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Quantum Mechanics 2 :-

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Identical Particles

$$N \text{ particles} \rightarrow (\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N, \vec{p}_1, \ldots, \vec{p}_N)$$

$$H \equiv H(\vec{r}_i, \vec{p}_i) \qquad i = 1(1)N.$$

For identical particles, $H(\vec{r}_1, \ldots, \vec{r}_N, \vec{p}_1, \ldots, \vec{p}_N)$ remains the same under $(\vec{r}_i, \vec{p}_i) \longrightarrow (\vec{r}_j, \vec{p}_j)$

$$H\psi = E\psi$$

$$\psi_n(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N) \qquad \vec{r}_i \longleftrightarrow \vec{r}_j$$

$\Rightarrow$ all ~~eigenvalu~~ functions with the same eigenvalue

Additional conditions

For bosons $\psi \rightarrow \psi$ under $\vec{r}_i \leftrightarrow \vec{r}_j$

fermions $\rightarrow$ $\psi \longrightarrow -\psi$ under $\vec{r}_i \leftrightarrow \vec{r}_j$

Consider two mutually non-interacting identical bosons moving in a central potential.

$$H = h(\vec{r}_1, \vec{p}_1) + h(\vec{r}_2, \vec{p}_2)$$

Single particle energy eigenfunctions

$$\psi_{nlm}(\vec{r}) = u_{nl}(r)\, Y_{lm}(\theta, \phi)$$

Non-identical particles

$$\psi_{n_1 l_1 m_1}(\vec{r}_1)\, \psi_{n_2 l_2 m_2}(\vec{r}_2)$$

Identical bosons :

$$\frac{1}{\sqrt{2}}\left(\psi_{n_1 l_1 m_1}(\vec{r}_1)\, \psi_{n_2 l_2 m_2}(\vec{r}_2) + \psi_{n_1 l_1 m_1}(\vec{r}_2)\, \psi_{n_2 l_2 m_2}(\vec{r}_1) \right)$$

For identical fermions:

$$\frac{1}{\sqrt{2}}\left(\psi_{n_1 l_1 m_1}(\vec{r_1})\, \psi_{n_2 l_2 m_2}(\vec{r_2}) - \psi_{n_1 l_1 m_1}(\vec{r_2})\, \psi_{n_2 l_2 m_2}(\vec{r_1})\right.$$

$$\left.\begin{array}{l}(n_1\ l_1\ m_1) = (1\ 0\ 0). \\ (n_2\ l_2\ m_2) = (1\ 0\ 0).\end{array}\right\}\ (\text{say})$$

1 - state for non-identical particles.

$\vec{L} = 0.$

(total angular momentum)

Identical bosons : 1 state, $\vec{L} = 0$

" fermions : no state.

(Exclusion principle)

Consider $(n_1\ l_1\ m_1) = (1, 1, m_1)$

$(n_2\ l_2\ m_2) = (1, 1, m_2)$

For non-identical particles : 9 states

$|\vec{L}| \equiv \underbrace{2, 1, 0.}_{}$

$(5+3+1)$ states

Identical bosons : 6 states ; $\vec{L} = 2\ \&\ 0.$

(Property of Clebsch-Gordon coefficients)

$$\sum_{m_1, m_2} C^{l\ l_1\ l_2}_{m\ m_1\ m_2}\ \psi_{n_1 l_1 m_1}(\vec{r_1})\ \psi_{n_2 l_2 m_2}(\vec{r_2})$$

$$C^{l\ l_1\ l_1}_{m\ m_1\ m_2} = (-1)^{l-2l_1}\, C^{l\ l_1\ l_1}_{m\ m_2\ m_1}$$

Identical fermions : 3 states, $L = 1.$ (Think!)

Suppose, we have N identical non-interacting particles.

Hamiltonian $h_{(N)} = \sum_i h(\vec{r_i}, \vec{p_i})$.

Suppose further that eigenstates of $h(\vec{r}, \vec{p})$ are $u_n(\vec{r})$

$$h(\vec{r}, \vec{p})\, u_n(\vec{r}) = e_n\, u_n(\vec{r}).$$

Basis states :

Bosons $\rightarrow$ $u_{\{n_1, \ldots, n_N\}}(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$

$$= \frac{1}{\sqrt{N!}} \left(u_{n_1}(\vec{r}_1)\, u_{n_2}(\vec{r}_2) \ldots u_{n_N}(\vec{r}_N) + \text{all permutations of } \vec{r}_1, \ldots, \vec{r}_N \right)$$

Fermions $\rightarrow$ $\frac{1}{\sqrt{N!}} \left(u_{n_1}(\vec{r}_1) \ldots u_{n_N}(\vec{r}_N) + (-1)^P \times \text{all permutations of } \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N \right)$

P : no. of changes (exchanges) needed to produce the permutation

1	2	3	$P = 0$		3	2	1	$P = 1$
2	1	3	$P = 1$	But	1	2	3	
1	3	2	$P = 1$			$\downarrow$		

Whether P is even or odd is uniquely defined. (Property of permutation group)

2 1 3 $\downarrow$
2 3 1 $\downarrow$
3 2 1 $\downarrow$

$P = 3$.

* A question about normalization $\rightarrow$ are all n_i's different? (NOT IN GENERAL !). So $\frac{1}{\sqrt{N!}}$ might not be the correct normalization.

Occupation number representation :-

u_1 occurs m_1 times
u_2 " m_2 " and so on ...

$m_1, m_2 \longrightarrow 0, 1, 2, \ldots$ (bosons)
$\longrightarrow 0, 1$ (fermions)

Label the state by $|m_1, m_2, \ldots \rangle$

$$N = \sum_i m_i$$

The $N!$ terms are grouped into $\left(\prod_i m_i !\right)$ identical terms. There are $\dfrac{N!}{\prod_i m_i !}$ different terms. (Groups)

$$\langle m_1, m_2, \ldots | m_1, m_2, \ldots \rangle = \frac{1}{N!}\left(\prod_i m_i !\right)^2 \times \frac{N!}{\prod_i m_i !}$$

$$= \prod_i m_i !$$

(*) Limitation :- Refers to ~~a~~ fixed numbers of particles.

Second quantization :-

One particle time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} = \hat{h} \Psi(\vec{r}, t)$$

· Think of this as a classical field equation (and quantize it)

$u_n(\vec{r})$: Complete set of basis states in $\vec{r}$ space

$\Psi(\vec{r}, t) \longrightarrow$ Think as a function of $\vec{r}$ at a fixed t.

$$\hat{h}\, u_n(\vec{r}) = e_n u_n(\vec{r})$$

$$\Psi(\vec{r}, t) = \sum_n a_n(t)\, u_n(\vec{r}).$$

Think as degrees of freedom of a classical system.

For arguments' sake, assume $n = 1$

$$i\hbar \sum_n \frac{\partial a_n}{\partial t} u_n(\vec{r}) = \sum_n e_n a_n(t) u_n(\vec{r}).$$

$$i\hbar \frac{\partial a_n}{\partial t} = e_n a_n(t)$$
$$-i\hbar \frac{\partial a_n^*}{\partial t} = e_n a_n^*(t)$$

Two equations

(Just like classical equations of motion.)

<u>Action</u> :-

$$\int dt \sum_n a_n^*(t) \left(i\hbar \frac{\partial a_n(t)}{\partial t} - e_n a_n(t) \right)$$

$$\delta(\text{Action}) = \int dt \sum_n \delta a_n^*(t) \left(i\hbar \frac{\partial a_n}{\partial t} - e_n a_n(t) \right) + \int dt \sum_n a_n^*(t) \left(i\hbar \frac{\partial}{\partial t} \delta a_n(t) - e_n \delta a_n(t) \right)$$

Think of a_n and a_n^* as independent variables. (Equivalent to equating real and imaginary parts of δa_n separately to zero.)

$$\delta(\text{Action}) = \int dt \sum_n \delta a_n^*(t) \underbrace{\left(i\hbar \frac{\partial a_n}{\partial t} - e_n a_n \right)}_{=0} + \int dt \sum_n \delta a_n(t) \underbrace{\left(-i\hbar \frac{\partial a_n^*}{\partial t} - e_n a_n^* \right)}_{=0}$$

→ We get back the equations of motion. (!)

<u>Exercise</u> : Show that action $= \int d^3r\, dt\, \psi^*(\vec{r}, t) \times \left(i\hbar \frac{\partial \psi}{\partial t} - \hat{h} \psi \right)$

$$\psi(\vec{r}, t) = \sum_n a_n(t)\, u_n(\vec{r})$$ ✓

→ This is called the <u>Schrödinger action</u> (Generates Schrödinger equation from variational principle)

$$L = \sum_n a_n^*(t) \left(i\hbar \dot{a}_n(t) - e_n a_n(t) \right)$$

$\dot{a}_n = \frac{da_n}{dt}$ by definition.

$$L = \int d^3r\, \psi(\vec{r},t)\left(i\hbar \frac{\partial \psi}{\partial t} - \hat{h}\psi\right) \quad \text{(In FT notation)}$$

$$\Pi_n = \frac{\partial L}{\partial \dot{a}_n} = i\hbar\, a_n^*$$

$$H = \sum_n \Pi_n \dot{a}_n - L$$

$$= \sum_n e_n a_n^* a_n .$$

$$= \int d^3r\, \psi^* \hat{h}\, \psi . \quad \text{(Find out !)}$$

$$\{a_n, \Pi_m\} = \delta_{mn}$$

$$i\hbar \{a_n(t), a_m^*(t)\} = \delta_{mn}$$

$$\Downarrow$$

(Prove)

$$i\hbar \{\psi(\vec{r},t), \psi^*(\vec{r}\,',t)\} = \delta^{(3)}(\vec{r} - \vec{r}\,')$$

Quantization :-

$$i\hbar [a_m, a_m^\dagger] = i\hbar\, \delta_{mn}$$

$$\Rightarrow [a_n, a_m^\dagger] = \delta_{mn}$$

$$\Rightarrow [\psi(\vec{r},t), \psi^\dagger(\vec{r}\,',t)] = \delta^{(3)}(\vec{r} - \vec{r}\,')$$

$$\hat{H} = \sum_n e_n a_n^\dagger a_n$$

$$= \int d^3r\, \psi^\dagger(\vec{r},t)\, \hat{h}\, \psi(\vec{r},t) .$$

(System of infinite no. of harmonic oscillators !)

$$\{a_n, a_m\} = 0 \quad ; \quad \{a_n^*, a_m^*\} = 0 .$$

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Hilbert space

Define $|0\rangle$ such that $a_n|0\rangle = 0 \ \forall \ n$.

Basis states : $a^\dagger_{n_1} a^\dagger_{n_2} \cdots a^\dagger_{n_N} |0\rangle$

We can also represent the same as

$$(a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2} \cdots |0\rangle \qquad m_1, m_2 = 0, 1, 2, \cdots$$

$$(a_1^\dagger)^2 a_2^\dagger |0\rangle = a_1^\dagger a_1^\dagger a_2^\dagger |0\rangle$$

Remember $|0\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle \cdots$

$$\hat{H} a^\dagger_{n_1} a^\dagger_{n_2} \cdots a^\dagger_{n_N} |0\rangle = (e_{n_1} + e_{n_2} + \cdots e_{n_N}) a^\dagger_{n_1} a^\dagger_{n_2} \cdots a^\dagger_{n_N} |0\rangle$$

Relations which can help –

$$\left\{ \begin{aligned} [\hat{H}, a_n^\dagger] &= e_n a_n^\dagger \\ [\hat{H}, a_n] &= -e_n a_n \end{aligned} \right\} \quad \text{Similar to SHO}$$

Next, we consider a bosonic system.

$$a^\dagger_{n_1} a^\dagger_{n_2} a^\dagger_{n_3} \cdots a^\dagger_{n_N} |0\rangle \longleftrightarrow u_{\{n_1, n_2, \cdots n_N\}} (\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N)$$

$$h_{(N)} = \sum_{i=1}^{N} h(\vec{r}_i, \vec{p}_i)$$

$$h_{(N)} u = (e_{n_1} + e_{n_2} + \cdots e_{n_N}) u$$

$$\langle m_1, m_2, \cdots | m_1, m_2, \cdots \rangle = \prod_i (m_i)!$$

(For multiparticle states)

$$\langle 0 | (a_1)^{m_1} (a_2)^{m_2} \cdots (a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2} \cdots |0\rangle$$

What is $a_1 (a_1^\dagger)^{m_1}$?

$$a_1 a_1^\dagger = 1 + a_1^\dagger a_1 \Bigg| \Rightarrow m_1 (a_1^\dagger)^{m_1 - 1}$$

So, this normalization is also $\left(\prod_i m_i!\right)$.

(*) Why not fermions? (Symmetries, restrictions on m_i's)

$$\hat{N} = \sum_{i=1}^{\infty} a_i^{\dagger} a_i \quad \text{(number operator)}$$

$$\hat{H} = \sum e_n a_n^{\dagger} a_n$$
$$= \int d^3r \, \hat{\psi}^{\dagger}(\vec{r}, t) \, \check{h} \, \hat{\psi}(\vec{r}, t)$$

First quantized operator :- $\vee$

Second " " :- $\wedge$

$$\check{h} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r})\right) \qquad \hat{\psi}(\vec{r}, t) = \sum_n \hat{a}_n(t) \, u_n(\vec{r})$$

$$\hat{N} = \int d^3r \, \hat{\psi}(\vec{r}, t)^{\dagger} \, \hat{\psi}(\vec{r}, t).$$

Suppose, we have N-particle Q.Mech. & $\check{O}_{(N)}$ is some operator in this QM.

Question :- What is the operator $\hat{O}$ such that

$$\langle 0 | a_{n_1} \cdots a_{n_N} \hat{O}_{(N)} a_{n_1'}^{\dagger} a_{n_2'}^{\dagger} \cdots a_{n_N'}^{\dagger} | 0 \rangle$$
$$= \int d^3r_1 \, d^3r_2 \cdots d^3r_N \, u^*_{\{n_1, \cdots n_N\}}(\vec{r}_1, \cdots, \vec{r}_N) \, \check{O}_{(N)} \, u_{\{n_1', \cdots, n_N'\}}(\vec{r}_1, \cdots, \vec{r}_N) \; ?$$

Only those operators are allowed which are symmetric under the exchange of two particles $(\vec{r}_i, \vec{p}_i) \longleftrightarrow (\vec{r}_j, \vec{p}_j)$

1- body operator

$$\check{b}_{(N)} = \sum_{i=1}^{N} \check{b}(\vec{r}_i, \vec{p}_i)$$

$$\hat{B} = \sum_{m,n} \langle n | \check{b} | m \rangle a_n^{\dagger} a_m$$

$$= \int d^3r \; \hat{\Psi}^\dagger(\vec{r},t) \; \check{b} \; \hat{\Psi}(\vec{r},t)$$

$$\hat{\Psi}^\dagger = \sum_n a_n^\dagger u_n^*(\vec{r}) \qquad \hat{\Psi} = \sum_m \hat{a}_m u_m(\vec{r})$$

Claim:- $\langle 0 | a_{n_1} \cdots a_{n_N} \hat{B} a_{n_1'}^\dagger \cdots a_{n_N'}^\dagger | 0 \rangle$

$$= \int d^3r_1 \cdots d^3r_N \; u^*_{\{n_1,\dots,n_N\}}(\vec{r}_1, \vec{r}_2, \cdots \vec{r}_N) \left(\sum_{i=1}^N \check{b}(\vec{r}_i, \vec{p}_i) \right) u_{\{n_1,\dots,n_N\}}(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)$$

Prove !

Power of second quantized formalism $\rightarrow$ no N dependence of second quantized operators (!)

The 1-body operator commutes with $\hat{N}$ (number operator)

One can also design non-(number conserving) operators.

2 - body operators

$\sum_{\substack{i=1\\j=1\\i\neq j}}^N \check{v}(\vec{r}_i, \vec{r}_j, \vec{p}_i, \vec{p}_j)$. [Find the corresponding second-quantized operator $\hat{V}$]

$$\int d^3r_1 \, d^3r_2 \; \Psi^\dagger(\vec{r}_1) \Psi^\dagger(\vec{r}_2) \check{v}(\vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2) \Psi(\vec{r}_1) \Psi(\vec{r}_2)$$

$$= \sum_{m,n,p,q} v_{mnpq} \, a_m^\dagger a_n^\dagger a_p a_q$$

$$v_{mnpq} = \int u_m^*(\vec{r}_1) \, u_n^*(\vec{r}_2) \; \check{v}(\vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2) \, u_p(\vec{r}_1) \, u_q(\vec{r}_2) \, d^3r_1 \, d^3r_2,$$

$\rightarrow$ Show the equality of matrix elements. (!)

Suppose $\check{b}_{(N)} = \sum_i \check{b}(\vec{r}_i, \vec{p}_i)$

$$\hat{B} = \sum_{m,n} b_{mn} a_m^\dagger a_n, \quad b_{mn} = \int u_m^*(\vec{r}) \, \check{b} \, u_n(\vec{r})$$

(Try for N = 1).

First quantized description : $\int d^3r\, u^*_{n_1}(\vec{r_1})\, \check{b}(\vec{r_1}, \vec{p_1})\, u_{n_1'}(r_1)$

Second quantized description : $\langle 0 | a_{n_1} \hat{B}\, a^\dagger_{n_1'} | 0 \rangle$

$$= \langle 0 | \underbrace{a_{n_1} \sum_{m\,n} b_{mn}\, a^\dagger_m}_{\delta_{n_1 m}} \underbrace{a_n\, a^\dagger_{n_1'}}_{\delta_{n n_1'}} | 0 \rangle$$

(The two matrix elements agree) $= b_{n_1 n_1'} \langle 0 | 0 \rangle = b_{n_1 n_1'}$ ✓

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N non-interacting identical bosons with Hamiltonian

$$\check{h}_{(N)} = \sum_{i=1}^{N} \check{h}_i$$
$$\Downarrow$$
$$\check{h}(\vec{r_i}, \vec{p_i})$$

2nd quantized theory

$$\Longleftrightarrow \quad \hat{H} = \int d^3r\, \hat{\psi}(\vec{r})^\dagger\, \check{h}\, \hat{\psi}(\vec{r})$$
$$= \sum_n e_n\, \hat{a}_n^\dagger a_n$$
$$\check{h} u_n = e_n u_n(\vec{r})$$
$$\hat{\psi}(\vec{r}) = \sum_n \hat{a}_n u_n(\vec{r})$$

Commutation relations :-

$$[a_m, a_n] = 0 = [a_m^\dagger, a_n^\dagger] \quad ; \quad [a_m, a_n^\dagger] = \delta_{mn}$$
$$\Longleftrightarrow \quad [\psi(\vec{r}), \psi^\dagger(\vec{r}\,')] = \delta^{(3)}(\vec{r} - \vec{r}\,')$$
$$[\psi(\vec{r}), \psi(\vec{r}\,')] = 0 = [\psi^\dagger(\vec{r}), \psi^\dagger(\vec{r}\,')]$$

$$\hat{N} \text{ (number operator)} = \sum_n a_n^\dagger a_n$$

$$\psi(\vec{r}) = \sum_n \hat{a}_n u_n(\vec{r}) \qquad = \int d^3r\, \psi^\dagger(\vec{r})\, \psi(\vec{r})$$

Hilbert space of the second quantized theory :-

Define $|0\rangle$ such that

$$a_n |0\rangle = 0 \quad \forall\, n \qquad \text{(0 particle state)}$$

$a_m^\dagger |0\rangle \longleftrightarrow$ 1 particle state with wavefunction $u_m(\vec{r}_1)$

$a_{m_1}^\dagger a_{m_2}^\dagger |0\rangle \longrightarrow$ 2 particle state

$$u_{\{m_1, m_2\}}(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}}\left(u_{m_1}(\vec{r}_1)\, u_{m_2}(\vec{r}_2) + u_{m_1}(\vec{r}_2)\, u_{m_2}(\vec{r}_1)\right)$$

A state of the form $a_{m_1}^\dagger a_{m_2}^\dagger \cdots a_{m_N}^\dagger |0\rangle$

$\Rightarrow$ N-particle state $u_{\{m_1, m_2, \ldots m_N\}}(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$

(Check: <u>Bosons</u>)

<u>Mapping of operators</u> :-

$$\check{b}_{(N)} = \sum_{i=1}^{N} \check{b}_i \Rightarrow \check{b}(\vec{r}_i, \vec{p}_i)$$

$$\hat{B} = \int d^3 r\, \hat{\Psi}(\vec{r})^\dagger\, \check{b}\, \hat{\Psi}(\vec{r}) = \sum_{m,n} b_{mn}\, a_m^\dagger a_n$$

(We work in Schrödinger picture)

$$b_{mn} \Longrightarrow \int d^3 r\, u_m^*(\vec{r})\, \check{b}\, u_n(\vec{r})$$

<u>2-body operators</u>

$$\check{v}_{(N)} = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \check{v}_{ij} \Rightarrow \check{v}(\vec{r}_i, \vec{p}_i, \vec{r}_j, \vec{p}_j)$$

$$\hat{V} = \int d^3 r_1\, d^3 r_2\, \hat{\Psi}^\dagger(\vec{r}_1)\, \hat{\Psi}^\dagger(\vec{r}_2)\, \check{v}\, \hat{\Psi}(\vec{r}_1)\, \Psi(\vec{r}_2)$$

We may rewrite as — $\hat{V} = \sum_{m,n,p,q} v_{mnpq}\, a_m^\dagger a_n^\dagger a_p a_q$

$$v_{mnpq} \Downarrow \int d^3 r_1\, d^3 r_2\, u_m^*(\vec{r}_1)\, u_n^*(\vec{r}_2)\, \check{v}(\vec{r}_1, \vec{p}_1, \vec{r}_2, \vec{p}_2)\, u_p(\vec{r}_1)\, u_q(\vec{r}_2)$$

<u>Interacting system</u>

$$\check{h}_{(N)} = \sum_{i=1}^{N} \check{h}_i + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \check{v}_{ij}$$

$$H = \int d^3 r\, \hat{\Psi}^\dagger(\vec{r})\, \check{h}\, \hat{\Psi}(\vec{r}) + \int d^3 r_1\, d^3 r_2\, \hat{\Psi}^\dagger(\vec{r}_1)\, \hat{\Psi}^\dagger(\vec{r}_2)\, \check{v}(\vec{r}_1, \vec{p}_1, \vec{r}_2, \vec{p}_2)\, \hat{\Psi}(\vec{r}_1)\, \hat{\Psi}(\vec{r}_2)$$

$$= \sum_n e_n\, a_n^\dagger a_n + \sum_{m,n,p,q} v_{mnpq}\, a_m^\dagger a_n^\dagger a_p a_q$$

Add to $\hat{H}$ $\sum_{m,n,p} \left(C_{mnp}\, a_m^\dagger a_n^\dagger a_p + h.c. \right)$

[doesn't conserve particle number]

$a_k^\dagger |0\rangle$ (Single-particle)

$$|m\rangle \Rightarrow |m\rangle + \lambda \sum_n C_{mn} |n\rangle + \ldots$$

$$\check{h} \Rightarrow \check{h} + \lambda \check{h}_1$$

$$C_{mn} \propto \langle m | \check{h}_1 | n \rangle$$

$$\langle 0 | a_m a_n \hat{H}_1 a_p^\dagger | 0 \rangle \neq 0$$

$$\propto C_{mnp}$$

Fermions

- There's no classical theory, whose quantization gives a theory for N fermions.
- There exists an ad-hoc principle.

$$\check{h}_{(N)} = \sum_{i=1}^{N} \check{h}_i \qquad \hat{H} = \int d^3r\, \hat{\Psi}^\dagger(\vec{r})\, \check{h}\, \Psi(\vec{r}) = \sum_n e_n a_n^\dagger a_n$$

(bosons)

Use same thing for bosons & fermions.

$$\{a_m, a_n\} = \{a_m^\dagger, a_n^\dagger\} = 0$$

$$\{A, B\} = AB + BA$$

$$\{a_m, a_n^\dagger\} = \delta_{mn}$$

$$\{\Psi(\vec{r}), \Psi^\dagger(\vec{r}\,')\} = \delta^{(3)}(\vec{r} - \vec{r}\,')$$

$$\{\Psi(\vec{r}), \Psi(\vec{r}\,')\} = \{\Psi^\dagger(\vec{r}), \Psi^\dagger(\vec{r}\,')\} = 0.$$

$\rightsquigarrow$ [NO CLASSICAL LIMIT]

Hilbert space

$$a_n |0\rangle = |0\rangle \quad \forall\, n$$

$|0\rangle \longrightarrow$ 0-particle state

$a_n^\dagger |0\rangle \longrightarrow$ 1-particle state $u_n(\vec{r})$

→ One can't construct a 'coherent state' with fermions (!)

$a^\dagger_{n_1} a^\dagger_{n_2} |0\rangle \rightarrow$ 2 particle state

$$\frac{1}{\sqrt{2}}\left(u_{n_1}(\vec{r_1})\, u_{n_2}(\vec{r_2}) - u_{n_1}(\vec{r_2})\, u_{n_2}(\vec{r_1})\right)$$

(Consistent)

Maps of 1, 2, 3, . . . body operators are identical to that for bosons. (Proof is not the same!)

26.01.2012

Relativistic Quantum Mechanics :-

Free particle SE : $i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi$.

$\hat{H}$ ↳ hamiltonian

Soln :- $\psi = \psi_0\, e^{-i(Et - \vec{p}\cdot\vec{x})/\hbar}$

ψ_0 → constant

We get $E\psi = \frac{\vec{p}^{\,2}}{2m}\psi$

Non-relativistic relation between E & p ← $E = p^2/2m$

- We want to modify SE in such a way as to give $E = \sqrt{p^2c^2 + m^2c^4}$ by imposing ψ_{free} as the solution.

$$\hat{H} = \sqrt{-c^2\hbar^2\nabla^2 + m^2c^4} \quad (?)$$

$$\rightarrow mc^2\left(1 - \frac{\hbar^2}{m^2c^2}\nabla^2\right)^{1/2}$$

$$= mc^2\left(1 - \frac{\hbar^2}{2m^2c^2}\nabla^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(\frac{\hbar^2}{2mc^2}\right)^2(\nabla^2)^2 + \dots\right)$$

One way of defining

$$\begin{cases} f(\vec{x}) = \frac{1}{(\sqrt{2\pi})^3}\int e^{-i\vec{k}\cdot\vec{x}}\, \tilde{f}(k)\, d^3k \\ \tilde{f}(\vec{k}) = \frac{1}{(\sqrt{2\pi})^3}\int e^{i\vec{k}\cdot\vec{x}}\, f(\vec{x})\, d^3x \end{cases}$$

$$\hat{H} f(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \; e^{-i\vec{k}\cdot\vec{x}} \tilde{f}(k) \times \sqrt{c^2\hbar^2k^2 + m^2c^4}$$

$$= \frac{1}{(2\pi)^3} \int d^3k \sqrt{c^2\hbar^2k^2+m^2c^4} \; e^{-i\vec{k}\cdot\vec{x}} \int d^3y \; e^{+i\vec{k}\cdot\vec{y}} f(y)$$

(we get back earlier result) expand this

- Non-local operator. (you need to infinite number derivatives of f at a point) Problems with causal

$$i\hbar \frac{\partial\Psi}{\partial t} = \sqrt{-c^2\hbar^2\nabla^2 + m^2c^4}\; \Psi \quad (x)$$

$$\left(i\hbar \frac{\partial\Psi}{\partial t}\right)\left(i\hbar \frac{\partial\Psi}{\partial t}\right) = \left(-c^2\hbar^2\nabla^2 + m^2c^4\right)\Psi$$

$$\Rightarrow \quad \hbar^2\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right)\Psi + m^2c^4\Psi = 0.$$

Klein-Gordon Equation.

(Relativistically invariant)

<u>$c=1$ unit</u> $\longrightarrow$ $\hbar^2\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\Psi + m^2\Psi = 0$

$$\Rightarrow \hbar^2\left(-\eta^{\mu\nu}\frac{\partial^2\Psi(x)}{\partial x^\mu \partial x^\nu}\right) + m^2\Psi(x) = 0 \rightarrow \boxed{\text{LOCAL EQUATION}}$$

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

<u>Lorentz transformations</u>:-

$x \longrightarrow \Lambda x$

Λ satisfies $\Lambda^T \eta \Lambda = \eta$

(Generalization of orthogonal group)

If $\Psi(x)$ is a solution, then $\Psi(\Lambda x)$ is also a solution for any Lorentz transformation matrix Λ. (Check!)

BUT, THERE ARE PROBLEMS (!)

Plane wave $\longrightarrow$ $\Psi = e^{-i(Et - \vec{p}\cdot\vec{x})/\hbar}$

$$\left(-E^2 + p^2\right)\psi + m^2\psi = 0$$

$$\Rightarrow \quad E^2 = p^2 + m^2$$

$$\Rightarrow \quad E = \pm\sqrt{p^2+m^2}$$ (Unbounded -ve energies !)

?

If we introduce V (pot.), +ve energy states start to mix with -ve energy states.

Conserved positive probability :-

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi \qquad \text{(Non-relativistic)}$$

It follows that

$$i\hbar \frac{\partial}{\partial t}\int \psi^* \psi \, d^3x = 0$$

$\sqrt{\ }$ operator $\longrightarrow$ conserves probability

But for KG equation, probability is not conserved

Ex. If we take $\frac{\partial}{\partial t}\int d^3x \left(\psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t}\psi\right) = 0$

But positivity (?) 0-component of prob. current

$\rightsquigarrow$ THIS PATH IS DANGEROUS !

We wanted $i\hbar \frac{\partial \psi}{\partial t} = \sqrt{-\hbar^2 c^2 \nabla^2 + m^2 c^4}$

We make ψ multi-component. (!) Then choose $\hat{H}$ such that $\hat{H}^2 = -\hbar^2 c^2 \nabla^2 + m^2 c^4$.

Remove non-locality

Take ψ to be n-dimensional column vector.

Try $\hat{H} = -i\hbar\, \tilde{\alpha}_i \frac{\partial}{\partial x^i} + \tilde{\beta}$

$\tilde{\alpha}_i$, $\tilde{\beta}$ are $n \times n$ matrices.

$$\hat{H}^2 = -\hbar^2 \tilde{\alpha}_i \tilde{\alpha}_j \cancel{+ \tilde{\alpha}_j \tilde{\alpha}_i} \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - i\hbar\left(\tilde{\alpha}_i \beta + \beta \tilde{\alpha}_i\right)\frac{\partial}{\partial x^i} + \tilde{\beta}^2$$

$$= -\frac{\hbar^2}{2}\left\{\tilde{\alpha}_i, \tilde{\alpha}_j\right\}\frac{\partial^2}{\partial x^i \partial x^j} - i\hbar\left\{\tilde{\alpha}_i, \beta\right\}\frac{\partial}{\partial x^i} + \tilde{\beta}^2$$

We want this to be

$$\left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right) \mathbb{1}_{n \times n}$$

$$\rightarrow \begin{cases} \left\{\tilde{\alpha}_i, \tilde{\alpha}_j\right\} = 2c^2 \delta_{ij} \\ \left\{\tilde{\alpha}_i, \beta\right\} = 0 \\ \tilde{\beta}^2 = m^2 c^4 \mathbb{1} \end{cases}$$

$\left\{\tilde{\alpha}_i, \beta\right\} = 0 \longrightarrow$ [no 1 component wavefn]

Write

$\tilde{\alpha}_i = c\alpha_i$

$\tilde{\beta}_i = c\beta_i$

Then $\left\{\alpha_i, \alpha_j\right\} = 2\delta_{ij}$

$\left\{\alpha_i, \beta\right\} = 0$

and $\beta^2 = \mathbb{1}$

$$H = -i\hbar c\, \alpha_i \frac{\partial}{\partial x^i} + mc^2 \beta$$

<u>30.01.2011</u>

$$i\hbar \frac{\partial \Psi}{\partial t} = -i\hbar\, \tilde{\alpha}_i \frac{\partial \Psi}{\partial x^i} + \tilde{\beta}\Psi = \hat{H}\Psi$$

$\Psi \rightarrow$ n-dim. column vector

$\tilde{\alpha}_i, \tilde{\beta} \rightarrow$ $n \times n$ matrices

If we look for a solution of the form

$$\Psi = \Psi_0 e^{i(Et - \vec{p}\cdot\vec{x})/\hbar}$$

$$\Rightarrow \quad E\Psi = \left(\tilde{\alpha}_i p_i + \tilde{\beta}\right)\Psi$$

$$\Rightarrow E^2 \Psi = \left(\tilde{\alpha}_i p_i + \tilde{\beta}\right)\left(\tilde{\alpha}_j p_j + \tilde{\beta}\right)\Psi$$

$$\Rightarrow \left(p^2 c^2 + m^2 c^4\right) \mathbb{1}_{n \times n} \Psi$$

We get constraints for $\tilde{\alpha}$, $\tilde{\beta}$.

$$\tilde{\alpha}_i = c\,\alpha_i \qquad \{\alpha_i, \alpha_j\} = 2\delta_{ij}$$
$$\tilde{\beta} = mc^2\beta$$
$$\{\alpha_i, \beta\} = 0 \quad , \quad \beta^2 = \mathbb{1}$$

Rescaling

$\beta^2 = \mathbb{1}$ implies that the eigenvalues of β are ± 1.

$\alpha_i^2 = \mathbb{1} \Rightarrow$ eigenvalues of each α_i are ± 1.

$$\begin{aligned} \mathrm{Tr}(\alpha_i) &= \mathrm{Tr}(\alpha_i \beta^2) \\ &= \mathrm{Tr}(\beta\alpha_i\beta) \\ &= -\mathrm{Tr}(\alpha_i\beta\beta) = -\mathrm{Tr}(\alpha_i) \end{aligned}$$
$$(\because \beta^2 = \mathbb{1})$$

So, $\mathrm{Tr}(\alpha_i) = 0$.

Similarly, $\mathrm{Tr}(\beta) = 0$.

$\rightarrow$ even dimensional matrices.

$\underbrace{\text{2d} \rightarrow \text{not possible}}$ (Pauli matrices)

$$\left\{\begin{aligned} \tilde{\alpha}_i &= c\,\alpha_i \\ \tilde{\beta} &= mc^2\beta \end{aligned}\right\}$$

If $m = 0$, β is not needed. $\rightsquigarrow$ Weyl Fermions (e.g. neutrinos)

4-dimensional matrices

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} ; \quad \beta = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}$$

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ \dots$$

Ex. $\rightarrow$ Check that $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$, $\beta^2 = \mathbb{1}$ and $\{\beta, \alpha_i\} = 0$.

$$\alpha_i\alpha_j + \alpha_j\alpha_i$$
$$= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_i \sigma_j & 0 \\ 0 & \sigma_i \sigma_j \end{pmatrix} + \begin{pmatrix} \sigma_j \sigma_i & 0 \\ 0 & \sigma_j \sigma_i \end{pmatrix}$$

$$= 2 \begin{pmatrix} \delta_{ij} I_{2\times 2} & 0 \\ 0 & \delta_{ij} I_{2\times 2} \end{pmatrix} = 2 \delta_{ij} I_{4\times 4}$$

$$\left.\begin{aligned} \alpha_i &\longrightarrow U \alpha_i U^{-1} \\ \beta &\longrightarrow U \beta U^{-1} \end{aligned}\right\} \text{Further choices}$$

Unitary $\longrightarrow$ $\left\{ \begin{array}{l} \text{preserve} \\ \text{hermiticity} \end{array} \right\}$ U: 4×4 unitary matrices

$\psi \longrightarrow U\psi$ (new solution!)

(Spinor representation)

Conservation of probability :-

$$i\hbar \frac{\partial \psi}{\partial t} = - i\hbar \underbrace{\tilde{\alpha}_i}_{= c\alpha_i} \frac{\partial \psi}{\partial x^i} + \underbrace{\tilde{\beta}}_{= mc^2 \beta} \psi$$

$$\therefore \frac{\partial \psi}{\partial t} = - \tilde{\alpha}_i \frac{\partial \psi}{\partial x^i} + \frac{\tilde{\beta}}{i\hbar} \psi$$

$$\frac{\partial}{\partial t} \psi^\dagger = - \frac{\partial \psi^\dagger}{\partial x^i} \tilde{\alpha}_i - \frac{1}{i\hbar} \psi^\dagger \tilde{\beta} \longrightarrow \text{row vector}$$

$$\therefore \frac{\partial}{\partial t} (\psi^\dagger \psi) = \frac{\partial \psi^\dagger}{\partial t} \psi + \psi^\dagger \frac{\partial \psi}{\partial t}$$

$$= \left(- \frac{\partial \psi^\dagger}{\partial x^i} \tilde{\alpha}_i \psi - \cancel{\frac{\tilde{\beta}}{i\hbar} \psi^\dagger \tilde{\beta} \psi} \right) - \psi^\dagger \alpha_i \frac{\partial \psi}{\partial x^i} + \cancel{\frac{1}{i\hbar} \psi^\dagger \tilde{\beta} \psi}$$

$$= - \frac{\partial}{\partial x^i} \left(\psi^\dagger \tilde{\alpha}_i \psi \right)$$

$$\therefore \frac{\partial}{\partial t} \int d^3x \, \psi^\dagger \psi = - \int d^3x \frac{\partial}{\partial x^i} \left(\psi^\dagger \alpha_i \psi \right)$$

$$= 0 .$$ (As long as $\psi^\dagger$ and ψ fa[ll] off sufficiently fast at [∞])

Set $c = 1$. $\qquad i\hbar \frac{\partial \Psi}{\partial t} = -i\hbar \alpha_i \frac{\partial \Psi}{\partial x^i} + m\beta\Psi$

$$i\hbar\beta \frac{\partial \Psi}{\partial t} = -i\hbar\, \beta\alpha_i \frac{\partial \Psi}{\partial x^i} + m\Psi \qquad (\beta^2 = \mathbb{1})$$

Define $\gamma^0 = i\beta$, $\quad \gamma^i = i\beta\alpha_i$

$\Rightarrow \quad \hbar\gamma^0 \frac{\partial \Psi}{\partial x^0} = -\hbar\gamma^i \frac{\partial \Psi}{\partial x^i} + m\Psi$

$x^0 = ct = t \Rightarrow \quad \hbar\gamma^\mu \frac{\partial \Psi}{\partial x^\mu} - m\Psi = 0.$

$$(\gamma^0)^2 = (i\beta)^2 = -\mathbb{1}$$

exercise $\rightarrow$ Show that

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad (!)$$

$$\eta = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}.$$

Suppose, Λ is a Lorentz transformation matrix.

$$\Lambda^T \eta \Lambda = \eta$$

Statement of Lorentz symmetry:

Naive $\longrightarrow$ If $\Psi(x)$ is a solution, then $\Psi(\Lambda x)$ is also a solution. $(\times)$

L.H.S.

$$\hbar\gamma^\mu \frac{\partial \Psi(\Lambda x)}{\partial x^\mu} - m\Psi(\Lambda x)$$

(Think of KG case $-(\Box^2 - m^2)\Psi = 0$)

$$y = \Lambda x$$

$$y^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\frac{\partial}{\partial x^\mu} = \frac{\partial y^\rho}{\partial x^\mu}\frac{\partial}{\partial y^\rho} = \Lambda^\rho{}_\mu \frac{\partial}{\partial y^\rho}.$$

??

$$LHS \to \hbar \gamma^\mu \Lambda^\rho_{\ \mu} \frac{\partial}{\partial y^\rho} \Psi(y) - m \Psi(y)$$

But Ψ satisfies $\hbar \gamma^\mu \frac{\partial \Psi(y)}{\partial y^\mu} - m\Psi(y) = 0$.

Doesn't work

$$\Psi(x) \longrightarrow \Psi(\Lambda x) \Rightarrow \text{scalar}$$

ansatz

Recall $\to$ Lorentz transformation of vector fields

$$A_\mu(x) \Rightarrow \Lambda^\rho_{\ \mu} A_\rho(\Lambda x)$$

vector index

Look for Lorentz symmetry transformations of the form

$$\Psi(x) \longrightarrow S\,\Psi(\Lambda x) \qquad \text{(understand!)}$$

$(n\times n)$ matrix $f^{\underline{n}}$ of Λ

LHS

$$= \hbar \gamma^\mu \frac{\partial}{\partial x^\mu} S\Psi(\Lambda x) - m S \Psi(\Lambda x)$$

Introduce $y = \Lambda x$

$$y^\rho = \Lambda^\rho_{\ \mu} x^\mu$$

$$\therefore \frac{\partial}{\partial x^\mu} = \Lambda^\rho_{\ \mu} \frac{\partial}{\partial y^\rho}$$

$$\therefore \hbar \gamma^\mu \Lambda^\rho_{\ \mu} \frac{\partial}{\partial y^\rho} S\Psi(y) - m S\Psi(y)$$

$$= S\left[\hbar \, S^{-1}\gamma^\mu S \, \Lambda^\rho_{\ \mu} \frac{\partial \Psi(y)}{\partial y^\rho} - m\Psi(y)\right]$$

matrix product

We want LHS to vanish if Ψ satisfies the Dirac equation.

We need $S^{-1}\gamma^\mu S \Lambda^\rho_{\ \mu} = \gamma^\rho$.

construct S ?

31.01.2012

$$\hbar \gamma^\mu \partial_\mu \Psi - m\Psi = 0.$$

$\partial/\partial x^\mu$

If $\Psi(x)$ is a solution, then $S\Psi(\Lambda x)$ is also a solution for any Lorentz Transformation Matrix Λ.

Q. → Can we find an S so that this is true?

$$\Psi(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{pmatrix}$$

We need

$$S^{-1}\gamma^\mu S \Lambda^\nu_{\ \mu} = \gamma^\nu$$

16 equations

Strategy

First solve the problem for infinitesimal Lorentz transformations.

$$\Lambda^\nu_{\ \mu} = \delta^\nu_{\ \mu} + \underbrace{\epsilon}_{\text{small number}} \omega^\nu_{\ \mu}$$

$$\Lambda^T \eta \Lambda = \eta$$

$$\Rightarrow \Lambda^\nu_{\ \mu}\eta_{\nu\sigma}\Lambda^\sigma_{\ \rho} = \eta_{\mu\rho}$$

(component form)

So, $$(\delta^\nu_{\ \mu} + \epsilon\omega^\nu_{\ \mu})\eta_{\nu\sigma}(\delta^\sigma_{\ \rho} + \epsilon\omega^\sigma_{\ \rho}) = \eta_{\mu\rho}$$

$$\cancel{\eta_{\mu\rho}} + \underbrace{\epsilon\omega^\nu_{\ \mu}\eta_{\nu\rho} + \epsilon\eta_{\mu\sigma}\omega^\sigma_{\ \rho}}_{\text{must cancel}} + O(\epsilon^2) = \cancel{\eta_{\mu\rho}}$$

$$\therefore \underbrace{\omega^\nu_{\ \mu}\eta_{\nu\rho}}_{\text{def.}} + \underbrace{\overset{\text{lowering}}{\eta_{\mu\sigma}}\omega^\sigma_{\ \rho}}_{\text{def.}} = 0$$

$$\omega_{\rho\mu} + \omega_{\mu\rho} = 0.$$

$\omega_{\alpha\beta} \rightarrow$ {antisymmetric}

$$\boxed{\omega^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}\omega_{\rho\sigma}} \qquad \boxed{\omega^{\mu\nu} = -\omega^{\nu\mu}} \text{ (too!)}$$

6 independent components → 3 rotations, → 3 boosts

Now, we want S to be infinitesimal if L.T. is infinitesimal

$$S = \mathbb{1} + \epsilon \mathbb{M} \rightarrow (4\times 4)$$

$$(\mathbb{1} - \epsilon \mathbb{M}) \gamma^\mu (\mathbb{1} + \epsilon \mathbb{M}) (\delta^\nu{}_\mu + \epsilon \omega^\nu{}_\mu) = \gamma^\nu$$

$$\cancel{\gamma^\nu} + \epsilon \Big\{ \underbrace{-M\gamma^\nu + \gamma^\nu M + \gamma^\mu \omega^\nu{}_\mu}_{\text{vanishes}} \Big\} + O(\epsilon^2) = \cancel{\gamma^\nu}$$

$$\Rightarrow \boxed{\mathbb{M}\gamma^\nu - \gamma^\nu \mathbb{M} = \gamma^\mu \omega^\nu{}_\mu} \rightarrow [M, \gamma^\rho] = \gamma^\mu \omega^\rho{}_\mu$$

Solution

$$\mathbb{M} = \frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu} \quad ; \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

Let's try to prove this (!)

Proof :- $[M, \gamma^\rho] = -\frac{1}{8} \omega_{\mu\nu} [[\gamma^\mu, \gamma^\nu], \gamma^\rho]$

$\mathbb{M} = \omega_{\mu\nu} \Sigma^{\mu\nu}$

Use the identity

exercise $[AB, C] = A\{B, C\} - \{A, C\}B$

$$= -\frac{1}{8} \omega_{\mu\nu} \, 2 \, [\gamma^\mu \gamma^\nu, \gamma^\rho] \, . \quad \text{(Understand !)}$$

$$= -\frac{1}{4} \omega_{\mu\nu} \left(\gamma^\mu \{\gamma^\nu, \gamma^\rho\} - \{\gamma^\mu, \gamma^\rho\} \gamma^\nu \right)$$

$$= -\frac{1}{4} \omega_{\mu\nu} \left(\gamma^\mu . 2\eta^{\nu\rho} - \gamma^\nu . 2\eta^{\mu\rho} \right)$$

$$= -\frac{1}{2} \gamma^\mu (-\omega_{\nu\mu}) \eta^{\nu\rho} + \frac{1}{2} \gamma^\nu \omega_{\mu\nu} \eta^{\mu\rho}$$

$$= \frac{1}{2} \gamma^\mu \omega^\rho{}_\mu + \frac{1}{2} \gamma^\nu \omega^\rho{}_\nu = \gamma^\mu \omega^\rho{}_\mu$$

Conclusion $\longrightarrow$ If $\Lambda^\nu{}_\mu = \delta^\nu{}_\mu + \epsilon \omega^\nu{}_\mu$ then

$$S = \mathbb{1} + \epsilon \frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}.$$

$$\lim_{N \to \infty} \left(1 + \frac{x}{N}\right)^N = e^x$$

We replace x by a matrix. (In this case)

$$\lim_{N \to \infty} \left(1 + \frac{1}{N} \omega\right)^N = e^{\omega}. \quad (\checkmark)$$

Corresponding $S = \lim_{N \to \infty} \left(1 + \frac{1}{N} \sigma^{\mu\nu} \omega_{\mu\nu}\right)^N$

$$= \exp\left(\frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}\right).$$

Some examples:-

1. Take $\omega_{01} = -\omega_{10} = -k$. $\omega_{\mu\nu} = 0$ otherwise. (Only Boost!)

$$\omega^0{}_1 = k_1 \omega^1{}_0 = k \qquad \omega = \begin{pmatrix} 0 & k & k & 0 \\ 0 & & & \\ & & & \\ & & & \end{pmatrix} = k$$

$$= k \sum_n$$

$$\Sigma^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Sigma^2 = \mathbb{1} \text{ (for given } n)$$

for all n

$$e^{\omega} = \exp(k\Sigma)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} k^n \Sigma^n$$

$$= \sum_{n \text{ even}} \frac{1}{n!} k^n \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$+ \sum_{n \text{ odd}} \frac{1}{n!} k^n \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \cancel{\cosh k} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \cosh k \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sinh k \times$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh k & \sinh k & 0 & 0 \\ \sinh k & \cosh k & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

<u>Corresponding S</u>

$$\exp\left(\frac{i}{4}\sigma_{\mu\nu}\,\omega^{\mu\nu}\right) = \exp\left(\frac{i}{4}\sigma_{01}\,\omega^{01} + \frac{i}{4}\sigma_{10}\omega^{10}\right)$$

$$= \exp\left(\frac{i}{4}k\,\sigma_{01} - \frac{i}{4}k\,\sigma_{10}\right)$$

$$= \exp\left(\frac{i}{2}k\,\sigma_{01}\right) = \exp\left(-\frac{i}{2}k\,\sigma^{01}\right)$$

$$\sigma^{01} = \gamma^0\gamma^1 - \gamma^1\gamma^0$$

$$= \frac{i}{2}\left(\gamma^0\gamma^1 + \gamma^0\gamma^1\right) = i\gamma^0\gamma^1$$

$$= i(i\beta)(i\beta\alpha_1) = -i\alpha_1 .$$

$$\alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \qquad \alpha_1^2 = \mathbb{1}$$

Finally, $S = \exp\left(-\frac{i}{2}k\,(-i\alpha_1)\right) = e^{\left(-k/2\,\alpha_1\right)}$.

$$= \sum_n \frac{1}{n!}\left(-\frac{k}{2}\right)^n \alpha_1^n = \cosh\left(-\frac{k}{2}\right)\mathbb{1} + \sinh\left(-\frac{k}{2}\right)\alpha_1$$

$$= \cosh\frac{k}{2}\,\mathbb{1} - \sinh\frac{k}{2}\,\alpha_1$$

$$= \begin{pmatrix} \cosh\frac{k}{2}\,\mathbb{1}_{2\times2} & -\sinh\frac{k}{2}\,\sigma_1 \\ -\sinh\frac{k}{2}\,\sigma_1 & \cosh\frac{k}{2}\,\mathbb{1}_{2\times2} \end{pmatrix}$$

For a composition of boosts,

$$\exp(a_i\alpha_i) = \exp\left(|\vec{a}|\,\frac{a_i}{|\vec{a}|}\,\alpha_i\right)$$

$$(a_i \alpha_i)^2 = \vec{a}^{\,2}\, \mathbb{1}$$

$$(a_i \alpha_i)^n = |\vec{a}|^n\, \mathbb{1} \quad (\text{even } n)$$
$$= |\vec{a}|^{n-1} a_i \alpha_i \quad (\text{odd } n)$$

Exercise :- Take $\omega_{12} = \theta$, $\omega_{21} = -\theta$ rest 0.
Find $\Lambda = \exp(\omega)$. Find corresponding S.
(turns out to be unitary)

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Relation to spin :-

Go back to Schrödinger equation $\longrightarrow$ has rotational invariance

$$\psi(x) \longrightarrow \psi(\Lambda x) \qquad \underbrace{\text{rotation}}\ \text{only} \qquad (\checkmark)$$

Consider infinitesimal Λ : $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon\, \omega^{\mu}{}_{\nu}$

$$\omega_{12} = 1, \quad \omega_{21} = -1$$

$$\Lambda \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ x^2 \\ -x^1 \\ 0 \end{pmatrix}$$

$$\therefore\ \psi(\Lambda x) = \psi(x) + \epsilon x^2 \frac{\partial \psi}{\partial x^1} - \epsilon x^1 \frac{\partial \psi}{\partial x^2} + O(\epsilon^2)$$

$$L_z \psi(x) \equiv L_3 \psi(x) = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \psi$$
$$= -i\hbar \left(x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} \right) \psi$$

$$\psi(\Lambda x) = \psi(x) - \frac{i\epsilon}{\hbar} L_z \psi$$

(Angular momentum as a generator of rotation)

Now, we are going to generalize.
(to Dirac field)

Again, choose infinitesimal Λ.

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon\, \omega^{\mu}{}_{\nu} . \qquad \omega_{12} = 1, \quad \omega_{21} = -1.$$

Define J_z using

$$S\cancel{S}\psi(\Lambda x) - \psi(x) = -\frac{i\epsilon}{\hbar} J_z \psi$$

Let's work out what J_z is (!)

$$S\psi(\Lambda x) = S\left(\mathbb{1} - \frac{i\epsilon}{\hbar} J_z\right)\psi$$

$$S = \mathbb{1} + \frac{i}{4}\epsilon\, \sigma^{\mu\nu}\omega_{\mu\nu}$$

$$\therefore\; S\psi(\Lambda x) = \left\{\mathbb{1} + \frac{i}{4}\epsilon\left(\sigma^{12} - \sigma^{21}\right)\right\}\left(\mathbb{1} - \frac{i\epsilon}{\hbar} L_z\right)\psi(x)$$

$$-i\hbar\left(x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}\right)$$

$$= \psi(x) + \frac{i}{4}\epsilon \times 2\sigma^{12}\psi - \frac{i\epsilon}{\hbar} L_z \psi$$

$$= -\frac{i\epsilon}{\hbar}\underbrace{\left[-\frac{1}{2}\hbar\sigma^{12} + L_z\right]}_{J_z}\psi + \psi(x)$$

{new contribution to angular momentum}

$$\sigma^{12} = \frac{i}{2}\left[\gamma^1, \gamma^2\right] = i\gamma^1\gamma^2$$

$$= i\left(i\beta\alpha_1\right)\left(i\beta\alpha_2\right) = \begin{pmatrix} 0_{2\times2} & \sigma_1 \\ \sigma_1 & 0_{2\times2} \end{pmatrix}\begin{pmatrix} 0_{2\times2} & \sigma_2 \\ \sigma_2 & 0_{2\times2} \end{pmatrix}$$

$$= -i\beta\alpha_1\beta\alpha_2$$

$$= i\beta^2\alpha_1\alpha_2$$

$$= i\alpha_1\alpha_2 \qquad = i\begin{pmatrix} \sigma_1\sigma_2 & 0 \\ 0 & \sigma_1\sigma_2 \end{pmatrix}$$

$$= -\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

$$\therefore J_z = \frac{1}{2}\hbar \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} + L_z .$$

$$= S_z + L_z$$

$$S_z \longrightarrow \frac{1}{2}\hbar \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \longrightarrow \left(\text{intrinsic to the particle}\right)$$

2 - spin-half representations.

<u>Exercise</u> $\rightarrow$ Show that

$$\left.\begin{aligned} S_x &= \frac{1}{2}\hbar \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \\ S_y &= \frac{1}{2}\hbar \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \end{aligned}\right\}$$

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

$$= \frac{1}{4}\hbar^2 \begin{pmatrix} \sigma_1^2 + \sigma_2^2 + \sigma_3^2 & 0 \\ 0 & \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \end{pmatrix}$$

$$= \frac{3}{4}\hbar^2 \mathbb{1}_{2\times 2}$$

Dirac equation : $\left(\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi = 0$

Goal : Look for solutions of the form

$$\psi = \psi_0 \, e^{-i/\hbar \left(E x^0 - \vec{p}\cdot\vec{x}\right)}$$

$$= \psi_0 \, e^{i/\hbar \, p_\mu x^\mu} .$$

$$\left(p_0 = E, \ \{p_1, p_2, p_3\} \equiv \vec{p}\right)$$

$$(i\cancel{\hbar} \gamma^\mu p_\mu - m)\psi_0 = 0 .$$

$$\Rightarrow \gamma^\mu p_\mu \psi_0 = -im\psi_0$$

$$\gamma^\nu p_\nu \gamma^\mu p_\mu \psi_0 = -im \underbrace{\gamma^\nu p_\nu \psi_0}_{(-im\psi_0)}$$

γ's and p's obviously 'commute' (!)

$$\Rightarrow \gamma^\mu \gamma^\nu p_\mu p_\nu \Psi_0 = -m^2 \Psi_0$$

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} p_\mu p_\nu$$

$$= \eta^{\mu\nu} p_\mu p_\nu$$

$$= -(p^0)^2 + \vec{p}^{\,2} = -m^2$$

$$\Rightarrow (p^0)^2 = p^2 + m^2$$

Next task:

Solve this equation

$$\vec{p} = 0$$

$$\{ i \underset{i\beta}{\underset{\parallel}{\gamma^0}} (-E) - m \} \Psi_0 = 0$$

$$(\beta E - m)\Psi_0 = 0$$

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad \therefore \quad \beta E = \begin{pmatrix} E & & & \\ & E & & \\ & & -E & \\ & & & -E \end{pmatrix}$$

$$\therefore \begin{pmatrix} (E-m) & & & \\ & (E-m) & & \\ & & (-E-m) & \\ & & & (-E-m) \end{pmatrix} \Psi_0 = 0.$$

$E = m$

two sols.

$$u_1(\vec{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-i/\hbar\, m x^0} \qquad u_2(\vec{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-\frac{i}{\hbar} m x^0}$$

$\vec{p} = 0$ $\qquad$ $\vec{p} = 0$

$$S_z = \frac{\hbar}{2} \qquad \underbrace{S_z \text{ eigenstates}} \qquad S_z = -\frac{\hbar}{2}$$

$E = -m$

$$v_1(\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+i/\hbar\, m x^0}; \qquad v_2(\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{i/\hbar\, m x^0}$$

$$S_z = \frac{\hbar}{2} \qquad\qquad S_z = -\frac{\hbar}{2}$$

Choose a Lorentz transformation matrix

$$\Lambda = e^{\omega} \qquad \omega^{01} = k, \quad \omega^{10} = -k$$

$$\omega^{\mu\nu} = 0 \quad \text{otherwise}$$

$$\| $$

$$\begin{pmatrix} \cosh k & \sinh k & 0 & 0 \\ \sinh k & \cosh k & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cosh k x^0 + \sinh k x^1 \\ \\ \\ \\ \end{pmatrix}$$

New solutions : $S \psi(\Lambda x)$

$$u_s(\vec{0}) e^{-i/\hbar \, m x^0} \longrightarrow S u_s(\vec{0}) \, e^{-i/\hbar \, m (\cosh k x^0 + \sinh k x^1)}$$

$$S \to \left(\cosh \frac{k}{2} \mathbb{1} - \sinh \frac{k}{2} \alpha_1\right)$$

?

$$e^{-i/\hbar (E x^0 - p^1 x^1)}$$

$$\left.\begin{aligned} E &= m \cosh k \\ p^1 &= - m \sinh k \end{aligned}\right\} \text{Identification. } (!)$$

Again, $E^2 = m^2 + (p^1)^2$.

$$\Rightarrow \sinh k = -\frac{p_1}{m}$$

Consider a free particle carrying momentum $\vec{p}$ (in any direction)

$$e^{i/\hbar (Et - \vec{p}\cdot\vec{x})}.$$

Replace α by $\dfrac{\vec{\alpha}\cdot\vec{p}}{|\vec{p}|}$

$$\sinh k = -\frac{|\vec{p}|}{m}.$$

$$\cosh k = \sqrt{1 + \sinh^2 k} = \frac{\sqrt{m^2 + p^2}}{m}$$

$$= E/m$$

$$2\sinh^2\frac{k}{2} + 1 = 2\cosh^2\frac{k}{2} - 1$$

$$\cosh^2\frac{k}{2} = \frac{1}{2}\left(1 + \frac{E}{m}\right) \qquad \cosh\frac{k}{2} = \frac{1}{\sqrt{2}}\sqrt{\frac{E+m}{m}}$$

$$\sinh\frac{k}{2} = (-)\frac{1}{\sqrt{2}}\sqrt{\frac{E-m}{m}}.$$ (Check choice of sign!)

$$\begin{aligned}
S &= \left(\cosh\frac{k}{2}\mathbb{1} - \sinh\frac{k}{2}\,\alpha_1\right)\\
&= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{E+m}{m}}\,\mathbb{1} + \sqrt{\frac{E-m}{m}}\,\alpha_1\right)\\
&= \sqrt{\frac{E+m}{2m}}\left(\mathbb{1} + \sqrt{\frac{E-m}{E+m}}\,\alpha_1\right)\\
&= \sqrt{\frac{E+m}{2m}}\left(\mathbb{1} + \frac{\sqrt{E^2-m^2}}{E+m}\,\alpha_1\right)\\
&= \sqrt{\frac{E+m}{2m}}\left(\mathbb{1} + \frac{\cancel{|\vec{p}|}}{E+m}\,\cancel{\alpha_1}\,\frac{\vec{\alpha}\cdot\vec{p}}{\cancel{|\vec{p}|}}\right)\\
&= \sqrt{\frac{E+m}{2m}}\left(\mathbb{1} + \frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right)
\end{aligned}$$

Two solutions :-

$$\underbrace{\sqrt{\frac{E+m}{2m}}\left(\mathbb{1} + \frac{1}{E+m}\,\vec{\alpha}\cdot\vec{p}\right)}_{u_s(\vec{p})} u_s(0)\, e^{-i/\hbar\,(Ex^0 - \vec{p}\cdot\vec{x})}$$

Two new sets :-

? ?

$$u_s(\vec{p})\; v_s(0)\; e^{i/\hbar\,(Ex^0 - \vec{p}\cdot\vec{x})}$$ (Understand!)

This describes solutions with energy $-E$ and momentum $-\vec{p}$.

(I use the same boost parameter)

$$E = -m \Rightarrow E = -\sqrt{\vec{p}^2 + m^2}$$