde{A}_j(t,\vec{p}) = \sum_{a=1}^{2} \widetilde{A}^{(a)}(t,\vec{p})\, \epsilon^{(a)}_j(\vec{p}) - i\, \widetilde{A}_{\|}(t,\vec{p})\, \frac{p_j}{|\vec{p}|}, \quad \widetilde{\Pi}_j(t,\vec{p}) = \sum_{a=1}^{2} \widetilde{\Pi}^{(a)}(t,\vec{p})\, \epsilon^{(a)}_j(\vec{p}) - i\, \widetilde{\Pi}_{\|}(t,\vec{p})\, \frac{p_j}{|\vec{p}|}$$

In these variables

$$R = \int d^3p \left[\frac{1}{2}\widetilde{\Pi}_i(t,-\vec{p})\widetilde{\Pi}_i(t,\vec{p}) + \frac{1}{4}\left\{p_i\widetilde{A}_j(t,-\vec{p}) - p_j\widetilde{A}_i(t,-\vec{p})\right\}\left\{p_i\widetilde{A}_j(t,\vec{p}) - p_j\widetilde{A}_i(t,\vec{p})\right\}\right]$$
$$= \frac{1}{2}\int d^3p \left[\sum_{a=1}^{2}\widetilde{\Pi}^{(a)}(t,-\vec{p})\widetilde{\Pi}^{(a)}(t,\vec{p}) + \widetilde{\Pi}_{\|}(t,-\vec{p})\widetilde{\Pi}_{\|}(t,\vec{p}) + \vec{p}^2\sum_{a=1}^{2}\widetilde{A}^{(a)}(t,-\vec{p})\widetilde{A}^{(a)}(t,\vec{p})\right]$$

The constraints take the form:

$$\widetilde{\Pi}_{\|}(t,\vec{p}) = 0, \qquad \widetilde{A}_{\|}(t,\vec{p}) = 0$$

The Poisson brackets:

$$\{A_i(t,\vec{r}), \Pi_j(t,\vec{r}\,')\}_{PB} = \delta_{ij}\delta^{(3)}(\vec{r}-\vec{r}\,')$$

$$\Rightarrow \quad \{\widetilde{A}_i(t,\vec{p}), \widetilde{\Pi}_j(t,\vec{p}\,')\}_{PB} = \delta_{ij}\,\delta^{(3)}(\vec{p}+\vec{p}\,')$$

Ex. Check that this gives

$$\{\widetilde{A}^{(a)}(t,\vec{p}), \widetilde{\Pi}^{(b)}(t,\vec{p}\,')\}_{PB} = \delta_{ab}\,\delta^{(3)}(\vec{p}+\vec{p}\,'), \quad \{\widetilde{A}_{\parallel}(t,\vec{p}), \widetilde{\Pi}_{\parallel}(t,\vec{p}\,')\}_{PB} = \delta^{(3)}(\vec{p}+\vec{p}\,')$$

$$\{\widetilde{A}^{(a)}(t,\vec{p}), \widetilde{\Pi}_{\parallel}(t,\vec{p}\,')\}_{PB} = 0, \quad \{\widetilde{A}_{\parallel}(t,\vec{p}), \widetilde{\Pi}^{(b)}(t,\vec{p}\,')\}_{PB} = 0$$

$\{\widetilde{A}, \widetilde{A}\}_{PB}$ and $\{\widetilde{\Pi}, \widetilde{\Pi}\}_{PB}$ vanish for all components of $\widetilde{A}$ and $\widetilde{\Pi}$.

We can now use the constraints

$$\widetilde{\Pi}_{\parallel}(t,\vec{p}) = 0, \qquad \widetilde{A}_{\parallel}(t,\vec{p}) = 0$$

to set $\widetilde{\Pi}_{\parallel}$ and $\widetilde{A}_{\parallel}$ to 0.

Ex. The Dirac brackets involving $\widetilde{A}^{(a)}(t,\vec{p})$ and $\widetilde{\Pi}^{(b)}(t,\vec{p}\,')$ are the same as their Poisson brackets since the constraints have vanishing Poisson bracket with $\widetilde{A}^{(a)}(t,\vec{p})$ and $\widetilde{\Pi}^{(b)}(t,\vec{p}\,')$.

Final form:

$$R = \frac{1}{2}\int d^3p \left[\sum_{a=1}^{2} \widetilde{\Pi}^{(a)}(t,-\vec{p})\widetilde{\Pi}^{(a)}(t,\vec{p}) + \vec{p}^{\,2}\sum_{a=1}^{2} \widetilde{A}^{(a)}(t,-\vec{p})\widetilde{A}^{(a)}(t,\vec{p})\right]$$

$$\{\widetilde{A}^{(a)}(t,\vec{p}), \widetilde{\Pi}^{(b)}(t,\vec{p}\,')\}_{DB} = \delta_{ab}\,\delta^{(3)}(\vec{p}+\vec{p}\,')$$

$$\{\widetilde{A}^{(a)}(t,\vec{p}), \widetilde{A}^{(b)}(t,\vec{p}\,')\}_{DB} = 0, \quad \{\widetilde{\Pi}^{(a)}(t,\vec{p}), \widetilde{\Pi}^{(b)}(t,\vec{p}\,')\}_{DB} = 0$$

$$R = \frac{1}{2}\int d^3p \left[\sum_{a=1}^{2} \widetilde{\Pi}^{(a)}(t,-\vec{p})\widetilde{\Pi}^{(a)}(t,\vec{p}) + \vec{p}^2 \sum_{a=1}^{2} \widetilde{A}^{(a)}(t,-\vec{p})\widetilde{A}^{(a)}(t,\vec{p})\right]$$

$$\{\widetilde{A}^{(a)}(t,\vec{p}), \widetilde{\Pi}^{(b)}(t,\vec{p}\,')\}_{DB} = \delta_{ab}\,\delta^{(3)}(\vec{p}+\vec{p}\,')$$

$$\{\widetilde{A}^{(a)}(t,\vec{p}), \widetilde{A}^{(b)}(t,\vec{p}\,')\}_{DB} = 0, \quad \{\widetilde{\Pi}^{(a)}(t,\vec{p}), \widetilde{\Pi}^{(b)}(t,\vec{p}\,')\}_{DB} = 0$$

We shall now quantize by regarding the Routhian as the Hamiltonian and the Dirac brackets as commutator bracket / i

$$H = \frac{1}{2}\int d^3p \left[\sum_{a=1}^{2} \widetilde{\Pi}^{(a)}(t,-\vec{p})\widetilde{\Pi}^{(a)}(t,\vec{p}) + \vec{p}^2 \sum_{a=1}^{2} \widetilde{A}^{(a)}(t,-\vec{p})\widetilde{A}^{(a)}(t,\vec{p})\right]$$

$$[\widetilde{A}^{(a)}(t,\vec{p}), \widetilde{\Pi}^{(b)}(t,\vec{p}\,')] = i\,\delta_{ab}\,\delta^{(3)}(\vec{p}+\vec{p}\,')$$

$$[\widetilde{A}^{(a)}(t,\vec{p}), \widetilde{A}^{(b)}(t,\vec{p}\,')] = 0, \quad [\widetilde{\Pi}^{(a)}(t,\vec{p}), \widetilde{\Pi}^{(b)}(t,\vec{p}\,')] = 0$$

– identical to a system containing a pair of zero mass K-G fields, one for each a.

– can be quantized as in K-G theory.

Define:

$$a^{(b)}(t,\vec{p}) = \frac{1}{\sqrt{2}}\{E_{\vec{p}}^{1/2}\widetilde{A}^{(b)}(t,\vec{p}) + i\,E_{\vec{p}}^{-1/2}\widetilde{\Pi}^{(b)}(t,\vec{p})\}, \qquad E_{\vec{p}} = |\vec{p}|$$

$$a^{(b)}(t,\vec{p})^\dagger = \frac{1}{\sqrt{2}}\{E_{\vec{p}}^{1/2}\widetilde{A}^{(b)}(t,-\vec{p}) - i\,E_{\vec{p}}^{-1/2}\widetilde{\Pi}^{(b)}(t,-\vec{p})\}$$

$$[a^{(b)}(t,\vec{p}), a^{(c)}(t,\vec{p}\,')^\dagger] = \delta_{bc}\,\delta^{(3)}(\vec{p}-\vec{p}\,'), \quad [a^{(b)}(t,\vec{p}), a^{(c)}(t,\vec{p}\,')] = 0 = [a^{(b)}(t,\vec{p})^\dagger, a^{(c)}(t,\vec{p}\,')^\dagger]$$

Then

$$H = \int d^3p \sum_{b=1}^{2} E_{\vec{p}}\, a^{(b)}(t,\vec{p})^\dagger\, a^{(b)}(t,\vec{p}),$$

$$[H, a^{(b)}(t,\vec{p})] = -E_{\vec{p}}\, a^{(b)}(t,\vec{p}), \quad [H, a^{(b)}(t,\vec{p})^\dagger] = E_{\vec{p}}\, a^{(b)}(t,\vec{p})^\dagger$$

$$H = \int d^3p \sum_{b=1}^{2} E_{\vec{p}}\, a^{(b)}(t,\vec{p})^\dagger\, a^{(b)}(t,\vec{p}),$$

$$[H, a^{(b)}(t,\vec{p})] = -E_{\vec{p}}\, a^{(b)}(t,\vec{p}), \quad [H, a^{(b)}(t,\vec{p})^\dagger] = E_{\vec{p}}\, a^{(b)}(t,\vec{p})^\dagger$$

In the Heisenberg picture, we construct the basis of states at $t = 0$ as usual.

Define $a^{(b)}(\vec{p}) = a^{(b)}(0,\vec{p})$, $a^{(b)}(\vec{p})^\dagger = a^{(b)}(0,\vec{p})^\dagger$, and define the vacuum states $|0\rangle$ via

$$a^{(b)}(\vec{p})|0\rangle = 0 \quad \text{for every } \vec{p},\ b = 1, 2$$

We now have two types of single particle states:

$$a^{(b)}(\vec{p})^\dagger|0\rangle, \qquad b = 1, 2$$

– carries energy $E_{\vec{p}}$

– carries momentum $\vec{p}$ (can be seen by first constructing the momentum operator using Noether procedure)

Since these have $E = |\vec{p}|$, they describe massless particles called photons.

The index b describes photon ‘polarization’.

$$a^{(b)}(\vec{p})^\dagger|0\rangle, \qquad b = 1, 2$$

– describe linearly polarized photons.

If we consider instead the linear combinations:

$$\left\{a^{(1)}(\vec{p})^\dagger \pm i\, a^{(2)}(\vec{p})^\dagger\right\}|0\rangle$$

then they describe circularly polarized photons

– eigenstates of the component of angular momentum along the photon momentum $\vec{p}$ with eigenvalues ± 1.

To see this we need to construct the angular momentum operators using Noether theorem and then find the action of

$$\epsilon^{ijk}\, p^i\, \mathcal{J}^{jk}/|\vec{p}|$$

on the states.

We shall now calculate the Feynman propagator for the photon:

$$D_{\mu\nu}(x,x') = \langle 0|T(A_\mu(t,\vec{r})A_\nu(t',\vec{r}\,'))|0\rangle$$

We shall proceed as in the case of K-G theory.

$$D_{\mu\nu}(x,x') = \int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3p'}{(2\pi)^{3/2}} e^{i\vec{p}.\vec{r}} e^{i\vec{p}\,'.\vec{r}\,'} \langle 0|T(\widetilde{A}_\mu(t,\vec{p})\widetilde{A}_\nu(t',\vec{p}\,'))|0\rangle$$

Note: $\widetilde{A}_\mu(t,\vec{p})$ satisfies:

$$\widetilde{A}_0(t,\vec{p}) = 0, \quad p_i\widetilde{A}_i(t,\vec{p}) = -i\,|\vec{p}|\,\widetilde{A}_\parallel(t,\vec{p}) = 0, \quad \epsilon_i^{(b)}(\vec{p})\widetilde{A}_i(t,\vec{p}) = \widetilde{A}^{(b)}(t,\vec{p}) \quad \text{for } b = 1,2$$

Strategy: Use these to express $\widetilde{A}_\mu(t,\vec{p})$, $\widetilde{A}_\nu(t',\vec{p}\,')$ in terms of $\widetilde{A}^{(b)}$, then in terms of $a^{(b)}$, $a^{(b)\dagger}$ for $b = 1,2$, and finally evaluate the matrix element.

$\widetilde{A}_\mu(t,\vec{p})$ satisfies:

$$\widetilde{A}_0(t,\vec{p})=0, \quad p_i\widetilde{A}_i(t,\vec{p})=0, \quad \epsilon_i^{(b)}(\vec{p})\widetilde{A}_i(t,\vec{p})=A^{(b)}(t,\vec{p}) \quad \text{for } b=1,2$$

Define the following contravariant four vectors:

$$n=(1,0,0,0), \quad \bar{p}=(0,\vec{p}), \quad \bar{\eta}^{(b)}(\vec{p})=(0,\vec{\epsilon}^{(b)}(\vec{p}))$$

Then we can rewrite the conditions on $\widetilde{A}_\mu$ as:

$$n.\widetilde{A}(t,\vec{p})=0, \quad \bar{p}.\widetilde{A}(t,\vec{p})=0, \quad \bar{\eta}^{(b)}(\vec{p}).\widetilde{A}(t,\vec{p})=\widetilde{A}^{(b)}(t,\vec{p})$$

Claim: The following relation holds:

$$* \ : \ \eta_{\mu\nu}=-n_\mu n_\nu+\frac{\bar{p}_\mu\bar{p}_\nu}{\vec{p}^2}+\sum_{b=1}^{2}\bar{\eta}_\mu^{(b)}(\vec{p})\,\bar{\eta}_\nu^{(b)}(\vec{p})$$

Proof. If A and B are two 4×4 matrices, then to prove $A=B$, it is enough to show that

$$(A_{\mu\nu}-B_{\mu\nu})v^\nu=0$$

for 4 linearly independent 4-vectors v.

Since $n,\bar{p},\bar{\eta}^{(1)}$ and $\bar{\eta}^{(2)}$ are linearly independent four vectors, it is enough to check that the contraction of both sides of * with these vectors give the same result.

Ex. Check this

$$\eta_{\mu\nu} = -n_\mu n_\nu + \frac{\bar{p}_\mu \bar{p}_\nu}{\vec{p}^2} + \sum_{b=1}^{2} \bar{\eta}^{(b)}_\mu(\vec{p})\, \bar{\eta}^{(b)}_\nu(\vec{p})$$

$$n.\widetilde{A}(t,\vec{p}) = 0, \quad \bar{p}.\widetilde{A}(t,\vec{p}) = 0, \quad \bar{\eta}^{(b)}(\vec{p}).\widetilde{A}(t,\vec{p}) = \widetilde{A}^{(b)}(t,\vec{p})$$

Using these we get

$$\widetilde{A}_\mu(t,\vec{p}) = \eta_{\mu\nu}\widetilde{A}^\nu(t,\vec{p}) = \sum_{b=1}^{2} \bar{\eta}^{(b)}_\mu(\vec{p})\widetilde{A}^{(b)}(t,\vec{p})$$

Therefore

$$D_{\mu\nu}(x,x') = \int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3p'}{(2\pi)^{3/2}} e^{i\vec{p}.\vec{r}} e^{i\vec{p}\,'.\vec{r}\,'} \langle 0|T(\widetilde{A}_\mu(t,\vec{p})\widetilde{A}_\nu(t',\vec{p}\,'))|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3p'}{(2\pi)^{3/2}} e^{i\vec{p}.\vec{r}} e^{i\vec{p}\,'.\vec{r}\,'} \sum_{b=1}^{2}\sum_{c=1}^{2} \bar{\eta}^{(b)}_\mu(\vec{p})\, \bar{\eta}^{(c)}_\nu(\vec{p}\,') \langle 0|T(\widetilde{A}^{(b)}(t,\vec{p})\widetilde{A}^{(c)}(t',\vec{p}\,'))|0\rangle$$

We now carry out the following steps.

1. Express $\widetilde{A}^{(b)}(t,\vec{p})$ in terms of $a^{(b)}(t,\vec{p})$, $a^{(b)}(t,-\vec{p})^\dagger$

2. Use

$$[H, a^{(b)}(t,\vec{p})] = -E_{\vec{p}}\, a^{(b)}(t,\vec{p}), \quad [H, a^{(b)}(t,\vec{p})^\dagger] = E_{\vec{p}}\, a^{(b)}(t,\vec{p})^\dagger$$

to get the time dependence of $a^{(b)}(t,\vec{p})$, $a^{(b)}(t,-\vec{p})^\dagger$:

$$a^{(b)}(t,\vec{p}) = a^{(b)}(\vec{p})\, e^{-iE_{\vec{p}}t}, \quad a^{(b)}(t,-\vec{p})^\dagger = a^{(b)}(-\vec{p})^\dagger\, e^{iE_{\vec{p}}t}$$

3. Carry out similar manipulations with $\widetilde{A}^{(c)}(t',\vec{p}\,')$

4. Use

$$\langle 0|a^{(b)}(\vec{p})a^{(c)}(-\vec{p}\,')^\dagger|0\rangle = \delta_{bc}\,\delta^{(3)}(\vec{p}+\vec{p}\,')$$

Ex. Show that this gives

$$\langle 0|T(\widetilde{A}^{(b)}(t,\vec{p})\widetilde{A}^{(c)}(t',\vec{p}\,'))|0\rangle = \frac{\delta_{bc}}{2E_{\vec{p}}}\delta^{(3)}(\vec{p}+\vec{p}\,')\left[H(t-t')e^{-iE_{\vec{p}}(t-t')} + H(t'-t)e^{iE_{\vec{p}}(t-t')}\right]$$

This gives

$$D_{\mu\nu}(x,x') = \int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3p'}{(2\pi)^{3/2}} e^{i\vec{p}.\vec{r}} e^{i\vec{p}\,'.\vec{r}\,'} \sum_{b=1}^{2}\sum_{c=1}^{2} \bar{\eta}_\mu^{(b)}(\vec{p})\,\bar{\eta}_\nu^{(c)}(\vec{p}\,')\,\langle 0|T(\widetilde{A}^{(b)}(t,\vec{p})\widetilde{A}^{(c)}(t',\vec{p}\,'))|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[H(t-t')e^{i\vec{p}.(\vec{r}-\vec{r}\,')-iE_{\vec{p}}(t-t')} + H(t'-t)e^{i\vec{p}.(\vec{r}-\vec{r}\,')+iE_{\vec{p}}(t-t')}\right] \sum_{b=1}^{2} \bar{\eta}_\mu^{(b)}(\vec{p})\,\bar{\eta}_\nu^{(b)}(\vec{p})$$

Use

$$\eta_{\mu\nu} = -n_\mu n_\nu + \frac{\bar{p}_\mu \bar{p}_\nu}{\vec{p}^2} + \sum_{b=1}^{2} \bar{\eta}_\mu^{(b)}(\vec{p})\,\bar{\eta}_\nu^{(b)}(\vec{p})$$

to write

$$\sum_{b=1}^{2} \bar{\eta}_\mu^{(b)}(\vec{p})\,\bar{\eta}_\nu^{(b)}(\vec{p}) = \eta_{\mu\nu} + n_\mu n_\nu - \frac{\bar{p}_\mu \bar{p}_\nu}{\vec{p}^2}$$

Using the same trick as in K-G theory, we can express $D_{\mu\nu}$ in more covariant form:

$$D_{\mu\nu}(x,x') = \int \frac{d^4p}{(2\pi)^4}\, e^{ip.(x-x')}\,\widetilde{D}_{\mu\nu}(p), \qquad \widetilde{D}_{\mu\nu}(p) = \left\{\eta_{\mu\nu} + n_\mu n_\nu - \frac{\bar{p}_\mu \bar{p}_\nu}{\vec{p}^2}\right\}\frac{i}{-p^2+i\epsilon}$$

Doing the p^0 integral by Cauchy's theorem, we recover the earlier result.

$D_{\mu\nu}$ plays the role of Δ_F in the Klein-Gordon theory.

Using $D_{\mu\nu}$ we can now calculate the Green's functions for any number of fields and their derivatives in Maxwell's theory

– use the same Feynman rules except that the propagators will be given by $D_{\mu\nu}$ instead of Δ_F.

31 Second quantization of the Dirac equation

Last time we calculated the Feynman propagator for the Maxwell's theory without source terms:

$$D_{\mu\nu}(x,x') = \langle 0|T(A_\mu(t,\vec{r})A_\nu(t',\vec{r}\,'))|0\rangle$$

Result:

$$D_{\mu\nu}(x,x') = \int \frac{d^4p}{(2\pi)^4}\, e^{ip.x}\, \widetilde{D}_{\mu\nu}(p), \qquad \widetilde{D}_{\mu\nu}(p) = \left\{\eta_{\mu\nu} + n_\mu n_\nu - \frac{\bar{p}_\mu \bar{p}_\nu}{\vec{p}^{\,2}}\right\} \frac{i}{-p^2 + i\epsilon}$$

$$\bar{p} = (0, \vec{p}), \quad n = (1,0,0,0)$$

Using $D_{\mu\nu}$ we can now calculate the Green's functions for any number of fields and their derivatives in the Maxwell's theory

– use the same Feynman rules except that the propagators will be given by $D_{\mu\nu}$ instead of Δ_F.

Example:

$$\langle 0|T(A_\mu(x)A_\nu(y)A_\rho(w)A_\sigma(z))|0\rangle = D_{\mu\nu}(x,y)D_{\rho\sigma}(w,z)+D_{\mu\rho}(x,w)D_{\nu\sigma}(y,z)+D_{\mu\sigma}(x,z)D_{\nu\rho}(y,w)$$

$$\langle 0|T(F_{\mu\nu}(x)F_{\rho\sigma}(y))|0\rangle = \partial_\mu\partial_\rho D_{\nu\sigma}(x,y)-(\mu\leftrightarrow\nu)-(\rho\leftrightarrow\sigma)+(\mu\leftrightarrow\nu,\rho\leftrightarrow\sigma)+\text{contact terms}$$

Contact terms: Terms proportional to $\delta(x^0-y^0)$ coming from derivatives of $H(x^0-y^0)$

A rewriting of $D_{\mu\nu}$:

$$\bar{p}_\mu = p_\mu + n.p\, n_\mu, \quad n=(1,0,0,0)$$

This gives

$$\bar{p}_\mu\bar{p}_\nu = p_\mu p_\nu + n.p\,(p_\mu n_\nu + p_\nu n_\mu) + (n.p)^2 n_\mu n_\nu$$

Ex. Check that

$$\begin{aligned}\widetilde{D}_{\mu\nu}(p) &= \left\{\eta_{\mu\nu} + n_\mu n_\nu - \frac{\bar{p}_\mu\bar{p}_\nu}{\vec{p}^2}\right\}\frac{i}{-p^2+i\epsilon} \\ &= \frac{i}{-p^2+i\epsilon}\left[\eta_{\mu\nu} - \frac{n.p}{\vec{p}^2}(p_\mu n_\nu + p_\nu n_\mu) - \frac{p_\mu p_\nu}{\vec{p}^2} + n_\mu n_\nu \frac{p^2}{\vec{p}^2}\right]\end{aligned}$$

Comments: In the calculation of the S-matrix, the terms proportional to p_μ and p_ν cancel after sum over diagrams due to gauge invariance.

The term proportional to $n_\mu n_\nu$ cancels the extra term in the Routhian proportional to J^0J^0 that we had found (to be seen later)

Therefore, we can effectively use $\widetilde{D}_{\mu\nu}(p) = i\,\eta^{\mu\nu}/(-p^2+i\epsilon)$

– Feynman propagator

Dirac equation

– describes relativistic electrons (also other fermions)

– described by a four component wave-function $\psi_\alpha(x)$ $(1 \le \alpha \le 4)$, represented as a four dimensional column vector:

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

Note: α's are not Lorentz vector indices and we'll not distinguish between upper and lower indices.

$\psi(x)$ satisfies the equation:

$$(i\,\gamma^\mu \partial_\mu - m)\psi = 0$$

γ^μ's for $0 \le \mu \le 3$ are 4×4 matrices, satisfying

$$\gamma^\mu \gamma^\nu + \gamma^\mu \gamma^\nu = -2\,\eta^{\mu\nu}, \quad \Rightarrow (\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1$$

$$(\gamma^0)^\dagger = \gamma^0, \qquad (\gamma^i)^\dagger = -\gamma^i$$

Explicit choice of these matrices is not important, so for now we shall proceed without making such a choice.

Our goal: First construct a classical field theory whose field equations give the Dirac equation

Then quantize this field theory, regarding the ψ_α's as describing fermions.

We shall do this following second quantization of Schrodinger equation, by writing the Dirac equation in a way that closely resembles Schrodinger equation.

$$(i\,\gamma^\mu\partial_\mu - m)\psi = 0 \quad \Rightarrow \quad (i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\psi = 0$$

Multiplying this by γ^0 from the left and using $(\gamma^0)^2 = 1$, we get

$$i\partial_0\psi = \widehat{h}\psi, \qquad \widehat{h} = -i\gamma^0\gamma^i\partial_i + m\,\gamma^0$$

Ex. Check that $\widehat{h}$ is hermitian, i.e.

$$\int d^3r\,\psi^\dagger\widehat{h}\chi = \int d^3r\,(\widehat{h}\psi)^\dagger\chi$$

In the new form, the Dirac equation looks like Schrodinger equation, except for the fact that ψ is a four component field instead of a single component field.

Corresponding action

$$\int d^4x\,\psi(x)^\dagger(i\partial_0 - \widehat{h})\psi(x) = \int d^4x\,\psi(x)^\dagger(i\partial_0 + i\gamma^0\gamma^i\partial_i - m\gamma^0)\psi(x) = \int d^4x\,\psi(x)^\dagger\gamma^0(i\,\gamma^\mu\partial_\mu - m)\psi$$

We define

$$\bar{\psi}(x) = \psi(x)^\dagger\gamma^0$$

and write the action as

$$\int d^4x\,\bar{\psi}(x)(i\,\gamma^\mu\partial_\mu - m)\psi$$

Corresponding Hamiltonian (following Schrodinger field theory)

$$H = \int d^3r\,\psi^\dagger(t,\vec{r})\,\widehat{h}\,\psi(t,\vec{r}) = \int d^3r\,\bar{\psi}(t,\vec{r})\,(-i\gamma^i\partial_i + m)\,\psi(t,\vec{r})$$

We shall now follow the same steps as in Schrodinger field theory.

1. Find the eigenstates $u_n(\vec{r})$ and the eigenvalues e_n of $\widehat{h}$.

In this case the eigenstates will be 4-dimensional column vector.

2. Expand $\psi(t, \vec{r})$ in the basis of the eigenfunctions as

$$\psi(t, \vec{r}) = \sum_n a_n(t)\, u_n(\vec{r})$$

3. Upon quantization, a_n's will satisfy the anti-commutation relations:

$$\{a_m(t), a_n(t)\} = 0, \quad \{a_m(t), a_n(t)^\dagger\} = \delta_{mn}, \quad \{a_m(t)^\dagger, a_n(t)^\dagger\} = 0$$

4. The Hamiltonian will be given by

$$H = \sum_n e_n\, a_n(t)^\dagger\, a_n(t)$$

The derivation of these relations follows the same one as in the case of Schrodinger field theory.

The only difference: In this case we shall have continuous energy eigenvalues

$\Rightarrow \sum_n$ will be replaced by momentum integration.

Step 1: Eigenstates and eigenvalues of $\widehat{h}$:

$$\widehat{h}u(\vec{r}) = e\,u(\vec{r}), \qquad \widehat{h} = -i\gamma^0\gamma^i\partial_i + m\,\gamma^0$$

Try

$$u(\vec{r}) = \tilde{u}\,e^{i\vec{p}.\vec{r}}, \qquad \tilde{u} : \vec{r} \text{ independent constant column vector}$$

This gives

$$\tilde{h}\,\tilde{u} = e\,\tilde{u}, \qquad \tilde{h} = \gamma^0\gamma^i p_i + m\gamma^0$$

Note 1: Since $\tilde{h}$ is a 4×4 matrix, we expect four eigenvalues for given $\vec{p}$.

Note 2: Since $\tilde{h}$ anti-commutes with $\gamma^j p_j$, $\gamma^j p_j\tilde{u}$ gives an eigenvector of $\tilde{h}$ with eigenvalue $-e$ $(H\,\gamma^j p_j\tilde{u} = -\gamma^j p_j H\,\tilde{u} = -e\,\gamma^j p_j\tilde{u})$

$\Rightarrow$ eigenvalues come in pairs with opposite signs.

Multiplying the eigenvalue equation by γ^0 and bringing the rhs to the lhs, we get

$$(-\gamma^0 e + \gamma^i p_i + m)\tilde{u} = 0 \quad \Rightarrow \quad (\gamma^\mu p_\mu + m)\tilde{u} = 0, \quad p^0 = e, \quad p_0 = -e$$

Multiplying this by $(-\gamma^\nu p_\nu + m)$ from the left and using $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\eta^{\mu\nu}$ we get

$$(p^2 + m^2)\tilde{u} = 0$$

This gives

$$e = p^0 = \pm E_{\vec{p}}, \qquad E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$$

Since the eigenvalues come in pairs with opposite sign, we must have two eigenvectors with eigenvalue $E_{\vec{p}}$ and two eigenvectors with eigenvalue $-E_{\vec{p}}$.

Call them $u(\vec{p}, s)$ and $v(-\vec{p}, s)$ with $s = 1, 2$.

Furthermore, since $\tilde{h}$ is hermitian, the eigenvectors of $\tilde{h}$ must be orthogonal.

$$u(\vec{p}, s)^\dagger u(\vec{p}, s') = \frac{E_{\vec{p}}}{m}\,\delta_{ss'}, \quad v(-\vec{p}, s)^\dagger v(-\vec{p}, s') = \frac{E_{\vec{p}}}{m}\,\delta_{ss'}, \quad u(\vec{p}, s)^\dagger v(-\vec{p}, s') = 0$$

$E_{\vec{p}}/m$ factor is our choice.

$$(-\gamma^0 e + \gamma^i p_i + m)\tilde{u} = 0$$

Solutions:

$$\tilde{u} = u(\vec{p}, s) \quad \text{with } e = E_{\vec{p}}, \qquad \tilde{u} = v(-\vec{p}, s) \quad \text{with } e = -E_{\vec{p}}$$

Therefore $u(\vec{p}, s)$ satisfies

$$(-\gamma^0 E_{\vec{p}} + \gamma^i p_i + m)\, u(\vec{p}, s) = 0 \quad \Rightarrow \quad (\gamma^\mu p_\mu + m)\, u(\vec{p}, s) = 0, \quad p^0 = E_{\vec{p}}$$

$v(\vec{p}, s)$ satisfies:

$$(\gamma^0 E_{\vec{p}} + \gamma^i p_i + m)\, v(-\vec{p}, s) = 0 \quad \Rightarrow \quad (-\gamma^\mu p_\mu + m)\, v(\vec{p}, s) = 0, \quad p^0 = E_{\vec{p}}$$

Normalizations:

$$u(\vec{p}, s)^\dagger u(\vec{p}, s') = \frac{E_{\vec{p}}}{m}\, \delta_{ss'}, \quad v(-\vec{p}, s)^\dagger v(-\vec{p}, s') = \frac{E_{\vec{p}}}{m}\, \delta_{ss'}, \quad u(\vec{p}, s)^\dagger v(-\vec{p}, s') = 0$$

The eigenstates of $\widehat{h}$ are:

$$u(\vec{p}, s; \vec{r}) = \frac{1}{(2\pi)^{3/2}}\, u(\vec{p}, s) e^{i\vec{p}.\vec{r}}, \qquad v(\vec{p}, s; \vec{r}) = \frac{1}{(2\pi)^{3/2}}\, v(\vec{p}, s)\, e^{-i\vec{p}.\vec{r}}$$

(Note: Changed $\vec{p}$ to $-\vec{p}$ in the second function)

This gives

$$\int d^3r\, u(\vec{p}, s; \vec{r})^\dagger u(\vec{p}', s'; \vec{r}) = u(\vec{p}, s)^\dagger u(\vec{p}', s')\, \delta^{(3)}(\vec{p} - \vec{p}') = \frac{E_{\vec{p}}}{m}\, \delta_{ss'}\, \delta^{(3)}(\vec{p} - \vec{p}')$$

$$\int d^3r\, v(\vec{p}, s; \vec{r})^\dagger v(\vec{p}', s'; \vec{r}) = v(\vec{p}, s)^\dagger v(\vec{p}', s')\, \delta^{(3)}(\vec{p} - \vec{p}') = \frac{E_{\vec{p}}}{m}\, \delta_{ss'}\, \delta^{(3)}(\vec{p} - \vec{p}')$$

$$\int d^3r\, u(\vec{p}, s; \vec{r})^\dagger v(\vec{p}', s'; \vec{r}) = u(\vec{p}, s)^\dagger v(\vec{p}', s')\, \delta^{(3)}(\vec{p} + \vec{p}') = 0$$

Note: Since $2E_{\vec{p}}\delta^{(3)}(\vec{p} - \vec{p}')$ is Lorentz invariant, these are Lorentz invariant normalizations.

$$\sum_n \quad \Rightarrow \quad \int d^3p \frac{m}{E_{\vec{p}}}$$

Step 2: Expand $\psi(t,\vec{r})$ in the basis of the eigenfunctions as

$$\psi(t,\vec{r}) = \sum_n a_n(t)\, u_n(\vec{r})$$

This becomes:

$$\psi(t,\vec{r}) = \int d^3p \frac{m}{E_{\vec{p}}} \sum_{s=1}^{2} \left[\sqrt{\frac{E_{\vec{p}}}{m}}\, b(t,\vec{p},s)\, u(\vec{p},s;\vec{r}) + \sqrt{\frac{E_{\vec{p}}}{m}}\, c(t,\vec{p},s)\, v(\vec{p},s;\vec{r}) \right]$$

$\sqrt{\frac{E_{\vec{p}}}{m}}\, b(t,\vec{p},s)$ and $\sqrt{\frac{E_{\vec{p}}}{m}}\, c(t,\vec{p},s)$ plays the role of a_n's.

Step 3: $\{a_m, a_n^\dagger\} = \delta_{mn}$ become:

$$\left\{ \sqrt{\frac{E_{\vec{p}}}{m}}\, b(t,\vec{p},s), \sqrt{\frac{E_{\vec{p}'}}{m}}\, b(t,\vec{p}',s')^\dagger \right\} = \delta_{ss'} \frac{E_{\vec{p}}}{m}\, \delta^{(3)}(\vec{p}-\vec{p}')$$

$$\Rightarrow \quad \left\{ b(t,\vec{p},s), b(t,\vec{p}',s')^\dagger \right\} = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}')$$

Note: In the Lorentz invariant normalization we are using, δ_{mn} is replaced by the Lorentz invariant delta function $\frac{E_{\vec{p}}}{m}\, \delta^{(3)}(\vec{p}-\vec{p}')$.

Similarly:

$$\left\{ c(t,\vec{p},s), c(t,\vec{p}',s')^\dagger \right\} = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}'), \quad \left\{ b(t,\vec{p},s), c(t,\vec{p}',s')^\dagger \right\} = 0, \quad \left\{ c(t,\vec{p},s), b(t,\vec{p}',s')^\dagger \right\} = 0$$

$$\{b(t,\vec{p},s), b(t,\vec{p}',s')\} = 0, \quad \{b(t,\vec{p},s), c(t,\vec{p}',s')\} = 0, \quad \{c(t,\vec{p},s), c(t,\vec{p}',s')\} = 0$$

$$\psi(t,\vec{r}) = \int d^3p \sqrt{\frac{m}{E_{\vec{p}}}} \sum_{s=1}^{2} \left[b(t,\vec{p},s)\, u(\vec{p},s;\vec{r}) + c(t,\vec{p},s)\, v(\vec{p},s;\vec{r}) \right]$$

Step 4: $H = \sum_n e_n a_n^\dagger a_n$ becomes

$$\begin{aligned} H &= \int d^3p\, \frac{m}{E_{\vec{p}}} \sum_{s=1}^{2} \left[E_{\vec{p}} \frac{E_{\vec{p}}}{m}\, b(t,\vec{p},s)^\dagger b(t,\vec{p},s) - E_{\vec{p}} \frac{E_{\vec{p}}}{m}\, c(t,\vec{p},s)^\dagger c(t,\vec{p},s) \right] \\ &= \int d^3p\, E_{\vec{p}} \sum_{s=1}^{2} \left[b(t,\vec{p},s)^\dagger b(t,\vec{p},s) - c(t,\vec{p},s)^\dagger c(t,\vec{p},s) \right] \end{aligned}$$

$$\psi(t,\vec{r}) = \int d^3p \sqrt{\frac{m}{E_{\vec{p}}}} \sum_{s=1}^{2} \left[b(t,\vec{p},s)\, u(\vec{p},s;\vec{r}) + c(t,\vec{p},s)\, v(\vec{p},s;\vec{r}) \right]$$

$$\left\{ b(t,\vec{p},s), b(t,\vec{p}\,',s')^\dagger \right\} = \delta_{ss'}\,\delta^{(3)}(\vec{p}-\vec{p}\,'), \quad \left\{ c(t,\vec{p},s), c(t,\vec{p}\,',s')^\dagger \right\} = \delta_{ss'}\,\delta^{(3)}(\vec{p}-\vec{p}\,'),$$

$$\left\{ b(t,\vec{p},s), c(t,\vec{p}\,',s')^\dagger \right\} = 0, \quad \left\{ c(t,\vec{p},s), b(t,\vec{p}\,',s')^\dagger \right\} = 0$$

$$\{ b(t,\vec{p},s), b(t,\vec{p}\,',s') \} = 0, \quad \{ b(t,\vec{p},s), c(t,\vec{p}\,',s') \} = 0, \quad \{ c(t,\vec{p},s), c(t,\vec{p}\,',s') \} = 0$$

$$H = \int d^3p\, E_{\vec{p}} \sum_{s=1}^{2} \left[b(t,\vec{p},s)^\dagger b(t,\vec{p},s) - c(t,\vec{p},s)^\dagger c(t,\vec{p},s) \right]$$

The analog of $a_n|0\rangle = 0$ now takes the form:

$$b(t,\vec{p},s)|0\rangle = 0, \quad c(t,\vec{p},s)|0\rangle = 0$$

However there is a problem.

It follows from the expression of H and the anti-commutation relations that

$$[H, b(t,\vec{p},s)] = -E_{\vec{p}}\, b(t,\vec{p},s), \quad [H, b(t,\vec{p},s)^\dagger] = E_{\vec{p}}\, b(t,\vec{p},s)^\dagger$$

$$[H, c(t,\vec{p},s)] = E_{\vec{p}}\, c(t,\vec{p},s), \quad [H, c(t,\vec{p},s)^\dagger] = -E_{\vec{p}}\, c(t,\vec{p},s)^\dagger$$

Therefore the action of $c(t,\vec{p},s)^\dagger$ lowers the energy of a state by $E_{\vec{p}}$

$\Rightarrow$ $|0\rangle$ defined this way is not the lowest energy state.

Solution: Regard c as the creation operator and $c^\dagger$ as the annihilation operator.

– possible since the anti-commutation relations remain unchanged under $c(t,\vec{p},s) \leftrightarrow c(t,\vec{p},s)^\dagger$.

Define

$$d(t,\vec{p},s) = c(t,\vec{p},s)^\dagger, \qquad d(t,\vec{p},s)^\dagger = c(t,\vec{p},s)$$

$$H = \int d^3p\, E_{\vec{p}} \sum_{s=1}^{2} \left[b(t,\vec{p},s)^\dagger b(t,\vec{p},s) + d(t,\vec{p},s)^\dagger d(t,\vec{p},s) \right] + \text{constant}$$

$$\psi(t,\vec{r}) = \int d^3p \sqrt{\frac{m}{E_{\vec{p}}}} \sum_{s=1}^{2} \left[b(t,\vec{p},s)\, u(\vec{p},s;\vec{r}) + d(t,\vec{p},s)^\dagger\, v(\vec{p},s;\vec{r}) \right]$$

$$\left\{ b(t,\vec{p},s), b(t,\vec{p}\,',s')^\dagger \right\} = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}\,'), \quad \left\{ d(t,\vec{p},s), d(t,\vec{p}\,',s')^\dagger \right\} = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}\,'),$$

$$\left\{ b(t,\vec{p},s), d(t,\vec{p}\,',s')^\dagger \right\} = 0, \quad \left\{ d(t,\vec{p},s), b(t,\vec{p}\,',s')^\dagger \right\} = 0$$

$$\{ b(t,\vec{p},s), b(t,\vec{p}\,',s') \} = 0, \quad \{ b(t,\vec{p},s), d(t,\vec{p}\,',s') \} = 0, \quad \{ d(t,\vec{p},s), d(t,\vec{p}\,',s') \} = 0$$

$$H = \int d^3p\, E_{\vec{p}} \sum_{s=1}^{2} [b(t,\vec{p},s)^\dagger b(t,\vec{p},s) + d(t,\vec{p},s)^\dagger d(t,\vec{p},s)]$$

The vacuum state $|0\rangle$ is defined via:

$$b(0,\vec{p},s)|0\rangle = 0, \qquad d(0,\vec{p},s)|0\rangle = 0$$

Now we have four types of single particle states, all carrying energy $E_{\vec{p}}$.

$$* \quad : \quad b(0,\vec{p},s)^\dagger|0\rangle, \qquad d(0,\vec{p},s)^\dagger|0\rangle, \qquad s = 1,2$$

Ex. The conserved momentum P_i is given by

$$P_i = \int d^3p\, p_i \sum_{s=1}^{2} [b(t,\vec{p},s)^\dagger b(t,\vec{p},s) + d(t,\vec{p},s)^\dagger d(t,\vec{p},s)]$$

$\Rightarrow$ the states in $*$ all carry momentum $\vec{p}$.

Therefore their masses are given by $\sqrt{E_{\vec{p}}^2 - \vec{p}^2} = m$.

We now have to find the physical interpretation of the four states.

32 Quantization of Dirac field

Expansion of the Dirac field:

$$\psi(t,\vec{r}) = \int d^3p \sqrt{\frac{m}{E_{\vec{p}}}} \sum_{s=1}^{2} \left[b(t,\vec{p},s)\, u(\vec{p},s;\vec{r}) + d(t,\vec{p},s)^\dagger\, v(\vec{p},s;\vec{r}) \right]$$

$$\left\{ b(t,\vec{p},s), b(t,\vec{p}\,',s')^\dagger \right\} = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}\,'), \quad \left\{ d(t,\vec{p},s), d(t,\vec{p}\,',s')^\dagger \right\} = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}\,'),$$

$$\left\{ b(t,\vec{p},s), d(t,\vec{p}\,',s')^\dagger \right\} = 0, \quad \left\{ d(t,\vec{p},s), b(t,\vec{p}\,',s')^\dagger \right\} = 0$$

$$\{ b(t,\vec{p},s), b(t,\vec{p}\,',s') \} = 0, \quad \{ b(t,\vec{p},s), d(t,\vec{p}\,',s') \} = 0, \quad \{ d(t,\vec{p},s), d(t,\vec{p}\,',s') \} = 0$$

$$H = \int d^3p\, E_{\vec{p}} \sum_{s=1}^{2} [b(t,\vec{p},s)^\dagger b(t,\vec{p},s) + d(t,\vec{p},s)^\dagger d(t,\vec{p},s)]$$

$$P_i = \int d^3p\, p_i \sum_{s=1}^{2} [b(t,\vec{p},s)^\dagger b(t,\vec{p},s) + d(t,\vec{p},s)^\dagger d(t,\vec{p},s)]$$

The vacuum state $|0\rangle$ is defined via:

$$b(0,\vec{p},s)|0\rangle = 0, \qquad d(0,\vec{p},s)|0\rangle = 0$$

Now we have four types of single particle states, all carrying energy $E_{\vec{p}}$ and momentum $\vec{p}$:

$$* \quad : \quad b(0,\vec{p},s)^\dagger|0\rangle, \qquad d(0,\vec{p},s)^\dagger|0\rangle, \qquad s = 1,2$$

Therefore their masses are given by $\sqrt{E_{\vec{p}}^2 - \vec{p}^2} = m$.

We now have to find the physical interpretation of the four states.

The Dirac action is invariant under

$$\psi(x) \to e^{i\theta}\psi(x), \qquad \psi(x)^\dagger \to e^{-i\theta}\psi(x)^\dagger$$

Corresponding conserved quantity (same derivation as in Schrodinger theory):

$$N = \int d^3r\, \psi(t,\vec{r})^\dagger \psi(t,\vec{r}) = \int d^3p \sum_{s=1}^{2} [b(t,\vec{p},s)^\dagger b(t,\vec{p},s) - d(t,\vec{p},s)^\dagger d(t,\vec{p},s)]$$

– has eigenvalue 1 for $b(0,\vec{p},s)^\dagger|0\rangle$ and -1 for $d(0,\vec{p},s)^\dagger|0\rangle$

If $-e$ is the electric charge carried by the electron, then we define the electric charge operator as:

$$Q = -e\,N = -e\int d^3p \sum_{s=1}^{2} [b(t,\vec{p},s)^\dagger b(t,\vec{p},s) - d(t,\vec{p},s)^\dagger d(t,\vec{p},s)]$$

– has eigenvalue $-e$ for $b(0,\vec{p},s)^\dagger|0\rangle$ and e for $d(0,\vec{p},s)^\dagger|0\rangle$

With this interpretation of Q, $b(0,\vec{p},s)^\dagger|0\rangle$ describes single electron state and $d(0,\vec{p},s)^\dagger|0\rangle$ describes single positron state.

Ex. Check that the expression for the conserved current J^μ is:

$$J^\mu = -e\,\bar{\psi}(x)\,\gamma^\mu\,\psi(x)$$

This gives

$$Q(t) = \int d^3r J^0(t,\vec{r}) = -e\int d^3r\,\psi(t,\vec{r})^\dagger\,\gamma^0\,\psi(t,\vec{r})$$

J^μ will be useful for coupling electromagnetic field to the Dirac field.

The remaining problem is to find an interpretation of the label s.

What do the two states $s = 1, 2$ represent?

For this we need to study the conserved charges $\mathcal{J}^{\mu\nu}$ associated with Lorentz transformation.

Recall that in a generic theory with field $\{\phi_r\}$, the infinitesimal Lorentz transformation takes the form:

$$\widetilde{\phi}_r(x) = \phi_r(x) + \epsilon\, \omega^{\mu}{}_{\nu}\, x^{\nu}\, \partial_{\mu}\phi_r(x) + \epsilon\, \omega_{\tau\nu}\, \Sigma^{\tau\nu}_{rs}\phi_s(x), \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

$\Sigma^{\tau\nu}_{rs}$ are specific constants encoding the Lorentz transformation laws of ϕ_r, e.g. for scalars, $\Sigma^{\tau\nu}_{rs} = 0$.

The corresponding conserved quantities:

$$\mathcal{J}^{\nu\tau} = \int d^3r M^{0\nu\tau}, \quad M^{\mu\nu\tau} = x^{\nu}T^{\mu\tau} - x^{\tau}T^{\mu\nu} - 2\sum_{r,s=1}^{n} \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_r)}\,\Sigma^{\tau\nu}_{rs}\phi_s$$

Ex. 1. Show that the Dirac action is invariant under Lorentz transformation, if in the transformation laws of ψ_{α}, we choose:

$$\Sigma^{\tau\nu}_{\alpha\beta} = c\,(\gamma^{\tau}\gamma^{\nu} - \gamma^{\nu}\gamma^{\tau})_{\alpha\beta}$$

for appropriate constant c. Find c.

2. Construct $\mathcal{J}^{\nu\tau}$.

3. From this we can get $J = \epsilon_{ijk}P^{i}\mathcal{J}^{jk}$ – the component of the angular momentum along the momentum.

4. Show that if we consider the states $b(0,\vec{p},s)^{\dagger}|0\rangle$ for $s = 1,2$, then appropriate linear combinations of the two states describe eigenstates of J with eigenvalues $\pm|\vec{p}|/2$

$\Rightarrow$ electrons have spin 1/2.

Similar analysis involving the states $d(0,\vec{p},s)^{\dagger}|0\rangle$ shows that positrons also have spin 1/2.

5. Show that under Lorentz transformation, $\bar{\psi}(x)\psi(x)$ transforms as a scalar and $\bar{\psi}(x)\gamma^{\mu}\psi(x)$ transforms as a contravariant vector.

We shall now define

$$T(\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)) = H(t_1 - t_2)\,\psi_\alpha(x_1)\bar{\psi}_\beta(x_2) - H(t_2 - t_1)\,\bar{\psi}_\beta(x_2)\,\psi_\alpha(x_1)$$

ans calculate

$$S_{F\alpha\beta}(x_1, x_2) = \langle 0|T(\psi_\alpha(x_1)\bar{\psi}_\beta(x_2))|0\rangle$$

Note the $-$ sign in the definition of the time ordered product

– related to the fact that we are considering a fermionic field.

Recall that $\bar{\psi} = \psi^\dagger\,\gamma^0$.

We proceed as in the case of scalar field theory.

Ex. Use the expression for H in terms of $b, b^\dagger, c, c^\dagger$ to check that

$$[H, b(t,\vec{p},s)] = -E_{\vec{p}}\,b(t,\vec{p},s), \quad [H, b(t,\vec{p},s)^\dagger] = E_{\vec{p}}\,b(t,\vec{p},s)^\dagger$$
$$[H, d(t,\vec{p},s)] = -E_{\vec{p}}\,d(t,\vec{p},s), \quad [H, d(t,\vec{p},s)^\dagger] = E_{\vec{p}}\,d(t,\vec{p},s)^\dagger$$

This gives

$$b(t,\vec{p},s) = b(0,\vec{p},s)\,e^{-iE_{\vec{p}}t}, \qquad b(t,\vec{p},s)^\dagger = b(0,\vec{p},s)^\dagger\,e^{iE_{\vec{p}}t}$$
$$d(t,\vec{p},s) = d(0,\vec{p},s)\,e^{-iE_{\vec{p}}t}, \qquad d(t,\vec{p},s)^\dagger = d(0,\vec{p},s)^\dagger\,e^{iE_{\vec{p}}t}$$

This gives

$$\psi_\alpha(t,\vec{r}) = \int d^3p \sqrt{\frac{m}{E_{\vec{p}}}} \sum_{s=1}^{2} \left[b(0,\vec{p},s)\,e^{-iE_{\vec{p}}t}\,u_\alpha(\vec{p},s;\vec{r}) + d(0,\vec{p},s)^\dagger\,e^{iE_{\vec{p}}t}\,v_\alpha(\vec{p},s;\vec{r}) \right]$$

$$\bar{\psi}_\beta(t,\vec{r}) = \int d^3p \sqrt{\frac{m}{E_{\vec{p}}}} \sum_{s=1}^{2} \left[b(0,\vec{p},s)^\dagger\,e^{iE_{\vec{p}}t}\,\bar{u}_\beta(\vec{p},s;\vec{r}) + d(0,\vec{p},s)\,e^{-iE_{\vec{p}}t}\,\bar{v}_\beta(\vec{p},s;\vec{r}) \right]$$

We now substitute into the expression for $S_{F\alpha\beta}$ and calculate the matrix elements using properties of $|0\rangle$ and the anti-commutation relations between $b, d, b^\dagger, d^\dagger$.

Recall our definitions:

$$u(\vec{p}, s; \vec{r}) = \frac{1}{(2\pi)^{3/2}}\, u(\vec{p}, s) e^{i\vec{p}.\vec{r}}, \qquad v(\vec{p}, s; \vec{r}) = \frac{1}{(2\pi)^{3/2}}\, v(\vec{p}, s)\, e^{-i\vec{p}.\vec{r}}$$

$$\Rightarrow \quad \bar{u}(\vec{p}, s; \vec{r}) = \frac{1}{(2\pi)^{3/2}}\, \bar{u}(\vec{p}, s) e^{-i\vec{p}.\vec{r}}, \qquad \bar{v}(\vec{p}, s; \vec{r}) = \frac{1}{(2\pi)^{3/2}}\, \bar{v}(\vec{p}, s)\, e^{i\vec{p}.\vec{r}}$$

Ex. Check that

$$\begin{aligned} S_{F\alpha\beta}(x_1, x_2) \; = \; & \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{m}{E_{\vec{p}}} \Big[H(t_1 - t_2) e^{i\vec{p}.(\vec{r}_1 - \vec{r}_2) - iE_{\vec{p}}(t_1 - t_2)} u_\alpha(\vec{p}, s) \bar{u}_\beta(\vec{p}, s) \\ & - H(t_2 - t_1) e^{-i\vec{p}.(\vec{r}_1 - \vec{r}_2) + iE_{\vec{p}}(t_1 - t_2)} v_\alpha(\vec{p}, s) \bar{v}_\beta(\vec{p}, s) \Big] \end{aligned}$$

Define

$$\not{p} = -\eta_{\mu\nu}\gamma^\mu p^\nu = \gamma^0 p^0 - \vec{\gamma}.\vec{p} \quad \Rightarrow \quad \not{p}\not{p} = p_\mu p_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} p_\mu p_\nu (\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = -p^2$$

Now we shall use some identities:

$$\sum_{s=1}^{2} u_\alpha(\vec{p}, s)\bar{u}_\beta(\vec{p}, s) = \frac{1}{2m}(\not{p} + m)_{\alpha\beta} \quad \text{with } p^0 = E_{\vec{p}}, \quad (m)_{\alpha\beta} = m\,\delta_{\alpha\beta}$$

$$\sum_{s=1}^{2} v_\alpha(\vec{p}, s)\bar{v}_\beta(\vec{p}, s) = \frac{1}{2m}(\not{p} - m)_{\alpha\beta} \quad \text{with } p^0 = E_{\vec{p}}$$

$$\bar{u}(\vec{p}, s)u(\vec{p}, s') = \delta_{ss'} = -\bar{v}(\vec{p}, s)v(\vec{p}, s'), \quad \bar{u}(\vec{p}, s)v(\vec{p}, s') = 0 = \bar{v}(\vec{p}, s)u(\vec{p}, s')$$

Proofs can be found in standard textbooks on relativistic quantum mechanics.

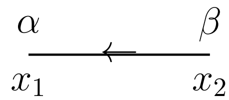
$$\begin{aligned} S_{F\alpha\beta}(x_1, x_2) \; &= \; \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\,E_{\vec{p}}} \Big[H(t_1 - t_2) e^{i\vec{p}.(\vec{r}_1 - \vec{r}_2) - iE_{\vec{p}}(t_1 - t_2)} (\not{p} + m)_{\alpha\beta} \\ & \qquad - H(t_2 - t_1) e^{-i\vec{p}.(\vec{r}_1 - \vec{r}_2) + iE_{\vec{p}}(t_1 - t_2)} (\not{p} - m)_{\alpha\beta} \Big] \\ &= \; \int \frac{d^4p}{(2\pi)^4}\, e^{ip.(x_1 - x_2)}\, \frac{i}{-p^2 - m^2 + i\epsilon}\, (\not{p} + m)_{\alpha\beta} \end{aligned}$$

$$S_{F\alpha\beta} = \int \frac{d^4p}{(2\pi)^4}\, e^{ip.(x_1 - x_2)} \left(\frac{i}{\not{p} - m + i\epsilon} \right)_{\alpha\beta} \quad \text{since} \quad (\not{p} + m)(\not{p} - m) = (-p^2 - m^2)$$

$$S_F(x_1, x_2) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip.(x_1-x_2)}\, \widetilde{S}_F(p)$$

$$\widetilde{S}_F(p) = \frac{i}{-p^2 - m^2 + i\epsilon}\,(\not{p} + m) = \left(\frac{i}{\not{p} - m + i\epsilon}\right)$$

When we draw Feynman diagrams, we represent $S_{F\alpha\beta}$ by an arrow pointing from β to α.

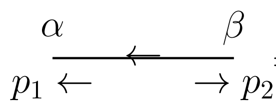


$= S_{F\alpha\beta}(x_1, x_2)$

We define the momentum space propagator as

$$\widetilde{S}_F(p_1, p_2) = \int d^4x_1\, d^4x_2\, e^{-ip_1.x_1 - ip_2.x_2}\, S_F(x_1, x_2) = \widetilde{S}_F(p_1)\,(2\pi)^4\, \delta^{(4)}(p_1 + p_2)$$

In momentum space, we represent the propagator as



$= \widetilde{S}_{F\alpha\beta}(p_1)\,(2\pi)^4\delta^{(4)}(p_1 + p_2)$

Note: Unlike scalar and vector propagators, the direction of momentum is important for fermions since $\widetilde{S}_F(p) \neq \widetilde{S}_F(-p)$.

Rule: If the arrow points from β to α, and the momentum along the arrow is p, then the factor is $\widetilde{S}_{F\alpha\beta}(p)$.

Ex. Check that

$$\langle 0|T(\psi_\alpha(x_1)\psi_\beta(x_2))|0\rangle = 0, \qquad \langle 0|T(\bar{\psi}_\alpha(x_1)\bar{\psi}_\beta(x_2))|0\rangle = 0$$

With the help of the propagator and the usual Feynman rules we can now calculate general Green's function of the form:

$$G_{\alpha_1\cdots\alpha_n;\beta_1,\cdots,\beta_n}(x_1,\cdots,x_n,y_1,\cdots y_n) = \langle 0|T(\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar{\psi}_{\beta_1}(y_1)\cdots\bar{\psi}_{\beta_n}(y_n))|0\rangle$$

Note that the Green's function vanishes unless there are equal number of ψ's and $\bar{\psi}$'s

– consequence of charge conservation.

We can express this as sum over Feynman diagrams

Difference from the theories studies earlier: The ψ's and $\bar{\psi}$'s have to be treated differently.

ψ: external vertex with the property that a propagator connected to the vertex must have its arrow towards the external vertex.

$\bar{\psi}$: external vertex with the property that a propagator connected to the vertex must have its arrow away from the external vertex.

Therefore a propagator connects a $\bar{\psi}$ to a ψ but never a ψ to a ψ or $\bar{\psi}$ to $\bar{\psi}$.

Sum over all possible ways of connecting the (y_j,β_j) to (x_i,α_i).

If (y_j,β_j) is connected to $(x_{P(j)},\alpha_{P(j)})$ where P is some permutation of $1,\cdots n$ and $P(j)$ is the j-th element of the permuted numbers.

$$G_{\alpha_1\cdots\alpha_n;\beta_1,\cdots,\beta_n}(x_1,\cdots,x_n,y_1,\cdots y_n) = \sum_P\prod_{j=1}^n S_{F\alpha_{P(j)}\beta_j}(x_{P(j)},y_j) \times \text{sign}$$

Determination of the sign for a given term:

1. Take the starting expression

$$\langle 0|T(\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar{\psi}_{\beta_1}(y_1)\cdots\bar{\psi}_{\beta_n}(y_n))|0\rangle$$

2. In a given term, if α_i is paired with β_j, connect them by a line on the top.

3. Do this for every pair that are connected by a propagator.

4. Count the total number of times the lines intersect.

5. If this is n then the sign is $(-1)^n$.

Example:

$$\langle 0|T(\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\bar{\psi}_{\beta_1}(y_1)\bar{\psi}_{\beta_2}(y_2))|0\rangle = -S_{F\alpha_1\beta_1}(x_1,y_1)S_{F\alpha_2\beta_2}(x_2,y_2)+S_{F\alpha_1\beta_2}(x_1,y_2)S_{F\alpha_2\beta_1}(x_2,y_1)$$

Note: To apply this rule, we need to first bring all the ψ's to the left of all the $\bar{\psi}$'s inside the time ordered product, picking up a minus sign for every exchange.

These rules follow from the usual manipulation involving creation and annihilation operators as in the Klein-Gordon theory, together with the observation that inside a time ordering, exchange of any pair of fermionic operators produces a minus sign.

33 Quantum electrodynamics

So far we have studied source free Maxwell's theory describing free photons and free Dirac theory describing non-interacting electrons and positrons.

Now we shall put the two theories together.

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} + \int d^4x\, \bar{\psi}(x)(i\,\gamma^\mu \partial_\mu - m)\psi$$

Note: We have not yet introduced any coupling between the electrons and photons

If we quantize this we shall get 6 types of single particle states;

1. 2 spin states of the electron $b(0,\vec{p},s)^\dagger|0\rangle$, $s=1,2$

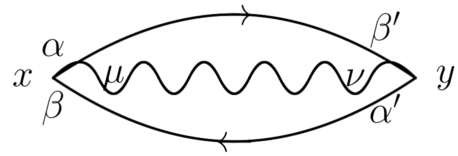
2. 2 spin states of the positron $d(0,\vec{p},s)^\dagger|0\rangle$, $s=1,2$

3. 2 spin states of the photon $a^{(c)}(0,\vec{p})^\dagger|0\rangle$, $c=1,2$

General multi-particle state contains electrons, positrons and photons, obtained by multiple application of the creation operators on the vacuum.

We can use the propagator of the Dirac field and the electromagnetic field to calculate Green's functions of operators made of A_μ, ψ and $\bar{\psi}$ e.g.

$$\langle 0|T(\bar{\psi}_\alpha(x)\gamma^\mu_{\alpha\beta}\psi_\beta(x)A_\mu(x)\ \bar{\psi}_{\alpha'}(y)\gamma^\nu_{\alpha'\beta'}\psi_{\beta'}(y)A_\nu(y))|0\rangle$$



$$-S_{F\beta'\alpha}(y,x)S_{F\beta\alpha'}(x,y)\gamma^\mu_{\alpha\beta}\gamma^\nu_{\alpha'\beta'}D_{\mu\nu}(x,y) = -Tr\{S_F(y,x)\gamma^\mu S_F(x,y)\gamma^\nu\}\,D_{\mu\nu}(x,y)$$

We now have to add extra terms that describe the coupling between the electron and the photon.

Recall the form of Maxwell's equation in presence of source terms:

$$\partial_\mu F^{\mu\nu} = -J^\nu, \qquad \partial_\nu J^\nu = 0$$

We shall consider the case where J^μ is the current produced by the electrons / positrons.

$$J^\mu(x) = -e\,\bar{\psi}(x)\gamma^\mu\psi(x)$$

Ex. This can be achieved by adding to the action a new term:

$$\int d^4x\, A_\mu J^\mu$$

Therefore we have

$$S = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(x)(i\,\gamma^\mu\partial_\mu - m)\psi + A_\mu J^\mu\right], \qquad J^\mu(x) = -e\,\bar{\psi}(x)\gamma^\mu\psi(x)$$

Note: Even in the presence of the new term in the action, the symmetry under

$$\psi(x) \to e^{i\theta}\psi(x), \quad \bar{\psi}(x) \to e^{-i\theta}\bar{\psi}(x)$$

remains intact

$\Rightarrow J^\mu$ still satisfies $\partial_\mu J^\mu = 0$.

What about gauge invariance: $A_\mu \to A_\mu(x) + \partial_\mu\Lambda(x)$?

Not a symmetry since $A_\mu J^\mu \to A_\mu J^\mu + \partial_\mu\Lambda\, J^\mu$

We cannot integrate by parts and use $\partial_\mu J^\mu = 0$ since $\partial_\mu J^\mu = 0$ is valid only when equations of motion hold.

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(x)(i\,\gamma^\mu \partial_\mu - m)\psi + A_\mu J^\mu \right], \qquad J^\mu(x) = -e\,\bar{\psi}(x)\gamma^\mu \psi(x)$$

Write the action as

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(x)\left\{ i\,\gamma^\mu (\partial_\mu + i\,e\,A_\mu) - m \right\} \psi \right]$$

Ex. Check that the action is invariant under

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x), \quad \psi(x) \to e^{-ie\Lambda(x)}\psi(x), \quad \bar{\psi}(x) \to e^{ie\Lambda(x)}\bar{\psi}(x)$$

We define 'covariant derivative'

$$D_\mu \psi = (\partial_\mu + ieA_\mu(x))\psi(x)$$

Then, under gauge transformation:

$$D_\mu\psi(x) \to (\partial_\mu + i\,e\,A_\mu + i\,e\,\partial_\mu\Lambda)e^{-ie\Lambda(x)}\psi(x) = e^{-ie\Lambda(x)} D_\mu\psi(x), \quad \bar{\psi}(x) D_\mu \psi(x) \to \bar{\psi}(x) D_\mu \psi(x)$$

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(x)\left\{ i\,\gamma^\mu\, D_\mu - m \right\} \psi \right]$$

remains invariant.

In general, consistent quantization of a gauge theory requires that the action must have exact gauge invariance, possibly with modified transformation laws of various fields.

Due to gauge symmetry, the equations of motion still do not fix the time evolution of the fields uniquely and we need to fix a gauge.

We shall now proceed to analyze this theory using the Routhian formalism.

$$L = \int d^3r \left[\frac{1}{2}(\partial_0 A_i - \partial_i A_0)(\partial_0 A_i - \partial_i A_0) - \frac{1}{4}F_{ij}F_{ij} + \bar{\psi}(x)(i\,\gamma^\mu\partial_\mu - m)\psi + A_\mu J^\mu\right]$$

$$J^\mu(x) = -e\,\bar{\psi}(x)\gamma^\mu\psi(x)$$

We shall treat A_i's and ψ_α's in the Hamiltonian formalism and A_0 in the Lagrangian formalism.

$$\Pi_i(t,\vec{r}) = \frac{\delta L}{\delta \dot{A}_i(t,\vec{r})} = \partial_0 A_i - \partial_i A_0, \quad \Pi^\psi_\alpha = \frac{\delta L}{\delta \dot{\psi}_\alpha} = i\,\psi^\dagger_\alpha$$

$$\begin{aligned} R &= \int d^3r \left[\Pi_i(t,\vec{r})\dot{A}_i(t,\vec{r}) + \Pi^\psi_\alpha(t,\vec{r})\dot{\psi}_\alpha(t,\vec{r}) - L\right] \\ &= \int d^3r \left[\frac{1}{2}\Pi_i(t,\vec{r})\Pi_i(t,\vec{r}) + \frac{1}{2}B_i(t,\vec{r})B_i(t,\vec{r}) - A_0\partial_i\Pi_i(t,\vec{r}) - A_\mu J^\mu\right] + H_D \end{aligned}$$

where

$$H_D = \int d^3r\,\bar{\psi}(t,\vec{r})\left(-i\gamma^i\partial_i + m\right)\psi(t,\vec{r})$$

$$B_i = \epsilon_{ijk}\partial_j A_k, \quad \Rightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

Equation of motion for A_0 gives

$$\frac{\partial}{\partial t}\frac{\delta R}{\delta \dot{A}_0} - \frac{\delta R}{\delta A_0} = 0 \quad \Rightarrow \quad \partial_i\Pi_i(t,\vec{r}) + J^0 = 0$$

Equations of motion for A_i and Π_i give:

$$\frac{\partial A_i}{\partial t} = \frac{\delta R}{\delta \Pi_i} = \Pi_i + \partial_i A_0$$

$$\frac{\partial \Pi_i}{\partial t} = -\frac{\delta R}{\delta A_i} = -\epsilon_{ijk}\partial_j B_k + J_i$$

Note: As before there is no equation that gives the time evolution of A_0

– consequence of gauge invariance under $A_\mu \to A_\mu + \partial_\mu\Lambda(x)$

We shall use Coulomb gauge $\vec{\nabla}.\vec{A} = 0$.

Strategy:

Express the Routhian using momentum space variables $\widetilde{A}_i(t,\vec{p})$, $\widetilde{\Pi}_i(t,\vec{p})$, $\widetilde{A}_0(t,\vec{p})$, $\widetilde{\psi}_\alpha(t,\vec{p})$, $\widetilde{\Pi}^\psi_\alpha(t,\vec{p})$.

$$A_i(t,\vec{r}) = \int \frac{d^3p}{(2\pi)^{3/2}}\, e^{i\vec{p}.\vec{r}}\, \widetilde{A}_i(t,\vec{p}) \qquad \text{etc.}$$

and then eliminate some of the variables using their equations of motion.

Introduce vectors $\vec{\epsilon}^{\,(a)}(\vec{p})$ for $a = 1, 2$ such that

$$\vec{p}.\vec{\epsilon}^{\,(a)}(\vec{p}) = 0, \qquad \vec{\epsilon}^{\,(a)}(\vec{p}).\vec{\epsilon}^{\,(b)}(\vec{p}) = \delta_{ab}, \qquad \vec{\epsilon}^{\,(a)}(-\vec{p}) = \vec{\epsilon}^{\,(a)}(\vec{p})$$

Define:

$$\widetilde{A}^{(a)}(t,\vec{p}) = \epsilon^{(a)}_j(\vec{p})\, \widetilde{A}_j(t,\vec{p}), \qquad \widetilde{\Pi}^{(a)}(t,\vec{p}) = \epsilon^{(a)}_j(\vec{p})\, \widetilde{\Pi}_j(t,\vec{p})$$

$$\widetilde{A}_\parallel(t,\vec{p}) = i\,\frac{p_j \widetilde{A}_j(t,\vec{p})}{|\vec{p}|}, \qquad \widetilde{\Pi}_\parallel(t,\vec{p}) = i\,\frac{p_j \widetilde{\Pi}_j(t,\vec{p})}{|\vec{p}|}$$

The reverse relations are:

$$\widetilde{A}_j(t,\vec{p}) = \sum_{a=1}^{2} \widetilde{A}^{(a)}(t,\vec{p})\, \epsilon^{(a)}_j(\vec{p}) - i\, \widetilde{A}_\parallel(t,\vec{p})\, \frac{p_j}{|\vec{p}|}, \quad \widetilde{\Pi}_j(t,\vec{p}) = \sum_{a=1}^{2} \widetilde{\Pi}^{(a)}(t,\vec{p})\, \epsilon^{(a)}_j(\vec{p}) - i\, \widetilde{\Pi}_\parallel(t,\vec{p})\, \frac{p_j}{|\vec{p}|}$$

We shall also define $\widetilde{J}^\mu(t,\vec{p})$ via

$$J^\mu(t,\vec{r}) = \int \frac{d^3p}{(2\pi)^{3/2}}\, e^{i\vec{p}.\vec{r}}\, \widetilde{J}^\mu(t,\vec{p})$$

and define:

$$\widetilde{J}^{(a)}(t,\vec{p}) = \epsilon^{(a)}_j(\vec{p})\, \widetilde{J}_j(t,\vec{p}), \qquad \widetilde{J}_\parallel(t,\vec{p}) = i\,\frac{p_j \widetilde{J}_j(t,\vec{p})}{|\vec{p}|}$$

This gives

$$\widetilde{J}_j(t,\vec{p}) = \sum_{a=1}^{2} \widetilde{J}^{(a)}(t,\vec{p})\, \epsilon^{(a)}_j(\vec{p}) - i\, \widetilde{J}_\parallel(t,\vec{p})\, \frac{p_j}{|\vec{p}|}$$

In these variables

$$R = \int d^3r \left[\frac{1}{2}\Pi_i(t,\vec{r})\Pi_i(t,\vec{r}) + \frac{1}{2}B_i(t,\vec{r})B_i(t,\vec{r}) - A_0\partial_i\Pi_i(t,\vec{r}) - A_\mu J^\mu\right] + H_D$$

$$= \frac{1}{2}\int d^3p \left[\sum_{a=1}^{2}\widetilde{\Pi}^{(a)}(t,-\vec{p})\widetilde{\Pi}^{(a)}(t,\vec{p}) + \widetilde{\Pi}_\parallel(t,-\vec{p})\widetilde{\Pi}_\parallel(t,\vec{p}) + \vec{p}^{\,2}\sum_{a=1}^{2}\widetilde{A}^{(a)}(t,-\vec{p})\widetilde{A}^{(a)}(t,\vec{p})\right]$$

$$-\int d^3p \left[\widetilde{A}_0(t,-\vec{p})\{|\vec{p}|\,\widetilde{\Pi}_\parallel(t,\vec{p}) + \widetilde{J}^0(t,\vec{p})\} + \widetilde{A}_\parallel(t,-\vec{p})\widetilde{J}_\parallel(t,\vec{p}) + \sum_{a=1}^{2}\widetilde{A}^{(a)}(t,-\vec{p})\widetilde{J}^{(a)}(t,\vec{p})\right] + H_D$$

Now examine the $\widetilde{A}_0$, $\widetilde{A}_\parallel$ and $\widetilde{\Pi}_\parallel$ equations of motion.

$$\partial_t\left(\frac{\delta R}{\delta(\partial_t\widetilde{A}_0)}\right) - \frac{\delta R}{\delta\widetilde{A}_0} \qquad \Rightarrow \qquad |\vec{p}|\,\widetilde{\Pi}_\parallel(t,\vec{p}) + \widetilde{J}^0(t,\vec{p}) = 0$$

$$\partial_t\widetilde{A}_\parallel(t,\vec{p}) = \frac{\delta R}{\delta\widetilde{\Pi}_\parallel(t,-\vec{p})} = \widetilde{\Pi}_\parallel(t,\vec{p}) - |\vec{p}|\,\widetilde{A}_0(t,\vec{p})$$

$$\partial_t\widetilde{\Pi}_\parallel(t,\vec{p}) = -\frac{\delta R}{\delta\widetilde{A}_\parallel(t,-\vec{p})} = \widetilde{J}_\parallel(t,\vec{p})$$

Note: $\widetilde{A}_i(t,-\vec{p})$ and $\widetilde{\Pi}_i(t,\vec{p})$ are conjugate variable since

$$\{A_i(t,\vec{r}),\Pi_j(t,\vec{r}\,')\}_{PB} = \delta^{(3)}(\vec{r}-\vec{r}\,')\,\delta_{ij} \quad \Rightarrow \quad \{\widetilde{A}_i(t,\vec{p}),\widetilde{\Pi}_j(t,\vec{p}\,')\}_{PB} = \delta^{(3)}(\vec{p}+\vec{p}\,')\,\delta_{ij}$$

Coulomb gauge condition $\vec{\nabla}.\vec{A} = 0$ gives

$$\widetilde{A}_\parallel = 0$$

First two equations of motion gives

$$\widetilde{\Pi}_\parallel(t,\vec{p}) = -\frac{1}{|\vec{p}|}\,\widetilde{J}^0(t,\vec{p}), \qquad \widetilde{A}_0(t,\vec{p}) = \frac{\widetilde{\Pi}_\parallel}{|\vec{p}|} = -\frac{1}{\vec{p}^{\,2}}\,\widetilde{J}^0(t,\vec{p})$$

The third equation now holds automatically since

$$\partial_\mu J^\mu = 0 \qquad \Rightarrow \qquad \partial_t\widetilde{J}^0 = -|\vec{p}|\,\widetilde{J}_\parallel$$

We use these equations to eliminate $\widetilde{A}_0, \widetilde{A}_\parallel$ and $\widetilde{\Pi}_\parallel$ from R.

$$R = \frac{1}{2}\int d^3p \left[\sum_{a=1}^{2} \widetilde{\Pi}^{(a)}(t,-\vec{p})\widetilde{\Pi}^{(a)}(t,\vec{p}) + \widetilde{\Pi}_{\parallel}(t,-\vec{p})\widetilde{\Pi}_{\parallel}(t,\vec{p}) + \vec{p}^{\,2}\sum_{a=1}^{2}\widetilde{A}^{(a)}(t,-\vec{p})\widetilde{A}^{(a)}(t,\vec{p})\right]$$

$$-\int d^3p \left[\widetilde{A}_0(t,-\vec{p})\{|\vec{p}|\,\widetilde{\Pi}_{\parallel}(t,\vec{p}) + \widetilde{J}^0(t,\vec{p})\} + \widetilde{A}_{\parallel}(t,-\vec{p})\widetilde{J}_{\parallel}(t,\vec{p}) + \sum_{a=1}^{2}\widetilde{A}^{(a)}(t,-\vec{p})\widetilde{J}^{(a)}(t,\vec{p})\right] + H_D$$

$$\widetilde{\Pi}_{\parallel}(t,\vec{p}) = -\frac{1}{|\vec{p}|}\,\widetilde{J}^0(t,\vec{p}), \qquad \widetilde{A}_0(t,\vec{p}) = -\frac{1}{\vec{p}^{\,2}}\,\widetilde{J}^0(t,\vec{p}), \qquad \widetilde{A}_{\parallel}(t,\vec{p}) = 0$$

Substituting this into the Routhian we get

$$R = \frac{1}{2}\int d^3p \left[\sum_{a=1}^{2} \widetilde{\Pi}^{(a)}(t,-\vec{p})\widetilde{\Pi}^{(a)}(t,\vec{p}) + \frac{1}{\vec{p}^{\,2}}\widetilde{J}^0(t,-\vec{p})\widetilde{J}^0(t,\vec{p}) + \vec{p}^{\,2}\sum_{a=1}^{2}\widetilde{A}^{(a)}(t,-\vec{p})\widetilde{A}^{(a)}(t,\vec{p})\right]$$

$$-\int d^3p \sum_{a=1}^{2}\widetilde{A}^{(a)}(t,-\vec{p})\widetilde{J}^{(a)}(t,\vec{p}) + H_D$$

Ex. Check that the Hamiltonian form of the equations of motion derived from this Routhian, together with the equations for $\widetilde{A}_0$, $\widetilde{A}_{\parallel}$ and $\widetilde{\Pi}_{\parallel}$, are equivalent to the equations of motion derived from the original Routhian.

(This includes equations of motion for $\widetilde{\psi}_\alpha$ and $\widetilde{\Pi}^{\psi}_{\alpha}$)

We treat R as the Hamiltonian and write

$$R = H_{free} + H_{int}$$

$$H_{free} = \frac{1}{2}\int d^3p \left[\sum_{a=1}^{2} \widetilde{\Pi}^{(a)}(t,-\vec{p})\widetilde{\Pi}^{(a)}(t,\vec{p}) + \vec{p}^{\,2}\sum_{a=1}^{2}\widetilde{A}^{(a)}(t,-\vec{p})\widetilde{A}^{(a)}(t,\vec{p})\right] + H_D$$

$$H_{int} = \int d^3p \left[\frac{1}{2}\frac{1}{\vec{p}^{\,2}}\widetilde{J}^0(t,-\vec{p})\widetilde{J}^0(t,\vec{p}) - \sum_{a=1}^{2}\widetilde{A}^{(a)}(t,-\vec{p})\widetilde{J}^{(a)}(t,\vec{p})\right]$$

$$R = H_{free} + H_{int}$$

$$H_{free} = \frac{1}{2}\int d^3p \left[\sum_{a=1}^{2} \widetilde{\Pi}^{(a)}(t,-\vec{p})\widetilde{\Pi}^{(a)}(t,\vec{p}) + \vec{p}^2 \sum_{a=1}^{2} \widetilde{A}^{(a)}(t,-\vec{p})\widetilde{A}^{(a)}(t,\vec{p})\right] + H_D$$

$$H_{int} = \int d^3p \left[\frac{1}{2}\frac{1}{\vec{p}^2}\widetilde{J}^0(t,-\vec{p})\widetilde{J}^0(t,\vec{p}) - \sum_{a=1}^{2} \widetilde{A}^{(a)}(t,-\vec{p})\widetilde{J}^{(a)}(t,\vec{p})\right]$$

We can now quantize it and calculate the Green's functions using perturbation theory following exactly the same procedure as for interacting scalar field theory.

$$\langle\Omega|T\left(\prod_{i=1}^{n}\widetilde{A}^{(a_i)}(t_i,\vec{k}_i)\prod_{j=1}^{m}\widetilde{\psi}_{\alpha_j}(t'_j,\vec{p}_j)\prod_{\ell=1}^{m}\widetilde{\bar{\psi}}_{\beta_\ell}(t''_\ell,\vec{q}_\ell)\right)|\Omega\rangle = \frac{\text{Numerator}}{\text{Denominator}}$$

$$\text{numerator} = \langle 0|T\left(\prod_{i=1}^{n}\widetilde{A}^{(a_i)I}(t_i,\vec{k}_i)\prod_{j=1}^{m}\widetilde{\psi}^I_{\alpha_j}(t'_j,\vec{p}_j)\prod_{\ell=1}^{m}\widetilde{\bar{\psi}}^I_{\beta_\ell}(t''_\ell,\vec{q}_\ell)\exp\left[-\int d\tau H^I_{int}(\tau)\right]\right)|0\rangle$$

$$\text{denominator} = \langle 0|T\left(\exp\left[-\int d\tau H^I_{int}(\tau)\right]\right)|0\rangle$$

$|0\rangle$: Vacuum of $H_{free}(t_0)$

I: Interaction picture field operators

The matrix elements in the numerator and the denominator are Green's functions in the free field theory

– can be computed using Feynman diagrams.

Note: H_{free} is identical to what we had in free Maxwell + Dirac theory

⇒ Feynman propagators used in perturbation theory are the same as the ones used in free Maxwell + Dirac theory.

$$R = H_{free} + H_{int}$$

$$H_{free} = \frac{1}{2}\int d^3p \left[\sum_{a=1}^{2} \widetilde{\Pi}^{(a)}(t,-\vec{p})\widetilde{\Pi}^{(a)}(t,\vec{p}) + \vec{p}^2 \sum_{a=1}^{2} \widetilde{A}^{(a)}(t,-\vec{p})\widetilde{A}^{(a)}(t,\vec{p})\right] + H_D$$

$$H_{int} = \int d^3p \left[\frac{1}{2}\frac{1}{\vec{p}^2}\widetilde{J}^0(t,-\vec{p})\widetilde{J}^0(t,\vec{p}) - \sum_{a=1}^{2} \widetilde{A}^{(a)}(t,-\vec{p})\widetilde{J}^{(a)}(t,\vec{p})\right]$$

In order to avoid dealing with the index a of the gauge fields in the propagator, we shall write this is a slightly different form.

Introduce the variable $\mathcal{A}_\mu(t,\vec{r})$ and its Fourier transform $\widetilde{\mathcal{A}}_\mu(t,\vec{p})$ as follows:

$$\widetilde{\mathcal{A}}_\mu(t,\vec{p}) = \sum_{a=1}^{2} \bar{\eta}_\mu^{(a)}(t,\vec{p}) A^{(a)}(t,\vec{p}), \qquad \bar{\eta}^{(a)}(\vec{p}) = (0, \vec{\epsilon}^{(a)}(\vec{p}))$$

Compare with the result in free Maxwell theory:

$$\widetilde{A}_\mu(t,\vec{p}) = \sum_{b=1}^{2} \bar{\eta}_\mu^{(b)}(\vec{p})\widetilde{A}^{(b)}(t,\vec{p})$$

In the interaction picture, the Feynman propagator for $\mathcal{A}_\mu^I$ will be the same as that of A_μ in the free Maxwell theory, given by $D_{\mu\nu}(x,y)$.

We can now write

$$H_{int} = \int d^3p \left[\frac{1}{2}\frac{1}{\vec{p}^2}\widetilde{J}^0(t,-\vec{p})\widetilde{J}^0(t,\vec{p}) - \widetilde{\mathcal{A}}_\mu(t,-\vec{p})\widetilde{J}^\mu(t,\vec{p})\right]$$

We can also write H_{int} in the position space as:

$$H_{int} = -\frac{1}{2}\int d^3r\, d^3r'\, \mathcal{G}(\vec{r},\vec{r}')\, J^0(t,\vec{r})\, J^0(t,\vec{r}') - \int d^3r\, \mathcal{A}_\mu(t,\vec{r})\, J^\mu(t,\vec{r})$$

$$\mathcal{G}(\vec{r},\vec{r}') = -\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}.(\vec{r}-\vec{r}')}\frac{1}{\vec{p}^2} = -\frac{1}{4\pi}\,\frac{1}{|\vec{r}-\vec{r}'|}$$

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$$\widetilde{\mathcal{A}}_\mu(t,\vec{p}) = \sum_{b=1}^{2} \bar{\eta}_\mu^{(b)}(\vec{p})\widetilde{A}^{(b)}(t,\vec{p}), \qquad J^\mu = -e\,\bar{\psi}(x)\gamma^\mu\psi(x)$$

$$H_{free} = \frac{1}{2}\int d^3p \left[\sum_{a=1}^{2}\widetilde{\Pi}^{(a)}(t,-\vec{p})\widetilde{\Pi}^{(a)}(t,\vec{p}) + \vec{p}^2\sum_{a=1}^{2}\widetilde{A}^{(a)}(t,-\vec{p})\widetilde{A}^{(a)}(t,\vec{p})\right] + H_D$$

$$\begin{aligned} H_{int} &= \int d^3p \left[\frac{1}{2}\frac{1}{\vec{p}^2}\widetilde{J}^0(t,-\vec{p})\widetilde{J}^0(t,\vec{p}) - \widetilde{\mathcal{A}}_\mu(t,-\vec{p})\widetilde{J}^\mu(t,\vec{p})\right] \\ &= -\frac{1}{2}\int d^3r\, d^3r'\, \mathcal{G}(\vec{r},\vec{r}')\, J^0(t,\vec{r})\, J^0(t,\vec{r}') - \int d^3r\, \mathcal{A}_\mu(t,\vec{r})\, J^\mu(t,\vec{r}) \end{aligned}$$

$$\mathcal{G}(\vec{r},\vec{r}') = -\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}.(\vec{r}-\vec{r}')}\frac{1}{\vec{p}^2} = -\frac{1}{4\pi}\,\frac{1}{|\vec{r}-\vec{r}'|}$$

From these we can derive the Feynman rules for computing Green's functions.

$\mathcal{A}_\mu$ propagator:

$$D_{\mu\nu}(x,y) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip.(x-y)}\, \widetilde{D}_{\mu\nu}(p),\ \widetilde{D}_{\mu\nu}(p) = \frac{i}{-p^2+i\epsilon}\left[\eta_{\mu\nu} - \frac{n.p}{\vec{p}^2}(p_\mu n_\nu + p_\nu n_\mu) - \frac{p_\mu p_\nu}{\vec{p}^2} + n_\mu n_\nu \frac{p^2}{\vec{p}^2}\right]$$

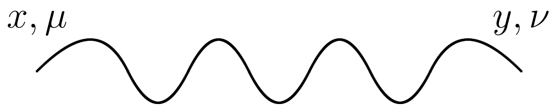
Fermion propagator:

$$S_{F\alpha\beta}(x_1,x_2) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip.(x_1-x_2)}\, \widetilde{S}_{F\alpha\beta}(p), \quad \widetilde{S}_{F\alpha\beta}(p) = \frac{i}{-p^2-m^2+i\epsilon}\,(\not{p}+m)_{\alpha\beta}$$

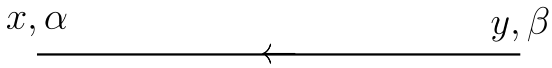
First term in H_{int} gives $\psi-\psi-\bar{\psi}-\bar{\psi}$ interaction vertex

Second term in H_{int} gives $\psi-\bar{\psi}-\mathcal{A}$ interaction vertex

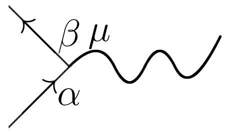
We shall now argue that the term proportional to $n_\mu n_\nu$ in $\widetilde{D}_{\mu\nu}$ cancels the effect of the $\psi-\psi-\bar{\psi}-\bar{\psi}$ interaction vertex in all Feynman diagrams



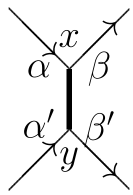
$D_{\mu\nu}(x,y)$



$S_{F\alpha\beta}(x,y)$



$-i\,(e\,\gamma^{\mu}_{\beta\alpha})$



$-i\,\left(-\frac{1}{2}\right)\,\delta(x^0-y^0)\,\mathcal{G}(\vec{x},\vec{y})\,e^2\,\gamma^0_{\beta\alpha}\gamma^0_{\beta'\alpha'}$

e: coupling constant in powers of which we expand.

Now, for every four fermion interaction vertex, we can consider the subdiagram of the Feynman diagram in which the thick line is replaced by a propagator.

Contribution from this part:

$$* \quad : \quad \frac{1}{2}\,D_{\mu\nu}(x,y)(-i\,e)^2\gamma^{\mu}_{\beta\alpha}\gamma^{\nu}_{\beta'\alpha'}$$

Consider the part of $D_{\mu\nu}$ coming from the $n_\mu n_\nu$ term in $\widetilde{D}_{\mu\nu}$

$$\int \frac{d^4p}{(2\pi)^4}e^{ip.(x-y)}\frac{i}{-p^2+i\epsilon}\frac{p^2 n_\mu n_\nu}{\vec{p}^2} = -i\,\delta(x^0-y^0)\int \frac{d^3p}{(2\pi)^3}e^{i\vec{p}.(\vec{x}-\vec{y})}\frac{n_\mu n_\nu}{\vec{p}^2} = i\,\delta(x^0-y^0)\,n_\mu\,n_\nu\,\mathcal{G}(\vec{x},\vec{y})$$

Therefore the $n_\mu n_\nu$ term in $*$ becomes

$$-\frac{i}{2}\,e^2\,\delta(x^0-y^0)\,\mathcal{G}(\vec{x},\vec{y})\gamma^0_{\beta\alpha}\gamma^0_{\beta'\alpha'}$$

– cancels the 4-fermion interaction vertex (**Check combinatoric factors**)

$\Rightarrow$ we can drop the term proportional to $n_\mu n_\nu$ in $\widetilde{D}_{\mu\nu}$ in the internal photon propagator and the $\psi-\psi-\bar{\psi}-\bar{\psi}$ interaction vertex

This leads to simpler Feynman rules:

1. Drop the four fermion interaction vertex.

2. Use the modified photon propagator for the internal photons:

$$D'_{\mu\nu}(x,y) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip.(x-y)}\, \widetilde{D}'_{\mu\nu}(p), \quad \widetilde{D}'_{\mu\nu}(p) = \frac{i}{-p^2+i\epsilon}\left[\eta_{\mu\nu} - \frac{n.p}{\vec{p}^2}(p_\mu n_\nu + p_\nu n_\mu) - \frac{p_\mu p_\nu}{\vec{p}^2}\right]$$

Ex. Compute the following Green's function to order e^2

$$\langle\Omega|T(\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\bar{\psi}_{\beta_1}(y_1)\bar{\psi}_{\beta_1}(y_1))|\Omega\rangle$$

$$\langle\Omega|T(\psi_\alpha(x)\bar{\psi}_\beta(y)\mathcal{A}_\mu(w)\mathcal{A}_\nu(z))|\Omega\rangle$$

Also compute the corresponding momentum space Green's functions.

Definition of general momentum space Green's function:

$$\begin{aligned}
&\widetilde{G}^{(n,m)}_{\mu_1,\cdots,\mu_n;\alpha_1,\cdots,\alpha_m;\beta_1,\cdots,\beta_m}(k_1,\cdots,k_n;p_1,\cdots,p_m;\ell_1,\cdots,\ell_m) \\
= &\int \prod_{i=1}^n d^4x_i \prod_{j=1}^m d^4y_j \prod_{s=1}^m d^4z_s\, e^{-i\left(\sum_{i=1}^n k_i.x_i + \sum_{j=1}^m p_j.y_j + \sum_{s=1}^m \ell_s.z_z\right)} \\
&\qquad \times \langle\Omega|T\left(\prod_{i=1}^n \mathcal{A}_{\mu_i}(x_i)\prod_{j=1}^m \psi_{\alpha_j}(y_j)\prod_{s=1}^m \bar{\psi}_{\beta_s}(z_s)\right)|\Omega\rangle
\end{aligned}$$

Ex. Write down the momentum space Feynman rules as in the case of scalar field theory.

Given the Green's functions, we can proceed as in scalar field theory to extract the physical masses of the particles and the expression for the S-matrix.

We begin with the fermion mass.

Use:

$$\langle\Omega|T(\psi_\alpha(x)\bar{\psi}_\beta(y))|\Omega\rangle = \begin{cases} \sum_n \langle\Omega|\psi_\alpha(x)|n\rangle \, \langle n|\bar{\psi}_\beta(y)|\Omega\rangle & \text{for } x^0 > y^0 \\ -\sum_n \langle\Omega|\bar{\psi}_\beta(y)|n\rangle \, \langle n|\psi_\alpha(x)|\Omega\rangle & \text{for } x^0 < y^0 \end{cases}$$

$\sum_n$ includes sum over all states but we shall focus on single particle states.

Charge conservation tells us that only the single electron states $|\vec{p}, s\rangle_-$ contribute to the top expression and the single positron states $|\vec{p}, s\rangle_+$ contribute to the second expression.

As in the case of scalar particles, constraint of Lorentz invariance and translation invariance fixes the form of these matrix elements in terms of the mass m_f of the single electron / positron state.

$$\langle\Omega|\psi_\alpha(x)|\vec{p}, s\rangle_- = \sqrt{Z_\psi}\, u_\alpha(\vec{p}, s; m_f) e^{ip.x} \frac{1}{(2\pi)^{3/2}}, \quad p^0 = \sqrt{\vec{p}^2 + m_f^2}, \quad (\not{p} - m_f)\, u(\vec{p}, s; m_f) = 0$$

$${}_-\langle\vec{p}, s|\bar{\psi}_\alpha(x)|\Omega\rangle = \sqrt{Z_\psi}\, \bar{u}_\alpha(\vec{p}, s; m_f) e^{-ip.x} \frac{1}{(2\pi)^{3/2}}$$

$$\langle\Omega|\bar{\psi}_\alpha(x)|\vec{p}, s\rangle_+ = \sqrt{Z_\psi}\, \bar{v}_\alpha(\vec{p}, s; m_f) e^{ip.x} \frac{1}{(2\pi)^{3/2}}, \quad p^0 = \sqrt{\vec{p}^2 + m_f^2}, \quad (\not{p} + m_f)\, v(\vec{p}, s; m_f) = 0$$

$${}_+\langle\vec{p}, s|\psi_\alpha(x)|\Omega\rangle = \sqrt{Z_\psi}\, v_\alpha(\vec{p}, s; m_f) e^{-ip.x} \frac{1}{(2\pi)^{3/2}}$$

Substituting these in the expression for the two point function we get the single particle contribution to the matrix element to be:

$$\langle\Omega|T(\psi_\alpha(x)\bar{\psi}_\beta(y))|\Omega\rangle = Z_\psi\, S_{F\alpha\beta}(x, y; m_f) + \cdots$$

$\cdots$ represent multi-particle contribution.

$$\langle\Omega|T(\psi_\alpha(x)\bar{\psi}_\beta(y))|\Omega\rangle = Z_\psi\, S_{F\alpha\beta}(x,y;m_f) + \cdots$$

Consider its Fourier transform:

$$\widetilde{G}^{(f)}_{\alpha\beta}(p) = \int d^4x\, e^{-ip.(x-y)}\langle\Omega|T(\psi_\alpha(x)\bar{\psi}_\beta(y))|\Omega\rangle$$

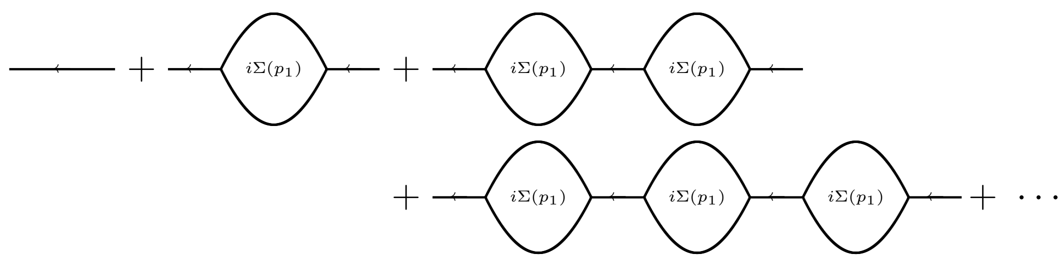
Using the known form of $S_{F\alpha\beta}(x,y;m_f)$ we get

$$\widetilde{G}^{(f)}_{\alpha\beta}(p) = \left(\frac{i\sqrt{Z_\psi}}{\not{p} - m_f}\right)_{\alpha\beta} + \cdots = \frac{i\sqrt{Z_\psi}(\not{p} + m_f)_{\alpha\beta}}{-p^2 - m_f^2 + i\epsilon} + \cdots$$

As in the case of scalar fields, we can find m_f in perturbation theory by resumming the series and looking for the poles of $\widetilde{G}^{(f)}_{\alpha\beta}(p)$.

Define by $i\Sigma_{\alpha\beta}(p)$ the 1PI 2-point function of fermions.

$\widetilde{G}^{(f)}_{\alpha\beta}(p) =$



$$\widetilde{G}^{(f)}(p) = \frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}i\Sigma(p)\frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}i\Sigma(p)\frac{i}{\not{p}-m}i\Sigma(p)\frac{i}{\not{p}-m} + \cdots = \frac{i}{\not{p}-m+\Sigma(p)}$$

Using Lorentz invariance one can argue that the general form of Σ is

$$\Sigma(p) = f(p^2)\not{p} - g(p^2)$$

for some functions f and g.

This gives

$$\widetilde{G}^{(f)}(p) = \frac{i}{\not{p}(1+f(p^2)) - m - g(p^2)} = i\frac{\not{p}(1+f(p^2)) + m + g(p^2)}{-p^2(1+f(p^2))^2 - (m+g(p^2))^2}$$

$$\widetilde{G}^{(f)}(p) = i\frac{\not{p}(1+f(p^2)) + m + g(p^2)}{-p^2(1+f(p^2))^2 - (m+g(p^2))^2}$$

The physical mass m_f is the location of the pole of $\widetilde{G}^{(f)}$ in $-p^2$ variable.

This leads to the equation:

$$m_f^2(1+f(-m_f^2))^2 - (m+g(-m_f^2))^2 = 0$$

Since f and g are small we can solve this iteratively by writing

$$m_f = (m+g(-m_f^2))/(1+f(-m_f^2))$$

Solutions are:

$$\begin{aligned} \text{0-th order} \quad &: \quad m_f = m \\ \text{1st order} \quad &: \quad m_f = (m+g(-m^2))/(1+f(-m^2)) \end{aligned}$$

etc.

Similarly Z_ψ can be calculated by evaluating the residue at $p^2 = -m_f^2$.

Both m_f and Z_ψ are UV divergent when expressed as function of m and e, and require suitable regularization beyond lowest order in perturbation theory.

Structure of the photon propagator:

$$\langle\Omega|T(\mathcal{A}_\mu(x)\mathcal{A}_\nu(y))|\Omega\rangle = \begin{cases} \sum_n \langle\Omega|\mathcal{A}_\mu(x)|n\rangle\,\langle n|\mathcal{A}_\nu(y)|\Omega\rangle & \text{for } x^0 > y^0 \\ \sum_n \langle\Omega|\mathcal{A}_\nu(y)|n\rangle\,\langle n|\mathcal{A}_\mu(x)|\Omega\rangle & \text{for } x^0 < y^0 \end{cases}$$

$\sum_n$ includes sum over all states but we shall focus on single particle states.

Charge conservation tells us that only the single photon states $|\vec{p}, c\rangle$ contribute to this expression.

(Note: This is a perturbative statement, ignoring existence of neutral bound states)

Furthermore, Lorentz invariance requires that the states are exactly massless i.e. $E_{\vec{p}} = |\vec{p}|$.

A massive spin one particle must have three states, not two.

As in the case of scalar particles, constraint of Lorentz invariance and translation invariance fixes the form of the matrix elements:

$$\langle\Omega|\mathcal{A}_\mu(x)|\vec{p}, c\rangle = \sqrt{Z_A}\,\bar{\eta}^{(c)}_\mu(\vec{p})\,e^{ip.x}\frac{1}{(2\pi)^{3/2}}, \quad p^0 = |\vec{p}|, \quad \bar{\eta}^{(c)} = (0, \vec{\epsilon}^{(c)})$$

$$\langle\vec{p}, c|\mathcal{A}_\mu(x)|\Omega\rangle = \sqrt{Z_A}\,\bar{\eta}^{(c)}_\mu(\vec{p})\,e^{-ip.x}\frac{1}{(2\pi)^{3/2}}$$

Substituting these in the expression for the two point function we get the single particle contribution to the matrix element to be:

$$\langle\Omega|T(\mathcal{A}_\mu(x)\mathcal{A}_\nu(y))|\Omega\rangle = Z_A\,D_{\mu\nu}(x,y) + \cdots$$

$\cdots$ represent multi-particle contribution.

The Fourier transformed Green's function is given by:

$$\begin{aligned} \widetilde{G}^A_{\mu\nu}(k) &= \int d^4x e^{-ik.(x-y)}\,\langle\Omega|T(\mathcal{A}_\mu(x)\mathcal{A}_\nu(y))|\Omega\rangle \\ &= Z_A\,\widetilde{D}_{\mu\nu}(k) + \cdots \end{aligned}$$

$$\widetilde{G}^A_{\mu\nu}(k) = \int d^4x e^{-ik.(x-y)} \langle\Omega|T(\mathcal{A}_\mu(x)\mathcal{A}_\nu(y))|\Omega\rangle$$
$$= Z_A \widetilde{D}_{\mu\nu}(k) + \cdots$$

Note: This implies that the two point Green's function of two photons continue to have pole at $k^2 = 0$

– can be checked explicitly in perturbation theory

If we denote by $i\,\Sigma^A_{\mu\nu}(k)$ the 1PI two point function in momentum space, then, as a consequence of gauge invariance:

$$\Sigma_{\mu\nu}(k) = h(k^2)\,(k_\mu k_\nu - k^2\,\eta_{\mu\nu})$$

for some function h.

Using this form to calculate $\widetilde{G}^A_{\mu\nu}(k)$, one can show that the pole in k^2 remains at $k^2 = 0$.

Note that $\Sigma_{\mu\nu}(k)$ can be regarded as the Fourier transform of

$$\langle\Omega|T(J^\mu(x)J^\nu(y))|\Omega\rangle$$

Since J^μ is gauge invariant, $\Sigma^{\mu\nu}$ is also gauge invariant.

35 S-matrix and symmetries of QED

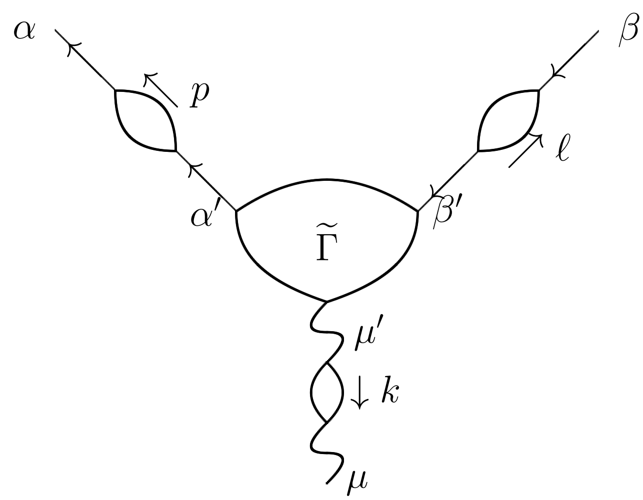
S-matrix of quantum electrodynamics (QED)

– can be analyzed in the same way as in the case of scalar field theory.

We shall state the final result without proof.

First define amputated Green's function $\widetilde{\Gamma}^{(n,m)}$ by factoring out the external two point functions from the momentum space Green's function:

$$\begin{aligned}
&\widetilde{G}^{(n,m)}_{\mu_1,\cdots,\mu_n;\alpha_1,\cdots,\alpha_m;\beta_1,\cdots,\beta_m}(k_1,\cdots,k_n;p_1,\cdots,p_m;\ell_1,\cdots,\ell_m) \\
=\ &\widetilde{\Gamma}^{(n,m)}_{\mu'_1,\cdots,\mu'_n;\alpha'_1,\cdots,\alpha'_m;\beta'_1,\cdots,\beta'_m}(k_1,\cdots,k_n;p_1,\cdots,p_m;\ell_1,\cdots,\ell_m) \\
&\qquad\times\prod_{i=1}^{n}\widetilde{G}^{A}_{\mu_i\mu'_i}(k_i)\prod_{j=1}^{m}\widetilde{G}^{(f)}_{\alpha_j\alpha'_j}(p_j)\prod_{s=1}^{m}\widetilde{G}^{(f)}_{\beta'_s\beta_s}(-\ell_s)
\end{aligned}$$



Goal: Calculate the S-matrix

$$ {}_{out}\langle \vec{k}_1, a_1, \cdots, \vec{k}_{n_1}, a_{n_1}; \vec{p}_1, s_1, \cdots, \vec{p}_{n_2}, s_{n_2}; \vec{\ell}_1, t_1, \cdots, \vec{\ell}_{n_3}, t_{n_3} | \vec{k}'_1, a'_1, \cdots, \vec{k}'_{m_1}, a'_{m_1}; \vec{p}'_1, s'_1, \cdots, \vec{p}'_{m_2}, s'_{m_2}; \vec{\ell}'_1, t'_1, \cdots, \vec{\ell}'_{m_3}, t'_{m_3} \rangle_{in} $$

$\vec{k}_1, a_1, \cdots, \vec{k}_{n_1}, a_{n_1}$: outgoing photon momenta / spin

$\vec{p}_1, s_1, \cdots, \vec{p}_{n_2}, s_{n_2}$: outgoing electron momentum / spin

$\vec{\ell}_1, t_1, \cdots, \vec{\ell}_{n_3}, t_{n_3}$: outgoing positron momentum / spin

$\vec{k}'_1, a'_1, \cdots, \vec{k}'_{m_1}, a'_{m_1}$: incoming photon momenta / spin

$\vec{p}'_1, s'_1, \cdots, \vec{p}'_{m_2}, s'_{m_2}$: incoming electron momentum / spin

$\vec{\ell}'_1, t'_1, \cdots, \vec{\ell}'_{m_3}, t'_{m_3}$: incoming positron momentum / spin

Charge conservation: $n_2 - n_3 = m_2 - m_3 \quad \Rightarrow \quad n_2 + m_3 = m_2 + n_3$

Result:

$$ {}_{out}\langle \vec{k}_1, a_1, \cdots, \vec{k}_{n_1}, a_{n_1}; \vec{p}_1, s_1, \cdots, \vec{p}_{n_2}, s_{n_2}; \vec{\ell}_1, t_1, \cdots, \vec{\ell}_{n_3}, t_{n_3} | \vec{k}'_1, a'_1, \cdots, \vec{k}'_{m_1}, a'_{m_1}; \vec{p}'_1, s'_1, \cdots, \vec{p}'_{m_2}, s'_{m_2}; \vec{\ell}'_1, t'_1, \cdots, \vec{\ell}'_{m_3}, t'_{m_3} \rangle_{in} $$

$$ || $$

$$ \widetilde{\Gamma}^{(n_1+m_1, n_2+m_3)}_{\mu_1,\cdots,\mu_{n_1},\nu_1,\cdots,\nu_{m_1};\alpha_1,\cdots,\alpha_{n_2},\beta'_1,\cdots,\beta'_{m_3};\beta_1,\cdots,\beta_{n_3},\alpha'_1,\cdots,\alpha'_{m_2}}(k_1,\cdots,k_{n_1},-k'_1,\cdots,-k'_{m_1};p_1,\cdots,p_{n_2},-\ell'_1,\cdots,-\ell'_{m_3};\ell_1,\cdots\,\ell_{n_3},-p'_1,\cdots,-p'_{m_2}) $$

$$ \times \left\{ \prod_{i=1}^{n_1} \frac{1}{(2\pi)^{3/2}} \bar{\eta}^{(a_i)\mu_i}(k_i) \right\} \left\{ \prod_{i=1}^{m_1} \frac{1}{(2\pi)^{3/2}} \bar{\eta}^{(a'_i)\nu_i}(k'_i) \right\} $$

$$ \times \left\{ \prod_{i=1}^{n_2} \frac{1}{(2\pi)^{3/2}} \bar{u}_{\alpha_i}(\vec{p}_i, s_i; m_f) \right\} \left\{ \prod_{i=1}^{m_3} \frac{1}{(2\pi)^{3/2}} \bar{v}_{\beta'_i}(\vec{\ell}'_i, t'_i; m_f) \right\} $$

$$ \times \left\{ \prod_{i=1}^{n_3} \frac{1}{(2\pi)^{3/2}} v_{\beta_i}(\vec{\ell}_i, t_i; m_f) \right\} \left\{ \prod_{i=1}^{m_2} \frac{1}{(2\pi)^{3/2}} u_{\alpha'_i}(\vec{p}'_i, s'_i; m_f) \right\} \times (Z_A)^{(n_1+m_1)/2} (Z_\psi)^{n_2+m_3} $$

$$ k_i^0 = |\vec{k}_i|, \quad k_i'^0 = |\vec{k}'_i|, \quad p_i^0 = \sqrt{\vec{p}^2 + m_f^2}, \quad p_i'^0 = \sqrt{\vec{p}'^2 + m_f^2}, \quad \ell_i^0 = \sqrt{\vec{\ell}^2 + m_f^2}, \quad \ell_i'^0 = \sqrt{\vec{\ell}'^2 + m_f^2} $$

The difference from scalar field theory arises from normalization and properties of the single particle states:

$$ \langle \Omega | \psi_\alpha(x) | \vec{p}, s \rangle_- = \sqrt{Z_\psi}\, u_\alpha(\vec{p}, s; m_f) e^{ip.x} \frac{1}{(2\pi)^{3/2}} $$

$$ \langle \Omega | \bar{\psi}_\alpha(x) | \vec{p}, s \rangle_+ = \sqrt{Z_\psi}\, \bar{v}_\alpha(\vec{p}, s; m_f) e^{ip.x} \frac{1}{(2\pi)^{3/2}} $$

$$ {}_-\langle \vec{p}, s | \vec{p}', s' \rangle_- = \frac{E_{\vec{p}}}{m_f} \delta_{ss'}\, \delta^{(3)}(\vec{p} - \vec{p}'), \qquad {}_+\langle \vec{p}, s | \vec{p}', s' \rangle_+ = \frac{E_{\vec{p}}}{m_f} \delta_{ss'}\, \delta^{(3)}(\vec{p} - \vec{p}') $$

$$ \langle \Omega | \mathcal{A}_\mu(x) | \vec{p}, c \rangle = \sqrt{Z_A}\, \eta_\mu^{(c)}(\vec{p})\, e^{ip.x} \frac{1}{(2\pi)^{3/2}} $$

$$ \langle \vec{p}, b | \vec{p}', c \rangle = 2E_{\vec{p}}\, \delta_{bc}\, \delta^{(3)}(\vec{p} - \vec{p}') $$

Symmetries of QED:

1. Gauge invariance: Already discussed

2. Rigid gauge transformation:

$$\widetilde{\psi}(x) = e^{-i\,e\,\lambda}\psi(x), \qquad \widetilde{\bar{\psi}}(x) = e^{i\,e\,\lambda}\bar{\psi}(x), \qquad \widetilde{A}_\mu(x) = A_\mu(x)$$

– corresponding Noether current gives $J^\mu = -e\,\bar{\psi}(x)\gamma^\mu\psi(x)$.

3. Lorentz transformation:

$$\widetilde{A}_\mu(x) = A_\nu(\Lambda x)\Lambda^\nu{}_\mu, \qquad \Lambda\eta\Lambda^T = \eta, \qquad \widetilde{\psi}_\alpha(x) = S_{\alpha\beta}\psi_\beta(\Lambda x)$$

For infinitesimal Lorentz transformation:

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon\,\omega^\mu{}_\nu, \quad S_{\alpha\beta} = \delta_{\alpha\beta} + \omega_{\nu\tau}\Sigma^{\nu\tau}_{\alpha\beta}$$

$$\Sigma^{\tau\nu}_{\alpha\beta} = c\,(\gamma^\tau\gamma^\nu - \gamma^\nu\gamma^\tau)_{\alpha\beta}, \qquad c = \text{a constant}$$

Ex. Show that for appropriate choice of c, this transformation leaves the QED action invariant.

Find the conserved charges associated with infinitesimal Lorentz transformation.

Discrete symmetries:

1. Parity

$$\widetilde{A}_0(t,\vec{r}) = A_0(t,-\vec{r}), \quad \widetilde{A}_i(t,\vec{r}) = -A_i(t,-\vec{r}), \quad \widetilde{\psi}_\alpha(t,\vec{r}) = P_{\alpha\beta}\psi_\beta(t,-\vec{r})$$

Goal: Show that for appropriate choice of the matrix P, this transformation leaves the QED action invariant.

Define $\vec{r}\,' = -\vec{r}$.

Ex. Check that the Maxwell part of the action is manifestly invariant under this.

We now check the invariance of the mass term in the Dirac action.

$$m\int dt\, d^3r\, \overline{\widetilde{\psi}}(t,\vec{r})\widetilde{\psi}(t,\vec{r}) = m\int dt\, d^3r\, \widetilde{\psi}(t,\vec{r})^\dagger\gamma^0\widetilde{\psi}(t,\vec{r})$$

$$= \; m\int dt\, d^3r\, \psi(t,-\vec{r})^\dagger P^\dagger\gamma^0 P\psi(t,-\vec{r}) = m\int dt\, d^3r'\, \psi(t,\vec{r}\,')^\dagger P^\dagger\gamma^0 P\psi(t,\vec{r}\,')$$

We want this to be equal to:

$$m\int dt\, d^3r\, \bar{\psi}(t,\vec{r})\,\psi(t,\vec{r}) = m\int dt\, d^3r\, \psi(t,\vec{r})^\dagger\,\gamma^0\,\psi(t,\vec{r})$$

This gives

$$P^\dagger\gamma^0 P = \gamma^0$$

Ex. Check that the invariance of the term involving $\bar{\psi}\gamma^\mu\partial_\mu\psi$ gives

$$P^\dagger\gamma^0\gamma^i P = -\gamma^0\gamma^i, \qquad P^\dagger P = I$$

Solution:

$$P = \gamma^0$$

Ex. Check that the full QED action is parity invariant.

2. Charge conjugation:

$$\widetilde{A}_\mu(x) = -A_\mu(x)$$

Since the action has a term $\int d^4x A_\mu(x) J^\mu(x)$, this means that

$$\widetilde{J}^\mu(x) = -J^\mu(x)$$

The transformation of ψ is such that the sign of the charge gets reversed

– transforms an electron state to a positron state and vice versa.

$$\widetilde{\psi}_\alpha(x) = C_{\alpha\beta}\bar{\psi}_\beta(x) \quad \Leftrightarrow \quad \widetilde{\psi}(x) = C\,\bar{\psi}(x)^T = C\,(\gamma^0)^T(\psi(x)^\dagger)^T = C(\gamma^0)^T\psi^*$$

$$\bar{\widetilde{\psi}}(x) = \widetilde{\psi}(x)^\dagger\gamma^0 = \psi^T(\gamma^0)^T C^\dagger\gamma^0$$

Invariance of the mass term requires

$$\bar{\widetilde{\psi}}(x)\widetilde{\psi}(x) = \bar{\psi}(x)\psi(x) \quad \Leftrightarrow \quad \psi^T(\gamma^0)^T C^\dagger\gamma^0 C(\gamma^0)^T\psi^* = \psi^\dagger\gamma^0\psi = -\psi^T(\gamma^0)^T\psi^*$$

Note: The $-$ sign comes from the fact that in the classical limit fields anti-commute.

$$\Rightarrow \quad (\gamma^0)^T C^\dagger\gamma^0 C(\gamma^0)^T = -(\gamma^0)^T \quad \Rightarrow \quad C^\dagger\gamma^0 C = -(\gamma^0)^T$$

$$\widetilde{J}^\mu(x) = -J^\mu(x) \quad \Rightarrow \quad \bar{\widetilde{\psi}}(x)\gamma^\mu\widetilde{\psi}(x) = -\bar{\psi}(x)\gamma^\mu\psi(x)$$

$$\Rightarrow \quad \psi^T(\gamma^0)^T C^\dagger\gamma^0\gamma^\mu C(\gamma^0)^T\psi^* = -\psi^\dagger\gamma^0\gamma^\mu\psi = \psi^T(\gamma^\mu)^T(\gamma^0)^T\psi^*$$

$$\Rightarrow \quad (\gamma^0)^T C^\dagger\gamma^0\gamma^\mu C(\gamma^0)^T = (\gamma^\mu)^T(\gamma^0)^T$$

$$\mu = 0 : \Rightarrow \; C^\dagger C = I, \qquad \mu = i : \Rightarrow \; C^\dagger\gamma^0\gamma^i C = (\gamma^0)^T(\gamma^i)^T \; \Rightarrow \; C^\dagger\gamma^i C = -(\gamma^i)^T$$

The specific choice of C satisfying this depends on the representation of the γ-matrices.

Ex. Check that the full QED action is invariant under charge conjugation once we choose C satisfying the above conditions.

3. Time reversal

$$\widetilde{\psi}_\alpha(t,\vec{r}) = \mathcal{T}_{\alpha\beta}\,\psi_\beta(-t,\vec{r}) \quad \Leftrightarrow \quad \widetilde{\psi}(t,\vec{r}) = \mathcal{T}\,\psi(-t,\vec{r})$$

$$\widetilde{\psi}(t,\vec{r})^\dagger = \psi(-t,\vec{r})^\dagger\,\mathcal{T}^\dagger$$

$$\widetilde{\bar{\psi}}(t,\vec{r}) = \psi(-t,\vec{r})^\dagger\,\mathcal{T}^\dagger\,(\gamma^0)^*$$

Note: Time reversal complex conjugates all numbers, including γ^μ's

Invariance of the mass term requires:

$$\int dt\,d^3r\,\widetilde{\bar{\psi}}(x)\widetilde{\psi}(x) = \int dt\,d^3r\,\bar{\psi}(x)\psi(x)$$

$$\Rightarrow \int dtd^3r\,\psi(-t,\vec{r})^\dagger\,\mathcal{T}^\dagger\,(\gamma^0)^*\mathcal{T}\,\psi(-t,\vec{r}) = \int dtd^3r\,\psi^\dagger(t,\vec{r})\gamma^0\psi(t,\vec{r}) = \int dtd^3r\,\psi^\dagger(-t,\vec{r})\gamma^0\psi(-t,\vec{r})$$

$$\Rightarrow \mathcal{T}^\dagger\,(\gamma^0)^*\mathcal{T} = \gamma^0$$

Invariance of the term $\propto i\,\bar{\psi}\gamma^i\partial_i\psi$ requires:

$$\int dt\,d^3r\,\widetilde{\bar{\psi}}(x)(-i)(\gamma^i)^*\partial_i\widetilde{\psi}(x) = \int dt\,d^3r\,\bar{\psi}(x)\,i\gamma^i\partial_i\psi(x)$$

$$\Rightarrow \int dt\,d^3r\,\psi(-t,\vec{r})^\dagger\,\mathcal{T}^\dagger\,(\gamma^0)^*(-i)(\gamma^i)^*\mathcal{T}\,\partial_i\psi(-t,\vec{r}) = \int dt\,d^3r\,\psi^\dagger(t,\vec{r})\gamma^0\,i\gamma^i\partial_i\psi(t,\vec{r})$$

$$\Rightarrow \mathcal{T}^\dagger\,(\gamma^0)^*(\gamma^i)^*\mathcal{T} = -\gamma^0\gamma^i \quad \Rightarrow \quad \mathcal{T}^{-1}(\gamma^i)^*\mathcal{T} = -\gamma^i$$

Invariance of the term $\propto i\,\bar{\psi}\gamma^0\partial_0\psi$ requires:

$$\int dtd^3r\widetilde{\bar{\psi}}(x)(-i)(\gamma^0)^*\partial_0\widetilde{\psi}(x) = \int dtd^3r\bar{\psi}(x)\,i\gamma^0\partial_0\psi(x)$$

$$\Rightarrow \int dtd^3r\,\psi(-t,\vec{r})^\dagger\,\mathcal{T}^\dagger\,(\gamma^0)^*(-i)(\gamma^0)^*\partial_0\mathcal{T}\,\psi(-t,\vec{r}) = \int dtd^3r\,\psi^\dagger(t,\vec{r})\gamma^0\,i\gamma^0\partial_0\psi(t,\vec{r})$$

$$\Rightarrow \int dtd^3r\,\psi(-t,\vec{r})^\dagger\,\mathcal{T}^\dagger\,(\gamma^0)^*(-i)(\gamma^0)^*\partial_0\mathcal{T}\,\psi(-t,\vec{r}) = -\int dtd^3r\,\psi^\dagger(-t,\vec{r})\gamma^0\,i\gamma^0\partial_0\psi(-t,\vec{r})$$

$$\Rightarrow \mathcal{T}^\dagger\,(\gamma^0)^*(\gamma^0)^*\mathcal{T} = \gamma^0\gamma^0 \quad \Rightarrow \quad \mathcal{T}^\dagger\mathcal{T} = I$$

Ex. Once we choose $\mathcal{T}$ satisfying these conditions, QED action is time reversal invariant if we take

$$\widetilde{A}_0(t,\vec{r}) = A_0(-t,\vec{r}), \quad \widetilde{A}_i(t,\vec{r}) = -A_i(-t,\vec{r})$$

Consider γ matrix representation:

$$\gamma^0 = \sigma_3 \otimes I, \quad \gamma^1 = i\sigma_2 \otimes I, \quad \gamma^2 = \sigma_1 \otimes i\sigma_1, \quad \gamma^3 = \sigma_1 \otimes i\sigma_2$$

$\sigma_1, \sigma_2, \sigma_3$: Pauli matrices

Ex. Check that $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\,\eta^{\mu\nu}$.

In this representation:

γ^0 is real and symmetric,

γ^1 and γ^3 are real and anti-symmetric

γ^2 is imaginary and symmetric

Now recall the conditions on C:

$$C^\dagger C = I, \qquad C^\dagger \gamma^\mu C = -(\gamma^\mu)^T \qquad \Rightarrow \qquad \gamma^\mu\, C = -C\,(\gamma^\mu)^T$$

Ex. Check that the following choice satisfies these equations:

$$C = i\,\gamma^0\,\gamma^2$$

The conditions on $\mathcal{T}$ are:

$$\mathcal{T}^\dagger\mathcal{T} = I, \qquad \mathcal{T}^{-1}(\gamma^i)^*\mathcal{T} = -\gamma^i, \qquad \mathcal{T}^{-1}(\gamma^0)^*\mathcal{T} = \gamma^0$$

$$\Rightarrow \quad (\gamma^i)^*\mathcal{T} = -\mathcal{T}\,\gamma^i, \qquad (\gamma^0)^*\mathcal{T} = \mathcal{T}\gamma^0$$

Ex. Check that the following choice satisfies these equations:

$$\mathcal{T} = i\gamma^1\gamma^3$$

36 Some other quantum field theories

1. Yukawa theory

– coupling of a scalar to a Dirac fermion

$$S = \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + g\,\phi\,\bar{\psi}\,\psi \right]$$

– describes interaction between scalar and fermion.

Feynman rules will now have a single interaction vertex with one scalar and two fermions, carrying a factor of the coupling constant g.

Green's function and S-matrix can be calculated following the usual procedure.

A slight variation of the action is:

$$S = \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + g\,\phi\,\bar{\psi}\,\gamma^5\psi \right], \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Ex. Check that under Lorentz transformation $\widetilde{\bar{\psi}}(x)\,\gamma^5\widetilde{\psi}(x) = \bar{\psi}(\Lambda x)\,\gamma^5\psi(\Lambda x)$

Therefore $\bar{\psi}\,\gamma^5\psi$ transforms as a scalar and the theory is Lorentz invariant.

Ex. Check that under parity transformation $\widetilde{\bar{\psi}}(x)\,\gamma^5\widetilde{\psi}(x) = -\bar{\psi}(t,-\vec{r})\,\gamma^5\psi(t,-\vec{r})$

Therefore the theory is parity invariant if we assign to ϕ the transformation

$$\widetilde{\phi}(t,\vec{r}) = -\phi(t,-\vec{r})$$

Such scalars are called pseudoscalars

Similarly if, under parity transformation, we need to assign to vectors opposite sign to what we had in QED, we call them pseudovectors.

2. Scalar electrodynamics

– describes the coupling of a charged scalar to electromagnetic field.

Charged scalar is described by a complex scalar field

$$S = \int d^4x \left[-\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi + \text{interactions}\right]$$

– can be regarded as a theory of two real scalar fields by writing

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

Note: The action has a symmetry under which

$$\widetilde{\phi}(x) = e^{i\theta}\phi(x), \qquad \widetilde{\phi}(x)^* = e^{-i\theta}\phi(x)^*$$

This corresponds to rotation in ϕ_1-ϕ_2 space.

Coupling to gauge field A_μ is achieved by making this into a local symmetry.

Assign gauge transformation law:

$$A_\mu \to A_\mu + \partial_\mu\Lambda(x), \quad \phi \to e^{-iq\Lambda(x)}\phi, \quad \phi^* \to e^{iq\Lambda(x)}\phi^*$$

and define covariant derivatives

$$D_\mu\phi = (\partial_\mu + i\,q\,A_\mu)\phi, \qquad D_\mu\phi^* = (\partial_\mu - i\,q\,A_\mu)\phi^*$$

Ex. Check that under gauge tranformation

$$D_\mu\phi \to e^{-iq\Lambda(x)}D_\mu\phi, \qquad D_\mu\phi^* \to e^{iq\Lambda(x)}D_\mu\phi^*$$

The gauge invariant coupling to the Maxwell field can now be obtained by replacing ∂_μ by D_μ in the action:

$$S = \int d^4x \left[-\eta^{\mu\nu}D_\mu\phi^*D_\nu\phi - m^2\phi^*\phi + \text{interactions}\right] + S_{maxwell}$$

Note: Interactions must preserve gauge invariance, e.g. $(\phi^*\phi)^2$ is allowed but not $\phi^3\phi^*$ or $\phi^{*3}\phi$.

3. Variation of Dirac theory.

We can have theories where we have half the number of fields.

(a) Majorana fermions:

We can impose the condition that the field is invariant under charge conjugation.

$$\psi = C\bar{\psi}^T$$

Ex. Check that both sides transform in the same way under Lorentz transformation

$\Rightarrow$ if we impose this in one inertial frame, it remains valid in all inertial frames.

Such particles are called Majorana particles.

Note: ψ must remain unchanged under gauge transformation since otherwise the two sides pick up opposite phases

$\Rightarrow D_\mu\psi = \partial_\mu\psi \Rightarrow$ we cannot couple ψ to gauge fields.

Majorana particles are charge neutral.

(b) Weyl fermions:

Since $(\gamma^5)^2 = 1$, γ^5 has eigenvalues ± 1 and we can impose the condition

$$\gamma^5 \psi = \psi \qquad \text{or} \qquad \gamma^5 \psi = -\psi$$

Ex. Check as before that this condition is consistent with Lorentz invariance.

The two signs correspond to 'left-handed' and 'right-handed' fermions.

Ex. Check that

$$\gamma^5 \psi = \pm \psi \qquad \Rightarrow \qquad \bar{\psi}\,\gamma^5 = \mp \bar{\psi}$$

This implies that for Weyl fermions:

$$\bar{\psi}\psi = 0$$

Proof.

$$\bar{\psi}\psi = \pm \bar{\psi}\gamma^5\psi = \pm(\mp\bar{\psi}\psi) = -\bar{\psi}\psi$$

Therefore Weyl fermions must be massless, but they can be coupled to Maxwell field since $\bar{\psi}\gamma^\mu\psi$ does not vanish.

Problem: In such theories the gauge symmetry becomes anomalous

– not a consistent theory.

We could cancel the anomaly by adding Weyl fermions of opposite charge.

An example of a theory with global anomalous symmetry:

Consider QED with massless fermions ($m = 0$).

This theory has a symmetry under:

$$\widetilde{\psi}(x) = e^{i\gamma^5\theta}\psi, \qquad \widetilde{A}_\mu(x) = A_\mu(x)$$

Ex. Check this for infinitesimal transformation (θ small).

This symmetry is anomalous.

The theory is consistent since the gauge symmetry is not anomalous, but the global symmetry described above is not present in the quantum theory.

Some general results in quantum field theory (without proof)

1. Spin-Statistics theorem:

In a Lorentz invariant theory, we must

– quantize integer spin fields as bosons i.e. commutators of fields vanish outside the light-cone (space-like separation)

– quantize (integer + 1/2) spin fields as fermions (anti-commutators vanish outside the light-cone)

e.g. for scalars we had found the general spectral decomposition formula:

$$\langle\Omega|\phi(x_1)\phi(x_2)|\Omega\rangle = \int_0^\infty du\, f(u)\, \Delta_+(x_1, x_2; \sqrt{u})$$

Therefore

$$\langle\Omega|[\phi(x_1), \phi(x_2)]|\Omega\rangle = \int_0^\infty du\, f(u)\, (\Delta_+(x_1, x_2; \sqrt{u}) - \Delta_+(x_2, x_1; \sqrt{u}))$$

– vanishes outside the light-cone.

But

$$\langle\Omega|\{\phi(x_1), \phi(x_2)\}|\Omega\rangle = \int_0^\infty du\, f(u)\, (\Delta_+(x_1, x_2; \sqrt{u}) + \Delta_+(x_2, x_1; \sqrt{u}))$$

– does not vanish outside the light-cone.

Similar spectral decomposition for Dirac fermions shows that the anti-commutators vanish outside the light-cone but the commutators do not vanish outside the light-cone.

Main point: Once we have proved spectral decomposition based on translation and Lorentz invariance, the (anti-)commutators are given by superposition of the free field results.

2. CPT theorem

In a local, Lorentz invariant quantum field theory, the combined transformation under C, P and T is always a symmetry even thought C, P and T may not be symmetries.

In nature C, P and T are violated but CPT is known to be a symmetry.

For more discussion you can consult textbooks

e.g. Bjorken and Drell section 15.14 gives a proof for interacting theory of scalar, Maxwell and Dirac fields.

Some final remarks:

The formalism we have used is not manifestly Lorentz invariant

Construction of the Hamiltonian gives a special role to time

– known as the canonical formalism

At the end the Green's functions are Lorentz covariant (except in gauge theories where gauge fixing may make the Green's function also Lorentz non-invariant although all physical quantities are Lorentz covariant)

This can be rectified using the path integral approach which gives a manifestly Lorentz covariant way of calculating the Green's functions.

However once we have constructed the Green's function, we have to invoke the results of canonical formalism to connect the results to physical observables like the S-matrix.