02.02.2012

$$\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} \Psi - m\Psi = 0 \quad \text{[Dirac Equation]}$$

Solutions :— $\vec{p} = 0 \Rightarrow \Psi = \Psi_0 \, e^{i p_\mu x^\mu / \hbar}$ Solutions of this form

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imx^0/\hbar}, \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imx^0/\hbar}$$

$$= u_1(\vec{0}) = u(\vec{0}, \tfrac{1}{2}) \qquad = u_2(\vec{0}) = u(\vec{0}, -\tfrac{1}{2})$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{imx^0/\hbar}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{imx^0/\hbar}$$

$$= v_1(\vec{0}) = v(\vec{0}, \tfrac{1}{2}) \qquad = v_2(\vec{0}) = v(\vec{0}, -\tfrac{1}{2})$$

What for $\vec{p} \neq 0$.

General solution

$$u(\vec{p}, s)\, e^{i/\hbar \, p_\mu x^\mu}$$

$$v(\vec{p}, s)\, e^{-i/\hbar \, p_\mu x^\mu}$$

$$u(\vec{p}, s) = \sqrt{\frac{E+m}{2m}} \left(1 + \frac{\vec{\alpha}\cdot\vec{p}}{2m}\right) u(\vec{0}, s)$$

$(E+m)$

$$p^0 = \sqrt{\vec{p}^{\,2} + m^2} \text{ by definition}$$

$$v(\vec{p}, s) = \sqrt{\frac{E+m}{2m}} \left(1 + \frac{\vec{\alpha}\cdot\vec{p}}{2m}\right) v(\vec{0}, s)$$

u's and v's are not spin eigenstates.

$u(\vec{p}, s)$ & $v(-\vec{p}, s)$ form a complete basis of states for a fixed $\vec{p}$ at $t = 0$.

(Compare non-relativistic case)

$$[(J_x, J_y, J_z), (p_x, p_y, p_z)] \neq 0. \quad \text{(Remember)}$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}. \quad \therefore \vec{\alpha}\cdot\vec{p} = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0_{2\times 2} & \begin{matrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{matrix} \\ \begin{matrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{matrix} & 0_{2\times 2} \end{pmatrix}$$

$$u\left(\vec{p}, \frac{1}{2}\right) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ p_z/2m \\ (p_x+ip_y)/2m \end{pmatrix}$$

Similarly, $u\left(\vec{p}, -\frac{1}{2}\right) = \begin{pmatrix} 0 \\ 1 \\ (p_x - ip_y)/2m \\ -p_z/2m \end{pmatrix}$

Change $\frac{p_x \pm ip_y}{2m}$ or $\frac{p_z}{2m}$ to $\frac{p_x \pm ip_y}{E+m}$ & $\frac{p_z}{E+m}$

$$v\left(\vec{p}, \frac{1}{2}\right) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} p_z/(E+m) \\ (p_x+ip_y)/(E+m) \\ 1 \\ 0 \end{pmatrix}, \quad v\left(\vec{p}, -\frac{1}{2}\right) = \begin{pmatrix} (p_x - ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix}$$

Let's calculate $u^\dagger\left(\vec{p}, \frac{1}{2}\right) u\left(\vec{p}, \frac{1}{2}\right)$

$$= \frac{E+m}{2m}\left(1 + 0 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2 + p_y^2}{(E+m)^2}\right)$$

$$= \frac{E+m}{2m}\left(1 + \frac{E^2 - m^2}{(E+m)^2}\right)$$

$$= \frac{E+m}{2m}\left(1 + \frac{E-m}{E+m}\right)$$

$$= \frac{\cancel{(E+m)}}{\cancel{2}m}\,\frac{\cancel{2}E}{\cancel{(E+m)}} = E/m \quad \left(\text{Cause} \rightarrow \text{Lorentz contraction !}\right)$$

If we want δ-function normalized wave function, they will be given by $\sqrt{\frac{m}{E}}\,\frac{1}{(2\pi\hbar)^{3/2}}\, e^{i(\vec{p}\cdot\vec{x} - Et)/\hbar}\, u(\vec{p}, s)$

& $\sqrt{\frac{m}{E}}\,\frac{1}{(2\pi\hbar)^{3/2}}\, e^{-i(\vec{p}\cdot\vec{x} - Et)/\hbar}\, v(\vec{p}, s)$

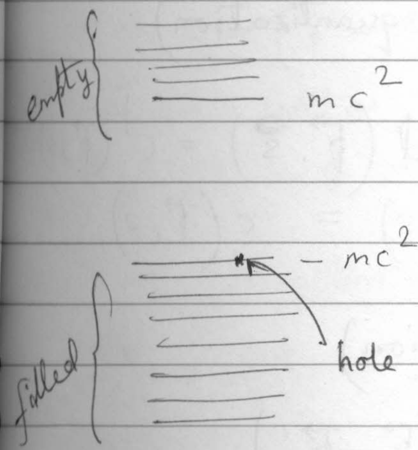
A general solution

$$\psi(\vec{x}, t) = \sum_s \int d^3p \sqrt{\frac{m}{E}}\,\frac{1}{(2\pi\hbar)^{3/2}} \times \Big[b(\vec{p}, s)\, u(\vec{p}, s) \times e^{i(\vec{p}\cdot\vec{x} - Et)/\hbar} + c(\vec{p}, s)\, v(\vec{p}, s) \times e^{-i(\vec{p}\cdot\vec{x} - Et)/\hbar} \Big]$$

$\Rightarrow$ b & c are arbitrary constants

(Compare non-relativistic counterpart)

(*) Let's think only about fermions



$$E = \sqrt{p^2c^2 + m^2c^4}$$

Consider an infinite particle system in which all the 'lower' family of states are filled and all the 'upper' ones are empty.

$\longrightarrow$ This defines 'vacuum'.

$\rightarrow$ Two types of ~~momentum~~ excitations.

$\longrightarrow$ antiparticle (!)

ENTER $\longrightarrow$ Second Quantization

$$\Psi = \sum_n a_n u_n(\vec{x}) \qquad \hat{H} = \sum_n e_n a_n^\dagger a_n$$

$$\{a_m, a_n^\dagger\} = \delta_{mn} \qquad \{a_m, a_n\} = 0 = \{a_m^\dagger, a_n^\dagger\}$$

$$\hat{H} = \sum_s \int d^3p \left[\sqrt{p^2+m^2}\, b^\dagger(\vec{p},s)\, b(\vec{p},s) - \sqrt{p^2+m^2}\, c^\dagger(\vec{p},s)\, c(\vec{p},s)\right]$$

$$\left\{ b(\vec{p},s), b^\dagger(\vec{p}',s') \right\} = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}')$$

$$\left\{ c(\vec{p},s), c^\dagger(\vec{p}',s') \right\} = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}')$$

$$b(\vec{p},s)|0\rangle = 0, \quad c(\vec{p},s)|0\rangle = 0.$$

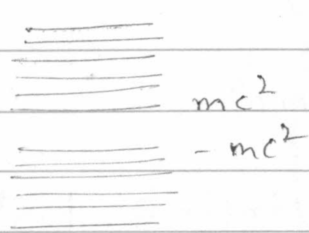
Definition of vacuum $\longrightarrow |0\rangle$.

Single particle states $\rightarrow b^\dagger(\vec{p},s)\,|0\rangle$

Energy $\rightarrow \sqrt{p^2+m^2}$.

$c^\dagger(\vec{p},s)\,|0\rangle$, Energy $\rightarrow -\sqrt{p^2+m^2}$.

("Naive" second quantization)



Define $d(\vec{p},s) = c^\dagger(\vec{p},s)$

$\therefore\ d^\dagger(\vec{p},s) = c(\vec{p},s)$

$\rightarrow$ "Feel" of fermions (anticommutation)

$$\left\{ d(\vec{p},s), d^\dagger(\vec{p}',s') \right\} = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}')$$

$$\hat{H} = \sum_s \int d^3p \left[\sqrt{\vec{p}^2+m^2}\, b^\dagger(\vec{p},s)\, b(\vec{p},s) - \sqrt{\vec{p}^2+m^2}\, \underbrace{d(\vec{p},s)\, d^\dagger(\vec{p},s)}_{= -d^\dagger(\vec{p},s)\, d(\vec{p},s) + \text{const}}\right]$$

infinite

$$= \sum_s \int d^3p \, \sqrt{\vec{p}^{\,2}+m^2} \left(b^\dagger(\vec{p},s)\, b(\vec{p},s) + d^\dagger(\vec{p},s)\, d(\vec{p},s) \right)$$

$$+ \text{ constant}$$

$$b(\vec{p},s)\,|0\rangle = 0, \qquad d(\vec{p},s)\,|0\rangle = 0.$$

$b^\dagger(\vec{p},s)\,|0\rangle$ = electron states of momentum $\vec{p}$, energy $\sqrt{\vec{p}^{\,2}+m^2}$.

$d^\dagger(\vec{p},s)\,|0\rangle$ = positron states of momentum $\vec{p}$, energy $\sqrt{\vec{p}^{\,2}+m^2}$.

(creating antiparticles)

11.02.2012

Non-relativistic particle in electromagnetic field

Restore $\hbar$, c. Our notation – $x^0 = ct$

$$p^0 = E/c \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

Classical Hamiltonian

$$H = \frac{1}{2m}\left(\vec{p} - e/c\,\vec{A}\right)^2 - e A_0 \qquad H = \frac{1}{2m}\left(p_k - \frac{eA_k}{c}\right)\left(p_k - \frac{eA_k}{c}\right)$$

$\vec{A} \longrightarrow$ vector potential $\qquad k = 1, 2, 3$

$A_0 \longrightarrow -\phi \longrightarrow$ electrostatic potential

Hamilton's equations :-

$$A_i \equiv A_i(\vec{x})$$

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} = \frac{1}{m}\left(p_i - e\frac{A_i}{c}\right)$$

$$\frac{dp^i}{dt} = -\frac{\partial H}{\partial x^i} = -\frac{1}{m}\left(p_k - \frac{eA_k}{c}\right)\left(-\frac{e}{c}\frac{\partial A_k}{\partial x^i}\right) + e\frac{\partial A_0}{\partial x^i}.$$

$$= \frac{dx^k}{dt}\frac{e}{c}\frac{\partial A_k}{\partial x^i} + e\frac{\partial A_0}{\partial x^i}$$

$$m \frac{dx^i}{dt} = p_i - \frac{e}{c} A_i$$

$$m \frac{d^2 x^i}{dt^2} = \frac{dp_i}{dt} - \frac{e}{c}\left(\frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x^k} \frac{dx^k}{dt} \right)$$

$$= \frac{e}{c} \frac{dx^k}{dt} \frac{\partial A_k}{\partial x^i} - \frac{e}{c} \frac{\partial A_i}{\partial t} - \frac{e}{c} \frac{\partial A_{\cancel{k}i}}{\partial x^k} \frac{dx^k}{dt} + e$$

$$= \underbrace{- e \frac{\partial \phi}{\partial x^i} - \frac{e}{c} \frac{\partial A_i}{\partial t}}_{eE_i} + \frac{e}{c} \frac{dx^k}{dt} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \right)$$

$$\vec{E} = - \vec{\nabla} \phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad E_i = - \partial_i \phi - \frac{1}{c} \frac{\partial A_0}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad B_i = \epsilon_{ijk}\, \partial_j A_k = \epsilon^{ijk}\, \partial_j A_k$$

$$\left(\partial_j A_k - \partial_k A_j \right) = \epsilon^{jkl} B_l$$

$$\therefore\ m \frac{d^2 \vec{x}}{dt^2} = e\left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right)$$

<u>Quantum Theory</u> :-

$$\hat{H} = \frac{1}{2m} \left(\hat{p}_i - \frac{e}{c} \hat{A}_i \right)\left(\hat{p}_i - \frac{e}{c} \hat{A}_i \right) - e \hat{A}_0$$

$$\hat{p}_i = - i\hbar \frac{\partial}{\partial x^i} \qquad [\hat{x}^i, \hat{p}^j] = i\hbar \delta_{ij}$$

Weak field (uniform magnetic field approximation)

Let's take $A_0 = 0$, $\vec{A} = \frac{1}{2}(\vec{B} \times \vec{r})$

$$A_i = \frac{1}{2} \epsilon^{imn} B_m x^n$$

(Particular choice of gauge)

$$\partial_i A_j - \partial_j A_i = \frac{1}{2} \epsilon^{jmi} B_m - \frac{1}{2} \epsilon^{imj} B_m$$

$$= \epsilon^{ijm} B_m \longrightarrow \text{taken as constant.}$$

$$\hat{H} = \frac{\vec{p}^{\,2}}{2m} - \frac{e}{2mc} \left(\hat{p}_i \hat{A}_i + \hat{A}_i \hat{p}_i \right) + O(B^2)$$

(We keep upto $O(B)$ terms.)

$$= \frac{p^2}{2m} - \frac{e}{2mc}\,\frac{1}{2}\left(\hat{p}_i\,\epsilon^{imn} B_m \hat{x}^n + \epsilon^{imn} B_m x^n \hat{p}_i\right)$$

$$[\hat{p}_i, \hat{x}^j] = -i\hbar\,\delta_{ij}$$

$$= \frac{p^2}{2m} - \frac{e}{4mc}\,\epsilon^{mni} B_m \left(\hat{x}^n \hat{p}_i + \hat{x}^n \hat{p}_i\right)$$

$$= \frac{p^2}{2m} - \frac{e}{2mc} B_m L_m \longrightarrow \text{angular momentum} \quad + O(B^2)$$

(Can also think as a non-relativistic limit)

$e/2mc\ \hat{L}_m \longrightarrow$ magnetic moment of orbiting electron

$$= \frac{p^2}{2m} - \mu_m B_m \Rightarrow \vec{\mu}\cdot\vec{B}$$

Gauge Invariance in Quantum Theory :-

Schrödinger Equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x^i} - \frac{e}{c}A_i(\vec{x},t)\right)\left(-i\hbar\frac{\partial}{\partial x^i} - \frac{e}{c}A_i(\vec{x},t)\right)\Psi - eA_0(\vec{x},t)\,\Psi.$$

- We'll use covariant notation.

$$x^0 = ct$$

$$\frac{i\hbar}{c}\frac{\partial\Psi}{\partial t} + \frac{e}{c}A_0(\vec{x},t)\,\Psi = \frac{1}{2mc}\left(-i\hbar\frac{\partial}{\partial x^i} - \frac{e}{c}A_i(\vec{x},t)\right)\left(-i\hbar\frac{\partial}{\partial x^i} - \frac{e}{c}A_i(\vec{x},t)\right)\Psi(\vec{x},t)$$

Qn. If we replace $A_i \longrightarrow A_i + \partial_i\Lambda$,

$A_0 \longrightarrow A_0 + \partial_0\Lambda \longrightarrow$ {Is this a symmetry?}

$$\Psi \longrightarrow e^{i\Lambda\frac{e}{\hbar c}}\,\Psi \quad \text{(condition for symmetry)}$$

$$\left(-i\hbar\frac{\partial}{\partial x^\mu} - \frac{e}{c}A_\mu - \frac{e}{c}\partial_\mu\Lambda\right) e^{i\Lambda\frac{e}{\hbar c}}\,\Psi$$

$$= e^{i\Lambda\frac{e}{\hbar c}}\left(-i\hbar\frac{\partial}{\partial x^\mu} - \frac{e}{c}A_\mu\right)\Psi$$

$$\Rightarrow \quad e^{i\Lambda \frac{e}{\hbar c}} \left(+ i\hbar \frac{\partial}{\partial x^0} + \frac{e}{c} A_0 \right) \psi = e^{i\Lambda \frac{e}{\hbar c}} \cdot \frac{1}{2mc} \times \left(-i\hbar \frac{\partial}{\partial x^i} - \frac{e}{c} A_i \right) \left(-i\hbar \frac{\partial}{\partial x^i} - \frac{e}{c} A_i \right)$$

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Return to free Dirac equation :-

(keep $\hbar$, c)

$$\left(\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - mc \right) \psi = 0 .$$

We use minimal coupling (through A_μ)

$$\underbrace{\left(\hbar \frac{\partial}{\partial x^\mu} - i \frac{e}{c} A_\mu \right)}_{\checkmark} \qquad \left\{ \gamma^\mu \left(\hbar \frac{\partial}{\partial x^\mu} - i \frac{e}{c} A_\mu \right) - mc \right\} \psi = 0$$

{Covariant derivative} $\longrightarrow$ Gauge invariance is manifest.

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda , \quad \psi \rightarrow e^{i e/\hbar c \Lambda} \psi$$

Moreover, we can add

$$c_1 F_{\mu\nu} F^{\mu\nu} + c_2 F_{\mu\nu} \sigma^{\mu\nu} \; - - - -$$

(non-minimal coupling)

- Any function of $F_{\mu\nu} F^{\mu\nu}$ can be added.

$$\left\{ \underset{\substack{\| \\ i\beta}}{\hbar \gamma^0} \frac{\partial}{\partial x^0} + \underset{\substack{\| \\ i\beta\alpha^i}}{\hbar \gamma^i} \frac{\partial}{\partial x^i} - i \frac{e}{c} \left(\underset{\substack{\| \\ i\beta}}{\gamma^0} A_0 + \underset{\substack{\| \\ i\beta\alpha^i}}{\gamma^i} A_i \right) - mc \right\} \psi = 0$$

$$\Rightarrow \boxed{ \begin{aligned} i\hbar \frac{\partial \psi}{\partial x^0} &= \cancel{-i\hbar \alpha^i \left(\frac{\partial}{\partial x^i} - \frac{e}{c} A_i \right)} \psi - \frac{e}{c} A_0 \psi + m \\ &\qquad \alpha^i \left(-i\hbar \frac{\partial}{\partial x^i} - \frac{e}{c} A_i \right) \psi \end{aligned} }$$

$$\psi = \begin{pmatrix} \hat{\phi} \\ \hat{\chi} \end{pmatrix} . \qquad \begin{aligned} \hat{\phi} &\longrightarrow \text{two component} \\ \hat{\chi} &\longrightarrow \text{two component} \end{aligned}$$

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$e^{-iEt/\hbar} \longrightarrow e^{-imc^2t/\hbar}$$

for $\hat{\phi}$ (non-relativistic case)

15.02.2012

Dirac Equation

$$\left(\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\psi = 0.$$

Non-relativistic limit : $c \longrightarrow \infty$.

Principle of Gauge Invariance $\longrightarrow$ suggests a minimal coupling

$$-i\hbar \frac{\partial}{\partial x^\mu} \longrightarrow -i\hbar\frac{\partial}{\partial x^\mu} + \frac{e}{c}A_\mu .$$

We rewrite Dirac equation as

$$\gamma^\mu\left(-i\hbar\frac{\partial}{\partial x^\mu} - \frac{e}{c}A_\mu\right)\psi + imc\psi = 0.$$

$A_\mu \longrightarrow$ vector potential.

Rewrite this in Hamiltonian formulation.

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

$$\gamma^0 = i\beta, \qquad \gamma^i = i\beta\alpha^i$$

$$i\beta\left(-i\hbar\frac{\partial}{\partial x^0} - \frac{e}{c}A_0\right)\psi + i\beta\alpha_i\left(-i\hbar\frac{\partial}{\partial x^i} - \frac{e}{c}A_i\right)\psi + imc\psi = 0.$$

(separating out 0 component)

Multiply both sides by $(-i\beta)\times c$ $\quad x^0 = ct$

$$\rightarrow c\left(-i\hbar\frac{\partial}{\partial x^0} - \frac{e}{c}A_0\right)\psi + c\alpha_i\left(-i\hbar\frac{\partial}{\partial x^i} - \frac{e}{c}A_i\right)\psi + \beta mc^2\psi = 0$$

$$\therefore\ i\hbar\frac{\partial\psi}{\partial t} = -eA_0\psi + c\alpha_i\left(-i\hbar\frac{\partial}{\partial x^i} - \frac{e}{c}A_i\right)\psi + mc^2\beta\psi$$

$$\phi \longrightarrow \begin{pmatrix}\hat{\phi}\\ \hat{\chi}\end{pmatrix}$$

$$\therefore\ i\hbar\begin{pmatrix}\partial\hat{\phi}/\partial t\\ \partial\hat{\chi}/\partial t\end{pmatrix} = -eA_0\begin{pmatrix}\hat{\phi}\\ \hat{\chi}\end{pmatrix} + mc^2\begin{pmatrix}\hat{\phi}\\ -\hat{\chi}\end{pmatrix} + \begin{pmatrix}-i\hbar\sigma^i c\dfrac{\partial\hat{\chi}}{\partial x^i} - e\sigma^iA_i\hat{\chi}\\ -i\hbar c\sigma^i\dfrac{\partial\hat{\phi}}{\partial x^i} - e\sigma^iA_i\hat{\phi}\end{pmatrix}$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}.$$ We now write the equations separa[tely] as

$$i\hbar \frac{\partial \hat{\phi}}{\partial t} = -eA_0 \hat{\phi} + mc^2 \hat{\phi} + c\left(-i\hbar \frac{\partial}{\partial x^i} - \frac{e}{c} A_i\right) \sigma_i$$

$$i\hbar \frac{\partial \hat{\chi}}{\partial t} = -eA_0 \hat{\chi} - mc^2 \hat{\chi} + c\left(-i\hbar \frac{\partial}{\partial x^i} - \frac{e}{c} A_i\right) \sigma_i \hat{\phi}.$$

Now take large c limit.

$$\hat{\phi} \sim \hat{\phi}_0 e^{-\frac{imc^2}{\hbar} t}, \quad \hat{\chi} \approx \hat{\chi}_0 e^{\frac{imc^2 t}{\hbar}}.$$

But, then $\hat{\phi}$ & $\hat{\chi}$ can't be taken as solutions of the Schrödinger equation.

We'll assume $\hat{\phi}_0$ & $\hat{\chi}_0$ to be slowly varying functions of time $\longrightarrow$ Justify (!)

<u>Define</u> :- $$\left.\begin{aligned} \hat{\phi} &= e^{-\frac{imc^2 t}{\hbar}} \phi \\ \hat{\chi} &= e^{-\frac{imc^2 t}{\hbar}} \chi \end{aligned}\right\}$$ (Take out phase)

$$i\hbar \frac{\partial \phi}{\partial t} = -eA_0 \phi + c\left(-i\hbar \frac{\partial}{\partial x^i} - \frac{e}{c} A_i\right) \sigma_i \chi.$$

$$i\hbar \frac{\partial \chi}{\partial t} = -eA_0 \chi + c\left(-i\hbar \frac{\partial}{\partial x^i} - \frac{e}{c} A_i\right) \sigma_i \phi - 2mc^2 \chi.$$

$\cdot$ ϕ, χ are slowly varying.

If ϕ & χ had been of same order $\longrightarrow$ $2mc^2 \chi$ domin[ates]

<u>Approximation</u> $\longrightarrow$ If ~~amplitude of~~ ϕ is of order 1, ~~amplitude~~ of χ is of order $1/c$.

Then we can adjust last two ~~terms~~ terms in the second equation.

$$\chi = \frac{1}{2mc}\left(-i\hbar\frac{\partial}{\partial x^i} - \frac{e}{c}A_i\right)\sigma_i\,\phi + O\left(\frac{1}{c^3}\right)$$

(has to hold!)

if $\chi \sim O(1/c)$

$\circledast$ Put this back in the ϕ-equation

$$i\hbar\frac{\partial\phi}{\partial t} = -eA_0\phi + \frac{1}{2m}\Big(\overbrace{-i\hbar\frac{\partial}{\partial x^i}}^{\hat{p}_i} - \frac{e}{c}A_i\Big)\Big(-i\hbar\frac{\partial}{\partial x^j} - \frac{e}{c}A_j\Big)\sigma_i\sigma_j\,\phi$$

$\phi \rightarrow$ two-component spinor

Use $\sigma_i\sigma_j = \delta_{ij} + i\,\epsilon^{ijk}\sigma_k$.

$$\therefore\ i\hbar\frac{\partial\phi}{\partial t} = -eA_0\phi + \frac{1}{2m}\left(\hat{p}_i - \frac{e\hat{A}_i}{c}\right)\left(p_i - \frac{e\hat{A}_i}{c}\right)\phi + \underbrace{\frac{i}{2m}\epsilon^{ijk}\left(-i\hbar\frac{\partial}{\partial x^i} - e/c\,A_i\right)\left(-i\hbar\frac{\partial}{\partial x^j} - \frac{e}{c}A_j\right)}_{\text{extra (!)}}\sigma_k\phi$$

$$\frac{i}{2m}\epsilon^{ijk}\left(-i\hbar\frac{\partial}{\partial x^i} - e/c\,A_i\right)\left(-i\hbar\frac{\partial}{\partial x^j} - e/c\,A_j\right)\sigma_k\phi$$

$$= \frac{ie}{2mc}\epsilon^{ijk}(+i\hbar)\underbrace{\left(\frac{\partial}{\partial x^i}A_j + A_i\frac{\partial}{\partial x^j}\right)}_{\left[\frac{\partial}{\partial x^i},\,A_j\right]}\sigma_k\phi$$

$= (-)\,A_j\dfrac{\partial}{\partial x^i}$

$\left[\dfrac{\partial}{\partial x^i}, A_j\right]$ commutator

$$= -\frac{e\hbar}{2mc}\epsilon^{ijk}\partial_i A_j\sigma_k\phi$$

$$= -\frac{e\hbar}{2mc}B_k\sigma_k\phi \quad (!)$$

$$i\hbar\frac{\partial\phi}{\partial t} = \hat{H}\phi$$

$$\hat{H}\phi = +\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 - eA_0\phi + (-)\frac{e\hbar}{2mc}B_k\sigma_k\phi.$$

spin operator (?)

$$-\frac{e}{mc}B_k\,S_k\,\phi$$

$S_k = \dfrac{\hbar}{2}\sigma_k$

Magnetic energy of a magnetic moment $\vec{\mu}$.

$$-\vec{\mu}\cdot\vec{B} \qquad \vec{\mu} = \frac{e}{mc}\vec{S} \quad \text{(spin moment of an electron)}$$

- Weak uniform $\vec{B}$ field limit

$$H = \frac{p^2}{2m} - \frac{e}{2mc}\vec{L}\cdot\vec{B}.$$

$\vec{L} \longrightarrow$ orbital angular momentum $(\vec{r}\times\vec{p})$

$$\therefore \vec{\mu}_{\text{total}} = \frac{e}{2mc}\left(\vec{L} + 2\vec{S}\right).$$

Weak uniform magnetic field limit

$$H = \frac{p^2}{2m} - \underbrace{\frac{e}{2mc}\left(\vec{L}\right.}_{\text{orbital}}\left.+ 2\vec{S}\right)\cdot\vec{B}.$$

QFT $\longrightarrow$ further corrections {electron gyromagnetic ratio (g_s)}

Relativistic H-atom

$$i\hbar\frac{\partial\Psi}{\partial t} = \vec{\alpha}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right) + \beta m - eA_0$$

Set $\hbar = c = 1$ from now on.

For hydrogen atom, we can set $\vec{A} = 0$.

$$-eA_0 = V(r) = -\frac{C}{r}. \longrightarrow \text{\{square of electric charge\}}$$

$$H = \vec{\alpha}\cdot\vec{p} + \beta m + V(r)$$

Central force problem

Here $[L_i, H] = 0$

$$[L_i, H] = [L_i, \vec{\alpha}\cdot\vec{p}]$$

$$= \alpha_j \,(i\underset{=1}{\hbar})\, \epsilon^{ijk} p_k = i\underset{=1}{\hbar}\, \epsilon^{ijk}\alpha_j p_k$$

$$[S_i, H] = ? \qquad S_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

$$[S_i, \beta] = 0, \quad [S_i, V(r)] = 0.$$

$$[S_i, \alpha_j p_j] = [S_i, \alpha_j] p_j \qquad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

$$[S_i, \alpha_j] = \frac{1}{2}\begin{pmatrix} 0 & [\sigma_i, \sigma_j] \\ [\sigma_i, \sigma_j] & 0 \end{pmatrix} \qquad [\sigma_i, \sigma_j] = 2i\epsilon^{ijk}\sigma_k$$

$$= i\epsilon^{ijk}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} = i\epsilon^{ijk}\alpha_k$$

$$\therefore \; [S_i, H] = i\epsilon^{ijk}\alpha_k p_j$$

$$\therefore \; [\underbrace{L_i + S_i}_{J_i \text{ (total angular momentum)}}, H] = 0. \quad (!)$$

<u>Exercise</u> $\longrightarrow$ $[J_i, J_j] = i\epsilon^{ijk} J_k.$

(Check!)

$\Rightarrow$ Eigenstates of H can be taken to be also eigenstates of J^2 and J_z (or J_3).

<u>Parity</u> :- Parity operator $\hat{P}$

$$\hat{P}\psi(x) = \psi(-x).$$

$$HP\psi = \left[-i\hbar\alpha^i \frac{\partial}{\partial x^i} + \beta m + V(r)\right]\psi(-\vec{r})$$

$$PH\psi = \left[\left(-i\hbar\alpha^i \frac{\partial}{\partial x^i} + \beta m + V(r)\right)\psi(\vec{r})\right]$$

$\vec{r} \to -\vec{r}$

$$= \left[i\hbar\alpha^i \frac{\partial}{\partial x^i} + \beta m + V(r)\right]\psi(-\vec{r})$$

So, $[P, H] \neq 0.$

Try $P\psi(\vec{x}) = \underbrace{U}_{\text{some matrix}} \psi(-\vec{x})$

$\underbrace{U^2 = \mathbb{1}}$.

We want $\quad HP\psi = PH\psi$

$$\left(-i\hbar\, \alpha^i \frac{\partial}{\partial x^i} + \beta m + V(\cancel{x})\right) U = U\left(i\hbar\, \alpha^i \frac{\partial}{\partial x^i} + \beta m +\right.$$

$\{ U \to \text{fixed matrix} \}$ $\qquad \beta U = U\beta$ ✓

$$\underbrace{\alpha^i U = -U\alpha^i}$$

So, $\quad [\beta, U] = 0, \qquad \{\alpha^i, U\} = 0.$

Possible choice : $U = \beta$.

$\Rightarrow \quad P\psi(\vec{r}) = \beta\psi(-\vec{r}) \qquad$ (Nice!)

Check if $[P, J_i] = 0$?

$J_i = L_i + S_i$. $\qquad P = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ & $\vec{r} \to -\vec{r}$

$\vec{L}$ is unchanged under $\vec{r} \to -\vec{r}$ $\qquad S_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$ ✓ (commutes)

$[L_i, P] = 0 \qquad\qquad [L_i, P] = 0, \ [S_i, P] =$

(Matrix part $\to$ no problem) $\qquad \Rightarrow [J_i, P] =$

$J_z, J^2, P \longrightarrow$ simultaneous eigenstates

(!)

$L^2, L_z \longrightarrow$ non-relativistic