$$\int \chi^* \tilde{h} \psi = \int (\tilde{h} \chi)^* \psi$$

$\nu \bar{\nu}$

(Imaginary $\beta$, real $\alpha^i$.)

Majorana spinors

Second quantized theory
$\longrightarrow$ describes a system in which a particle is its own antiparticle.

2.04.2012

Tutorial :-

Hamiltonian for problem 2 :

$$\sum_m e_m u_m(\vec{r}_1) \int d^3 r_1' \, u_m^*(\vec{r}_1') \psi(\vec{r}_1', \vec{r}_2)$$

$$+ \sum_m e_m u_m(\vec{r}_2) \int d^3 r_2' \, u_m^*(\vec{r}_2') \psi(\vec{r}_1, \vec{r}_2')$$

$$+ \sum_{m,n,p,q} V_{mnpq} \, u_m(\vec{r}_1) u_n(\vec{r}_2) \int d^3 r_1' \, d^3 r_2' \, u_p^*(\vec{r}_1') u_q^*(\vec{r}_2') \psi(\vec{r}_1', \vec{r}_2') .$$

We are given $\sum_n e_n a_n^\dagger a_n + \sum_{m,n,p,q} V_{mnpq} a_m^\dagger a_n^\dagger a_p a_q$

$e_n^* = e_n$

$$\downarrow$$

$$\sum_{q,p,m,n} V^*_{qpnm} \, a_m^\dagger a_n^\dagger a_p a_q .$$

$$\therefore \quad V^*_{qpnm} = V_{mnpq}$$

(3.) $L = \dfrac{m\dot{x}^2}{2} - \dfrac{m\omega^2 x^2}{2} + f(t)\, x$

$\Rightarrow \quad \ddot{x} + \omega^2 x = f(t)/m .$

$x_h(t) = c_1 \sin \omega t + c_2 \cos \omega t$

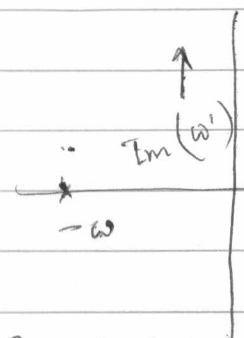
$x(t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} d\omega' \, e^{i\omega' t} \tilde{x}(\omega') \implies \tilde{x}(\omega') = -\dfrac{1}{m} \dfrac{\tilde{f}(\omega')}{\omega'^2 - \omega^2}$

$f(t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} d\omega' \, e^{i\omega' t} \tilde{f}(\omega')$

$$x_p(t) = -\frac{1}{m\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \, \frac{\tilde{f}(\omega')}{\omega'^2 - \omega^2} \, e^{i\omega' t}$$

$$= -\frac{1}{\sqrt{2\pi}\; m} \int_{-\infty}^{\infty} d\omega' \, \frac{e^{i\omega' t}}{\omega'^2 - \omega^2} \int_{-\infty}^{\infty} \frac{dt'}{\sqrt{2\pi}} \, e^{-i\omega' t'} f(t')$$

$$= -\frac{1}{2\pi m} \int_{-\infty}^{\infty} dt' \, f(t') \int_{-\infty}^{\infty} d\omega' \, \frac{e^{i\omega'(t-t')}}{\omega'^2 - \omega^2 + i\epsilon}$$




$$= -\frac{1}{2\pi m} \int_{-\infty}^{\infty} dt' \, f(t') \left( -\frac{2\pi i}{2\omega} \right)$$

$$\left[ e^{-i\omega(t-t')} \, \Theta(t-t') + e^{i\omega(t-t')} \, \Theta(t-t' \right.$$

13.04.2012

Tutorial :-

Prove $\sum_{s=\uparrow,\downarrow} u_{s,\alpha}(\vec{p}) \, \bar{u}_{s,\beta}(\vec{p}) = \frac{i\gamma^\mu p_\mu + m}{2m}$

- Rest frame $\longrightarrow$ Manifestly true $\quad \frac{1}{2}(\mathbb{1} + \beta)_{\alpha\beta}$
- Apply a Lorentz boost ($\vec{0}$ to $\vec{p}$ in momentum)

$$u(\vec{p}) = \sqrt{\frac{E+m}{2m}} \left( \mathbb{1} + \frac{\vec{\alpha}\cdot\vec{p}}{E+m} \right) u(\vec{0}).$$

$$\bar{u} = u^\dagger \cancel{\gamma^0} \beta$$

$$= u^\dagger \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

$$\alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$u_{s,\alpha}(\vec{p}) = \left[ \sqrt{\frac{E+m}{2m}} \left( \mathbb{1} + \frac{\vec{\alpha}\cdot\vec{p}}{E+m} \right) \right]_{\alpha\eta} u_{s,\eta}(0)$$

$$u_{s,\beta}(\vec{p}) = \left[ \sqrt{\frac{E+m}{2m}} \left( \mathbb{1} + \frac{\vec{\alpha}\cdot\vec{p}}{E+m} \right) \right]_{\beta\lambda} u_{s,\lambda}(0)$$

$$\bar{u}_{s,\beta}(\vec{p}) = u^{\dagger}_{s,\tau}(\vec{p})\,\beta_{\tau\beta}$$

$$= u^{\dagger}_{\lambda,s}(0)\left[\sqrt{\frac{E+m}{2m}}\left(\mathbb{1}+\frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right)\right]_{\lambda\tau}\beta_{\tau\beta}$$

$$\therefore \sum_{s} u_{s,\alpha}(\vec{p})\,\bar{u}_{s,\beta}(\vec{p})$$

$$= \left[\sqrt{\frac{E+m}{2m}}\left(\mathbb{1}+\frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right)\right]_{\alpha\eta}\left[\sqrt{\frac{E+m}{2m}}\left(\mathbb{1}+\frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right)\right]_{\lambda\tau}\beta_{\tau\beta} \times \frac{1}{2}\left(\mathbb{1}+\beta\right)_{\eta\lambda}$$

$$= \frac{E+m}{2\cdot 2m}\left[\left(\mathbb{1}+\frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right)\left(\mathbb{1}+\beta\right)\left(\mathbb{1}+\frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right)\beta\right]_{\alpha\beta}$$

$$= \frac{E+m}{4m}\left[\left(1+\frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right)\left(\mathbb{1}+\beta\right)\left(\mathbb{1}-\frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right)\right]_{\alpha\beta}$$

$$= \frac{E+m}{4m}\left[\left(\mathbb{1}+\frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right)\left(\mathbb{1}-\frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right)+\left(\mathbb{1}+\frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right)\left(\mathbb{1}+\frac{\vec{\alpha}\cdot\vec{p}}{E+m}\right)\beta\right]$$

$$= \frac{E+m}{4m}\left[\left\{\mathbb{1}-\frac{E^2-m^2}{(E+m)^2}\right\}+\left\{\mathbb{1}+\frac{2\vec{\alpha}\cdot\vec{p}}{E+m}+\frac{E^2-m^2}{(E+m)^2}\right\}\beta\right]$$

$$= \frac{\cancel{E+m}\,1}{4m}\left[\left\{(\cancel{E}+m)-(\cancel{E}-m)\right\}+\left\{(E+\cancel{m})+2\vec{\alpha}\cdot\vec{p}+E-\cancel{m}\right\}\beta\right]$$

$$= \frac{1}{4m}\left[2m+2E\beta-2\beta\,\vec{\alpha}\cdot\vec{p}\right]$$

$$= \frac{1}{2m}\left[-iE\gamma^0+i\,\vec{p}\cdot\vec{\gamma}+m\right]$$

$$= \frac{1}{2m}\left[i\gamma^{\mu}p_{\mu}+m\right].\quad \text{(Proved)}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$(\gamma^5)^2 = \mathbb{1}.$$

$$-\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\left\{\frac{1}{2}\left(\mathbb{1}+\gamma^5\right)\right\}^2 = \frac{1}{2}\left(\mathbb{1}+\gamma^5\right).$$

$$[\gamma^5, \sigma^{\mu\nu}] = 0.$$

$$S = \exp\left[\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right] \Rightarrow [\gamma^5, S] = 0.$$

$P^2 = P \rightarrow$ projection operator $\quad \frac{1}{2}(1+\gamma^5)\Psi = \pm\Psi.$

cut down ~~with~~ components $\rightsquigarrow$ $\gamma^5\Psi = \pm\Psi$

- Is this consistent with Dirac equation?

$$(\hbar\gamma^\mu\partial_\mu - m)\Psi = 0.$$

~~Does $\gamma^5$ satisfy $[\hat{H}, \gamma^5] = 0$~~ ?

$$(\hbar\gamma^\mu\gamma^5\partial_\mu - \gamma^5 m)\Psi$$

$$= -\gamma^5(\hbar\gamma^\mu\partial_\mu + m)\Psi = -\gamma^5 \, 2m\Psi$$

$$\rightsquigarrow -2m\Psi = 0 \qquad m = 0.$$

(*) Weyl projection. $\qquad$ massless

18.04.2012

Dirac : $\hbar\gamma^\mu\frac{\partial\Psi}{\partial x^\mu} - m\Psi = 0$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}.$$

- Majorana representation $\rightarrow$ specific choice of $\gamma$ matrices for which $\gamma^\mu$ is real $\forall\ \mu$.

$\Rightarrow$ $\beta$ is imaginary, $\alpha^i$'s are real.

$$\alpha^1 = \begin{pmatrix} 0_{2\times2} & -\sigma_1 \\ -\sigma_1 & 0_{2\times2} \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned}\{\alpha^i, \alpha^j\} &= 2\delta^{ij} \\ \{\alpha^i, \beta\} &= 0\end{aligned}}$$

$\beta^2 = \mathbb{1}$.

Such a choice of $\gamma$ matrices form Majorana representation

<u>Majorana particle</u> :-

$\psi = \psi^*$ in Majorana representation

$\gamma^\mu$ : Majorana representation

$\tilde{\gamma}^\mu = U \gamma^\mu U^{-1}$ : Other representation

$\tilde{\psi} \longrightarrow$ wave function in this representation.

$\tilde{\psi} = U\psi$.

$\psi = U^{-1}\tilde{\psi}$

Majorana spinor : $\psi^* = \psi$

$(U^{-1})^* \tilde{\psi}^* = U^{-1}\tilde{\psi}$

In Majorana representation, $\Rightarrow \tilde{\psi}^* = \boxed{U^* U^{-1} \tilde{\psi}}$

$C = \mathbb{1}$.

Charge conjugation operator ($C$)

Ex. Find $C$ in the original representation we used.

<u>Dirac spinor in Majorana representation</u> :-

$$\hbar \gamma^\mu \frac{\partial \psi}{\partial x^\mu} - m\psi = 0$$

$(C = 1)$.

Sol$^n$:
$$u(\vec{p}, s)\, e^{-i(Et - \vec{p}\cdot\vec{x})/\hbar}$$
$$v(\vec{p}, s)\, e^{+i(Et - \vec{p}\cdot\vec{x})/\hbar}$$

If $\psi$ is a solution, so is $\psi^*$ (real $\gamma^\mu$'s)

$\therefore$ $u^*(\vec{p}, s)\, e^{i(Et - \vec{p}\cdot\vec{x})/\hbar}$ is a solution to Dirac equation.

$$E = \sqrt{(\vec{p})^2 + m^2}$$

- Can take this to be $\rightarrow v(\vec{p}, s)\, e^{i(Et - \vec{p}\cdot\vec{x})/\hbar}$

(Linear combination)

Second quantized form :- ① ~~Naive~~

$$\Psi(\vec{x}, t) = \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi\hbar)^{3/2}} \sqrt{\frac{m}{E}} \Big\{ b(\vec{p}, s)\, u(\vec{p}, s)\, e^{i(\vec{p}\cdot\vec{x} - Et)/\hbar} + c(\vec{p}, s)\, v(\vec{p}, s)\, e^{-i(\vec{p}\cdot\vec{x} - Et)/\hbar}$$

$$H = \sum_{s} \int d^3 p \sqrt{\vec{p}^{\,2} + m^2} \times \Big\{ b(\vec{p}, s)^\dagger\, b(\vec{p}, s) - c(\vec{p}, s)^\dagger\, c(\vec{p}, s)$$

Define $d(\vec{p}, s) = c^\dagger(\vec{p}, s)$, $d^\dagger(\vec{p}, s) = c(\vec{p}, s)$ (Particle-hole transformation)

$$H = \sum_{s} \int d^3 p \sqrt{\vec{p}^{\,2} + m^2} \Big\{ b(\vec{p}, s)^\dagger\, b(\vec{p}, s) + d(\vec{p}, s)^\dagger\, d(\vec{p}, s) + \text{const.}$$

In the Majorana spinor case,

$$\Psi(\vec{x}, t) = \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi\hbar)^{3/2}} \sqrt{\frac{m}{E}} \Big\{ b(\vec{p}, s)\, u(\vec{p}, s)\, e^{i(\vec{p}\cdot\vec{x} - Et)/\hbar} + d^\dagger(\vec{p}, s)\, v(\vec{p}, s)\, e^{-i(\vec{p}\cdot\vec{x} - Et)/\hbar}$$

$v(\vec{p}, s) = u^*(\vec{p}, s) \longrightarrow$ Impose

Impose the Majorana condition $\rightarrow \Psi^*(\vec{x}, t) = \Psi(\vec{x}, t)$

Second quantized theory

$\Psi^\dagger(\vec{x}, t) = \Psi(\vec{x}, t)$ [Field operator becomes hermitian]

$$\| \quad \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi\hbar)^{3/2}} \sqrt{\frac{m}{E}} \Big\{ b^\dagger(\vec{p}, s)\, v(\vec{p}, s)\, e^{-i(\vec{p}\cdot\vec{x} - Et)/\hbar} + d(\vec{p}, s)\, u(\vec{p}, s)\, e^{i(\vec{p}\cdot\vec{x} - Et)/\hbar}$$

$\rightsquigarrow \boxed{\begin{aligned} b^\dagger &= d^\dagger \\ b &= d \end{aligned}} \Rightarrow$ [particle - identified with antiparticle]

Define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$$(\gamma^5)^2 = i\gamma^0\gamma^1\gamma^2\gamma^3 \times i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= (i^2)(-1)^3(\gamma^0)^2(-1)^2\underbrace{\gamma^2\gamma^3\gamma^2\gamma^3}_{(-1)(\gamma^2)^2(\gamma^3)^2}$$

$$= (+1)(+1)(+1)(+1)(+1) \times 1 \times 1 = 1.$$

Eigenvalues of $\gamma^5$ are $\pm 1$.

Ex: 2 eigenvalues $+1$, 2 " $-1$

Furthermore, $\gamma^5$ anti-commutes with any $\gamma^\mu$.

So, $\gamma^5$ is a Lorentz scalar. $\left[\gamma^5, \underset{\overset{\parallel}{\sigma^{\mu\nu}}}{\Sigma^{\mu\nu}}\right] = 0.$

So, Weyl spinor $\longrightarrow$ $\gamma^5\psi = \psi$ or $\gamma^5\psi = -\psi$

(Project into a 2-dim. subspace).

$\boxed{\text{Is this consistent ?}}$ $\longrightarrow$ Lorentz invariance is okay. (✓)

If $\gamma_5\psi = \psi$,

then $\gamma_5 S\psi = S\gamma_5\psi$

$= S\psi.$

Dirac equation

$$\hbar\gamma^\mu\frac{\partial\psi}{\partial x^\mu} - m\psi = 0.$$

$$\Rightarrow \gamma^5\left(\hbar\gamma^\mu\frac{\partial\psi}{\partial x^\mu} - m\psi\right) = 0.$$

$$\Rightarrow -\hbar\gamma^\mu\frac{\partial\psi}{\partial x^\mu} - m\psi = 0.$$

compatible? only for $m = 0$.

- You can impose Weyl condition only if your particles are massless.

Go back to our original representation:

$$\gamma^0 = i\beta = i\begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix} \qquad \gamma^i = i\beta\alpha^i$$

$$= i\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= i\cancel{(-1)}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\cancel{(-1)}\begin{pmatrix} -\sigma_2\sigma_3 & 0 \\ 0 & -\sigma_2\sigma_3 \end{pmatrix} = i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}.$$

$$= i\begin{pmatrix} 0 & -i\mathbb{1} \\ -i\mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Consider $\vec{p} = (p_x = 0,\ p_y = 0,\ p_z = \pm E)$ $\quad E > 0$.

$$u\left(\vec{p}, \tfrac{1}{2}\right) \propto \cancel{\sqrt{\frac{E+m}{2m}}}\begin{pmatrix} 1 \\ 0 \\ p_z/E \\ 0 \end{pmatrix} = \cancel{\sqrt{\frac{E+m}{2m}}}\begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{pmatrix}$$

(have to normalize differently)

$$u\left(\vec{p}, -\tfrac{1}{2}\right) \propto \begin{pmatrix} 0 \\ 1 \\ 0 \\ \mp 1 \end{pmatrix}$$

$$\gamma^5 u\left(\vec{p}, \tfrac{1}{2}\right) = \pm\begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{pmatrix}$$

(Immediately check!)

$$\gamma^5 u\left(\vec{p}, -\tfrac{1}{2}\right) = \mp\begin{pmatrix} 0 \\ 1 \\ 0 \\ \mp 1 \end{pmatrix}.$$

(Immediately check!)

$$\gamma^5 \psi = \psi$$

$\longrightarrow$ spin component & momentum component are positive

$$\gamma^5 \psi = -\psi$$

$\Rightarrow$ −ve z-axis $\rightarrow$ spin

& moving along −ve z-axis.

For $\nu$'s $\longrightarrow$ spin is anti-aligned to momentum.

<u>Majorana</u> $\longrightarrow$ Particles with both spins but antiparticles $\equiv$ particles

<u>Weyl</u> $\longrightarrow$ Particles with spin along $\vec{p}$
Antiparticles with spin along $-\vec{p}$

- Massless particles

We assert $\longrightarrow$ Zero mass Majorana spinor is equivalent to a Weyl spinor.

<u>Majorana representation</u>

Define $\gamma^5 = \text{- - -}$. We can have a Weyl spinor such that $\gamma^5 \psi = \psi$.

Define a new spinor $\boxed{\chi = \psi + \psi^*.}$

$$\left( \hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - \cancel{m} \right) \psi = 0 \qquad \hbar \gamma^\mu \frac{\partial}{\partial x^\mu} \chi = 0.$$

$$\left( \hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - \cancel{m} \right) \psi^* = 0.$$

$\chi \longrightarrow$ doesn't ~~not a Weyl spinor~~

(Majorana representation)

4 real components $\Big\}$ satisfy Weyl projection condition.

- We've made a Weyl spinor into a Majorana spinor.

- Physically, declare antiparticle as a particle having spin along $-\vec{p}$.

19.04.2012

$$\psi^* = -\gamma_2 \psi \checkmark \qquad \text{Define } \tilde{\psi} = e^{i\phi}\psi$$

$$\cancel{\psi^* = i\gamma_2 \psi} \qquad \tilde{\psi}^* = e^{-i\phi}\psi^*$$

One can choose this phase via a redefinition of the field.

$$= e^{-i\phi}(-\gamma_2)\psi$$
$$= e^{-2i\phi}(-\gamma_2)\tilde{\psi}$$

iħ

- Ambiguity in defining the Majorana representation. (existence of arbitrary phase)

## Time-dependent Perturbation theory :-

$$H(t) = H_0 + H_{int}(t)$$

$$i\hbar \frac{\partial \psi_s}{\partial t} = H(t)\psi_s \longrightarrow \text{Schrödinger picture}$$

$$\psi_s(t) = T\left[\exp\left(-\frac{i}{\hbar}\int_0^t dt'\, H(t')\right)\right]\psi_s(0)$$

$$\Downarrow$$

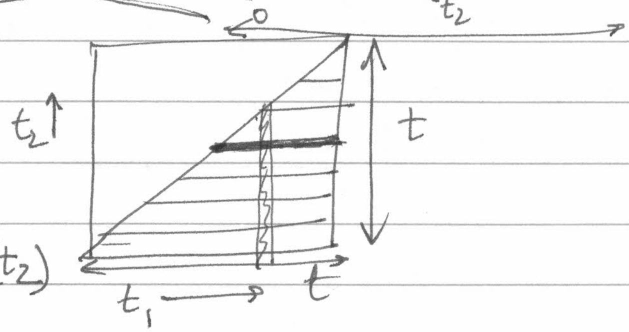
$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n \int_0^t dt_1 \cdots \int_0^t dt_n\, T\left[H(t_1)H(t_2)\cdots H(t_n)\right] \times \psi_s(0)$$

$$= \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_0^t dt_1\, H(t_1)\int_0^{t_1} dt_2\, H(t_2) \cdots \int_0^{t_{n-1}} dt_n\, H(t_n)\, \psi_s(0)$$

$$\frac{1}{2!}\int_0^t dt_1 \int_0^t dt_2\, T\left[H(t_1)H(t_2)\right)$$

$$= \frac{1}{2!}\left\{\int_0^t dt_1 \int_0^{t_1} dt_2\, H(t_1)H(t_2) + \int_0^t dt_2 \int_{t_2}^t dt_1\, H(t_1)H(t_2)\right.$$

$$\int_0^t dt_1 \int_0^{t_1} dt_2$$




$$= \int_0^t dt_1 \int_0^{t_1} dt_2\, H(t_1)H(t_2)$$

$$i\hbar \frac{\partial \psi_s}{\partial t} = (i\hbar) \sum_{n=1}^{\infty} \left(\frac{-i}{\hbar}\right)^n H(t) \int_0^{t} dt_2 \int_0^{t_2} dt_3 \cdots H(t_2)\, H(t_3) \cdots H(t_n)\, \psi_s(0)$$

$$= (i\hbar)\left(-\frac{i}{\hbar}\right) H(t) \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^{n-1} \int_0^{t} dt_1 \int_0^{t_1} dt_2 \cdots H(t_1) \cdots H(t_{n-1})\, \psi_s(0)$$

$$= H(t)\, \psi_s(t) \,. \quad \text{(Proved)}$$

$$\psi_H(t) \equiv \psi_s(0) = \left\{ T \exp\left(-\frac{i}{\hbar} \int_0^t dt'\, H(t')\right) \right\}^{-1} \psi_s(t) \,.$$

(Heisenberg representation)

- Operators must transform too. $\longrightarrow$ Becomes time-dependent

Define

INTERACTION PICTURE

$$\psi_I(t) = \exp\left(\frac{i}{\hbar} H_0 t\right) \psi_s(t)$$

$$i\hbar \frac{\partial}{\partial t} \psi_I(t) = \exp\left(\frac{i}{\hbar} H_0 t\right) \cdot \left(i\hbar \cdot \frac{i H_0}{\hbar}\right) \psi_s(t) + \exp\left(\frac{i}{\hbar} H_0 t\right) \left(H_0 + H_{int}(t)\right) \psi_s(t) \,.$$

$$= \underbrace{\exp\left(\frac{i}{\hbar} H_0 t\right) H_{int}(t) \exp\left(-\frac{i}{\hbar} H_0 t\right)}_{H_I(t)} \psi_I(t)$$

$$\underbrace{i\hbar \frac{\partial}{\partial t} \psi_I(t) = H_I(t)\, \psi_I(t)}$$

$$\psi_I(t) = T \exp\left(-\frac{i}{\hbar} \int_0^t H_I(t')\, dt'\right) \psi_I(0) \,.$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar}\right)^n \int_0^t dt_1 \cdots \int_0^t dt_n\, T\left(H_I(t_1) \cdots H_I(t_n)\right) \psi_I(0)$$

$\boxed{\text{Dyson series}}$ (Can use as a perturbation expansion)

Eigenstates of $H_0$ : $H_0 |m\rangle = E_m |m\rangle$

$\langle n | \Psi_I(t) \rangle$ $\boxed{\text{eigenstate of } H_0}$

$\Psi_I(0) = |m\rangle$

$= \langle n | m \rangle + \left(-\frac{i}{\hbar}\right) \int_0^t dt_1 \langle n | H_I(t_1) | m \rangle$

$H_I = e^{i/\hbar\, H_0 t} H_{int} e^{-i/\hbar\, H_0 t}$

write $H_I$ in terms of $H_0$ & Hint,

$$= \langle n | m \rangle - \frac{i}{\hbar} \int_0^t dt_1\, e^{i/\hbar (E_m - E_n) t_1} \langle n | H_{int}(t_1) | m \rangle$$

Suppose $H_{int}(t_1) = \begin{cases} V & \text{for } 0 < t_1 < T \\ 0, & \text{outside this range} \end{cases}$

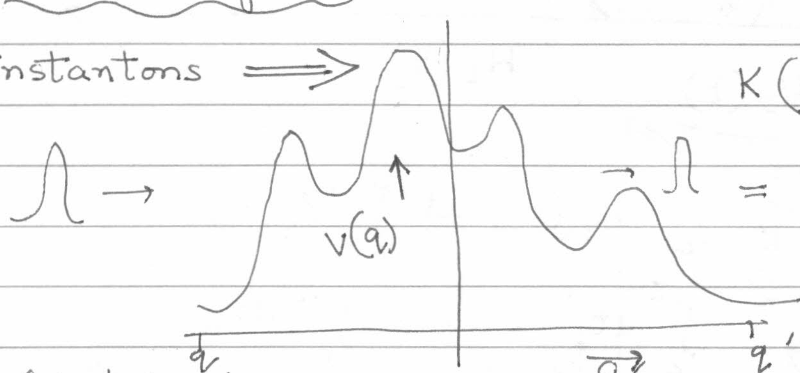
$$\Rightarrow \delta_{mn} - \frac{i}{\hbar} \left( \int_0^T e^{i/\hbar (E_m - E_n) t_1} \right) \times \langle n | V | m \rangle$$

$$= \delta_{mn} - \frac{i}{\hbar} \frac{e^{i/\hbar (E_m - E_n) T} - 1}{i/\hbar (E_m - E_n)} \langle n | V | m \rangle$$

(Transition amplitude from m to n.)

## Path Integrals :-

(*) Instantons $\Longrightarrow$




$K(q, t; q', t')$

$= \langle q' | e^{-i/\hbar\, H(t'-t)} | q \rangle$

$\int \Psi(q) |q\rangle dq$

$\langle q' | e^{-i/\hbar\, H(t'-t)} | q \rangle$

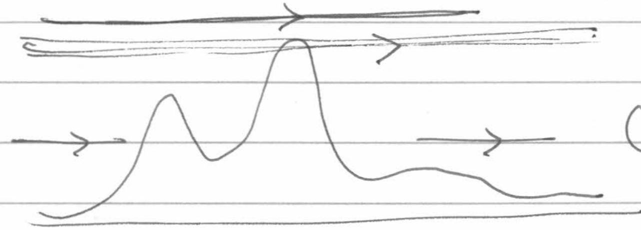
$\int \langle q' | e^{-i/\hbar\, H(t'-t)} | q \rangle \Psi(q)\, dq.$

# Classical approximation

Particle is at $q$ at time $t$. Find the classical path so that it's at $q'$ at time $t'$.

$$e^{iS/\hbar} = e^{i/\hbar \int_t^{t'} dt'' \, L(t'')} \Big|_{\text{path}} .$$

into $e^{-i/\hbar \, H t}$



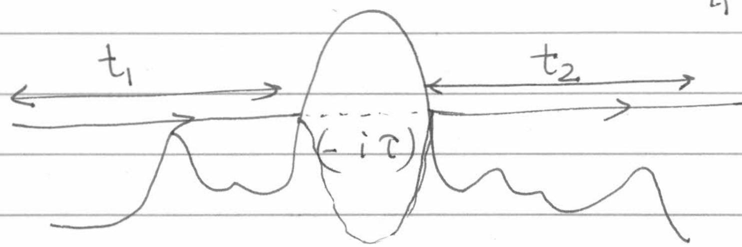
($x \longrightarrow$ not classically admissible)

• Large $(t'-t)$ behaviour

• Introduce Complex $t$, $t'$.

$$K(q, t \, ; \, q', t' - i\tau) = \exp(\quad) \times \text{phase}$$

$$t_1 + t_2 = t' - t$$




$$\mathcal{L} = \frac{m\dot{q}^2}{2} - V(q)$$

$$t = -i\tau$$

$$= -\left\{ \frac{1}{2} m \left( \frac{dq}{d\tau} \right)^2 + V(q) \right\}$$

$\therefore$ Classical path $\longrightarrow$ $\left\{ \begin{array}{l} \text{EL Equations with potential} \\ \quad -V(q) . \end{array} \right.$

$\rightsquigarrow$ WKB answer