01.09.2011

Angular momentum operators :-

$$L_x = i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right)$$

$$L_y = i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right)$$

$$L_z = i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

$$\left\{ \begin{aligned} L_+ &= L_x + iL_y \\ L_- &= L_x - iL_y \end{aligned} \right\}$$

$$\hat{L}^2 = L_x^2 + L_y^2 + L_z^2 .$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(r)$$

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{2mr^2}\hat{L}^2\psi + V(r)\psi$$

$$[\hat{H}, L_x] = [\hat{H}, L_y] = [\hat{H}, L_z] = [\hat{H}, \hat{L}^2] = 0 .$$

$$[\hat{L}^2, L_x] = [\hat{L}^2, L_y] = [\hat{L}^2, L_z] = 0 .$$

$\Rightarrow$ Eigenstates of $\hat{H}$ can be taken to be eigenstates of $\hat{L}^2$ and $\hat{L}_z$.

$$\left\{ \begin{aligned} L_z \psi_{\lambda,m} &= m\hbar\, \psi_{\lambda,m} \\ \hat{L}^2 \psi_{\lambda,m} &= \lambda\hbar^2 \psi_{\lambda,m} \end{aligned} \right\}$$

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{\lambda\hbar^2}{2mr^2}\psi + V(r)\psi$$

$\Rightarrow$ Solutions with energy $E_{n,l}$ as $\psi_{n,\lambda,m}$

What we proved based on the L_x, L_y, L_z commutation relations and the positivity of the norm —

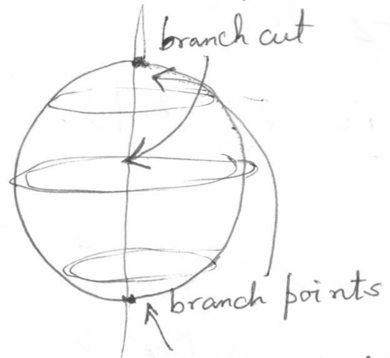
$$\lambda = l(l+1)$$

$l =$ integer or integer $+ 1/2$.

$m = -l, -l+1, \ldots, l-1, l.$

But, <u>we proved more</u> $\Rightarrow$ $L_z = -i\hbar \frac{\partial}{\partial \phi}$

($m \rightarrow$ integer)



If $\psi = e^{im\phi} f(r, \theta)$

and m = half-integer

(*) If we want ψ to be non-singular, m = integer <u>only</u>

$\psi(\theta, \phi)$ has branch points at $\theta = 0, \pi$

$$\psi_{n,\lambda,m} = e^{im\phi} f_{n,\lambda,m}(r, \theta)$$

$$\hat{L}^2 \psi = \hbar^2 \left\{ -\frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} - \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \psi}{\partial\theta}\right) \right\}$$

$$\Rightarrow \cancel{\hbar^2} \left\{ \frac{m^2}{\sin^2\theta} f_{n,\lambda,m} - \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f_{n,\lambda,m}}{\partial\theta}\right) \right\}$$

$$= \cancel{\hbar^2}\, l(l+1) f_{n,\lambda,m}$$

$x = \cos\theta$

$$\boxed{-\frac{\partial}{\partial x}\left((1-x^2) \frac{\partial f_{n,\lambda,m}}{\partial x}\right) + \frac{m^2}{1-x^2} f_{n,\lambda,m} = l(l+1) f_{n,\lambda,m}}$$

Solutions : $P_l^m(x) \times$ [Associated Legendre Polynomials]

$g_{n,\lambda}(r)$

<u>Legendre polynomials</u>

$$P_l(x) = P_l^{m=0}(x)$$

$$-\frac{d}{dx}\left((1-x^2) \frac{dP_l(x)}{dx}\right) = l(l+1) P_l(x).$$

$$Y_{lm}(\theta,\phi) = e^{im\phi} P_l^m(\cancel{\theta}\cos\theta)$$

$$L_z\, Y_{lm}(\theta,\phi) = m\hbar\, Y_{lm}(\theta,\phi)$$

$$\hat{L}^2\, Y_{lm}(\theta,\phi) = \hbar^2 l(l+1)\, Y_{lm}(\theta,\phi)$$

$$\psi_{n,l,m}(r,\theta,\phi) = g_{n\lambda}(r)\, Y_{lm}(\theta,\phi)$$

[g is a solution to

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg}{dr}\right) + \frac{\hbar^2}{2mr^2} l(l+1) g(r) = \cancel{E_{n,l}\,g(r)}\, E_{n,l}\, g(r)$$]

We know that, $\int \psi_{n,l,m}(r,\theta,\phi)^* \psi_{n',l',m'}(r,\theta,\phi)\, r^2 \sin\theta\, d\theta\, d\phi\, dr$

$$= \delta_{nn'}\delta_{ll'}\delta_{mm'}$$

$$= \int g_{n,\lambda}(r)^* g_{n',\lambda'}(r)\, r^2\, dr \int \sin\theta\, d\theta\, d\phi\, Y_{lm}^*(\theta,\phi)\, Y_{lm}(\theta,\phi)$$

Normalize Y_{lm}'s such that

$$\int \sin\theta\, d\theta\, d\phi \oint \left(Y_{lm}(\theta,\phi)\right)^* Y_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$L_+ = L_x + iL_y = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_- = L_x - iL_y = \hbar e^{-i\phi}\left(-\partial/\partial\theta + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_+ \psi_{\lambda, m} = \hbar \sqrt{l(l+1) - m(m+1)}\ \psi_{\lambda, m+1}$$

$$L_- \psi_{\lambda, m} = \hbar \sqrt{l(l+1) - m(m-1)}\ \psi_{\lambda, m-1}$$

$$L_+ \psi_{lm}(\theta, \phi) = \hbar \sqrt{l(l+1) - m(m+1)}\ Y_{l, m+1}(\theta, \phi)$$

$$L_- \psi_{lm}(\theta, \phi) = \hbar \sqrt{l(l+1) - m(m-1)}\ Y_{l, m-1}(\theta, \phi)$$

$$Y_{l,0} = P_l(\cos\theta)$$

$$Y_{l,1}(\theta, \phi) = \cancel{\hbar\sqrt{l(l+1)}}\ \frac{1}{\hbar\sqrt{l(l+1)}} \left\{ \hbar e^{i\phi} \left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi} \right) P_l \right\}$$

$$= \frac{1}{\cancel{\hbar}\sqrt{l(l+1)}}\ e^{i\phi}\ \frac{\partial P_l(\cos\theta)}{\partial\theta}$$

$$\underline{e^{i\phi}\ P_l^{1}(\cos\theta)}$$

$$\therefore\quad P_l^{1}(\cos\theta) = \frac{1}{\sqrt{l(l+1)}} \frac{\partial P_l(\cos\theta)}{\partial\theta}$$

Notation

$$\psi_{n,l,m} \rightarrow |n, l, m\rangle$$

$$\langle \psi_{n,l,m} | \psi_{n',l',m'} \rangle = \langle n, l, m | n', l', m' \rangle$$

$$L_z |n, l, m\rangle = m\hbar |n, l, m\rangle$$

$$L_+ |n, l, m\rangle = \hbar \sqrt{l(l+1) - m(m+1)}\ |n, l, m+1\rangle$$

$$L_- |n, l, m\rangle = \hbar \sqrt{l(l+1) - m(m-1)}\ |n, l, m-1\rangle$$

General form

some numbers

$$\hat{L}_i |n, l, m\rangle = \hbar \sum_{m'=-l}^{l} D^{l,i}_{m', m}\ |n, l, m'\rangle$$

$$\left\{ \begin{array}{l} i = x, y, z \\ \text{or } +, -, z \end{array} \right\}$$

Commutation relations

$$[L_i, L_j] = i\hbar \sum_k f_{ijk} L_k$$

if $i, j \longrightarrow x, y, z$, $f_{ijk} = \epsilon_{ijk}$

$+, -, z$ basis $\Rightarrow i f_{z++} = 1$

$$\begin{cases} [L_z, L_+] = \hbar L_+ \\ [L_z, L_-] = -\hbar L_- \end{cases} \Rightarrow f_{z++} = -i$$

$$\hat{L}_i \hat{L}_j |n, l, m\rangle = \hat{L}_i \hbar \sum_{m'=-l}^{l} D^{l, j}_{m', m} |n, l, m'\rangle$$

$$= \hbar \sum_{m'=-l}^{l} D^{l, j}_{m', m} \hbar \sum_{m''=-l}^{l} D^{l, i}_{m'', m'} |n, l, m''\rangle$$

$$= \hbar^2 \sum_{m''=-l}^{l} \sum_{m'=-l}^{l} D^{l, i}_{m'', m'} D^{l, j}_{m', m} |n, l, m''\rangle$$

$$(D^{l, i} D^{l, j})_{m'', m} |n, l, m''\rangle$$

Apply $\hat{L}_j \hat{L}_i \longrightarrow (D^{l, j} D^{l, i})_{m'', m}$

So, $[L_i, L_j] |n, l, m\rangle = \hbar^2 \sum_{m''=-l}^{l} ([D^{l, i}, D^{l, j}])_{m'' m} |n, l, m''\rangle$

$$i\hbar \sum_k f_{ijk} L_k |n, l, m\rangle$$

$$= i\hbar \sum_k f_{ijk} \hbar \sum_{m''=-l}^{l} D^{l, k}_{m'', m} |n, l, m''\rangle$$

$$= \hbar^2 \sum_{m''=-l}^{l} i \sum_k f_{ijk} D^{l, k}_{m'', m} |n, l, m''\rangle$$

$$\Rightarrow ([D^{l, i}, D^{l, j}])_{m'', m} = i \sum_k f_{ijk} D^{l, k}_{m'', m}$$

$$\left[D^{l,i}, D^{l,j}\right] = i \sum_{k} f_{ijk} D^{l,k}$$

The matrices $D^{l,i}$ form representation of the angular momentum algebra.

<u>If</u> $l = 0$, $\quad D^{l,i} = 0 \;\; \forall\, i$

(Trivial representation)

For $l = \frac{1}{2}$, $\quad |n, \frac{1}{2}, \frac{1}{2}\rangle$ & $|n, \frac{1}{2}, -\frac{1}{2}\rangle$

$$L_z |n, \tfrac{1}{2}, \tfrac{1}{2}\rangle = \frac{\hbar}{2} |n, \tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$L_z |n, \tfrac{1}{2}, -\tfrac{1}{2}\rangle = -\frac{\hbar}{2} |n, \tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$D^{1/2, z} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$$L_+ |n, \tfrac{1}{2}, \tfrac{1}{2}\rangle = 0$$

$$L_+ |n, \tfrac{1}{2}, -\tfrac{1}{2}\rangle = \hbar \sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right) + \tfrac{1}{2}\left(1 - \tfrac{1}{2}\right)}\; |n, \tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$= \hbar\, |n, \tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$D^{1/2, +} = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$D^{1/2, -} = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

<u>Exercise</u> (Do this for $l = 1$ and verify the commutation relations)

02.09.2011

$$\hat{L}_i \,|n, l, m\rangle$$

$$= \hbar \sum_{m'=-l}^{l} D^{l,i}_{m',m} \,|n, l, m\rangle$$

$$[\hat{L}_i, \hat{L}_j] = i\hbar \sum_k f_{ijk} \hat{L}_k$$

$$\Rightarrow [D^{l,i}, D^{l,j}] = i \sum_k f_{ijk} D^{l,k}.$$ (Proved earlier)

$D^{l,i}$ form a representation of the algebra $[\hat{L}_i, \hat{L}_j] = i\hbar \sum_k f_{ijk} \hat{L}_k$.

Given any set of matrices $R^i_{\alpha\beta}$ such that $[R^i, R^j] = i \sum_k f_{ijk} R^k$ we say that R^i's form a representation.

Other representations (?)

$$R^i = \begin{pmatrix} D^{l,i} & 0 \\ 0 & D^{l',i} \end{pmatrix}$$

$$R^i R^j = \begin{pmatrix} D^{l,i} D^{l,j} & 0 \\ 0 & D^{l',i} D^{l',j} \end{pmatrix}$$

$$R^j R^i = \begin{pmatrix} D^{l,j} D^{l,i} & 0 \\ 0 & D^{l',j} D^{l',i} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} AC & 0 \\ 0 & BD \end{pmatrix}}$$

$$\therefore R^{l,i} R^{l,j} - R^{l,j} R^{l,i} = \begin{pmatrix} [D^{l,i}, D^{l,j}] & 0 \\ 0 & [D^{l',i}, D^{l',j}] \end{pmatrix}$$

$$= \begin{pmatrix} i \sum_k f_{ijk} D^{l,k} & 0 \\ 0 & i \sum_k f_{ijk} D^{l',k} \end{pmatrix}.$$

$$= i \sum_k f_{ijk} \begin{pmatrix} D^{l,k} & 0 \\ 0 & D^{l',k} \end{pmatrix}$$

$$\begin{pmatrix} A+B & 0 \\ 0 & C+D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}$$

* Trivial way to construct new representations.

[BLOCK DIAGONAL FORM]

Irreducible representations

A representation means

$$\hat{L}_i \longrightarrow R^i \text{ acting on some vector space.}$$

We say a representation is irreducible if we can't form a representation by considering the action of R^i on a subspace of the vector space.

$$R^i = \begin{pmatrix} D^{l,i} & 0 \\ 0 & D^{l',i} \end{pmatrix}$$

$(2l+1+2l'+1) \times (2l+1+2l'+1)$

Dimension of vector space

{This is reducible}

$$\tilde{R}^i = U R^i U^{-1}. \qquad R^i = \begin{pmatrix} D^{l,i} & 0 \\ 0 & D^{l',i} \end{pmatrix}$$

$U \longrightarrow$ arbitrary unitary matrix

$$\left.\begin{aligned} \tilde{R}^i \tilde{R}^j &= U R^i R^j U^{-1} \\ \tilde{R}^j \tilde{R}^i &= U R^j R^i U^{-1} \end{aligned}\right\} \text{This is also a representation.}$$

$$\therefore [\tilde{R}^i, \tilde{R}^j] = U [R^i, R^j] U^{-1}$$

$$= i \sum_k f_{ijk} \tilde{R}^k.$$

Previously, a subspace of the form $\begin{pmatrix} a_1 \\ \vdots \\ a_{2l+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ could be taken

Now, we can choose $U \begin{pmatrix} a_1 \\ \vdots \\ a_{2l+1} \\ \vdots \\ 0 \end{pmatrix}$ as the subspace.

$\therefore \tilde{R}$ is also a reducible representation.

L_+, L_- ——— $\to D^{l's}$ form an irreducible representation.

$$[L_i, A] = B_i$$
$$[L_i, B_j] = C_{ij}$$

Consider instead a set of operators $A_1, \ldots, A_n$ such that

$$[L_i, \hat{A}_s] = \hbar \sum_{s'=1}^{n} M^i_{s's} \hat{A}_{s'}$$

$$[\hat{L}_i, [\hat{L}_j, \hat{A}_s]] = [\hat{L}_i, \hbar \sum_{s'} M^j_{s's} \hat{A}_{s'}]$$

numbers

$$= \hbar \sum_{s'} M^j_{s's} [\hat{L}_i, A_{s'}]$$

$$= \hbar^2 \sum_{s'} M^j_{s's} \sum_{s''} M^i_{s''s'} \hat{A}_{s''}$$

$$= \hbar^2 \sum_{s''} (M^i M^j)_{s''s} \hat{A}_{s''}$$

$$\therefore [L_i, [L_j, A_s]] - [L_j, [L_i, A_s]]$$
$$= \hbar^2 \sum_{s''} [M^i, M^j]_{s''s} A_{s''}$$
$$= [[L_i, L_j], A_s] \quad \text{(By Jacobi Identity}$$

$$= i\hbar \sum_k f_{ijk} [L_k, A_s]$$

$$= i\hbar \sum_k f_{ijk} \sum_{s''} \hbar M^k_{s''s} A_{s''}$$

What do we get ?

$$\sum_{s''} [M^i, M^j]_{s''s} A_{s''}$$

$$= i \sum_k f_{ijk} \sum_{s''} M^k_{s''s} A_{s''}$$

has to be true for all s + $A_{s''}$ are all independent operators. So, we obtain,

$$\Rightarrow \quad [M^i, M^j] = i \sum_k f_{ijk} M^k$$

s'' & s sum from 1 to n.

$A_1, \ldots, A_n$ form an irreducible representation if M^i's form an irreducible representation.

$(\hat{x}, \hat{y}, \hat{z})$ form a representation

$$[L_z, \hat{x}] = + i\hbar \hat{y}$$

$$[L_z, \hat{y}] = - i\hbar \hat{x}$$

$$[L_z, z] = 0.$$

$$M^z = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \xrightarrow{\text{irreducible}} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

$l = 1$ representation

$$\left.\begin{aligned} x^{+} &= \frac{1}{\sqrt{2}}(x + iy) \\ x^{-} &= \frac{1}{\sqrt{2}}(x - iy) \end{aligned}\right\} \text{(Change of basis)}$$

$$[L_z, x^{+}] = \frac{1}{\sqrt{2}}(i\hbar y + \hbar x)$$

$$= \frac{1}{\sqrt{2}}\hbar(x + iy)$$

$$= \hbar x^{+}.$$

$$[L_z, x^{-}] = -\hbar x^{-}, \quad [L_z, z] = 0.$$

(can be easily checked!)

$$M^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{x}^2,\ \hat{y}^2,\ \hat{z}^2,\ (\hat{x}\hat{y} + \hat{y}\hat{x}),\ \ldots$$

$$\Downarrow$$

$$(\hat{x}^{+})^2,\ x^{+}z,\ (x^{+}x^{-} + x^{-}x^{+}),\ z^2,\ x^{-}z,\ (\hat{x}^{-})^2$$

$$\left\{\begin{aligned} [L_z, (x^{+})^2] &= 2\hbar (x^{+})^2. \\ [L_z, x^{+}z] &= \hbar x^{+} z. \\ [L_z, x^{+}x^{-} + x^{-}x^{+}] &= 0. \\ [L_z, z^2] &= 0. \\ [L_z, x^{-}z] &= -\hbar x^{-} z. \\ [L_z, (x^{-})^2] &= -2\hbar (x^{-})^2. \end{aligned}\right\}$$

$l = 2$ & $l = 0$

$$\begin{pmatrix} +2 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

(2 times reducible)

$$\boxed{D^{l=2} \oplus D^{l=0}}$$

Suppose, L_z eigenvalues are —

~~n_1 eigenvalue — $10\hbar$~~

~~n_2 eigenvalue —~~

$$\left.\begin{array}{lcl} n_5 & \longrightarrow & -5\hbar \\ n_4 & \longrightarrow & -4\hbar \\ n_3 & \longrightarrow & -3\hbar \\ n_2 & \longrightarrow & -2\hbar \\ n_1 & \longrightarrow & -\hbar \\ n_0 & \longrightarrow & 0 \\ n_1 & \longrightarrow & \hbar \\ \vdots & & \vdots \\ & & 5\hbar \end{array}\right\}$$

n_0 times reducible

$l = 5 \xrightarrow{\text{appears}} n_5$ times

$l = 4 \xrightarrow{\text{appears}} (n_4 - n_5)$,,

$l = 3 \xrightarrow{\text{appears}} (n_3 - n_4)$,,

$l = 2 \xrightarrow{\text{appears}} (n_2 - n_3)$,,

$l = 1 \xrightarrow{\text{appears}} (n_1 - n_2)$,,

$l = 0 \xrightarrow{\text{appears}} (n_0 - n_1)$,,

$\boxed{n_0}$

(*) Speciality of angular momentum algebra. (SU(2) rotation group.)

07.09.2011

Tutorial :- (Prof. A. Sen)

$$[L_i, L_j] = i\hbar \sum_k f_{ijk} L_k .$$

Representation → Collection of matrices $R^i_{s's}$ such that

$$[R^i, R^j] = i \sum_k f_{ijk} R^k$$

$$\left.\begin{array}{l} [L_i, A_s] = \hbar \sum_{s'=1}^{n} R^i_{s's} A_{s'} \\ [R^i, R^j] = i \sum_k f_{ijk} R^k \end{array}\right\} (\checkmark)$$

(R^i) $\quad B_1 \rightarrow$ linear combination of $A_1, \dots A_n$

$\vdots$

$B_m \qquad m < n$

If (R^i) is a representation and we can find a smaller set of operators under which the operation is closed, (R^i) is reducible.

$$R_0^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$R^3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longleftrightarrow L^1, L^2, L^3 \checkmark$$

But (R^i)'s form a reducible representation.

Consider the matrices

$$\left\{ \begin{aligned} U R^x U^{-1} &= \tilde{R}^x \\ U R^y U^{-1} &= \tilde{R}^y \\ U R^z U^{-1} &= \tilde{R}^z \end{aligned} \right\} \qquad \text{Diagonalize } R_0^z \quad R^z = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

$$D^l \qquad R^{l,i} = \begin{pmatrix} D^{l_1,i} & & \\ & D^{l_2,i} & \\ & & D^{l_3,i} \end{pmatrix} \quad \text{(By ~~usio~~ using some } U\text{)}.$$

(Collection of irreducible representations).

$$R^z = \begin{pmatrix} l_1 & & & & & & \\ & l_1 - 1 & & & & & \\ & & \ddots & & & & \\ & & & -l_1 & & & \\ & & & & l_2 & & \\ & & & & & \ddots & \\ & & & & & & -l_2 \end{pmatrix} \qquad U = \begin{pmatrix} U_1 & & \\ & U_2 & \\ & & \ddots \end{pmatrix}$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \to \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$$

ordering of eigenva

R_z : n_2 eigenvalues of 2

n_1 " " 1

n_0 " " 0

n_1 " " -1

n_2 " " -2

$l = 2$ max.

$l = 2 \Rightarrow n_2$ times

$l = 1 \longrightarrow (n_1 - n_2)$ times

$l = 0 \longrightarrow (n_0 - n_1)$ "

$$R^z = \begin{pmatrix} \underbrace{D^{2,i} \quad D^{2i} \quad \cdots}_{n_2} & & \\ & \underbrace{D^{1,i} \quad D^{1,i} \quad \cdots}_{(n_1 - n_2)} & \\ & & \underbrace{D^{0,i} \quad D^{0,i} \quad \cdots}_{(n_0 - n_1)} \end{pmatrix}$$

Unitary representations :-

(*) Hermitian operators go to Hermitian matrices.

R^x, R^y, R^z are hermitian matrices.

$(R^z)^\dagger = R^z$, $(R^+)^\dagger = R^-$.
(not hermitian).

$$[L_i, A_s] = \hbar \sum_{s'=1}^{n} R^i_{s's} A_{s'}$$

$$R_i = W (R^i)_{standard} W^{-1}$$

$$= W (R^i)_{standard} W^\dagger$$

$$[L_i, A_s] = \hbar \sum_{s'} \left(W R^i_{standard} W^\dagger\right)_{s's} A_{s'}$$

$$= \hbar \sum_{s', s'', s'''} W_{s's''} \left(R^i_{standard}\right)_{s''s'''} (W^\dagger)_{s''',s} A_{s'}$$

Define $B_{s''} = \sum_{s'} W_{s's''} A_{s'}$

$$[L_i, B_r] = \sum_s W_{sr} [L_i, A_s]$$

$$= \hbar \sum_{s, s', s'', s'''} W_{sr} W_{s's''} \left(R^i_{standard}\right)_{s''s'''} (W^\dagger)_{s''' s} A_{s'}$$

Note that $\boxed{\sum_s (W^\dagger)_{s''' s} W_{sr} = \delta_{rs''}}$

$$= \hbar \sum_{s', s''} W_{s's''} \left(R^i_{standard}\right)_{s'' r} A_{s'}$$

$$= \hbar \sum_{s''} B_{s''} \left(R^i_{standard}\right)_{s'' r}$$

$$[L_i, B_r] = \hbar \sum_{s''} B_{s''} \left(R^i_{standard}\right)_{s'' r}$$

R^z, $R^{\pm}$

$$\left\{\begin{array}{l} [R^z, R^+] = R^+ \\ [R^z, R^-] = -R^- \\ [R^+, R^-] = 2R_z \end{array}\right\}$$ matrices

$\left\{ R^z \text{ is a Hermitian matrix.} \right\}$

- One can always diagonalize R^z.

→ Choose a basis in which R^z is diagonal.

Define $R^2 = R_x^2 + R_y^2 + R_z^2$

$$= R_z^2 + \frac{1}{2}(R_+ R_- + R_- R_+).$$

(Hermitian matrix)

$[R^2, R_z] = 0$. (can be proved).

* We can simultaneously diagonalize R^2 and R_z by same basis transformation.

$$\left\{\begin{array}{l} R_z v_m = m\, v_m \\ R^2 v_m = \lambda\, v_m \end{array}\right\}$$

$R_+ v_m = \sqrt{\lambda - m(m+1)}\; v_{m+1}$

$\lambda = l(l+1) \longrightarrow$ as proved earlier

$(v_1, v_2) = v_1^\dagger v_2$ $\quad$ $(R_+ v_1, v_2) = (v_1, R_- v_2)$

{These matrices are just the D^l's.}

$\lambda = l(l+1)$

~~In this subspace~~

$$\left\{\begin{array}{l} R_+ v_{l,m} \\ = \sqrt{l(l+1) - m(m+1)}\; v_{l,m+1} \\ = \sum_{m'} D^{l,+}_{m',m}\, v_{l,m'} \end{array}\right\}$$

$$\left\{\begin{array}{l} R_- v_{l,m} \\ = \sqrt{l(l+1) - m(m-1)}\; v_{l,m-1} \\ = \sum_{m'} D^{l,-}_{m',m}\, v_{l,m'} \end{array}\right\}$$

R_x, R_y, R_z

$$R_z : \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & -1 \end{pmatrix}$$

u_0, v_0

$u_0^\dagger v_0 = 0$, $u_0^\dagger u_0 = 1$

$= v_0^\dagger v_0$

$R_z u_0 = 0$, $R_z v_0 = 0$

$$R^2 : \begin{pmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 2 \end{pmatrix}$$

$R^2 u_0 = 2u_0$, $R^2 v_0 = 2v_0$

u_0 & v_0 are 2-d vectors

Define $u_1 = \dfrac{1}{\sqrt{1(1+1) - 0(0+1)}} R_+ u_0$

$$u_{-1} = \frac{1}{\sqrt{1(1+1) - 0(0-1)}} R_- u_0$$

$$\left[\begin{aligned} v_1 &= \frac{1}{\sqrt{2}} R_+ v_0 \\ v_{-1} &= \frac{1}{\sqrt{2}} R_- v_0 \end{aligned} \right]$$

$R_i \, (u_{-1}, u_0, u_1, v_{-1}, v_0, v_1)$

$$\begin{bmatrix} u_{-1}, u_0, u_1 \\ v_{-1}, v_0, v_1 \end{bmatrix} = \begin{pmatrix} D^{1,i} & \\ & D^{1,i} \end{pmatrix} (u_{-1}, u_0, u_1, v_{-1}, v_0, v_1)$$

$$\left\{ \begin{aligned} R_i u_m &= \sum_{m'} D^{l,i}_{m',m} u_{m'} \\ R_i v_m &= \sum_{m'} D^{l,i}_{m',m} v_{m'} \ldots \end{aligned} \right\}$$

Choose a new basis with basis vectors

$$(u_1, u_0, u_{-1}, v_1, v_0, v_{-1})$$

$$\begin{pmatrix} D^{1,i} & 0 \\ 0 & D^{1,i} \end{pmatrix} u$$

Consider x^i, p^j bilinears

$$9 = 5 + 3 + 1$$

$x^+, x^-, x^z, p^+, p^-, p_z$

$$[L_z, x^+ p^+] = 2\hbar\, x^+ p^+$$
$$[L_z, x^+ p^z] = \hbar\, x^+ p^z$$
$$[L_z, z p^+] = \hbar\, z p^+$$
$$[L_z, x^+ p^-] = 0$$
$$[L_z, x^- p^+] = 0$$
$$[L_z, z p^z] = 0$$

$l = 2$

$l = 1$

$l = 0$

$\boxed{l=1}$

$(x^+ p^- + x^- p^+ + z p^z)$

$(x^+ p^z + z p^+)$

$(x^- p^z + z p^-)$

$L_z = 0$ of $l = 1$ state

$$\text{const.}\,(x^+ p^- - x^- p^+) \longrightarrow L_z = 0 \qquad L^+ \quad x^+ p^- + z$$

$$= L_z$$

$L_z = 1$ component of $l = 1$ state

$$\text{Const.}\,(x^+ p^z - z p^+) \longrightarrow$$

$L_z = -1$ component of $l = 1$ state

$$\text{Const.}\,(x^- p^z - z p^-) \longrightarrow$$

$l = 0 \longrightarrow L_z = 0$ state (rotationally symmetric)

P. T. O.

$A_s \qquad s = \frac{1}{2}, -\frac{1}{2}$

$$\left\{ \begin{aligned} [L_i, A_s] &= \hbar \sum_{s'=\frac{1}{2},-\frac{1}{2}} D^{\frac{1}{2},i}_{s's} A_{s'} \\ [L_i, B_s] &= \hbar \sum_{s'=\frac{1}{2},-\frac{1}{2}} D^{\frac{1}{2},i}_{s',s} B_{s'} \end{aligned} \right\}$$

Construct $A_s B_r$

$$\boxed{\begin{aligned} s &= \tfrac{1}{2}, -\tfrac{1}{2} \\ r &= \tfrac{1}{2}, -\tfrac{1}{2} \end{aligned}}$$

4-dimensional representation

Find the D^{l}'s.

$$[L_i, A_s B_r] = \hbar \sum_{s' r'} R^{i}_{s'r' : sr} A_{s'} B_{r'}$$

Find the particular linear combinations which transform according to the D^{l}'s.

08.09.2011 — Quantum Mechanics :-

Results of problem given in previous tutorial

$$l = 1 \left\{ \begin{aligned} m &= 1 \qquad A_{\frac{1}{2}} B_{\frac{1}{2}} \\ m &= 0 \qquad \frac{1}{\sqrt{2}} \left(A_{\frac{1}{2}} B_{-\frac{1}{2}} + A_{-\frac{1}{2}} B_{\frac{1}{2}} \right) \\ m &= -1 \qquad A_{-\frac{1}{2}} B_{-\frac{1}{2}} \end{aligned} \right.$$

$$l = 0, \quad m = 0 \qquad \frac{1}{\sqrt{2}} \left(A_{\frac{1}{2}} B_{-\frac{1}{2}} - A_{-\frac{1}{2}} B_{\frac{1}{2}} \right)$$

$$\sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m}} W^{l \; l_1, l_2}_{m \; m_1, m_2} A_{l_1 m_1} B_{l_2 m_2} = Q_{l,m}$$

such that these operators transform by the matrix D^{l}.

Find the W's for

$$A_{l_1 m_1} \; B_{l_2 m_2} = \sum_{l,m} V^{l\, l_1\, l_2}_{m\, m_1\, m_2} \; \mathcal{B}_{l,m}.$$

(one can invert the relations)

For the earlier example,

$$A_{\frac{1}{2}} \; B_{\frac{1}{2}} = \mathcal{B}_{1,1}$$

$$\frac{1}{\sqrt{2}}\left(A_{\frac{1}{2}} \; B_{-\frac{1}{2}} + A_{-\frac{1}{2}} \; B_{\frac{1}{2}} \right) = \mathcal{B}_{1,0}$$

and - - - - .

$$A_{l_1 m_1} \; |n_2 \; l_2 \; m_2\rangle$$

$$\hat{L}_i \, A_{l_1 m_1} \; |n_2 \; l_2 \; m_2\rangle = [\hat{L}_i, A_{l_1 m_1}] \; |n_2 \; l_2 \; m_2\rangle + A_{l_1 m_1} \; \hat{L}_i \; |n_2 \; l_2 \; m_2\rangle$$

$$= \hbar \sum_{m_1'} D^{l_1, i}_{m_1', m_1} \; A_{l_1 m_1'} \; |n_2 \; l_2 \; m_2\rangle + \hbar \sum_{m_2'} D^{l_2, i}_{m_2', m_2} \; A_{l_1 m_2'} \; |n_2 \, l_2 \, m_2'\rangle$$

$$= \hbar \sum_{m_1', m_2'} \underbrace{\left(D^{l_1, i}_{m_1', m_1} \; \delta_{m_2' m_2} + \delta_{m_1' m_1} \; D^{l_2, i}_{m_2' m_2} \right)} \; A_{l_1 m_1'} \; |n_2, l_2, m_2'\rangle$$

Compare with $\left[L_i, \; A_{l_1 m_1} B_{l_2 m_2} \right]$

$$= [L_i, \; A_{l_1 m_1}] \; B_{l_2 m_2} + A_{l_1 m_1} \; [L_i, \; B_{l_2 m_2}]$$

$$= \hbar \sum_{m_1', m_2'} \underbrace{\left(D^{l_1, i}_{m_1' m_1} \; \delta_{m_2' m_2} + \delta_{m_1' m_1} \; D^{l_2, i}_{m_2' m_2} \right)}^{*} \; A_{l_1 m_1'} \; B_{l_2 m_2'}$$

(*) The matrices are the same.

Action of $\hat{L}_i$ ~~acting~~ on product of two operators is similar to the action on $A|n \; l \; m\rangle$.

$$[L_i, A_{l,m}] = \sum_{m'} D^{l,i}_{m'm} A_{l,m'}$$

Take $A_{lm} |n, l=0, m=0\rangle$ (spherically symmetric state)

$L_i A_{lm} |n, l=0, m=0\rangle$

$= [L_i, A_{lm}] |n, l=0, m=0\rangle$

$= \hbar \sum_{m'} D^{l,i}_{m'm} \underbrace{\left(A_{lm'} |n, l=0, m=0\rangle\right)}_{\text{state with angular momentum numbers } \boxed{l, m}}$

— x — x — x — x — x — x — x —

Wigner - Eckart Theorem :-

$\langle n, l, m | A_{l_1 m_1} | n_2, l_2, m_2 \rangle$ → reduced matrix elements

$= \underbrace{C^{l\, l_1\, l_2}_{m\, m_1\, m_2}}_{\text{Clebsch-Gordon coefficients}} \underbrace{\langle n, l \| A_{l_1} \| n_2, l_2 \rangle}_{\text{some function of } n, n_2, l_1, l_2, l \text{ and the operator } A}$ symbol

~~Cle~~ Clebsch- ~~Jordan~~ Gordon coefficients

Proof :- $\langle n, l, m | A_{l_1 m_1} | n_2\, l_2\, m_2 \rangle$

$|n\, l\, m\rangle = N_{lm} (L_+)^{m+l} |n\; l\; -l\rangle$

normalisation

$= N^*_{lm} \langle n, l, -l | (L_-)^{m+l} A_{l_1 m_1} |n_2\, l_2\, m_2\rangle$

$m = m_1 + m_2$

Check with L_z

$\left\{ C^{l\; l_1\; l_2}_{m\; m_1\; m_2} = 0 \text{ unless } m = m_1 + m_2 \right\}$

Take $m = m_1 + m_2$ right from the beginning.

$$= N^*_{l,m} \langle n, l, -l \,|\, (L_-)^{m+l-1} \left\{ [L_-, A_{l_1 m_1}] \,|n_2, l_2, m_2\rangle + A_{l m_1} L_- \,|n_2, l_2, m_2\rangle \right\}$$

$$\doteq N^*_{l m} \langle n, l, -l \,|\, L_-^{m+l-1} \left\{ \hbar\sqrt{l_1(l+1) - m_1(m_1-1)}\; A_{l_1, m_1-1} \,|n_2, l_2, m_2\rangle + \hbar\sqrt{l_2(l_2+1) - m_2(m_2-1)}\; A_{l m_1} \,|n_2, l_2, m_2-1\rangle \right\}$$

Repeat this process . . . . .

$$= \boxed{\langle n \, l, -l| \sum_{\substack{m_1', m_2' \\ m_1'+m_2' = m_1+m_2 - (m+l) = -l.}} K(l, m, l_1, m_1, l_2, m_2, m_1', m_2') \, A_{l_1 m_1'} \,|n_2, l_2, m_2'\rangle}$$

Halfway through!

$$|n_2, l_2, m_2'\rangle = \tilde{N}_{l_2 m_2'} (L_+)^{m_2'+l_2} \,|n_2, l_2, -l_2\rangle$$

$$\langle n, l, -l \,|\, A_{l_1, m_1'} (L_+)^{m_2'+l_2} \,|n_2, l_2, -l_2\rangle$$

$$= \left([A_{l_1, m_1'}, L_+] + L_+ A_{l_1 m_1'} \right) L_+^{m_2'+l_2-1}$$

$$= -\hbar\sqrt{l_1(l_1+1) - m_1'(m_1'+1)}\; A_{l_1, m_1'+1} + 0.$$

Keep doing this . . . . .

$$A_{l_1, m_1'+m_2'+l_2} = A_{l_1, l_2-l}$$

At the end

$$\propto \langle n, l, -l \,|\, A_{l_1, l_2-l} \,|\, n_2, l_2, -l_2\rangle.$$

$$\times \sum_{m_1', m_2'} f(l, m, l_1, m_1, l_2, m_2, m_1', m_2')$$

$$\doteq C^{l\, l_1\, l_2}_{m\, m_1\, m_2} \times \underbrace{\langle n\, l \,\|\, A_{l_1} \,\|\, n_2, l_2\rangle}_{\text{reduced matrix element}}$$

$$\frac{C^{111}_{000}}{C^{111}_{01-1}} = ?$$

$$\frac{\langle n,1,0 | A_{1,0} | n_2,1,0\rangle}{\langle n,1,0 | A_{1,1} | n_2,1,-1\rangle}$$

$-[L_-, A_{1,1}] \quad |n_2,1,-1\rangle$

$+L_-A_{11} = \frac{1}{\sqrt{2}} L_- |n_2,1,0\rangle$

$$\frac{\langle n,1,0 | A_{1,0} | n_2,1,0\rangle}{\langle n,1,0 | A_{1,1} | n_2,1,-1\rangle}$$

$\langle n,1,0 / A_{1,0}, n_2, 1, 0\rangle$

$\langle n,1,0 / A_{1,0} | n_2,1,0\rangle$

$= \langle n,1,-1 / L_- \frac{1}{\sqrt{2}} A_{1,0} | n_2,1,0\rangle$

$= \frac{1}{\sqrt{2}} \langle n,1,-1 / \left\{ \frac{\sqrt{2}}{\sqrt{2}} A_{1,-1} / n_2,1,0\rangle + A_{1,0} \cdot \frac{\sqrt{2}\,1}{\sqrt{2}} | n_2,1,-1\rangle \right\}$

write as $\cancel{\sqrt{2}} A_{1,-1} \frac{L_+}{\cancel{\sqrt{2}}} |n_2,1,-1\rangle$

$= -\sqrt{2} A_{1,0} \quad |n_2,1,-1\rangle$

$\cancel{-[L_+, A_{1,-1}] + L_+ A_{1,-1}}$

$= 0.$ (Lucky !)

09. 08. 2011

Collection of results

① $\langle n, l, m | A_{l,m_1} | n_2, l_2, m_2\rangle$

W-E Theorem $\quad = C^{l\ l_1\ l_2}_{m\ m_1\ m_2} \langle n, l \| A_{l,l_1} \| n_2, l_2\rangle$

universal coefficients

② If we consider the set of operators

$A_{l_1 m_1} \quad B_{l_2 m_2}$ for $-l_1 \le m_1 \le l_1$ &

$-l_2 \le m_2 \le l_2$

can be rearranged as

$$C_{\ell m} = \sum_{m_1, m_2} W(\ell_1, m_1, \ell_2, m_2; \ell, m) A_{l_1 m_1} B_{l_2 m_2}$$

such that $\left[L_i, C_{\ell m}\right] = \hbar \sum_{m'} D^{\ell, i}_{m' m} C_{l m'}$

(irreducible representation)

$(2\ell_1+1)(2\ell_2+1) \longrightarrow$ operators in the beginning

$= \boxed{\sum_{\ell} (2\ell+1)} \longrightarrow$ operators in the end

• Think of W as a matrix ~~where m_1, m_2 are the rows and l_1, l_2's, the columns.~~

[W is $(2\ell_1+1) \otimes (2\ell_2+1) \times (2\ell_1+1)(2\ell_2+1)$ dimensional matrix.]

$V = W^{-1}$ (say) $= W^{\dagger}$ (W is unitary)

$$\sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} V(\ell, m; \ell_1, m_1, \ell_2, m_2) W(\ell_1, m_1, \ell_2, m_2; \ell', m')$$

$$= \delta_{\ell \ell'} \delta_{mm'} .$$

• Left and right inverses are the same

$$A_{l_1 m_1} B_{l_2 m_2} = \sum_{l, m} V(\ell, m; \ell_1, m_1, l_2, m_2) C_{\ell m} .$$

$$A_{l_1 m_1} | n_2, l_2, m_2 \rangle$$

$$\cancel{=} \sum_{m_1, m_2} W(\ell_1, m_1, \ell_2, m_2; \ell, m) A_{l_1 m_1} | n_2, l_2, m_2 \rangle$$

$$= | n_2, (A), \ell, m \rangle$$

$$A_{l_1 m_1} |n_2, l_2, m_2\rangle = \sum_{l', m'} V(l', m'; l_1, m_1, l_2, m_2) \, |n_2, (A), l', m'\rangle$$

$$\langle n, l, m | A_{l_1, m_1} | n_2, l_2, m_2\rangle$$

$$= \sum_{l', m'} V(l', m'; l_1, m_1, l_2, m_2) \, \langle n, l, m | n_2, (A), l', m'\rangle = \delta_{ll'} \delta_{mm'} f(n, n_2, (A), l, \ldots)$$

[Here $|n_2, (A), l, m\rangle = \sum c_n |n, l, m\rangle$ — c_n depends on n_2 & A — no depen...]

$$\langle n, l, m-1 | n_2, (A), l, m-1\rangle$$

$$= \langle n, l, m-1 | L_- \, n_2, (A), l, m\rangle \frac{1}{\hbar\sqrt{l(l+1) - m(m-1)}}$$

$$= \langle n, l, m | n_2, (A), l, m\rangle \frac{\cancel{\hbar\sqrt{l(l+1)-m(m-1)}}}{\cancel{\hbar\sqrt{l(l+1)-m(m-1)}}}$$

$$\therefore C^{l \, l_1 \, l_2}_{m \, m_1 \, m_2} \langle n, l \| A_{l_1} \| n_2, l_2\rangle$$

$$= V(l, m; l_1, m_1, l_2, m_2) \, f(n, n_2, (A), l)$$

- We can fix the normalisation of $C^{l \, l_1 \, l_2}_{m \, m_1 \, m_2}$ by taking $C^{l \, l_1 \, l_2}_{m \, m_1 \, m_2} = V(l, m; l_1, m_1, l_2, m_2)$

$$= \left(W(l_1, m_1, l_2, m_2; l, m)\right)$$

$l = ?$. We know that $\sum_l (2l+1) = (2l_1+1)(2l_2+1)$

$l_1 + l_2$: once

$l_1 + l_2 - 1$: twice

$l_1 + l_2 - 2$: thrice

$\vdots$

$l_1 \quad l_1 - 1 \quad \ldots \quad -l_1$

$l_2 \quad \ldots \quad -l_2$

Allowed l values

$$l_1 + l_2\,,\quad l_2 + l_1 - 1,\quad \dots\dots,\quad |l_1 - l_2|$$

$$\sum_{l_1-l_2}^{l_1+l_2} (2l+1) = \cancel{2\,\frac{(l_1+l_2)(l_1+l_2+1)}{2}}\ 2l_1(2l_2+1) + (2l_2+1)$$

$$= (2l_1+1)(2l_2+1)\ \checkmark$$

$$\langle n, l, m \,|\, A_{l_1 m_1} \,|\, n_2, l_2, m_2 \rangle$$

See <u>Clebsch - Gordon coefficients</u>

$$= \langle n, l, -l \,|\, A_{l_1,\, l_2 - l} \,|\, n_2, l_2, -l_2 \rangle$$

$$\Rightarrow \quad l_1 \geq |l_2 - l|$$

$$-l_1 \leq l_2 - l \leq l_1$$

Add $l + l_1$ on both sides — Add $l - l_1$ on both sides

$$l \leq l_1 + l_2 \qquad\qquad l \geq l_2 - l_1$$

$$A_{l_1,\, l_2 - l} = \underbrace{[L_+, [L_+, [L_+, \dots\dots}_{l_2 - l + l_1} A_{l_1,-l_1}]]] \times \text{constant}$$

$$\Rightarrow \langle n, l, -l \,|\, A_{l_1,-l_1} \,|\, n_2, l_2, l_1 - l \rangle$$

$$-l_2 \leq l_1 - l \leq l_2 \qquad \text{(constraint)}$$

$$l \leq l_1 + l_2 \qquad\qquad l \geq l_1 - l_2$$

See next page!