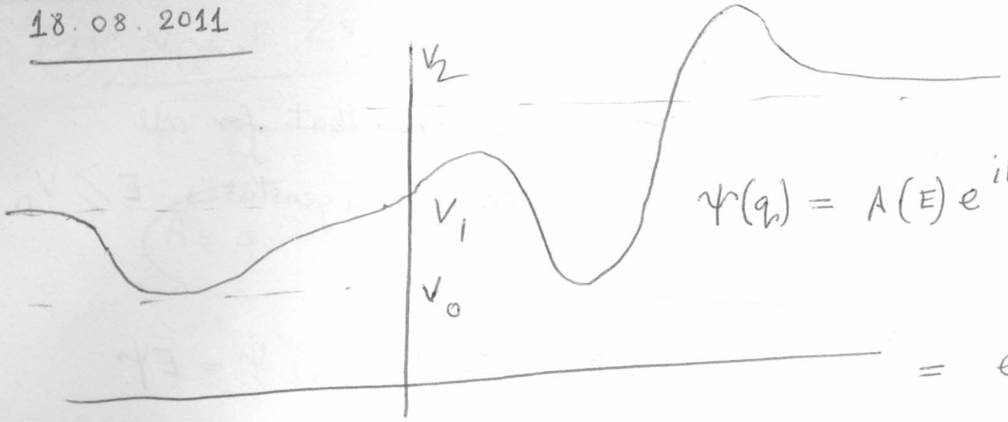
18. 08. 2011



$$\psi(q) = A(E)e^{ik_1 q} + B(E)e^{-ik_1 q} \quad \text{as } q \to -\infty$$

$$= e^{ik_2 q} \quad \text{as } q \to \infty$$

$A(E)\, e^{ik_1 q}$:- Incident wave

Density : $|A(E)|^2$

$B(E)\, e^{-ik_1 q}$ → reflected wave

Density : $|B(E)|^2$

$e^{ik_2 q}$ → transmitted wave (density 1.) .

$|\psi(q)|^2 \delta q$ = Prob. of finding a particle between q and $q + \delta q$

For large $-ve$ q, average number of particles / length

$$\frac{1}{L}\int_{-\Lambda}^{-\Lambda+L} |\psi(q)|^2 dq$$

$$= \frac{1}{L}\int_{-\Lambda}^{-\Lambda+L}\left(|A(E)|^2 + |B(E)|^2 + A^* B\, e^{-2ik_1 q} + B^* A\, e^{+2ik_1 q}\right) dq$$

$$= |A(E)|^2 + |B(E)|^2$$

Average momentum

$$\frac{1}{L}\int_{-\Lambda}^{-\Lambda+L} dq\, \psi^*(q)\left(-i\hbar \frac{\partial \psi}{\partial q}\right)$$

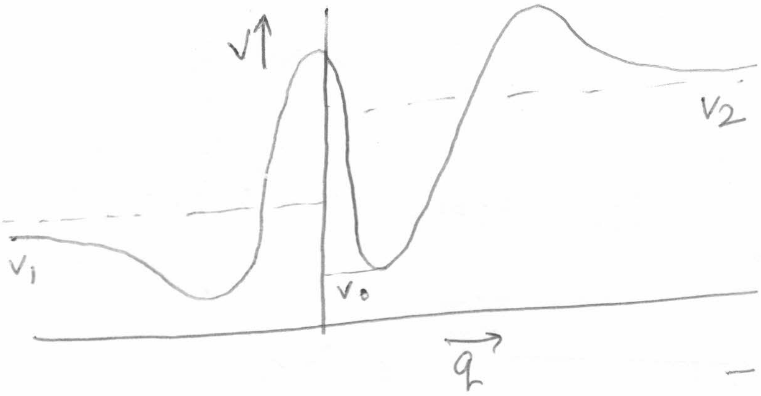
$$= \hbar k_1 |A|^2 - \hbar k_1 |B|^2$$

Calculate avge. energy

get $E(|A|^2 + |B|^2)$

$|A|^2 \to$ density of incident wave

$|B|^2 \to$ density of reflected wave



① Show that for all energy eigenstates $E \geqslant V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dq^2} + V(q)\,\psi = E\psi$$

① $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dq^2} = (E - V(q))\,\psi(q)$$

$$\times \int_{-\infty}^{\infty} \psi^*(q)\, dq$$

$$\text{LHS} = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \psi^*(q)\frac{d^2\psi}{dq^2}\, dq$$

$$= \frac{\hbar^2}{2m}\int dq\, \frac{d\psi^*}{dq}\frac{d\psi}{dq} - \frac{\hbar^2}{2m}\underbrace{\left[\psi^*\frac{\partial\psi}{\partial q}\right]_{-\infty}^{\infty}}_{=0}$$

$$\geqslant 0$$

$$\text{RHS} = \int_{-\infty}^{\infty} dq\, \psi^*(q)\,(E - V(q))\,\psi(q)$$

If $E < V_0$, RHS < 0 ~~for E < V₀~~

So, contradiction

$$-\frac{\hbar^2}{2m}\frac{d^2\psi^*}{dq^2} + V(q)\,\psi^* = E\psi^*$$

Define $$J = \left(\psi^*\frac{d\psi}{dq} - \frac{d\psi^*}{dq}\psi\right).$$

$$\frac{dJ}{dq} = \psi^*\frac{d^2\psi}{dq^2} - \frac{d^2\psi^*}{dq^2}\psi = 0$$

(Using Schrödinger eqn)

$$J(q)\Big|_{q=-\infty} = J(q)\Big|_{q=\infty}$$

① $E < V_1, V_2$

As $q \to \infty$, $\psi(q) \longrightarrow e^{-K_2 q}$

$J \longrightarrow 0$

As $q \to -\infty$ $\quad \psi(q) \to B(E)\, e^{K_1 q}$ $\quad (\because A = 0)$

$J \to 0.$

② $V_1 < E < V_2$

As $q \to \infty$, $\psi(q) \to e^{-\kappa_2 q}$

$J \to 0$

As $q \to -\infty$

$$\psi(q) \longrightarrow A e^{ik_1 q} + B e^{-ik_1 q}$$

$$J = \psi^* \frac{d\psi}{dq} - \frac{d\psi^*}{dq}\psi$$

$$= \left(A^* e^{-ik_1 q} + B^* e^{+ik_1 q}\right)\left(ik_1 A e^{ik_1 q} - ik_1 B e^{-ik_1 q}\right)$$

$$- \left(A e^{ik_1 q} + B e^{-ik_1 q}\right)\left(-ik_1 A^* e^{-ik_1 q} + ik_1 B^* e^{+ik_1 q}\right)$$

$$= 2ik_1\left(|A|^2 - |B|^2\right).$$

But $J = 0$ for $q \to \infty$.

$$|A|^2 = |B|^2$$

③ $E > V_1, V_2$

$\psi(q) \longrightarrow e^{ik_2 q}$ as $q \to \infty$

$\longrightarrow A e^{ik_1 q} + B e^{-ik_1 q}$ as $q \to -\infty$

$J \longrightarrow 2ik_1\left(A^* A - B^* B\right)$

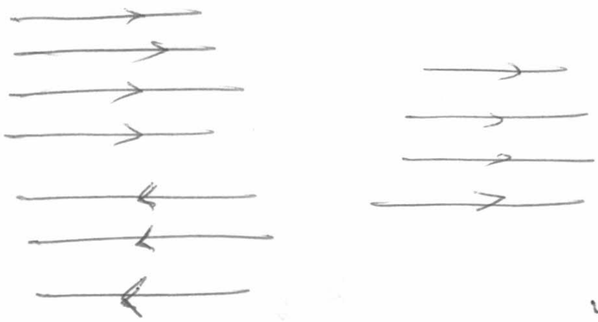
$q \to \infty \longrightarrow J = 2ik_2$

$$\Rightarrow A^* A - B^* B = \frac{k_2}{k_1}$$

or, $\boxed{|A|^2 = |B|^2 + \frac{k_2}{k_1} \times 1}$ → average density in the transmitted wave

$|A|^2$ → average density in the incident wave

$|B|^2$ → average density in the reflected wave



$$n_{incident} \times \frac{\hbar k_1}{m}$$

$$- \; n_{reflected} \times \frac{\hbar k_1}{m}$$

$$= n_{transmitted} \times \frac{\hbar k_2}{m}$$

✓

number / area passing through unit time → (✓)

$$1 = \frac{|B|^2}{|A|^2} + \frac{k_2}{k_1}\left|\frac{1}{A}\right|^2$$

or, $$1 = |r(E)|^2 + \frac{k_2}{k_1}|t(E)|^2$$

$|r(E)|^2 \Downarrow R(E)$ (fraction which gets reflected back)

$\frac{k_2}{k_1}|t(E)|^2 \Downarrow T(E)$ (fraction which gets transmitted)

Oscillation theorem

Suppose, we arrange the energy eigenvalues as $E_0 < E_1 < E_2 < \ldots$ & the corresponding wavefunctions are $\psi_0(q), \psi_1(q), \ldots$ Then, $\psi_n(q)$ has precisely n nodes. (n zeros other than $\pm\infty$).

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Non-degeneracy of bound states :-

→ only one eigenfunction for a given energy

Assume the contrary

$$\left\{\begin{array}{l} -\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_1}{dq^2} + V\psi_1 = E\psi_1 \\[2ex] -\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_2}{dq^2} + V\psi_2 = E\psi_2 \end{array}\right\}$$

Ex. Show that $\frac{d}{dq}\left(\psi_1 \frac{d\psi_2}{dq} - \psi_2 \frac{d\psi_1}{dq}\right) = 0$.

$$\therefore \quad \psi_1 \frac{d\psi_2}{dq} - \psi_2 \frac{d\psi_1}{dq} = C \quad \text{(Independent of } q\text{)}$$

What is C ? $C = 0$

$$\boxed{\text{At } q = \infty, \quad \psi_1, \psi_2 \longrightarrow 0}$$

$$\therefore \quad \psi_1 \frac{d\psi_2}{dq} = \psi_2 \frac{d\psi_1}{dq}$$

$$\text{or,} \quad \frac{1}{\psi_1}\frac{d\psi_1}{dq} = \frac{1}{\psi_2}\frac{d\psi_2}{dq}$$

$$\text{or,} \quad \ln \psi_1 = \ln \psi_2 + k \quad (k = \text{integration constant})$$

$$\Rightarrow \boxed{\psi_1 = e^k \psi_2}$$

$\Rightarrow$ ~~ψ_1, ψ_2 are basi~~ ψ_1, ψ_2 describe the same state.

Oscillation theorem

Suppose, the eigenvalues of the Hamiltonian are
$E_0 < E_1 < E_2 < \cdots$ & the eigenfunctions are
$\psi_0, \psi_1, \psi_2, \cdots$

Then, ψ_n has n nodes. (n points where it vanishes)

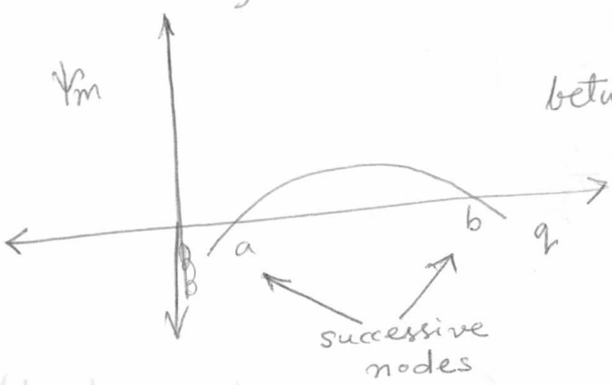
Suppose, ψ_m and ψ_n are two eigenfunctions of energy E_m and E_n with $E_m < E_n$.

Then, we'll show that ψ_n has larger number of nodes than ψ_m.

$$\left\{ \begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi_m}{dq^2} + V(q)\psi_m &= E_m \psi_m \\ -\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dq^2} + V(q)\psi_n &= E_n \psi_n \end{aligned} \right\}$$

Ex. Show that $\dfrac{d}{dq}\left(\psi_n \dfrac{d\psi_m}{dq} - \psi_m \dfrac{d\psi_n}{dq}\right) = (E_m - E_n)\psi_m \psi_n \times \left(\dfrac{-2m}{\hbar^2}\right)$

We'll show that between any two nodes of ψ_m there is a node of ψ_n.



Assume that ψ_m is positive between a and b. Then

$$\left[\ \frac{d\psi_m}{dq} > 0 \text{ at } q = a \quad \text{and} \quad \frac{d\psi_m}{dq} < 0 \text{ at } q = b\ \right]$$

We now integrate $\dfrac{d}{dq}\left(\psi_n \dfrac{d\psi_m}{dq} - \psi_m \dfrac{d\psi_n}{dq}\right)$ from a to b.

$$\int \frac{d}{dq}\left(\psi_n \frac{d\psi_m}{dq} - \psi_m \frac{d\psi_n}{dq}\right)dq = -\frac{2m}{\hbar^2}(E_m - E_n)\int_a^b \psi_m \psi_n \, dq$$

$$\text{LHS} = \psi_n \frac{d\psi_m}{dq}\Big|_a^b$$

Suppose, ψ_n has no zero between a and b.

Let's assume that $\psi_n > 0$ in [a, b].

$$\left[\begin{array}{l} \text{L.H.S.} = \text{negative } (< 0). \\ \text{R.H.S.} > 0. \end{array}\right\} \left.\begin{array}{l} > 0 \\ < 0 \end{array}\right] (\text{if } \psi_n \text{ is } -\text{ve})$$

So, there's a contradiction.

$\therefore$ ψ_n must change sign between a & b.

x : nodes of ψ_m

o : nodes of ψ_n

○ > # ×

General proof :-

Define $F_2 = \frac{1}{f_2}\frac{\partial f_2}{\partial q}$, $G_2 = \frac{1}{g_2}\frac{\partial g_2}{\partial q}$

Why partial derivatives? ⇝ [f_2, g_2 depends on energy E]

Ex

○ : F_2 and G_2 satisfy the same 1st order differential equation

$$F = \frac{1}{\psi}\frac{d\psi}{dq}$$

$$\frac{\partial F}{\partial q} = \frac{1}{\psi}\frac{\partial^2\psi}{\partial q^2} - \frac{1}{\psi^2}\left(\frac{\partial\psi}{\partial q}\right)^2$$

$$= -\frac{2m}{\hbar^2}\left(E - V(q)\right) - F^2$$

Ex.

(1.) For generic E, $F_2 \to -\kappa_2$ as $q \to \infty$

$\to -\kappa_1$ as $q \to -\infty$

$G_2 \to \kappa_2$ for $q \to \infty$

$\to \kappa_1$ for $q \to -\infty$

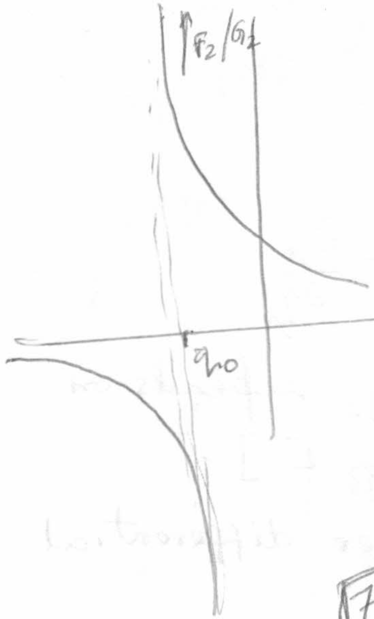
(2.) If E = energy of a bound state, then

$F_2 = G_2 \longrightarrow -\kappa_2$ as $q \to \infty$

$\longrightarrow \kappa_1$ as $q \to -\infty$.

(3.) If F_2 (or G_2) has singularity of the form $\left[\frac{A}{(q-q_0)^n} + \cdots\right]$ then $A = 1$, $n = 1$.

(use diff. eq.)



(4.) If $F_2 = G_2$ at any one point, then $F_2(q) = G_2(q)$.

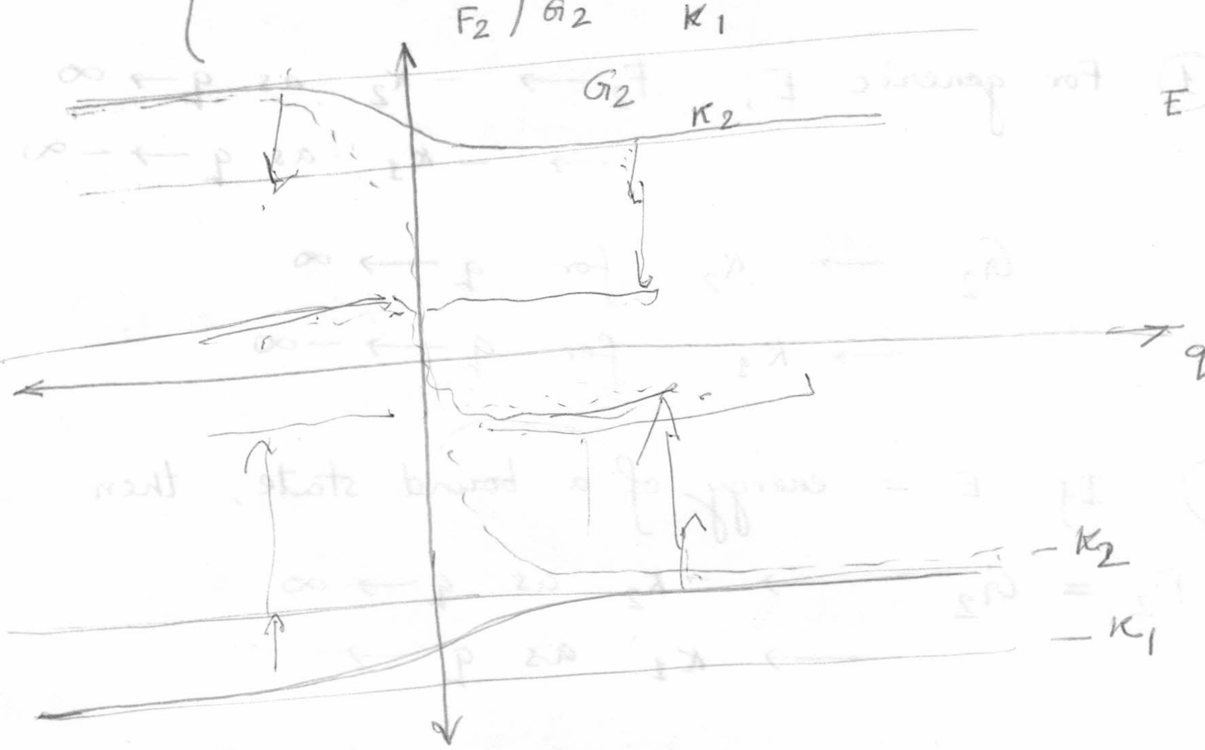
(5.) F_2 (or G_2) goes as $\frac{1}{q-q_0}$ $(\simeq)$ implies

$\boxed{7}$ f_2 (or g_2) $\simeq C(q-q_0)$.

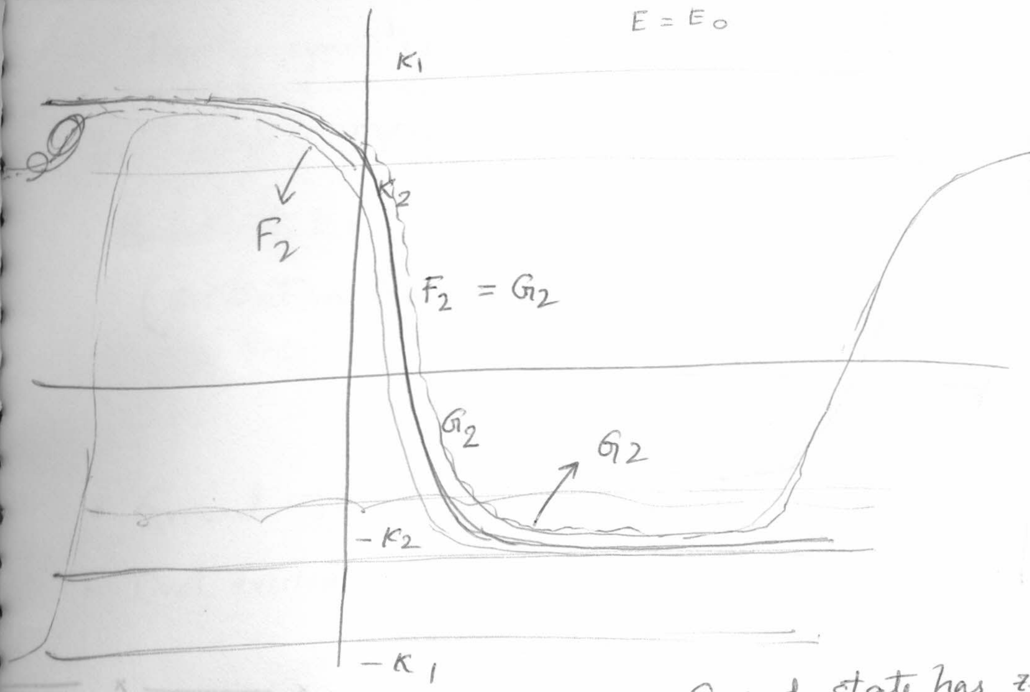
[near $q = q_0$]

(5.) If $E < V_0$, then $F_2(q) < 0$, $G_2(q) > 0$

(6.)
$$\left\{ \begin{array}{l} \frac{\partial}{\partial E} F_2(q, E) > 0 \\ \frac{\partial}{\partial E} G_2(q, E) < 0 \end{array} \right\}$$



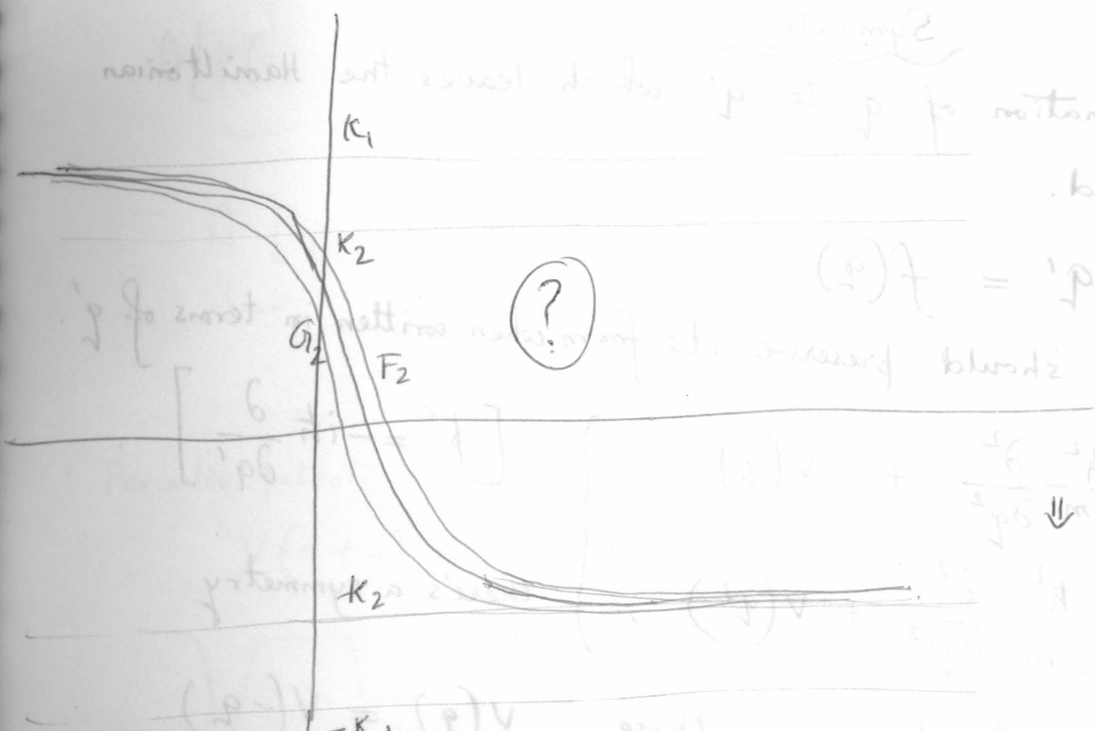
What happens if $E = E_0$?

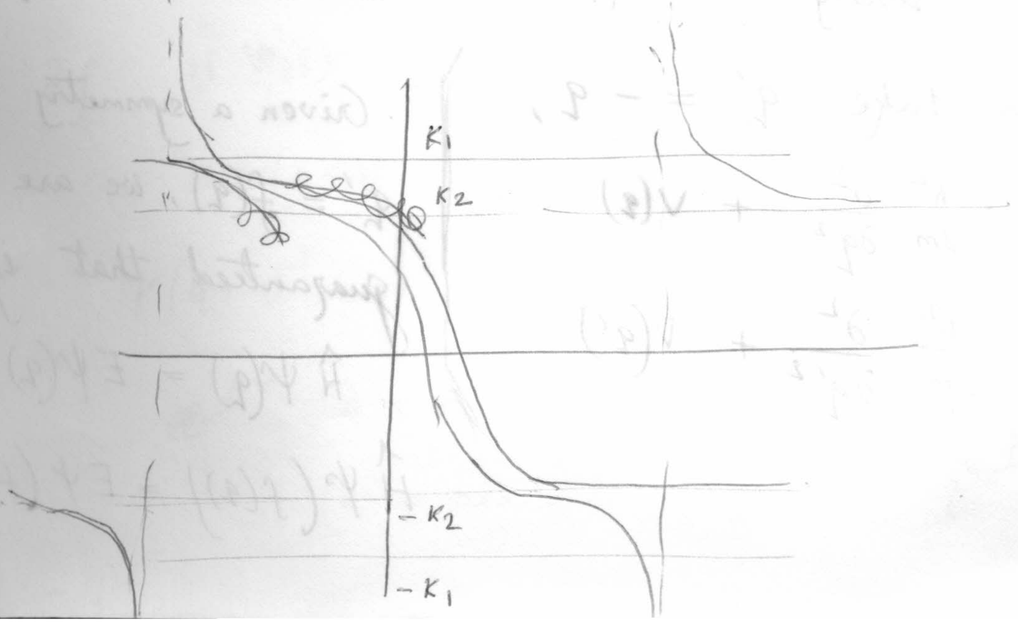


$$\epsilon\, e^{\kappa_2 q} + e^{-\kappa_2 q}$$

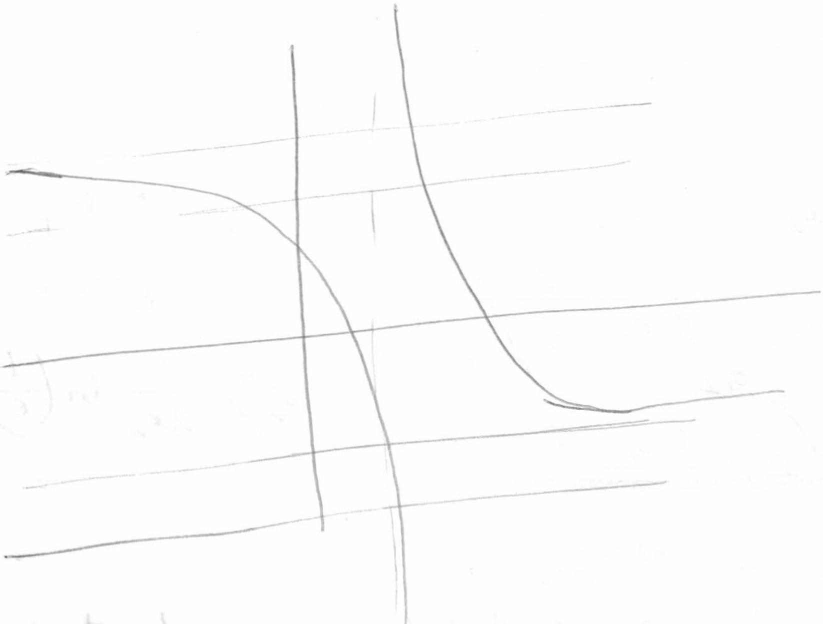
$$q > \frac{1}{2\kappa_2} \ln\left(\frac{1}{\epsilon}\right)$$

→ Ground state has zeros only at $\pm\infty$.





$E = E_1$



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Symmetry

→ Transformation of q to q' which leaves the Hamiltonian unchanged.

$$q' = f(q)$$

$\hat{H}$ should preserve its form when written in terms of q'.

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)$$

$$\left[p' = -i\hbar\frac{\partial}{\partial q'}\right]$$

If $H' = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial q'^2} + V(q')$, there's a symmetry

Example: parity. Suppose, $V(q) = V(-q)$

If we take $q' = -q$,

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q'^2} + V(q')$$

Given a symmetry $q' = f(q)$, we are guaranteed that if

$\hat{H}\psi(q) = E\psi(q)$ th

$\hat{H}\psi(f(q)) = E\psi(f(q))$

Parity symmetry

If $\hat{H}\psi(q) = E\psi(q)$, then $\hat{H}\psi(-q) = E\psi(-q)$.

Bound states : $\psi(-q) = \lambda\psi(q)$ for some λ

(Oscillation theorem) $\quad = \lambda^2\psi(-q)$

$\lambda = \pm 1 \rightarrow$ {eigenvalues of parity operator}

- Ground state $\longrightarrow$ even parity
- First excited " $\longrightarrow$ odd "

(no. of nodes)

$\boxed{\psi(q) \neq \lambda\psi(-q)}$

What if "scattering states" are there?

$$\hat{H}\underbrace{[\psi(q) + \psi(-q)]}_{\text{even}} = E[\psi(q) + \psi(-q)]$$

- Not useful in scattering

$$\hat{H}\underbrace{[\psi(q) - \psi(-q)]}_{\text{odd}} = E[\psi(q) - \psi(-q)]$$

Another example

- Periodic potential

$V(q+a) = V(q)$

$q' = q + a$

$$\hat{H} = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)\right.$$

$$\checkmark \quad = \left.-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q'^2} + V(q')\right\}$$

If $H\psi(q) = E\psi(q)$, then

$H\psi(q+a) = E\psi(q+a)$

$\psi(q+a) = \lambda\psi(q)$ (One can choose.)

$\underbrace{|\lambda| \text{ must be } 1.}_{\text{(normalization)}}$

$e^{\pm i}\alpha \quad \lambda = e^{i\alpha}$

($\alpha \rightarrow$ real)

$\psi(q) = e^{i\alpha q/a}\,\phi(q)$ [Def^{n} of $\phi(q)$]

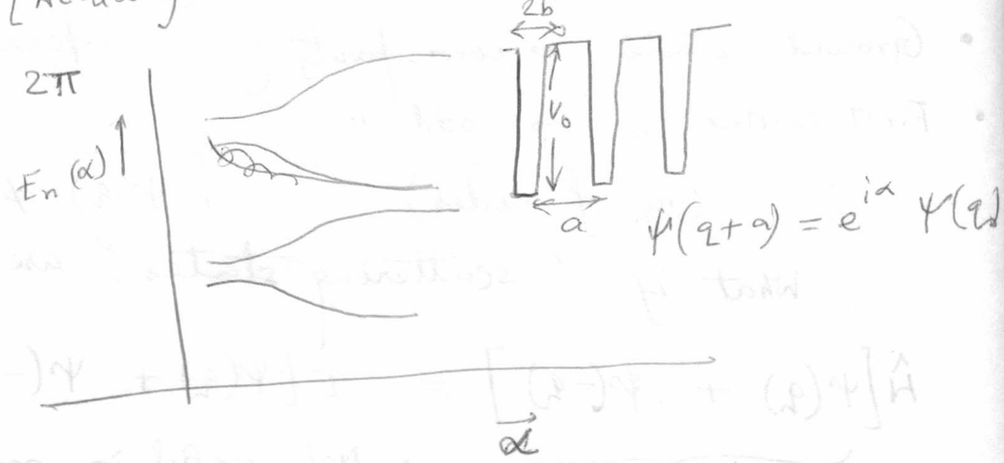
$\therefore \quad \phi(q+a) = \phi(q)$.

- Solve SE in a period. Use $\psi(q+a) = \psi(q)$.

Fix one α. Vary α over a continuous range.

[Actually continuous spectrum]

$0 \le \alpha \le 2\pi$



Delta function pot.

$$V(q) = a\,\delta(q-q_0)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dq^2} + a\,\delta(q-q_0)\,\psi = E\psi$$

Away from q_0, we have free particle.

Near q_0, integrate the eq. from $(q_0-\epsilon)$ to $(q_0+\epsilon)$.

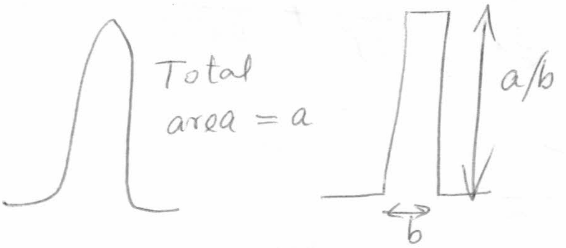
$$-\frac{\hbar^2}{2m}\int_{q_0-\epsilon}^{q_0+\epsilon}\frac{d^2\psi}{dq^2}\,dq = a\int_{q_0-\epsilon}^{q_0+\epsilon} dq\,\delta(q-q_0)\,\psi(q) = E\int_{q_0-\epsilon}^{q_0+\epsilon}\psi(q)\,dq$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\left(\left.\frac{d\psi}{dq}\right|_{q_0+\epsilon} - \left.\frac{d\psi}{dq}\right|_{q_0-\epsilon}\right) + a\,\psi(q_0) = O(\epsilon)$$

$$\left.\frac{d\psi}{dq}\right|_{q_0+\epsilon} - \left.\frac{d\psi}{dq}\right|_{q_0-\epsilon} = \frac{2ma}{\hbar^2}\psi(q_0)$$

$(\epsilon \to 0.)$

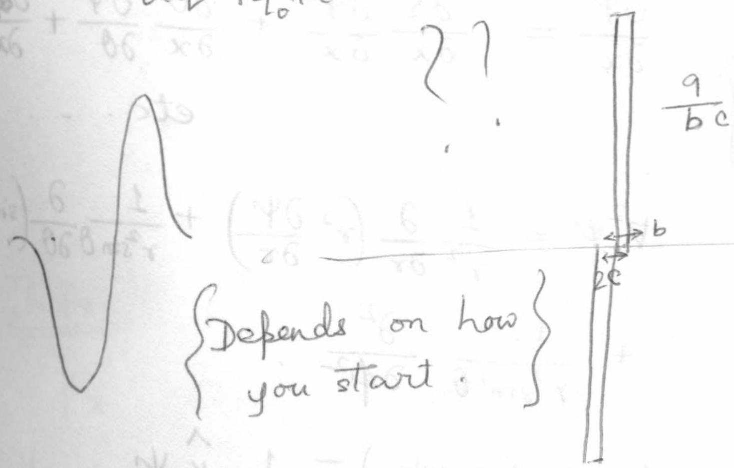
$$\psi\Big|_{q_0+\epsilon} = \psi\Big|_{q_0-\epsilon} \quad \text{for } \epsilon \longrightarrow 0$$



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dq^2} + a\,\delta'(q-q_0)\,\psi = E\psi$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\left\{\frac{d\psi}{dq}\Big|_{q_0+\epsilon} - \frac{d\psi}{dq}\Big|_{q_0-\epsilon}\right\} - a\,\psi'(q_0) = 0 .$$

$$\Rightarrow \quad \frac{d\psi}{dq}\Big|_{q_0+\epsilon} - \frac{d\psi}{dq}\Big|_{q_0-\epsilon} = \frac{2m}{\hbar^2}\,a\left(-\frac{d\psi}{dq}\right)\Big|_{q_0}$$



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$\psi_0 \quad f_2(E) \quad \& \quad f_2(E+\delta E)$

$$\frac{d}{dq}\left(f_2(E)\,\frac{df_2(E+\delta E)}{dq} - f_2(E+\delta E)\,\frac{df_2(E)}{dq}\right) = \int_{\infty}^{q}$$

divide by $f_2(E)\, f_2(E+\delta E) = F_2(E+\delta E) - F_2(E)$

3-dimensional systems

Three coordinates : x, y, z

$\Rightarrow$ operators : $\hat{x}, \hat{y}, \hat{z}$

$$\hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}, \quad \hat{p}_y = \frac{\hbar}{i}\frac{\partial}{\partial y}, \quad \hat{p}_z = \frac{\hbar}{i}\frac{\partial}{\partial z}$$

$$H = \frac{\vec{p}^2}{2m} + V(x,y,z) \Rightarrow \text{Quantize}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(\hat{x}, \hat{y}, \hat{z})$$

Energy eigenvalue equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(x,y,z)\psi = E\psi$$

$$\nabla^2 \equiv \partial_x^2 + \partial_y^2 + \partial_z^2 \quad \text{(Laplacian operator)}$$

$\longrightarrow$ Central potential $\quad V = V(r)$

Spherical polar coordinates

$$\left\{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \qquad \frac{\partial\psi}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial\psi}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial\psi}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial\phi}$$

etc. . . .

Using these, show that,

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{1}{r^2}\hat{K}\psi$$

$$\hat{K}\psi = \boxed{-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}}$$

operator

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\hbar^2}{2mr^2}\hat{K}\psi + V\psi.$$

Now, we consider central potentials. $[V = V(r)]$

Then : $\hat{H}\hat{K}\psi = \hat{K}\hat{H}\psi$.

$$[\hat{H}, \hat{K}] = 0.$$

Choose ψ to be an eigenstate of $\hat{K}$.

$$\hat{K}\psi = \lambda\psi$$

$$\Rightarrow \quad \hat{H}\psi = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \left[\frac{\hbar^2\lambda}{2mr^2} + V\right]\psi$$

$\Rightarrow$ One dimensional problem (effectively)

$[r \longrightarrow 0 \text{ to } \infty]$

We want to find eigenfunctions and eigenvalues of $\hat{K}$.

__Definitions__ : Angular momentum operators

$$\vec{L} = \vec{r} \times \vec{p} \quad \text{(classically)}$$

$$L_x = y p_z - z p_y, \ldots$$

$$\hat{L}_z = x\hat{p}_y - y\hat{p}_x \qquad \hat{L}_x\psi = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\psi$$

$$\hat{L}_x = y\hat{p}_z - z\hat{p}_y \qquad \hat{L}_z\psi = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\psi$$

$$\hat{L}_y = -(x\hat{p}_z - z\hat{p}_x)$$

__Define__ $\hat{L}^2\psi = \left(\hat{L}_x\hat{L}_x\psi + \hat{L}_y\hat{L}_y\psi + \hat{L}_z\hat{L}_z\psi\right)$

Ex: Check the following

$$\left\{\begin{array}{ll} [\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z & [\hat{L}_x, \hat{L}^2] = 0 \\ [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x & [\hat{L}_y, \hat{L}^2] = 0 \\ [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y & [\hat{L}_z, \hat{L}^2] = 0. \end{array}\right\}$$

Define $\left\{\begin{array}{l} L_+ = L_x + iL_y \\ L_- = L_x - iL_y \end{array}\right\}$

Check

$$\left\{\begin{array}{l} [L_+, L_-] = 2\hbar L_z \\ [L_z, L_+] = \hbar L_+ \\ [L_z, L_-] = -\hbar L_- \end{array}\right\}$$

__Ex__ Check that

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}, \quad L_- = e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_+ = e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}^2\psi = \left\{\frac{1}{2}\left(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+\right) + \hat{L}_z\hat{L}_z\right\}\psi$$

$$= -\hbar^2\left(\left[\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)\right]\right.$$

[Use the above formulae]

$$= \hbar^2\hat{K}\psi \qquad \hat{K}\psi = \lambda\psi$$

$$\hat{L}^2\psi = \lambda\hbar^2\psi$$

Use $[\hat{L}^2, \hat{L}_x] = 0$, $[\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z]$

$\Rightarrow [\hat{L}^2, L_+] = 0$, $[\hat{L}^2, L_-] = 0$.

Choose them to be eigenstates of $\hat{L}_z$ with eigenvalue $m\hbar$. (m = real number)

Call this eigenstate $\psi_{\lambda,m}$. This satisfies

$$\boxed{\begin{aligned}\hat{L}_z\psi_{\lambda,m} &= m\hbar\,\psi_{\lambda,m}\\ \hat{L}^2\psi_{\lambda,m} &= \lambda\hbar^2\psi_{\lambda,m}\end{aligned}}$$

$$\hat{L}_z\hat{L}_+\psi_{\lambda,m} = \left\{[\hat{L}_z, \hat{L}_+] + L_+L_z\right\}\psi_{\lambda,m}.$$

$$= \hbar\hat{L}_+\psi_{\lambda,m} + m\hbar\hat{L}_+\psi_{\lambda,m}$$

$$= (m+1)\hbar L^+\psi_{\lambda,m}$$

$$\left\{\begin{aligned} L_+ &\longrightarrow \text{raising operator}\\ L_- &\longrightarrow \text{lowering } \text{"}\end{aligned}\right\}$$

$$\hat{L}^2\hat{L}_+\psi_{\lambda,m} = \left\{[\hat{L}^2, \hat{L}_+] + \hat{L}_+\hat{L}^2\right\}\psi_{\lambda,m}$$

$$= \lambda\hbar^2\hat{L}_+\psi_{\lambda,m}.$$

$$\langle L_+ \psi_{\lambda,m} | L_+ \psi_{\lambda,m} \rangle = \cancel{\psi_{\lambda,m} (L_+)^\dagger L} \quad \langle \psi_{\lambda,m} | (L_+)^\dagger L_+ \psi_{\lambda,m} \rangle$$

$$\left\{ \begin{aligned} L_- L_+ - L_+ L_- &= -2\hbar L_z \\ L_- L_+ + L_+ L_- &= 2(\hat{L}^2 - \hat{L}_z^2) \\ \therefore \quad L_- L_+ &= \hat{L}^2 - \hat{L}_z^2 - \hbar \hat{L}_z \end{aligned} \right\}$$

$$= \langle \psi_{\lambda,m} | L_- L_+ \psi_{\lambda,m} \rangle$$

$$= \hbar^2 (\lambda - m^2 - m) \underbrace{\langle \psi_{\lambda,m} | \psi_{\lambda,m} \rangle}_{= 1}$$

If $\lambda = m(m+1)$, then $L_+ \psi_{\lambda,m} = 0$

If $\lambda > m(m+1)$, then $L_+ \psi_{\lambda,m} \neq 0$.

$$\psi_{\lambda,m+1} = \frac{1}{\hbar\sqrt{\lambda - m^2 - m}} L_+ \psi_{\lambda,m}$$

$\longrightarrow$ normalized

If $\lambda < m(m+1)$: not possible (non-normalizable solutions)

If $\lambda > m(m+1)$, we can construct a new eigenstate of (L^2, L_z) of ev. $(\lambda\hbar^2, (m+1)\hbar)$.

$$L_+ \psi_{\lambda,m+1} = 0 \text{ if } \lambda = (m+1)(m+2).$$
$$\neq 0 \text{ '' } \lambda > (m+1)(m+2).$$

$\Rightarrow$ define $\psi_{\lambda,m+2}$

(not possible if $\lambda < (m+1)(m+2)$.)

Step 1. : $\lambda \geqslant m(m+1)$

Step 2. $\lambda \geqslant (m+1)(m+2)$

$\vdots$ $\quad \lambda \geqslant (m+2)(m+3)$

$(m+p) \implies \lambda \overset{\text{must be}}{\underset{\text{(for integer } p>0)}{\Longrightarrow}} (m+p)(m+p+1)$.

$\lambda = (m+p)(m+p+1)$ for some non-negative integer

$l = m+p \longrightarrow$ definition of l.

Repeat this with L_-

$L^- \psi_{\lambda, m} \Rightarrow$ 1 $\overset{\text{Exercise}}{\text{Show that}}$ this has (L^2, L_z) e.v.

$(\lambda\hbar^2, (m-1)\hbar)$

2 $\left(L_- \psi_{\lambda,m}, L_- \psi_{\lambda,m}\right) = \hbar^2 \left(\lambda - m(m-1)\right)$.

Possibilities

1 $\lambda = m(m-1) \Rightarrow L_- \psi_{\lambda,m} = 0$

2 $\lambda < m(m-1) \Rightarrow$ not possible

3 $\lambda > m(m-1) \Rightarrow \psi_{\lambda, m-1} = \dfrac{1}{\hbar\sqrt{\lambda - m(m-1)}} L_- \psi_{\lambda,m}$

Keep repeating with $\psi_{\lambda, m-1}, \psi_{\lambda, m-2}, \ldots$

After q^{th} step, we'll get $\lambda \geq (m-q)(m-q+1)$

$\lambda = l(l+1)$

At some value of q,

$$\lambda = (q-m)(q-m-1)$$
$$= l(l+1)$$

Quadratic in q

Solutions $q - m = l+1, -l$
$= m+p+1, -m-p$

$\boxed{q > 0, \quad p > 0}$ $\therefore q - m = m + p + 1$

or, $q = 2m + p + 1$

or, $2m = q - p - 1$

$2l - 2p = q - p - 1$ $\quad (p \geq 0,\ q \geq 1)$

or, $2l = p + q - 1 = \text{integer} \geq 0.$

l = non-negative (either integer or (integer $+\frac{1}{2}$))

$l - m \longrightarrow$ integer (non-negative)

$\therefore$ m is integral.

$m : l, l-1, l-2, \ldots \ldots -l.$

$l \rightarrow$ also an integer

LIMIT OF ALGEBRA

$$L_z \psi_{\lambda,m} = m\hbar \psi_{\lambda,m}$$

$$-i\cancel{\hbar} \frac{\partial}{\partial \phi} \psi_{\lambda,m} = m\cancel{\hbar} \psi_{\lambda,m}$$

$$\psi = \boxed{e^{-im\phi} f(r,\theta)}$$

$m \rightarrow$ integer

$$F_2 = \frac{1}{f_2}\frac{\partial f_2}{\partial q} \quad 2$$

$$\frac{\partial F_2}{\partial q} = -\frac{1}{f_2^2}\left(\frac{\partial f_2}{\partial q}\right) + \frac{1}{f_2}\frac{\partial^2 f_2}{\partial q^2}$$

26.08.2011 Tutorial :- (Prof. A. Sen)

$f_2(E)$, $f_2(E+\delta E)$

$$\frac{\partial}{\partial q}\left[f_2(E) \frac{\partial f_2}{\partial q}(E+\delta E) - f_2(E+\delta E) \frac{\partial f_2(E)}{\partial q} \right]$$

$$= f_2(E) \frac{\partial^2 f_2}{\partial q^2}(E+\delta E) - f_2(E+\delta E) \frac{\partial^2 f_2(E)}{\partial q^2}$$

$$= -\frac{2m}{\hbar^2}(\delta E) f_2(E) f_2(E+\delta E). \quad \text{(By Schrödinger eq}^{\underline{n}}\text{)}$$

or, $$\frac{1}{f_2(E+\delta E)}\frac{\partial^2 f_2(E+\delta E)}{\partial q^2} - \frac{1}{f_2(E)}\frac{\partial^2 f_2(E)}{\partial q^2} = -\frac{2m}{\hbar^2}(\delta E)$$

$$F_2 = \frac{1}{f_2}\frac{\partial f_2}{\partial q} \quad \therefore \frac{\partial F_2}{\partial q} = \frac{1}{f_2}\frac{\partial^2 f_2}{\partial q^2} - \left(\frac{1}{f_2}\frac{\partial f_2}{\partial q}\right)^2$$

$$\therefore \frac{\partial}{\partial q}\left[F_2(E+\delta E) - F_2(E)\right] + F_2^2(E+\delta E) - F_2^2(E) = -\frac{2m}{\hbar^2}\,\delta E$$

$$\text{or,} \quad \frac{\partial}{\partial q}\left[F_2(E+\delta E) - F_2(E)\right] + \frac{\partial F_2^2(E)}{\partial E}\,\delta E = -\frac{2m}{\hbar^2}\,\delta E$$

$$\text{or,} \quad \frac{\partial}{\partial q}\frac{\left[F_2(E+\delta E) - F_2(E)\right]}{\delta E} = -\left[\frac{2m}{\hbar^2} + 2F_2(E)\frac{\partial F_2(E)}{\partial E}\right]$$

Taking limit $\delta E \to 0$, we have

$$\frac{\partial}{\partial q}\frac{\partial F_2(E)}{\partial E} = -2\int^{q}\left[\frac{m}{\hbar^2} + F_2(E)\frac{\partial F_2(E)}{\partial E}\right]dq' + P(E)$$

(?)

OR

$$f_2(E)\left[-\frac{2m}{\hbar^2}\left(V(q) - \overline{E+\delta E}\right) f_2(E+\delta E)\right] - f_2(E+\delta E)\left[\frac{2m}{\hbar^2}\left(V(q)-E\right) f_2(E)\right] = -\frac{2m}{\hbar^2}\,\delta E\, f_2(E)\, f_2(E+\delta E)$$

$$f_2(E)\frac{\partial f_2(E+\delta E)}{\partial q} - f_2(E+\delta E)\frac{\partial f_2(E)}{\partial q} = -\int_{\infty}^{q}\frac{2m}{\hbar^2}\,\delta E\, f(E)\, f(E+\delta E)\, dq'$$

$$f_2 = e^{-\kappa_2 q} \rightsquigarrow \text{asymptotically}$$

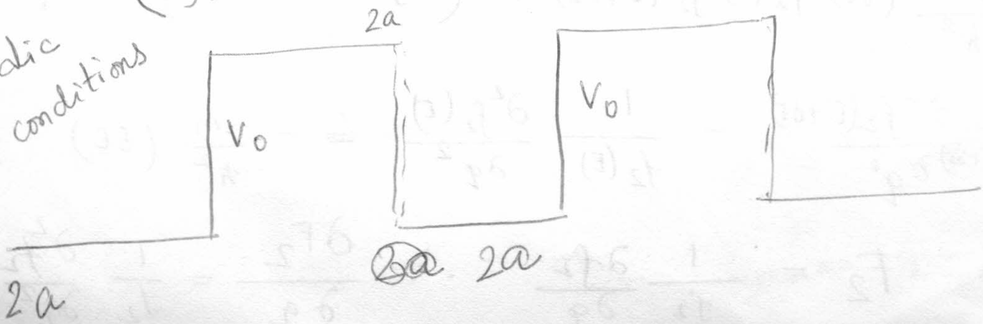
$$\Rightarrow F_2(E+\delta E) - F_2(E) = -\frac{\delta E}{f_2(E)\, f_2(E+\delta E)}\int_{\infty}^{q}\frac{2m}{\hbar^2} f_2(E)\, f_2(E+\delta E)\, dq$$

$$\Rightarrow \frac{\partial F_2}{\partial E} = -\frac{1}{f_2^2(E)}\frac{2m}{\hbar^2}\int_{\infty}^{q} f_2^2(E)\, dq'$$

$$(\delta E \to 0) \quad = \frac{1}{f_2^2(E)}\frac{2m}{\hbar^2}\int_{q}^{\infty} f_2^2(E)\, dq'$$

(f_2 is real here)

Periodic bdy. conditions



$V = 0$ for $-2a$ to 0, $2a$ to $4a$, $6a$ to $8a$, ..
$= -V_0$ " 0 to $2a$, $4a$ to $6a$, ..

$$\psi(x+a) = e^{i\alpha}\,\psi(x)$$

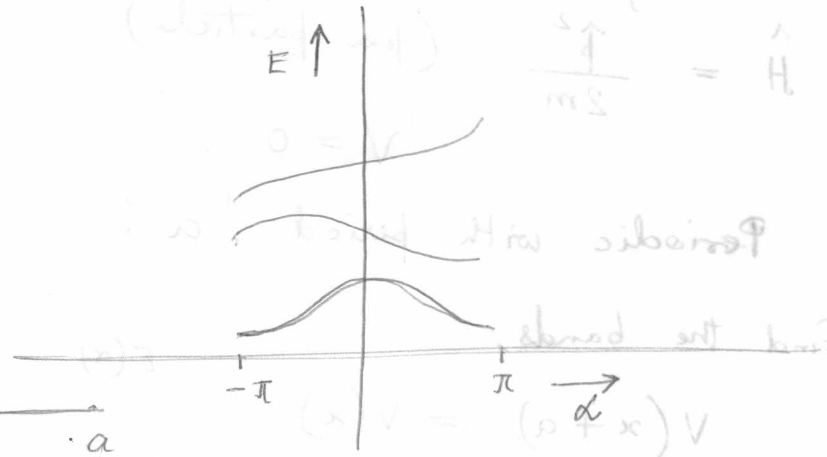
$\psi(x)$ & $\psi(x+a)$ are both energy eigenfunctions

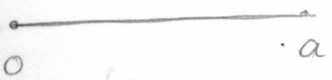
$$\boxed{\psi(x+a) = \lambda\,\psi(x)}$$

If not, try linear combinations

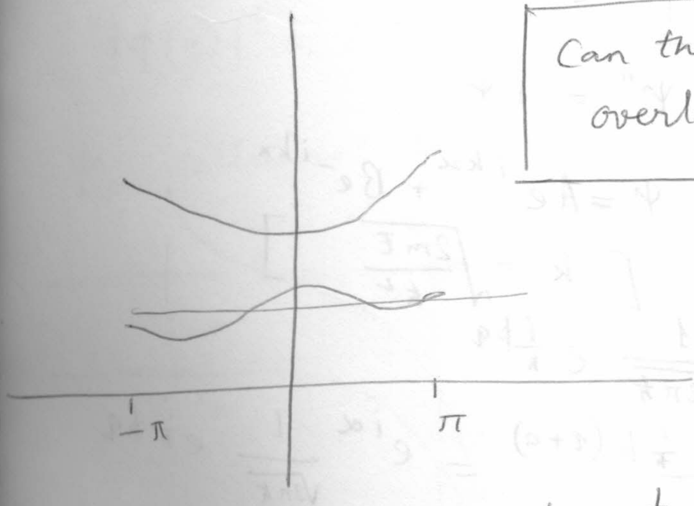
$$e^{-2i\beta}\,\psi(x-2a) + e^{-i\beta}\,\psi(x-a) + \cdots + e^{i\beta}\,\psi(x+a) + \cdots$$

For some β, these combinations may be zero.





Can the bands overlap ?

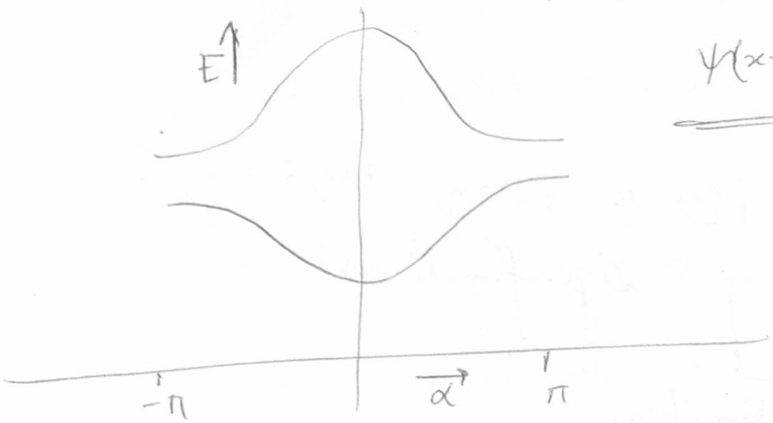


We can't have more than two solutions for any α.

Can't cut more than twice.

$\psi \longrightarrow \psi^*$ ✓

$\boxed{E(\alpha) \text{ is always a symmetric fn of } \alpha}$



$$\psi(x+a) = e^{i\alpha}\psi(x)$$

$$\left\{ \begin{array}{l} H\psi = E\psi \\ H\psi^* = E\psi^* \end{array} \right\}$$

For every E, we have two solutions.

- Therefore, the bands can't overlap.
- They can touch at $\alpha = 0, \pm\pi$

Take $\hat{H} = \frac{\hat{p}^2}{2m}$ (free particle)

$$V = 0.$$

Periodic with period 'a'.

Find the bands. $E(\alpha)$

$$V(x+a) = V(x)$$

$$\boxed{\psi(x+a) = e^{i\alpha}\psi(x)} \quad (\text{for any } \alpha)$$

$$\frac{\hat{p}^2}{2m}\psi = E\psi \qquad -\frac{\hbar^2}{2m}\psi'' = E\psi$$

$$\text{or, } \psi = Ae^{ikx} + Be^{-ikx}$$

$$\cancel{\psi = \frac{2mE}{p^2}\psi} \qquad \left[k = \sqrt{\frac{2mE}{\hbar^2}} \right]$$

$$u_p(q) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}pq}$$

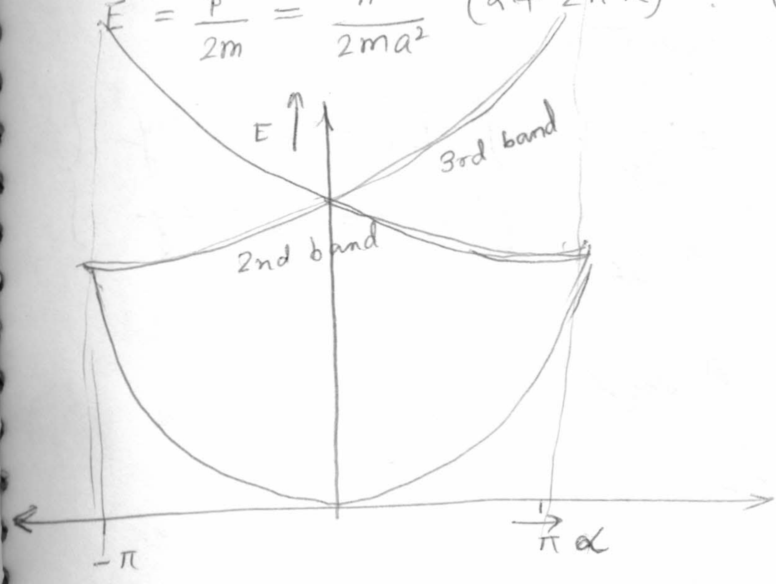
$$u_p(q+a) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}p(q+a)} = e^{i\alpha}\frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}pq}$$

$$\alpha = \frac{pa}{\hbar}$$

$$p = \frac{\hbar\alpha}{a} + \frac{(2m\pi)\hbar}{a}$$

$$E = \left(\frac{\hbar^2\alpha^2}{a^2} + \frac{4m^2\pi^2\hbar^2}{a^2} + \frac{4m\pi\hbar^2\alpha}{a^2} \right)$$

$$E = \frac{p^2}{2m} = \frac{\hbar^2}{2ma^2}(\alpha + 2n\pi)^2. \quad (n = \text{integer})$$



1st problem

$\psi_0 = \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L}$. First measure position.
Next, " energy.

Calculate probability ~~that the particle is within~~ That first measurement gives the result between $L/4$ to $L/2$ and Second one gives the ground state energy.

$P_1 = |\psi_0|^2 dq$

$|\phi(q)|^2 = \delta(q - q_0)$.

$E \geq E_0$

ψ_0 collapses to ϕ_0