07.03.2012

Path integral formulation :-

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}).$$ Suppose, we use Heisenberg picture

(*) States are time-independent.

Operators are time-dependent.

$|q\rangle \rightarrow$ eigenstates of $\hat{q}$ with eigenvalue q.

$$\langle q | q' \rangle = \delta(q - q').$$

- Easily generalized to multiple dimensions

Define $K(q', t'; q'', t'') = \langle q'' | e^{-iH(t''-t')/\hbar} | q$

- This has all the information of the quantum system

Exercise

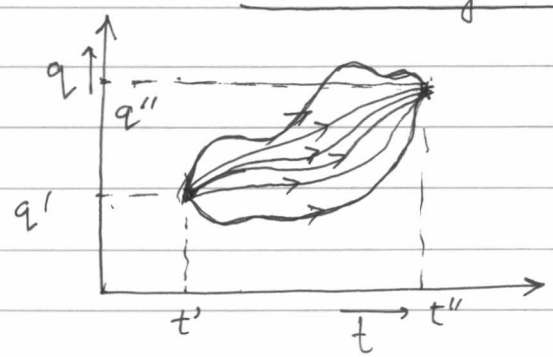
Check that $i\hbar \frac{\partial}{\partial t''} K(q', t'; q'', t'')$

$$= \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q''^2} + V(q'') \right\} K(q', t'; q'', t'')$$

$$\lim_{t'' \to t'} K(q', t'; q'', t'') = \delta(q'' - q')$$

$K \rightarrow$ kernel

Path Integral Prescription :-



$$K(q', t'; q'', t'') = \sum_{\text{paths}} e^{i S_{path}/\hbar}$$

S_{path} → actional along that path

$$S_{path} = \int_{path} dt \left[\frac{1}{2} m \left(\frac{dq}{dt} \right)^2 - V(q) \right]$$

(functional)

- What does it mean to sum over "all paths" ?

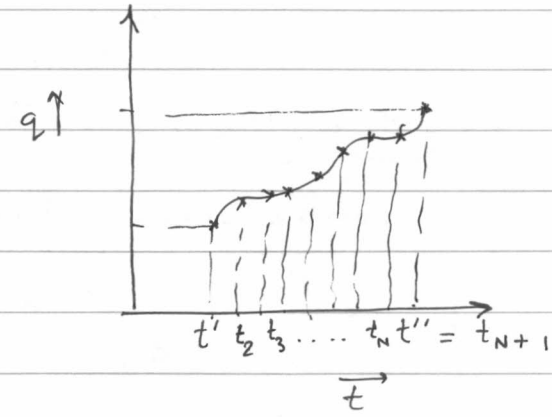
$$S = \int_{t'}^{t''} dt \left[\frac{1}{2} m \left(\frac{dq}{dt} \right)^2 - V(q) \right]$$

$$q(t') = q' , \quad q(t'') = q''.$$

$$\int_{t'}^{t''} f(t)\, dt = \lim_{\substack{N \to \infty \\ \Delta \to 0 \\ N\Delta = t''-t'}} \Delta \sum_{k=0}^{N-1} f(t' + k\Delta)$$

$$\Delta = \frac{t''-t'}{N}$$

$$S = \lim_{\substack{N \to \infty \\ \Delta \to 0 \\ N\Delta = t''-t'}} \Delta \sum_{k=0}^{N-1} L(t' + k\Delta)$$



$$q(t_i) = q_i$$

$$L_i = L\left(t' + (i-1)\Delta\right) \qquad t_i = t' + (i-1)\Delta$$

$$= -V(q_i) + \frac{1}{2} m \left(\frac{q_{i+1} - q_i}{\Delta} \right)^2$$

- We'll eventually take $\Delta \to 0$ limit

$$S = \Delta \sum_{i=1}^{N} \left\{ \frac{1}{2} m \left(\frac{q_{i+1} - q_i}{\Delta} \right)^2 - V(q_i) \right\}$$

$$N\Delta = t'' - t'.$$

Path integral prescription :-

$$K(q', t' ; q'', t'') = \lim_{\substack{\Delta \to 0 \\ N \to \infty \\ \Delta N = (t''-t')}} \int dq_2 \cdots dq_N \exp\left[\frac{i\Delta}{\hbar} \sum_{i=1}^{N} \left\{ \frac{1}{2} m \left(\frac{q_{i+1} - q_i}{\Delta} \right)^2 - V(q_i) \right] \right]$$

$\times \; N$

normalization factor (indep. of t', t'', q', q'').

q_{i+1} & q_i can be largely different from each other

$\rightarrow$ Integrand becomes small for large fluctuations

Original definition :-

$$K(q', t'; q'', t'') = \langle q'' | e^{iH(t''-t')/\hbar} | q' \rangle$$

$$\frac{t''-t'}{N} = \Delta \quad \text{for some large no. } N.$$

$$= \langle q'' | \underbrace{e^{-i\Delta H/\hbar} e^{-i\Delta H/\hbar} \cdots e^{-i\Delta H/\hbar}}_{N \text{ times}} | q' \rangle$$

$\cdots H(t'+\Delta)\, H(t')$

If $\underbrace{H = H(t)}$, this way of writing holds.

Use completeness relation $\Rightarrow \int dq\, |q\rangle\langle q| = \mathbb{1}$

$$\int dq_2\, dq_3 \cdots dq_N \langle q'' | e^{-iH\Delta/\hbar} | q_N \rangle \langle q_N | e^{-iH\Delta/\hbar} | q_{N-1} \rangle \cdots \langle q_2 | e^{-iH\Delta/\hbar} | q' \rangle$$

$q'' = q_{N+1}$, $q' = q_1$

- Still we have not assumed that Δ is small.

Typical term

$$\cdots \langle q_{i+1} | e^{-iH\Delta/\hbar} | q_i \rangle \langle q_i | e^{-iH\Delta/\hbar} | q_{i-1} \rangle$$

$$e^{-iH\Delta/\hbar} = e^{-i\left(\frac{\hat{p}^2}{2m} + V(\hat{q})\right)\frac{\Delta}{\hbar}}.$$

$$e^{A+B} = e^A e^B \times e^{[A,B]} \rightarrow \text{factor of } \Delta^2$$

$O(\Delta^2) \rightarrow$ drop

$$e^{-i\left(\frac{\hat{p}^2}{2m}\right)} e^{-iV(\hat{q})\frac{\Delta}{\hbar}}$$

$$= e^{-i\sum_{i=1}^{N} V(q_i)\frac{\Delta}{\hbar}} \langle q_{N+1} | e^{-i\Delta/\hbar\, \frac{\hat{p}^2}{2m}} | q_N \rangle$$

$$\langle q_N | e^{-i\frac{\Delta}{\hbar}\frac{\hat{p}^2}{2m}} | q_{N-1} \rangle \cdots \langle q_{i+1} | e^{-i\frac{\Delta}{\hbar}\frac{\hat{p}^2}{2m}} | q_i \rangle \langle q_i | e^{-i\frac{\Delta}{\hbar}\frac{\hat{p}^2}{2m}} | q_{i-1} \rangle \cdots$$

$$\cdots \langle q_{i+1} | e^{-i\frac{\Delta}{\hbar}\frac{\hat{p}^2}{2m}} | q_i \rangle$$

$$= \int dp_i\, \langle q_{i+1} | e^{-i\frac{\Delta}{\hbar}\frac{\hat{p}^2}{2m}} | p_i \rangle \langle p_i | q_i \rangle$$

$$= \int dp_i\, e^{-i\frac{\Delta}{\hbar}\frac{p_i^2}{2m}} \langle q_{i+1} | p_i \rangle \langle p_i | q_i \rangle$$

$$= \int dp_i \, e^{-i\frac{\Delta}{\hbar}\frac{p_i^2}{2m}} \, A e^{i p_i q_{i+1}/\hbar} \, e^{-i p_i q_i/\hbar}$$

$$= A \int dp_i \, e^{-\frac{i}{2m}\frac{\Delta}{\hbar} p_i^2 + i p_i (q_{i+1} - q_i)/\hbar}$$

$$= A \int dp_i \, e^{-\frac{i}{2m}\frac{\Delta}{\hbar}\left\{ p_i^2 - 2m \frac{(q_{i+1} - q_i)}{\Delta} p_i \right\}}$$

{Complete the square $\underbrace{\left\{ p_i - m\frac{(q_{i+1} - q_i)}{\Delta} \right\}}_{u_i}^2 - m^2 \left(\frac{q_{i+1} - q_i}{\Delta}\right)^2$

u_i (new integration variable)

$$= A \left(\int du_i \, e^{-\frac{i}{2m}\frac{\Delta}{\hbar} u_i^2} \right) e^{\frac{i}{2} m \frac{\Delta}{\hbar} \times \left(\frac{q_{i+1} - q_i}{\Delta}\right)^2}$$

$$= (AB) \, e^{\frac{i}{2} m \frac{\Delta}{\hbar} \left(\frac{q_{i+1} - q_i}{\Delta}\right)^2}.$$ (This is just one factor)

<u>Final result</u>

$$K(q', t'; q'', t'') = (AB)^N \int \exp\left\{ e^{-i\frac{\Delta}{\hbar}\sum_{i=1}^{N} V(q_i)} \right\} \exp\left\{ \frac{i}{2} m \frac{\Delta}{\hbar} \sum_{i=1}^{N} \left(\frac{q_{i+1} - q_i}{\Delta}\right)^2 \right\} dq_2 \cdots dq_N$$

$q' \to q_1$, $q'' \to q_{N+1}$

$\mathcal{N}$ (normalization factor)

$$= \mathcal{N} \int dq_2 \cdots dq_N \exp\left[i/\hbar \, \Delta \sum_{i=1}^{N} \left\{ \frac{1}{2} m \left(\frac{q_{i+1} - q_i}{\Delta}\right)^2 - V(q_i) \right\} \right]$$

(*) <u>QFT</u> Now $\to$ {just an alternative}

$$\frac{dq_i}{dt} \to \frac{q_{i+1} - q_i}{\Delta}$$

or, $\frac{q_i - q_{i-1}}{\Delta}$ ----

Complicated Hamiltonians $\longrightarrow$ Result depends on method of discretization ---.

$p \cdot q$ (products)

$pq \longrightarrow$?

$$e^{\overset{A}{\frac{p^2}{2m}}}\, e^{\overset{B}{q}} = e^{\overset{A+B}{qp \;\; qp^2}}$$

$$H(q, p^2) = f(q)\, p^2 + \cdots$$

(No unique quantization)

$$e^{f(p)\, g(q)} \longrightarrow \text{order } \Delta \text{ corrections}$$

$$\langle q'' | e^{-iH(t''-t')/\hbar} | q' \rangle$$

$$\langle q'' | e^{-iH(t''-t_2)/\hbar} \hat{q}(t_2) e^{-iH(t_2-t_1)/\hbar} \hat{q}(t_1) e^{-iH(t_1-t')/\hbar} | q' \rangle$$

$$= \mathcal{N} \int dq_2 \cdots dq_N \; e^{iS/\hbar}\, q(t_1) \cdots q(t_n)$$

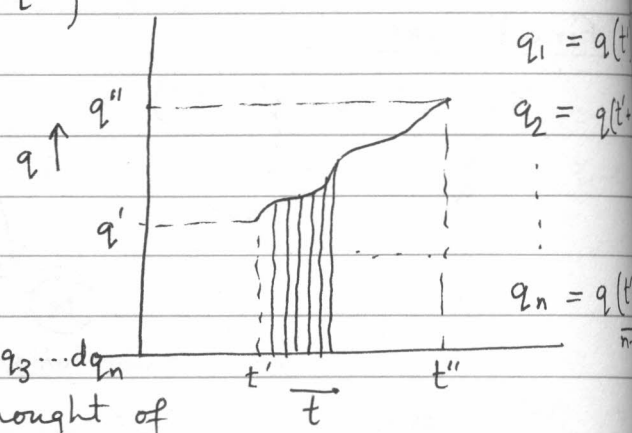
<u>16.03.2012</u>

$$\langle q'' | e^{-iH(t''-t)/\hbar} | q' \rangle$$

$$= K(q', t'; q'', t'')$$

$$= \int [\mathcal{D}q]\, e^{iS/\hbar}$$

sum over paths



$$[\mathcal{D}q] = \mathcal{N}\, dq_2\, dq_3 \cdots dq_n$$

- This measure can be thought of as this

$q(t_i) \longrightarrow q_i$ (at $t = t' + i\Delta$)

$S \longrightarrow$ discretized version

$$\dot{q}(t) = \frac{q_{i+1} - q_i}{\Delta} \quad \text{at} \quad t = t' + i\Delta$$

$\mathcal{N} \longrightarrow$ overall normalization

can be fixed by the requirement $K(q', t'; q'', t'') \to \delta(q''-q$ (for $t'' \to t'$)

Consider the object $\langle q'' | e^{-iH(t''-\tau_l)/\hbar} q(\tau_l)\, e^{-iH(\tau_l-\tau_{l-1})/\hbar} q(\tau_{l-1}) \cdots e^{-iH(\tau_2-\tau_1)/\hbar} q(\tau_1)\, e^{-iH(\tau_1-t')/\hbar} | q' \rangle$

τ's $\longrightarrow$ time intervals

$t'' > \tau_l > \tau_{l-1} > \cdots > \tau_1 > t'$.

$q(\tau) \rightarrow$ Heisenberg representation of operator q.

$$= e^{+iH\tau/\hbar}\, q\, e^{-iH\tau/\hbar}$$

(q at $\tau = 0$)

We want to calculate

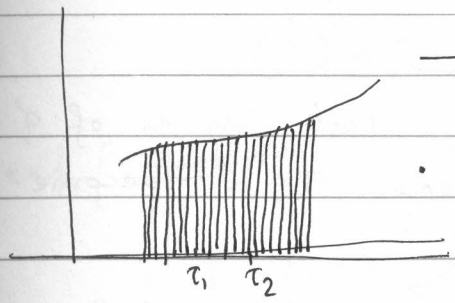
$$\langle q'' | e^{-iHt''/\hbar}\, q(\tau_\ell)\, q(\tau_{\ell-1}) \cdots q(\tau_1)\, e^{iHt'/\hbar} | q' \rangle$$

$$= \langle q'' | e^{-iH(t''-\tau_\ell)/\hbar}\, q\, e^{-iH(\tau_\ell-\tau_{\ell-1})/\hbar}\, q\, e^{-iH(\tau_{\ell-1}-\tau_{\ell-2})/\hbar} \cdots e^{-iH(\tau_2-\tau_1)/\hbar}\, q\, e^{-iH(\tau_1-t')/\hbar} | q' \rangle$$

$$= \int \mathcal{D}q\, e^{iS/\hbar}\, q(\tau_1)\, q(\tau_2) \cdots q(\tau_\ell)$$

(A classical integral)

$\longrightarrow$ any ordering possible



- Concept of time-ordering (TIME-ORDERED PRODUCT)

$$\langle q'' | e^{-iHt''/\hbar}\, T\big(q(\tau_1) \cdots q(\tau_\ell)\big)\, e^{iHt'/\hbar} | q' \rangle$$

$$\Downarrow$$

$$\Theta(\tau_n - \tau_{n-1})\, \Theta(\tau_{n-1} - \tau_{n-2}) \cdots \Theta(\tau_2 - \tau_1)\, q(\tau_n) \cdots q(\tau_1)$$

$+ (n! - 1)$ terms related by permutation of $\tau_1, \cdots \tau_n$

$\Theta = 1$ if $x > 0$
$= 0$ if $x < 0$ } Heaviside function

$$H = \frac{p^2}{2m} + V(q)$$

$$\langle q'' | e^{-iHt''/\hbar}\, T\Big(m \frac{dq(\tau_\ell)}{d\tau_\ell}\, q(\tau_{\ell-1}) \cdots q(\tau_1) \Big) e^{iHt'/\hbar} | q' \rangle$$

$$\frac{dq}{dt} = \frac{p(\tau)}{m}$$

$$p(\tau) = m \frac{dq(\tau)}{d\tau}$$

$$= m \frac{d}{d\tau_\ell} \Big(\qquad \Big)$$

$$+ \sum \delta(\tau_\ell - \tau_i)\, \Theta \ldots \Theta \ldots \Theta$$

$\tau_1, \tau_2, \tau_3, \tau_4$ $\qquad\qquad$ $\Theta(\tau_2-\tau_1)\,\Theta(\tau_1-\tau_3)$

$\delta(\tau_1-\tau_2)\,\Theta(\tau_2-\tau_3)\,\Theta(\tau_3-\tau_4)$ $\qquad\qquad$ $\Theta(\tau_3-\tau_4)$

$\Theta(\tau_1-\tau_2)\,\Theta(\tau_2-\tau_3)\,\Theta(\tau_3-\tau_4) \Rightarrow \delta(\tau_1-\tau_2)\,\Theta(\tau_2-\tau_3)\,\Theta$

$q(\tau_1)q(\tau_2)\,q(\tau_3)\,q(\tau_4)$ $\qquad\qquad$ $q(t_2)\,q(t_1)\,q(t_3)\,q(t_4)$.

Two terms.

$[q(\tau_1), q(\tau_2)] = 0$ as there's $\delta(\tau_1-\tau_2)$

$\rightsquigarrow$ one p operator

What is $\dfrac{d}{d\tau_2}\langle\,|\,T\left(p(\tau_1)\,q(\tau_2)\cdots q(\tau_n)\right|\,\rangle$

$\longrightarrow$ $p(\tau_1)\,q(\tau_2)\cdots - p(\tau_2)\,q(\tau_1)\cdots$

$\longrightarrow$ $i\hbar$ terms. (with less no. of q's)

- Complication ~~to~~ comes when some τ_i's become equal.

$$\langle q''|e^{-iH(t''-t')/\hbar}|q'\rangle$$

$$= \sum_{m,n}\langle q''|n\rangle\langle n|e^{-iH(t''-t')/\hbar}|m\rangle\langle m|q'\rangle$$

write

$$= \sum_n \langle q''|n\rangle\langle n|q'\rangle\, e^{-iE_n(t''-t')/\hbar}.$$

$t \to -i\Lambda$. $\qquad$ $e^{-\Lambda E_n/\hbar}$

(Wick rotation) $\qquad$ $\Lambda \longrightarrow \infty$

$\longrightarrow$ Get E_n's.

(at least ground state)

- Numerical studies.

21.03.2012

$$K(q',t';q'',t'') = \langle q'',t''|q',t'\rangle$$

$$= \langle q''|\exp\left(-i\frac{H}{\hbar}(t''-t')\right)|q'\rangle$$

$$= \int[\mathcal{D}q]\,e^{iS/\hbar} = \mathcal{N}\int dq_2\cdots dq_n \exp\left[\frac{i}{\hbar}\epsilon\sum_{i=1}^{n}\left\{\frac{(q_{i+1}-q_i)^2}{2\epsilon} - V(q_i)\right\}\right]$$

$\epsilon = t'' - t'$. $\qquad \underbrace{q_1 = q_0' \;,\; q_{n+1} = q''}$

$$\underbrace{N = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-n/2}}_{\text{(can be determined independently)}}$$

<u>free particles</u> $\longrightarrow$ $N \int dq_2 \cdots dq_n \exp\left[\frac{i}{\hbar\epsilon}\sum_{i=1}^{n}(q_{i+1}-q_i)^2\right]$

Consider the integral : $\int dx \exp\left[\frac{i}{\hbar\epsilon}\left\{a(x-y)^2 + b(x-z)^2\right\}\right]$

$$(a+b)x^2 - 2x(ay+bz) + ay^2 + bz^2$$

$$= (a+b)\left\{x^2 - 2\frac{ay+bz}{a+b}x + \frac{ay^2+bz^2}{a+b}\right\}$$

$$\left(x - \frac{ay+bz}{a+b}\right)^2 - \left(\frac{ay+bz}{a+b}\right)^2 + \frac{ay^2+bz^2}{a+b}.$$

$$\int dx \exp\left[\frac{i(a+b)}{\hbar\epsilon}\left(x - \frac{ay+bz}{a+b}\right)^2 - \frac{1}{(a+b)}\frac{i}{\hbar\epsilon}\left\{(ay+bz)^2 - (a+b)(ay^2+bz^2)\right\}\right]$$

$$= \int dx \exp\left[\frac{i(a+b)}{\hbar\epsilon}\left(x - \frac{ay+bz}{a+b}\right)^2\right] \times$$

$$\exp\left[-\frac{i}{\hbar\epsilon}\frac{1}{(a+b)}\left\{(ay+bz)^2 - (a+b)(ay^2+bz^2)\right\}\right]$$

$$a^2y^2 + b^2z^2 + 2abyz - a^2y^2 - abz^2 - aby^2 - b^2z^2$$

$$= \int dx \exp\left[\frac{i(a+b)}{\hbar\epsilon}\left(x - \frac{ay+bz}{a+b}\right)^2\right] \times \exp\left[\frac{i}{\hbar\epsilon}\frac{ab}{a+b}(y-z)^2\right]$$

$\uparrow$ convergence . .

· Concept of analytic continuation.

$$\int_{-\infty}^{\infty} \exp\left[-\frac{a+b}{i\hbar\epsilon}\left(x - \frac{ay+bz}{a+b}\right)^2\right] dx$$

$\Downarrow$ ~~pt~~ pretend to be real

$$= \sqrt{\frac{\pi i\hbar\epsilon}{a+b}}.$$

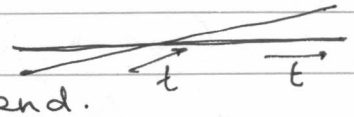
Imagine $\left(\frac{a+b}{i\hbar\epsilon}\right) = \cancel{x+i\eta}\; -iR$ (purely imaginary)

Add a small real part (+ve)

$$= -iR + \eta.$$

→ This makes the integral meaningful.

$$\underbrace{a+b \longrightarrow a+b+i\eta}$$



Take $\eta \to 0+$ in the end.

$$\int dx \, \exp\left[\frac{i\,\cancel{(a+b)}}{\hbar\epsilon}\left\{a(x-y)^2 + b(x-z)^2\right\}\right]$$

$$= \sqrt{\frac{\pi\hbar i\epsilon}{a+b}} \exp\left[\frac{i}{\hbar\epsilon}\,\frac{ab}{a+b}\,(y-z)^2\right]$$

<u>Path integral</u> :- $N\int dq_2 \cdots dq_n \exp\left[-\frac{i}{2\hbar\epsilon}\, m \sum_{i=1}^{n} (q_i - q_{i-1}\right.$

$$a = b = \frac{m}{2} \quad \text{here}$$

$$= \int dq_3 \cdots dq_n \, N \sqrt{\frac{\pi i\hbar\epsilon}{m}} \exp\left[\frac{i}{2\hbar\epsilon}\, m\left\{\frac{(q_1-q_3)^2}{2} + (q_3-q_4)^2 + (q_4-q_5)^2 + \cdots\right.\right.$$

$$= N\sqrt{\frac{\pi\hbar i\epsilon}{m}}\sqrt{\frac{\pi\hbar i\epsilon}{m}\times\frac{4}{3}} \exp\left[\frac{i}{\hbar}\,\frac{m}{2\epsilon}\left\{\frac{1}{3}(q_1-q_4)^2 + (q_4-q_5)^2 + \cdots\right.\right.$$

$$= \cdots\cdots = \exp\left[\frac{i}{\hbar}\,\frac{m}{2\epsilon}\,\frac{(q_1-q_{n+1})^2}{n}\right]$$

$$= \tilde{N} \exp\left[\frac{i}{\hbar}\,\frac{m}{2}\,\frac{(q''-q')^2}{t''-t'}\right]$$

$$= N\left(\sqrt{\frac{\pi i\hbar\epsilon\times 2}{m}}\right)^{n-1}\sqrt{\prod_i \frac{1}{(a_i+b_i)}}$$

$$\sqrt{\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\cdots\frac{n-1}{n}} = \sim\sqrt{\frac{1}{n}}$$

So, eventually, path integral

$$= \left(\sqrt{\frac{2\pi i\hbar\epsilon}{m}}\right)^{-n}\left(\sqrt{\frac{2\pi i\hbar\epsilon}{m}}\right)^{n-1} \times \frac{1}{\sqrt{n}} \exp\left[\frac{i}{\hbar}\,\frac{m}{2}\,\frac{(q''-q'}{(t''-}\right.$$

$$= \sqrt{\frac{m}{2\pi i \hbar \epsilon n}} \exp\left[\cdot \cdot \right] = \sqrt{\frac{m}{2\pi i \hbar (t''-t')}} \exp\left[\frac{i}{\hbar}\frac{m}{2}\frac{(q''-q')^2}{(t''-t')}\right]$$

(*) What should N depend on ? (not q'', q')

$\longrightarrow \phi(t''-t')$

(Assuming time-independent Hamiltonian)

In general, path integral

$$K(q',t';q'',t'') = \underbrace{f(t''-t')}_{\text{unknown}} \underbrace{g(q',t';q'',t'')}_{\text{known}}$$

(how have we divided time slices)

$$K(q',t';q'',t'')$$
$$= \langle q'', t'' | q', t' \rangle$$

$$\int \langle q'', t'' | q, t \rangle \langle q, t | q', t' \rangle \, dq$$

$$= K(q',t';q'',t'')$$

$$f(t''-t')\, f(t-t') \int dq\, g(q,t;q'',t'')\, g(q',t';q,t)$$

$$= f(t''-t')\, g(q',t';q'',t'')$$

$$f(t''-t) \longrightarrow e^{iC(t''-t)} f(t''-t)$$

(satisfies)

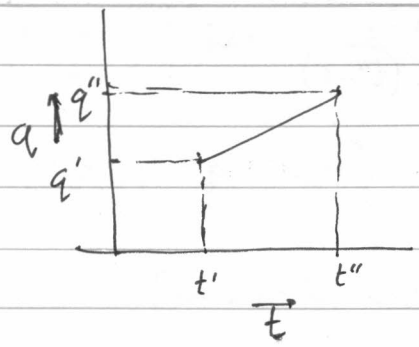
constant added to the Hamiltonian

* Try to fix the normalization using this

<u>Exercise</u> :- Take $K(q',t';q'',t'')$

$$= f(t''-t') \exp\left[\frac{i}{2} m/\hbar \frac{(q''-q')^2}{t''-t'}\right]$$

and fix $f(t''-t')$.



$$S_{cl.} = \frac{1}{2} m \left(\frac{q'' - q'}{t'' - t'}\right)^2 \times (t'' - t')$$

$$\therefore e^{iS_{cl.}/\hbar} = e^{\left[\frac{im}{2\hbar}\left(\frac{q''-}{(t''-}\right.\right.}$$

Introduce new variables $u(t)$ such that

$$u(t) = q(t) - \underbrace{q_{cl}(t)}_{\text{classical path}}$$

$$\int [\mathcal{D}q]\, e^{iS[q]/\hbar} = \int [\mathcal{D}u]\, e^{i/\hbar\, S(q_{cl}+u)}$$

(generally valid)

$$= \int [\mathcal{D}u]\, e^{i/\hbar \{S[q_{cl}] + O(u^2) + \cdots\}}$$

$$\boxed{\left.\frac{\partial S}{\partial q}\right|_{q_{cl}} = 0.}$$

$\approx (\;) \times$ Gaussian Integral (exact for free particle)

$$= e^{i/\hbar\, S[q_{cl}]} \int \mathcal{D}u\, e^{i/\hbar\, [O(u^2) + \cdots]}$$

expansion in powers of

- $\hbar \to 0$ limit
- Dispersion $\rightsquigarrow \sqrt{\hbar}$ [oscillatory for large u]

$$\underbrace{u(t') = 0, \quad u(t'') = 0}_{\text{(Boundary conditions)}}$$

<u>Free particle</u>

$$S = \frac{1}{2} m \int \dot{q}^2\, dt. \qquad S(q_{cl} + u)$$

$$= \frac{1}{2} m \int \dot{q}_{cl}^2\, dt + \frac{1}{2} m \int \dot{u}^2\, dt$$

$$\therefore K(q', t'; q'', t'') = e^{i/\hbar\, S(q_{cl.})} \int \mathcal{D}u\, e^{i/\hbar\, S(u)}$$

independent of classical path (spl. for quadratic actions)

$$\boxed{u(t') = 0 \;;\; u(t'') = 0}$$

(*) {Also happens for SHO}

$$u = \sum_{n=1}^{\infty} a_n \sin\left(\frac{(t''-t')}{t''-t'} n\pi\right)\sqrt{\frac{2}{(t''-t')\pi}} .$$

$$\left[\frac{(q''-q')^2}{(t''-t')}\right]$$

$$\int \mathcal{D}u \longrightarrow N \int \prod_n da_n \ldots$$ (another method of calculation)

Think of Fourier decomposition.

23.03.2012

Euclidean continuation :-

$$\int (\mathcal{D}q)\, e^{iS} = K(q', t'; q'', t'')$$

$$K(q', t'; q'', t'') = \langle q'' | e^{-iH(t''-t')} | q \rangle$$

Evaluate this for imaginary time

$$K = \langle q'' | e^{-H(\tau''-\tau')} | q' \rangle \qquad \left.\begin{aligned} t'' &= -i\tau'' \\ t' &= -i\tau' \end{aligned}\right\}$$

$\xrightarrow{\tau''-\tau' \to \infty}$ (Insert a complete set of energy eigenstates)

$$\sum_n |n\rangle\langle n|$$

$$= \sum_n \overbrace{\langle q'' | n \rangle}^{\psi_n(q'')} \overbrace{\langle n | q' \rangle}^{\psi_n^*(q')} e^{-E_n(\tau''-\tau')}$$

$$= \langle q'' | n=0 \rangle e^{-iE_0(\tau''-\tau')} \langle n=0 | q' \rangle$$

(One can extract the ground state energy and wavefunction)

- Next, subtract ground state contribution and try to find next higher energy eigenvalue and eigenfunction.

- To fix the normalization –

calculate
$$\frac{\int dq_2 \ldots dq_n e^{iS/\hbar}}{\int dq_2 \ldots dq_n e^{iS_0}} = \ldots$$ (without ambiguity)

S_0 → free particle

Analytic continuation in path integral :-

$$S = \int_{t'}^{t''} dt \quad L$$

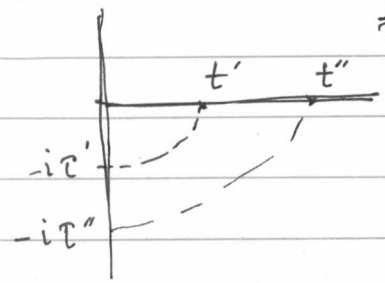
$$= \int_{t'}^{t''} dt \left[\frac{1}{2} m \left(\frac{dq}{dt}\right)^2 - V(q) \right]$$

$t = -i\tau$

$$= \int_{\tau'}^{\tau''} (-i\, d\tau) \left[-\frac{1}{2} \left(\frac{dq}{d\tau}\right)^2 - V(q) \right]$$

τ is real here

t has become imaginary



$$\therefore \left(e^{iS/\hbar}\right) = e^{1/\hbar \int_{\tau'}^{\tau''} \left[\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 - V(q)\right]}$$

Convergent path integral as long as $V(q)$ increases . . . .

$$\left.\begin{aligned} q(\tau') &= q' \\ q(\tau'') &= q'' \end{aligned}\right\} \text{(say)} \int \exp\left[-\frac{1 \cdot m\epsilon}{\hbar \cdot 2} \sum_{i=1}^{n} \left(\frac{q_{i+1} - q_i}{\epsilon}\right)^2 - \frac{1}{\hbar}\epsilon \sum_i \right.$$

$$dq_2 \cdot \text{—} \cdot dq_n$$

Harmonic Oscillator

- Speciality of Quadratic Lagrangians

$$(q', t') \longrightarrow (q'', t'')$$

$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m\omega^2 q^2 . \quad \text{(Lagrangian)}$$

Solution is $q = A \cos \omega t + B \sin \omega t$

$K(q', t' ; q'', t'')$ depends only on $t'' - t'$.

(Translational invariance in time)

- Choose $t' = 0$. $\boxed{t'' = T}$

$\therefore$ Boundary conditions : $q' = q'$ at $t = 0$
$= q''$ at $t = T$

$$\left\{\begin{aligned} q' &= A \\ q'' &= A \cos \omega T + B \sin \omega T \end{aligned}\right.$$

$$A = q' , \quad B = \left(q'' - q' \cos \omega T\right) / \sin \omega T .$$

$$S_{cl.} = \int_0^T \frac{1}{2} m\omega^2 \left[A^2 \sin^2 \omega t + B^2 \cos^2 \omega t - 2AB \sin\omega t \cos\omega t \right] - A^2 \cos^2 \omega t - B^2 \sin^2 \omega t - 2AB \sin\omega t \cos\omega t \Big]$$

$$= \frac{1}{2} m\omega^2 \int_0^T \left[-A^2 \cos 2\omega t + B^2 \cos 2\omega t - 2AB \sin 2\omega t \right]$$

$$= \frac{1}{2} m\omega^2 \times \frac{1}{2\omega} \left[-A^2 \sin 2\omega T + B^2 \sin 2\omega T + 2AB \cos 2\omega T - 1 \right]$$

Now put in A and B as functions of q', q'', T

One finally gets

Exercise

$$S = \frac{m\omega}{2 \sin \omega T} \left\{ \left((q')^2 + (q'')^2 \right) \cos \omega T - 2 q' q'' \right\}$$

Now, take T to $-i\tau$.

$$\sin \omega T = \frac{e^{i\omega T} - e^{-i\omega T}}{2i}$$

$$= \frac{1}{i} \sinh (\omega \tau).$$

$$\cos \omega T \longrightarrow \cosh (\omega \tau)$$

$$\therefore \underset{\text{(action)}}{S_{cl.}} = \frac{im\omega}{2 \sinh(\omega\tau)} \left\{ (q')^2 + (q'')^2 \right\} \cosh(\omega\tau) - 2q'q'' \Big]$$

$$K(q', t'; q'', t'') = f(\tau)\, e^{iS}$$

$$= f(\tau)\, e^{-\frac{m\omega}{2 \sinh(\omega\tau)} \left\{ \left((q')^2 + (q'')^2 \right) \cosh(\omega\tau) - 2q'q'' \right\}}$$

Exercise : Calculate $f(\tau)$

(Use completeness relation)

a) $\int K \times K = K$

b) Ratio with free particle result

Now, $K(q', 0; q'', -i\tau)$

$$= \langle q'' | e^{-H\tau} | q' \rangle$$

$$= \sum_n \langle q'' | n \rangle \langle n | q' \rangle e^{-E_n \tau}.$$

$$\langle q'' | 0 \rangle \langle 0 | q' \rangle \, e^{-E_0 \tau}$$

$$= \underset{\tau \to \infty}{f(\tau)} \; e^{-\frac{m\omega}{2\hbar}(q'^2 + q''^2)}$$

$$S \to S/\hbar \qquad \langle 0 | q' \rangle = C e^{-\frac{m\omega}{2\hbar} q'^2}$$

As $\tau \to \infty$, $f(\tau) \sim e^{-\frac{1}{2}\hbar\omega\tau/\hbar}$ (Gaussian !)

This gives ground state energy $e^{-1/2\,\omega\tau}$

$$\frac{K(q', 0 \,;\, q'', -i\tau)}{K(\tilde{q}', 0, \tilde{q}'', -i\tau)} = \frac{\sum_n \langle q'' | n \rangle \langle n | q' \rangle e^{-E_n \tau/\hbar}}{\sum_n \langle \tilde{q}'' | n \rangle \langle n | \tilde{q}' \rangle e^{-E_n \tau/\hbar}}$$

$$\cong \frac{\langle q'' | 0 \rangle \langle 0 | q' \rangle e^{-E_0 \tau/\hbar} \left\{ 1 + \frac{\langle q'' | 1 \rangle \langle 1 | q' \rangle}{\langle q'' | 0 \rangle \langle 0 | q' \rangle} e^{-(E_1 - E_0)\tau/\hbar} + \dots \right\}}{q'' \to \tilde{q}'', \; q' \to \tilde{q}'}$$

(large τ)

$$= \frac{\langle q'' | 0 \rangle \langle 0 | q' \rangle}{\langle \tilde{q}'' | 0 \rangle \langle 0 | \tilde{q}' \rangle} \left\{ 1 + \frac{\langle q'' | 1 \rangle \langle 1 | q' \rangle}{\langle q'' | 0 \rangle \langle 0 | q' \rangle} e^{-(E_1 - E_0)\tau/\hbar} - \frac{\langle \tilde{q}'' | 1 \rangle \langle 1 | \tilde{q}' \rangle}{\langle \tilde{q}'' | 0 \rangle \langle 0 | \tilde{q}' \rangle} e^{-(E_1 - E_0)\tau/\hbar} + \dots \right\}$$

Take $\tilde{q}'' = \tilde{q}' = 0$ drops out

$$\frac{\langle q'' | 0 \rangle \langle 0 | q' \rangle}{\langle \tilde{q}'' = 0 | 0 \rangle \langle 0 | \tilde{q}' = 0 \rangle} \left\{ 1 + \frac{\langle q'' | 1 \rangle \langle 1 | q' \rangle}{\langle q'' | 0 \rangle \langle 0 | q' \rangle} e^{-i(E_1 - E_0)\tau/\hbar} \right\}$$

$$= \exp\left[-\frac{m\omega}{2 \sinh(\omega\tau)} \left\{ (q')^2 + (q'')^2 \right\} \cosh \omega t - 2q'q'' \right.$$

$$\frac{\cosh \omega\tau}{\sinh \omega\tau} = \frac{e^{\omega\tau} + e^{-\omega\tau}}{e^{\omega\tau} - e^{-\omega\tau}} = \left(1 + e^{-2\omega\tau}\right)\left(1 - e^{-2\omega\tau}\right)$$

$$= 1 + 2e^{-2\omega\tau} + \dots$$

$$\exp\left(-\frac{m\omega}{e^{\omega\tau}}\left(-2q'q''\right)\right) \times \exp\left(-\frac{1}{2}m\omega\left(q'^2+q''^2\right)\right)$$

$$= \exp\left(2m\omega q'q''\, e^{-\omega\tau} + \dots\right) \times \exp\left(-\frac{1}{2}m\omega\left(q'^2+q''^2\right)\right)$$

$$= \exp\left(-\frac{1}{2}m\omega\left(q'^2+(q'')^2\right)\right)\left[1 + 2m\omega q'q''\, e^{-\omega\tau} + \dots\right]$$

Compare with

$$1 + \frac{\langle q''|1\rangle\langle 1|q'\rangle}{\langle q''|0\rangle\langle 0|q'\rangle}\, e^{-(E_1-E_0)\tau/\hbar}$$

$$\Rightarrow \left\{\begin{array}{l} \underbrace{E_1 - E_0 = \hbar\omega} \\ \dfrac{\langle 1|q'\rangle}{\langle 0|q'\rangle} = \sqrt{2m\omega}\, q'. \end{array}\right\}$$

11.04.2012

Some aspects of Dirac equation :-

$$\beta = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix} \qquad \alpha^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$(3+1)$ dim. $\qquad c = 1$ units

$$\gamma^0 = i\beta, \quad \gamma^i = i\beta\alpha^i$$

Dirac eq$^{\underline{n}}$ $\longrightarrow$ $\left(\hbar\gamma^\mu \dfrac{\partial}{\partial x^\mu} - m\right)\Psi = 0.$

$$\Psi = \exp\left(i p_\mu x^\mu/\hbar\right)\tilde{\Psi}$$

$$\left(i\gamma^\mu \not{p}_\mu - m\right)\tilde{\Psi} = 0$$

$$E = \sqrt{\vec{p}^{\,2} + m^2}$$

Solutions :- $\quad u_r(\vec{p})\, e^{-i(Et - \vec{p}\cdot\vec{x})/\hbar}$

$\qquad v_r(\vec{p})\, e^{i(Et - \vec{p}\cdot\vec{x})/\hbar}$

$$\left\{\begin{aligned} u_r(\vec{p}) &= \sqrt{\frac{E+m}{2m}}\left(1 + \frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right) u_r(\vec{0}) \\ v_r(\vec{p}) &= \sqrt{\frac{E+m}{2m}}\left(1 + \frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right) v_r(\vec{0}) \end{aligned}\right\}$$

$$u_r(\vec{0}) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix}.$$

$$v_r(\vec{0}) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

$$u_r^\dagger(\vec{p})\, u_r(\vec{p}) = \frac{E+m}{2m}\, u_r(\vec{0})^\dagger \left(1 + \frac{\vec{\alpha}\cdot\vec{p}}{2m}\right)^2 u_r(\vec{0})$$

$$= \left\{1 + \frac{2\vec{\alpha}\cdot\vec{p}}{E+m} + \frac{(\vec{\alpha}\cdot\vec{p})^2}{(E+m)^2}\right.$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

[term gives zero contribution]

$$\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$= 0.$$

$$(\vec{\alpha}\cdot\vec{p})^2$$

$$= \alpha^i p^i \alpha^j p^j$$

$$= p^i p^j \frac{1}{2}\{\alpha_i, \alpha_j\}$$

$(\because\ p^i p^j$ is symmetric in $i, j)$

$$= p^i p^j \delta_{ij}$$

$$= \vec{p}^{\,2}$$

$$\therefore \underbrace{u_r(\vec{p})^\dagger u_r(\vec{p})}_{\text{not a Lorentz invariant}} = \frac{E+m}{2m}\left\{1 + \frac{E^2 - m^2}{(E+m)^2}\right\} = \frac{2E}{\cancel{(E+m)}} \times \frac{\cancel{(E+m)}}{2m}$$

$$= E/m.$$

$\rightsquigarrow$ $\psi(x)^\dagger \psi(x)$ is not a Lorentz scalar.

<u>Lorentz transformation law</u>

$$\psi(x) \longrightarrow S\psi(\Lambda x) \qquad \underbrace{x^\mu \longrightarrow \Lambda^\mu{}_\nu x^\nu}_{\text{Lorentz transformation}}$$

General form of $\Lambda \longrightarrow \exp(\omega)$

In order that Λ describes Lorentz transformations, $\omega^\mu{}_\nu$ was defined such that $\Lambda = \exp(\omega)$

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \qquad S = \exp\left(\frac{i}{4}\sigma_{\mu\nu}\,\omega^{\mu\nu}\right)$$

$$\boxed{\omega_{\mu\nu} = \eta_{\mu\rho}\,\omega^\rho{}_\nu} \qquad \boxed{\omega^{\mu\nu} = \omega^\mu{}_\rho\,\eta^{\rho\nu}}$$

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] = \exp\left(\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}\right).$$

$$\psi^\dagger(x)\psi(x) \longrightarrow \psi(\Lambda x)^\dagger \underbrace{S^\dagger S}_{} \psi(\Lambda x)$$

$\left\{\text{If } S^\dagger S = \mathbb{1}, \text{ we would have had no problem.}\right\}$

$$S^\dagger = \exp\left[-\frac{i}{2}(\sigma^{\mu\nu})^\dagger \omega_{\mu\nu}\right]$$

$$\boxed{(\sigma^{\mu\nu})^\dagger \neq (\sigma^{\mu\nu})}. \qquad (\sigma^{\mu\nu})^\dagger = -\frac{i}{2}\left[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger\right]$$

$$= \frac{i}{2}\left[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger\right]$$

- γ^μ & γ^ν are not hermitian $\longrightarrow$ root of the fact that $S^\dagger \neq S^{-1}$.

(non-unitarity)

$$(\sigma^{ij})^\dagger = \sigma^{ij} \quad ; \quad (\sigma^{0i})^\dagger = -\sigma^{0i}$$

$$\left[\gamma^0 \longrightarrow \text{not hermitian}\right] \rightsquigarrow \text{failure of } S^\dagger S = \mathbb{1}.$$

<u>Remedy</u> :- Define $\bar{\Psi} = \Psi^\dagger \beta = -i\, \Psi^\dagger \gamma^0$.

<u>Claim</u> $\Rightarrow$ $\bar{\Psi}(x)\, \Psi(x)$ is a Lorentz scalar.

Proof $\Rightarrow$ $\bar{\Psi}(x)\, \Psi(x) = \Psi^\dagger(x)\, \beta\, \Psi(x)$

$$\rightarrow \Psi^\dagger(\Lambda x)\, S^\dagger \beta\, S\, \Psi(\Lambda x)$$

(under Lorentz transformation)

$$= \Psi^\dagger(\Lambda x)\, \beta\, \beta^{-1} S^\dagger \beta\, S\, \Psi(\Lambda x)$$

$$= \bar{\Psi}(\Lambda x)\, \underbrace{\beta^{-1} S^\dagger \beta}\, S\, \Psi(\Lambda x)$$

We have to prove that $\boxed{\beta^{-1} S^\dagger \beta = S^{-1}}$.

$$S = \exp\left[\frac{i}{4} \sigma^{\mu\nu} \omega_{\mu\nu}\right]. \qquad S^\dagger = \exp\left[-\frac{i}{4} (\sigma^{\mu\nu})^\dagger \omega_{\mu\nu}\right]$$

$$\beta^{-1} S^\dagger \beta = \exp\left[-\frac{i}{4} \beta^{-1} (\sigma^{\mu\nu})^\dagger \beta\, \omega_{\mu\nu}\right]$$

We had already seen that $(\sigma^{ij})^\dagger = \sigma^{ij}$.

$$\therefore \beta^{-1} (\sigma^{ij})^\dagger \beta = \beta^{-1} (\sigma^{ij}) \beta = \sigma^{ij}.$$

(β commutes with σ^{ij}).

$$(\sigma^{i0})^\dagger = -\sigma^{i0}.$$

$$\therefore \beta^{-1} (\sigma^{i0})^\dagger \beta = -\beta^{-1} (\sigma^{i0}) \beta$$

$$= \sigma^{i0}.$$

$\therefore \beta^{-1} S^{\dagger} \beta = \exp\left[-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}\right] = S^{-1}$ for all indices.

Hence, $\bar{\psi}(x)\psi(x)$ is a Lorentz scalar.

Ex. Check that
$$\left\{\begin{aligned} \bar{u}_s(\vec{p})\, u_{s'}(\vec{p}) &= \delta_{ss'} \\ \bar{v}_s(\vec{p})\, v_{s'}(\vec{p}) &= -\delta_{ss'} \end{aligned}\right\}$$

$\bar{u}_s(\vec{p})\, v_{s'}(\vec{p}) = 0$ These are all Lorentz invariants. (as they should be!)

$\boxed{\bar{\psi}(x)\gamma^{\mu}\psi(x)} \longrightarrow$ transforms as a vector.

$$\psi^{\dagger}(x)\beta\gamma^{\mu}\psi(x) \longrightarrow \psi^{\dagger}(\Lambda x) S^{\dagger}\beta\gamma^{\mu} S\psi(\Lambda x)$$
$$\longrightarrow \boxed{\Lambda^{\mu}{}_{\nu}\,\bar{\psi}(\Lambda x)\gamma^{\nu}\psi(\Lambda x)}$$
(Check!)

Prove that $\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$ transforms as a $\begin{pmatrix}2\\0\end{pmatrix}$ tensor under Lorentz transformations.

Completeness relation

$$\sum_s \left\{ u_{s,\alpha}(\vec{p})\,\bar{u}_{s,\beta}(\vec{p}) - v_{s,\alpha}(\vec{p})\,\bar{v}_{s,\beta}(\vec{p}) \right\} = \delta_{\alpha\beta}$$

$\alpha, \beta = 1, 2, 3, 4$ (spinor index)

(Manifestly valid in rest frame)

General frame

$$u_{s,\alpha}(\vec{p}) = S_{\alpha\alpha'}\, u_{s,\alpha'}(\vec{p}=0)$$
$$\bar{u}_{s,\beta}(\vec{p}) = \left(u_s^{\dagger}(\vec{p})\,\cancel{\gamma^0}\beta\right)_{\beta}$$
$$= \left(u_s^{\dagger}(\vec{0})\, S^{\dagger}\beta\right)_{\beta} = \left(u_s^{\dagger}\beta S^{-1}\right)_{\beta}.$$

$$\therefore \quad \bar{u}_{s,\beta}(\vec{p}) = \bar{u}_{s,\beta'}(\vec{0})\,(S^{-1})_{\beta'\beta}.$$

So, $$\sum_s \left\{ u_{s,\alpha}(\vec{p})\,\bar{u}_{s,\beta}(\vec{p}) - v_{s,\alpha}(\vec{p})\,\bar{v}_{s,\beta}(\vec{p}) \right\}$$

$$= S_{\alpha\alpha'} \left\{ u_{s,\alpha'}(\vec{0})\,u_{s,\beta'}(\vec{0}) - v_{s,\alpha'}(\vec{0})\,\bar{v}_{s,\beta'}(\vec{0}) \right\} (S^{-1})_{\beta'\beta}$$

$$= \boxed{S_{\alpha\alpha'}\,\delta_{\alpha'\beta'}\,(S^{-1})_{\beta'\beta}} = \delta_{\alpha\beta}.$$

(Proved !)

Exercise

$$\sum_s u_{s,\alpha}(\vec{p})\,\bar{u}_{s,\beta}(\vec{p}) = \left\{ \frac{i\gamma^\mu p_\mu + m\mathbb{1}}{2m} \right\}_{\alpha\beta}$$

$$\sum_s v_{s,\alpha}(\vec{p})\,\bar{v}_{s,\beta}(\vec{p}) \qquad \cancel{\sum_s v_{s,\alpha}(\vec{p})\,\bar{v}_{s,\beta}(\vec{p})}$$

$$= \left\{ \frac{i\gamma^\mu p_\mu - m\mathbb{1}}{2m} \right\}_{\alpha\beta}. \qquad p_\mu = (p_0, \vec{p}) \qquad p_0 = -p^0 = -E.$$

One gets $S^{-1}\left(\frac{1}{2}(\mathbb{1}+\beta)\right)S$ on Lorentz transformi the zero momentum result.

Trailer $$\left(\hbar\,\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi = 0.$$

Suppose, that in some representation, all γ^μ's are real. (Still must obey the Clifford algebra –

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu},$$

$$(\gamma^0)^\dagger = -\gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i \;)$$

- We can now choose ψ to be real. (Unlike Schrödinger equation)

$$S = \exp\left(\frac{i}{4}\,\omega_{\mu\nu}\,\sigma^{\mu\nu}\right) \qquad \sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$$

- Show that we can find real γ^μ's which satisfy the required relations.