10/8/05

Books :-

1. Percival & Richards (Intro. to dynamics)
2. Goldstein
3. Arnold (mathematical methods in cl. mech.)

N particles in 1-dimension.

Coordinates: $(q^1, q^2, \ldots, q^N) = \vec{q}$

forces $\vec{f} = \vec{f}(\vec{q}, \dot{\vec{q}}, t)$

$$\frac{d^2 q^i}{dt^2} = f^i(\vec{q}, d\vec{q}/dt, t) \qquad (i = 1, 2, \ldots, N)$$

We will try to convert into a set of 1st order eqns.

Introduce new variables:-

$$v^i = \frac{dq^i}{dt}$$

$$v^i(t) = dq^i/dt$$

To beginning with, think of q & v as indep. variables.

$$\left.\begin{aligned} \frac{dq^i}{dt} &= v^i \\ \frac{dv^i}{dt} &= f^i(\vec{q}, \vec{v}, t) \end{aligned}\right\} \quad \begin{aligned} &x^1 = q^1, x^2 = q^2, \ldots x^N = q^N, \\ &x^{N+1} = v_1, x^{N+2} = v_2, \ldots, x^{2N} = v_N \end{aligned}$$

Hence, we write $\boxed{\dfrac{dx^i}{dt} = F^i(\vec{x}, t)}$.

$$\cancel{F^i(\vec{x}, t) = f^i(\vec{q}, \vec{v}, t)}$$

$$F^1(\vec{x}, t) = v^1 = x^{N+1}$$
$$F^2(\vec{x}, t) = v^2 = x^{N+2}$$
$$\ldots$$
$$F^N(x, t) = v^N = x^{2N}$$
$$F^{N+1}(\vec{x}, t) = f^1(\vec{q}, \vec{v}, t)$$

Think of these as eqns. of v_i, but not as defns of v_i's

They are indep. before u impose the eqns.

(2)

In this form, these eqns have more general applicability than Newtonian form.

Suppose we have N variables $q^1, \ldots, q^N$ each satisfying m^{th} order differential eqn.

$$\frac{d^m q^i}{dt^m} = f^i\left(\vec{q}, \frac{d\vec{q}}{dt}, \ldots, \frac{d^{m-1}\vec{q}}{dt^{m-1}}, t\right)$$

More general system than a Newtonian system

This sort of eqns can also be cast into a set of 1st order eqns, by defining

$$\frac{dq^i}{dt} = v^i_{(1)}, \quad \frac{d^2 q^i}{dt^2} = v^i_{(2)}, \ldots$$

$$\frac{d^{m-1} q^i}{dt^{m-1}} = v^i_{(m-1)}$$

$$\frac{dv^i_{(m-1)}}{dt} = f^i\left(\vec{q}, \vec{v}_{(1)}, \vec{v}_{(2)}, \ldots, \vec{v}_{(m-1)}, t\right)$$

$$\frac{dq^i}{dt} = v^i_{(1)}$$

$$\frac{dv^i_{(1)}}{dt} = v^i_{(2)}$$

$$\vdots$$

$$\frac{dv^i_{(m-2)}}{dt} = v^i_{(m-1)}$$

We define a new vector

$x^1 = q^1, \ldots, x^N = q^N$,

$$x^{2N+1} = v^1_{(2)}, \ldots, x^{3N} = v^N_{(2)}$$

$$\vdots$$

$$\frac{dx^i}{dt} = F^i(\vec{x}, t), \quad i = 1, \ldots, mN \qquad x^{mN} = v^N_{(m-1)}$$

[Only the n last set is non-trivial, the other sets are trivial].

$$\frac{dx^i}{dt} = F^i(\vec{x}, t) \qquad i = 1, \ldots, n$$

$[n = mN]$

$\Rightarrow$ known as the n-th order dynamical system.

~~Autonomous dyn. system with an explicit dependence on t~~

If $F^i(\vec{x}, t)$ indep. of t, then it is an autonomous dynamical system.

(not subject to ext. environment changing with time)

t-dep. $\rightarrow$ external environment changing with time

$\therefore$ in this case, $\frac{dx^i}{dt} = F^i(\vec{x})$

If we have an n-th order non-autonomous system, one can show that it is equivalent to an $(n+1)$-th order autonomous system, i.e., if $\frac{dx^i}{dt} = F^i(\vec{x}, t) \quad i = 1, \ldots, n$

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then ~~so~~ let $x^1, \ldots, x^{n+1}$,

$$\frac{dx^{n+1}}{dt} = t \rightsquigarrow \text{we define } t \text{ like this}$$

$$\frac{dx^i}{dt} = F^i(\vec{x}, x^{N+1}) \quad \text{for } i = 1, \ldots, n$$

We are including time as one of the variables of system — technical way — not much physical about it

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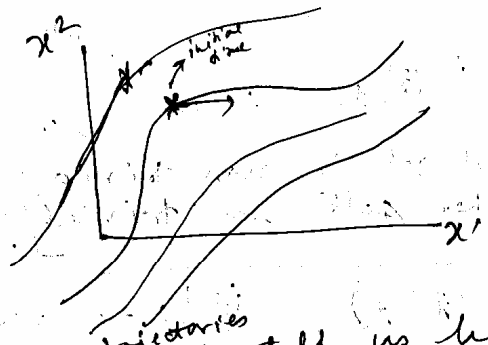
For this, for the time being we can think of autonomous systems only.

$$dx^i/dt = F^i(\vec{x}) \qquad i=1,\dots,n$$



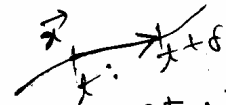
Symmetries & transfns are not at present considered — t as parameter etc.

These trajectories don't tell us how much time it takes from one point to another. They don't (only the pictures) contain total info about the ~~solns~~ solns of diff. eqns — but can give lots of info.

The tangent vector at any point $\vec{x}$ is parallel to $(F^1(\vec{x}), \dots, F^N(\vec{x}))$

~~the vector~~ space labelled by $(x^1, \dots, x^N) \equiv$ phase space

These lines are the <u>flow lines</u> in phase space.



At $t+\delta t$, the ith coord. will be $x^i + F^i(\vec{x})\,dt$

tangent vector is along $F^i(\vec{x})\,\delta t$

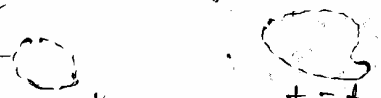
2 flow lines can't intersect — there is a same tangent vector.

We can visualise this as the streamline flow of a fluid in the phase space. (How fast the fluid is flowing isn't apparent from the diag.)

We can ask more imp. Ques. about the fluid, which has implications in the dynamical system.

Q) Is this fluid incompressible?

([Incomp. fluid → ρ changing])



$t = t_0$; $t = t_1$

Non-intersection of flow lines doesn't intersect ~~comp~~ incompressibility

In this context of our problem, the 2 volumes must be the same.

See how the volume at $t=t_0$ changes at $t=t_1$ — almost surely it changes its shape (for non-trivial flow). If the vol. is the same for every possible time, → then incompressible.

We can ask Are these diff. eqns such that they preserve the volume? — Even without solving the Diff. eqn, we can answer it.)

→ In general, it is ~~not~~ compressible, but in ~~certain~~ special cases, it is incompressible.

[Some sp. prop. of some diff. eqns will ensure incompressibility.]

~~Examples of dynami~~

$\frac{dx^i}{dt} = f^i(x,t)$

⑥

Examples of dynamical systems

① Newtonian system.

② Radioactive decay of Nuclei

$$dN/dt = -\alpha N$$

③ Growth of biological population

$$dN/dt = \underbrace{(B(N)}_{\text{birth rate}} - \underbrace{D(N)}_{\text{death rate}})N$$

for many species int. with each other, each dN/dt will be fn. of all other $B(N)$'s & $D(N)$'s

④ Chemical reactions.

A, B, C, D, E

$A + B \rightleftharpoons C + D$ $\quad B + C \rightleftharpoons A + E$

$$dN_A/dt = f_1(N_A, N_B, \ldots, N_E)$$

$$dN_B/dt = f_2(N_A, N_B, \ldots, N_E)$$

(Any time-dep. process can be thought of a dyn. system — Also, t doesn't necessarily have to be time).

We can learn the general features of the eqns, even without solving them explicitly.

Simplest system

1st order autonomous system.

$$dx/dt = F(x)$$

$$dt = \frac{dx}{F(x)}$$

$$\int_{t_0}^{t} dt = \int_{x}^{x} \frac{dx}{F(x)} \qquad x = x_0 \text{ at } t = t_0$$

... it analytically.

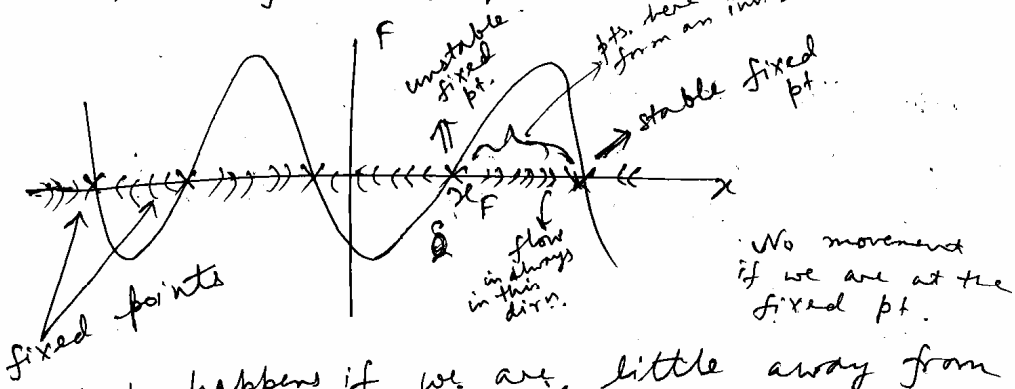
But what general features do the solutions have?

General features of the soln :—

Fixed point

Defn. → If $F(x_F) = 0$, then x_F is called a fixed point. [fixed points are the zeroes of $F(x)$]

It's a fixed pt., because $x = x_F$ is a soln of the eqn.



What happens if we are little away from a fixed pt.?

Define a new variable $y = x - x_F$

Now, $dy/dt = dx/dt = F(x) = F(x_F + y)$

$\Rightarrow dy/dt = \underbrace{F(x_F)}_{0} + y F'(x_F) + \cdots$

$\Rightarrow dy/dt = y F'(x_F)$

$\Rightarrow y = A e^{F'(x_F) t}$

→ integrating const. (small)

If $F' > 0$, we go away from the fixed pt. → called unstable fixed pts.

If $F' < 0$, " " towards → known as stable fixed pts.

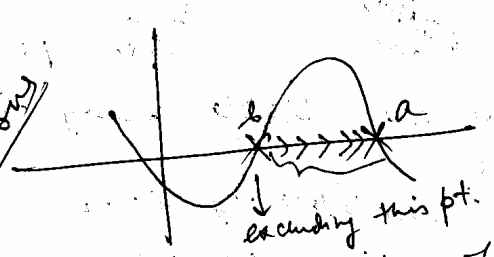
(8) The stable fixed pts. are like sinks
& un " " " " " sources
→ in the fluid-flow picture
(fluid coming from sources & going into sinks).

Invariant set of points:
Is a set such that a point in that set remains in the set at all finite time (both +ve & -ve)

[The fixed pts. ~~themselves are~~ itself is an inv. set]

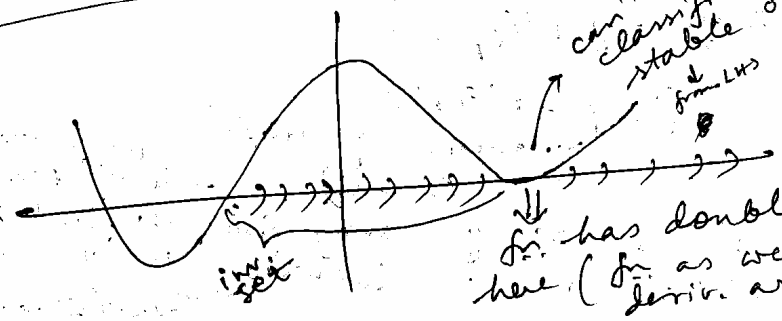
Domain of attraction of a fixed point:
Set of points which reach the fixed point as $t \to \infty$

1st order autonomous system



excluding this pt. — domain of attraction of fixed pt. a

Domain of attraction of fixed pt. b is the fixed pt. itself.

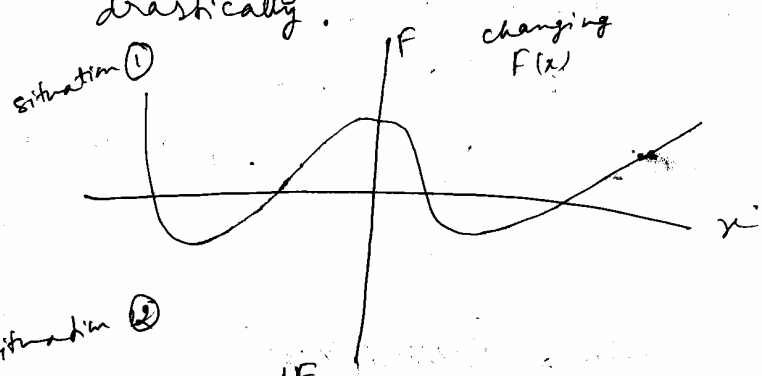


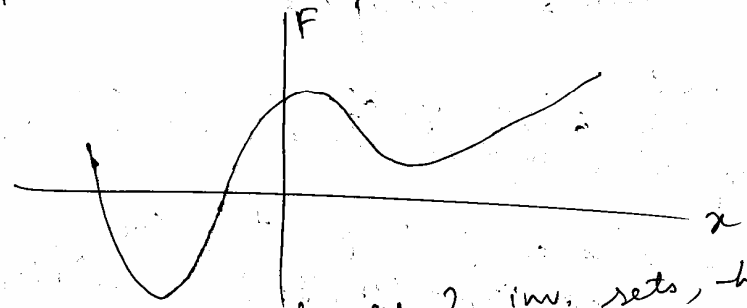
can it be classified as stable or unstable
from LHS
from RHS

fn. has double zero here (fn. as well as its deriv. are zero)

These are structurally unstable situations

If we change $F(x)$ to $F(x) + \epsilon f(x)$ (small perturbation of $F(x)$), then for arbitrarily small ϵ, the structure of invariant set can change drastically.





what were 2 inv. sets, here merged into one inv. set — structure changed drastically.

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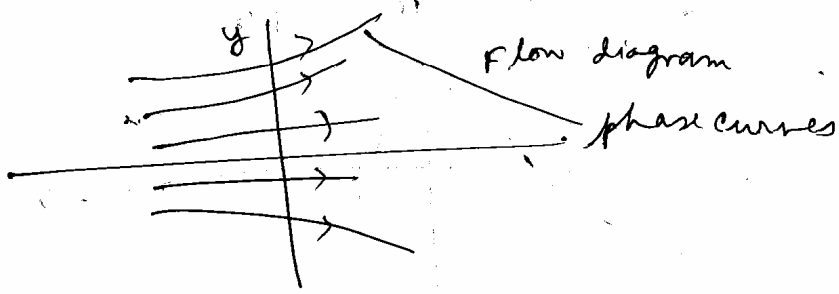
Newtonian systems

2nd-order autonomous system

$$\left.\begin{aligned} dx/dt &= F_1(x,y) \\ dy/dt &= F_2(x,y) \end{aligned}\right\} \frac{dy}{dx} = \frac{F_2(x,y)}{F_1(x,y)}$$

Solution Curves in phase space



"Velocity" vector $= \left(F_1(x,y), F_2(x,y)\right)$

↓ Rate of change of x & y

The overall mag. of the velocity is not contained in these curves, but dirn. is contained.

Suppose there is a fixed point at $\vec{r}_F = (x_F, y_F)$ (Neither x_F & y_F should change)

$$\therefore \quad F_1(x_F, y_F) = 0, \quad F_2(x_F, y_F) = 0$$

Suppose u are little away from the fixed pt; what will be the time evolution?

⟹ Motion near a fixed point

$$(x,y) \simeq (x_F, y_F)$$

$$dx/dt = F_1(x,y) = \overset{=0}{F_1(x_F, y_F)} + \left.\frac{\partial F_1}{\partial x}\right|_{\vec{r}_F} (x - x_F) + \left.\partial F_1/\partial y\right|_{\vec{r}_F} (y - y_F) + \cdots$$

Similarly $\frac{d}{dt}(y-y_F) = c(x-x_F) + d(y-y_F) + \cdots$ — (2)

$\left[\text{where } c = \frac{\partial f_2}{\partial x}\Big|_{\vec{r}_F} \;\&\; d = \frac{\partial f_2}{\partial y}\Big|_{\vec{r}_F}\right]$

$\boxed{\text{(1) \& (2) are the eqns we need to solve}}$

We adopt the matrix notation for solving :—

$$\frac{d}{dt}\begin{pmatrix} x-x_F \\ y-y_F \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x-x_F \\ y-y_F \end{pmatrix}$$

The 4 parameters control / govern the eqn. and the fixed pt.

The solns depend on the eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Suppose λ is an eigenvalue of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then, $\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$

$\Rightarrow ad - \lambda(a+d) + \lambda^2 - bc = 0$

$\Rightarrow \lambda^2 - \tau\lambda + \delta = 0$ $\left[\tau = a+d = \text{trace},\ \delta = ad-bc = \text{determinant}\right]$

$\Rightarrow \lambda = \tau/2 \pm \sqrt{\tau^2/4 - \delta} = \lambda_1, \lambda_2$

Note :- $\lambda_1 + \lambda_2 = \tau$ & $\lambda_1\lambda_2 = \delta$

Various cases :-

~~1) λ_1, λ_2 are real [when τ^2 >~~

(1) $\tau^2 > 4\delta \rightarrow \lambda_1, \lambda_2$ real & $\lambda_1 \neq \lambda_2$

(2) $\tau^2 < 4\delta \rightarrow \lambda_1, \lambda_2$ complex & $\lambda_1 \neq \lambda_2$ & $\lambda_1 = \lambda_2^*$

(3) $\tau^2 = 4\delta \rightarrow \lambda_1 = \lambda_2$

Case (1) $\Rightarrow$ (a) $\lambda_1, \lambda_2 > 0 \leftrightarrow \tau > 0,\ \delta > 0$

(b) $\lambda_1, \lambda_2 < 0 \leftrightarrow \tau < 0,\ \delta > 0$

(c) $\lambda_1 > 0,\ \lambda_2 < 0 \leftrightarrow \delta < 0$

will determine nature of fixed points

Case (2) $\rightarrow$ a) $\tau > 0$
b) $\tau < 0$
c) $\tau = 0$

Case 1 $\tau^2 > 4\delta$; λ_1, λ_2 ~~real~~ real & $\lambda_1 \neq \lambda_2$

Eigenvectors $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \quad (1)$$

[Note :- a, b, c, d are real $\therefore$ f_1 & f_2 are real]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \quad (2)$$

real solns for $\alpha_1, \beta_1, \alpha_2, \beta_2$
[$\because$ a, b, c, d & λ_1, λ_2 are all real]

$$\boxed{\frac{d}{dt}\begin{pmatrix} x - x_p \\ y - y_p \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x - x_p \\ y - y_p \end{pmatrix}}$$

My working --

LHS $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} a\alpha_1 + b\beta_1 & a\alpha_2 + b\beta_2 \\ c\alpha_1 + d\beta_1 & c\alpha_2 + d\beta_2 \end{pmatrix}$

RHS $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\lambda_1 & \alpha_2\lambda_2 \\ \beta_1\lambda_1 & \beta_2\lambda_2 \end{pmatrix}$

from (1) & (2),

$a\alpha_1 + b\beta_1 = \alpha_1\lambda_1$
$a\alpha_2 + b\beta_2 = \alpha_2\lambda_2$

Define $A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$

Then, we'll get $\begin{pmatrix} a & b \\ c & d \end{pmatrix} A = A \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

This shows that we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} A^{-1}$$

$$\therefore \frac{d}{dt}\begin{pmatrix} x - x_p \\ y - y_p \end{pmatrix} = A \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} A^{-1} \begin{pmatrix} x - x_p \\ y - y_p \end{pmatrix}$$

Define $\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = A^{-1}\begin{pmatrix}x-x_F\\ y-y_F\end{pmatrix}$ $\cancel{\Rightarrow \frac{d}{dt}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}}$

Then, we have,

$$A^{-1}\frac{d}{dt}\begin{pmatrix}x-x_F\\ y-y_F\end{pmatrix} = A^{-1}A\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}A^{-1}\begin{pmatrix}x-x_F\\ y-y_F\end{pmatrix}$$

As long as $\lambda_1 \neq \lambda_2$, A is non-singular & A^{-1} exists

$$\Rightarrow \frac{d}{dt}\left[A^{-1}\begin{pmatrix}\cdot\\ \cdot\end{pmatrix}\right] = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}.$$

[$\because$ A^{-1} is a const. matrix]

$$\Rightarrow \frac{d}{dt}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}$$

$$\therefore \frac{d\tilde{x}}{dt} = \lambda_1\tilde{x}, \quad d\tilde{y}/dt = \lambda_2\tilde{y}.$$

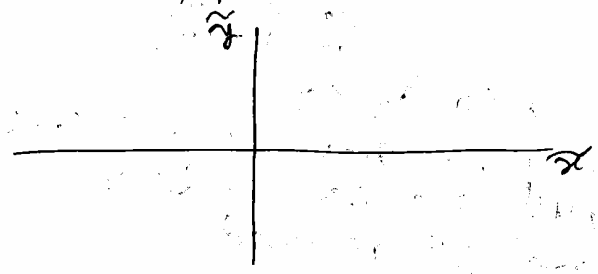
$$\Rightarrow \tilde{x} = \tilde{x}_0 e^{\lambda_1 t}, \quad \tilde{y} = \tilde{y}_0 e^{\lambda_2 t} \longrightarrow$$

Valid as long as $\tilde{x}, \tilde{y}$ are small

[because of our earlier approx. of being very near to the fixed pt.]

Case 1(a) :- $\lambda_1 > 0$, $\lambda_2 > 0$

$\tilde{x}, \tilde{y}$ increase with t



My —

$\lambda_2 t = \ln(\tilde{y}/\tilde{y}_0)$

& $\lambda_1 t = \ln(\tilde{x}/\tilde{x}_0)$

$\therefore \lambda_2/\lambda_1 = \frac{\ln(\tilde{y}/\tilde{y}_0)}{\ln(\tilde{x}/\tilde{x}_0)}$

Expanding, $\lambda_2/\lambda_1 \approx \frac{\tilde{y}/\tilde{y}_0}{\tilde{x}/\tilde{x}_0}$

$\ln(\tilde{x}/\tilde{x}_0)^{\lambda_2/\lambda_1} = \ln(\tilde{y}/\tilde{y}_0)$

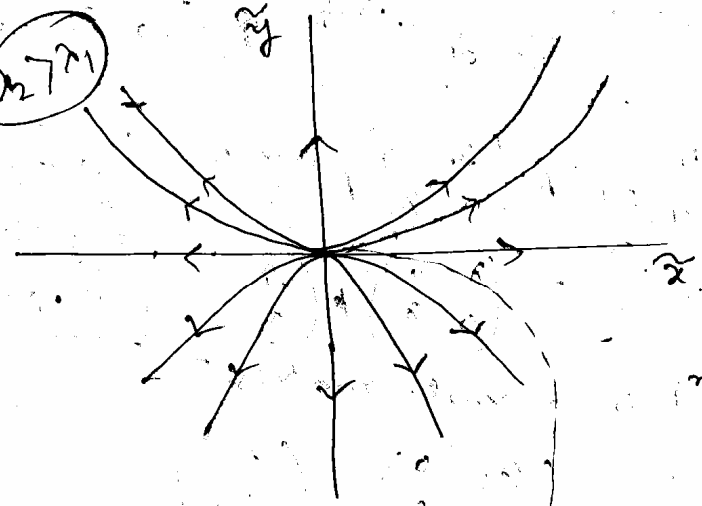
$\Rightarrow \tilde{y}/\tilde{y}_0 = (\tilde{x}/\tilde{x}_0)^{\lambda_2/\lambda_1}$

[We are to eliminate t to plot $\tilde{y}$ vs $\tilde{x}$]

After eliminating, we'll get $\tilde{y}/\tilde{y}_0 = \left(\frac{\tilde{x}}{\tilde{x}_0}\right)^{\lambda_2/\lambda_1}$ $\Rightarrow \tilde{y} \propto \tilde{x}^{\lambda_2/\lambda_1}$

Either $\lambda_2 > \lambda_1$ or $\lambda_2 < \lambda_1$.

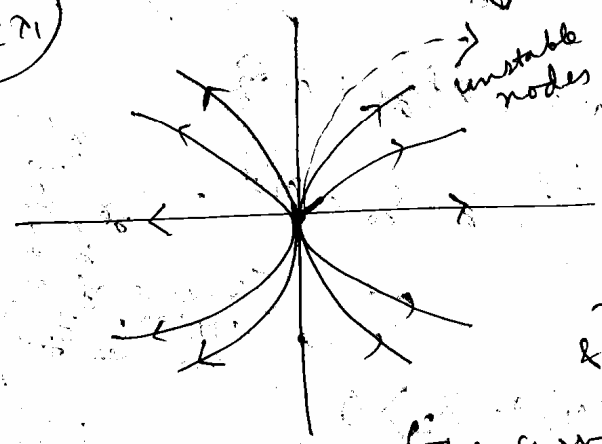
$\lambda_2 > \lambda_1$



$\tilde{x} = 0$ means $\tilde{x}_0 = 0$

unstable nodes [because things flow out from the fixed pt.]

$\lambda_2 < \lambda_1$



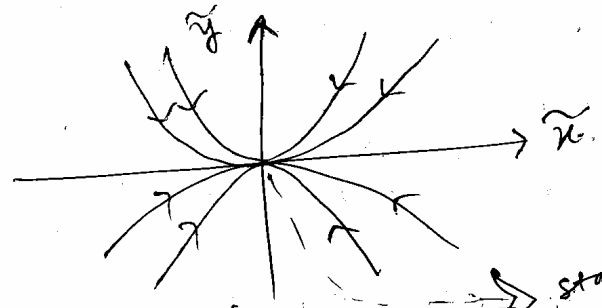
$\frac{d\tilde{y}}{d\tilde{x}} \propto \tilde{x}^{\lambda_2/\lambda_1 - 1}$

$\tilde{y} = 0$ means $\tilde{y}_0 = 0$ & $\tilde{y} = 0$ for all t

(The curves are singular — they are singular near the origin — non-analytic at the origin)

Case 1 (b) :- $\lambda_1 < 0$, $\lambda_2 < 0$

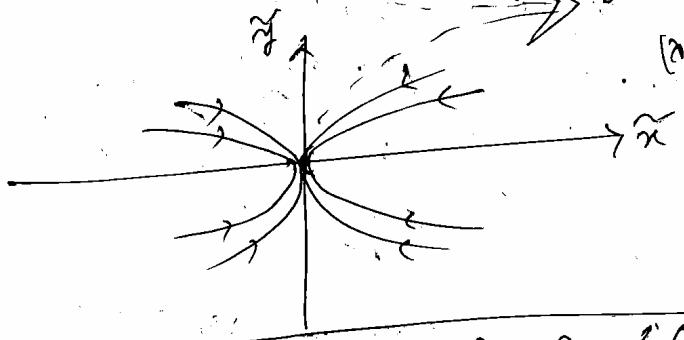
The phase diagrams will look exactly the same, except that the arrows get reversed.



$|\lambda_2| > |\lambda_1|$

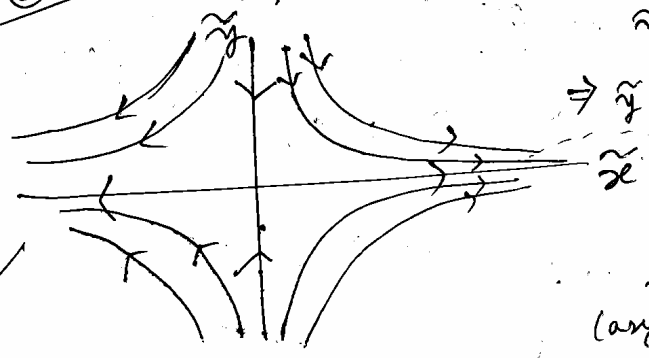
stable nodes

$|\lambda_2| < |\lambda_1|$.



All flow lines pass through fixed pts.

Case 1 (c) $\lambda_1 > 0$ & $\lambda_2 < 0$



$\tilde{y} \propto \tilde{x}^{\lambda_2/\lambda_1} = \tilde{x}^{-|\lambda_2/\lambda_1|}$

$\Rightarrow \tilde{y}\, \tilde{x}^{|\lambda_2/\lambda_1|} = \text{const.}$

$\tilde{y} \downarrow$ with time

$\tilde{x} \uparrow$ (asymptotically with $\tilde{x}$)

All flow lines don't reach the fixed pts.

If $\lambda_1 = -\lambda_2$, then the curves are rectangular hyperbolae.

(Qualitatively, the curves look like ~~like~~ hyperbolae if $\lambda_1 \neq -\lambda_2$.)

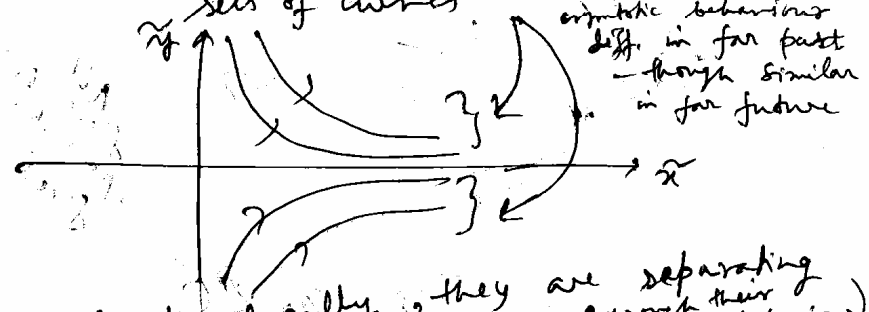
These are called <u>Hyperbolic fixed points</u>.

The only flow lines reaching the fixed

Separatrix :- Phase curves which meet a hyperbolic fixed pt.

→ These lines separate flow lines with very different asymptotic behaviour.

e.g. → The $\tilde{x}$ axis separates the 2 sets of curves.



At least locally, they are separating the 2 diff. sets of curves (through their asym. behaviour).

Globally, these flow lines may become closed curves & behaviour near fixed pt. isn't enough to predict this.

Case 2 :- $\lambda_1, \lambda_2 = \tau/2 \pm \sqrt{\tau^2/4 - \delta}$

$$\tau^2 < 4\delta \qquad (\delta > 0)$$

λ_1, λ_2 complex, $\lambda_1 = \lambda_2^*$

$\lambda_1 \lambda_2 = \delta$, $\lambda_1 = \lambda_2^*$

$$\lambda_1 = \sqrt{\delta}\, e^{i\omega}, \quad \lambda_2 = \sqrt{\delta}\, e^{-i\omega}$$

Because λ_1 & λ_2 are complex, the eigenvectors themselves will be complex.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \Rightarrow \alpha_1, \beta_1 \text{ complex}$$

" α_2, β_2

As a result, the diagonalizing matrix will also be complex.
Now, $\tilde{x}$ & $\tilde{y}$ will both be complex & we need a 4-dim. for plotting.
We want sth simpler.

$$\boxed{\text{The soln. is NOT TO DIAGONALIZE } \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

Define :- $\alpha = \tau/2 = \dfrac{a+d}{2}$ & $\omega = \sqrt{\delta - \tau^2/4} = \sqrt{ad-bc+(a+d)^2/4}$
$= \sqrt{-bc - \dfrac{(a-d)^2}{4}}$

Then, eigenvalues are $\alpha \pm i\omega$

Let's define $A^{-1} = \begin{pmatrix} c & \alpha - a \\ 0 & \omega \end{pmatrix}$

Ex: Show that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} A = A \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}$

or $A^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix} A^{-1}$

Define :- $\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = A^{-1} \begin{pmatrix} x - x_F \\ y - y_F \end{pmatrix}$

$$\therefore \frac{d}{dt}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = A^{-1} \frac{d}{dt}\begin{pmatrix} x - x_F \\ y - y_F \end{pmatrix}$$

$$= A^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x - x_F \\ y - y_F \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix} A^{-1} \begin{pmatrix} x - x_F \\ x - y_F \end{pmatrix}$$

The original matrix is a similarity transf. of $\begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}$, having the same e. values as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\frac{d\tilde{x}}{dt} = \alpha\tilde{x} - \omega\tilde{y}$$
$$d\tilde{y}/dt = \omega\tilde{x} + \alpha\tilde{y}$$

These eqns are still coupled & solns can't be read out easily

Introduce new variables

$$\tilde{x} = r\cos\theta \quad \& \quad \tilde{y} = r\sin\theta$$

Now, $dr/dt = \partial r/\partial\tilde{x}\; d\tilde{x}/dt + \partial r/\partial\tilde{y}\; d\tilde{y}/dt$

$r = \sqrt{\tilde{x}^2 + \tilde{y}^2}$

$$= \tilde{x}/r\,(\alpha\tilde{x} - \omega\tilde{y}) + \tilde{y}/r\,(\omega\tilde{x} + \alpha\tilde{y})$$

$$= \frac{\alpha\tilde{x}^2 + \alpha\tilde{y}^2}{r}$$

$$= \alpha r$$

$\boxed{\text{Ex. Check that } d\theta/dt = \omega}$

(we can derive at this also by the complex variable root & take the mag. & phase parts — But in this manner, we never have to talk about the complex variables)

---

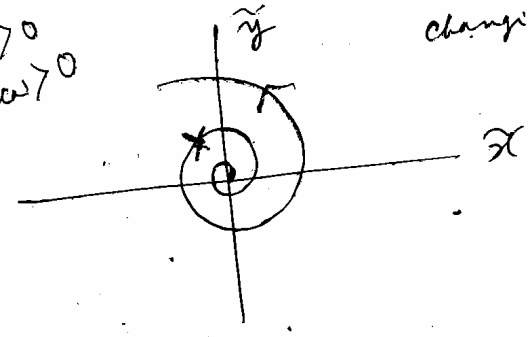
soln.) $\theta = \omega t + \theta_0$

$r = r_0 e^{\alpha t}$

a) $\alpha > 0$ (spiralling outward) → called Unstable spiral

b) $\alpha < 0$ → called stable spiral.

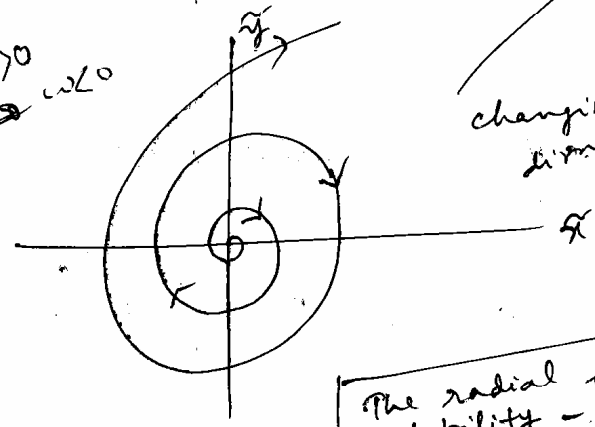
$\alpha > 0$
$\omega > 0$



changing the arrow dirn, we'll get $\alpha < 0$ & $\omega < 0$
↳ stable spiral

Unstable spirals

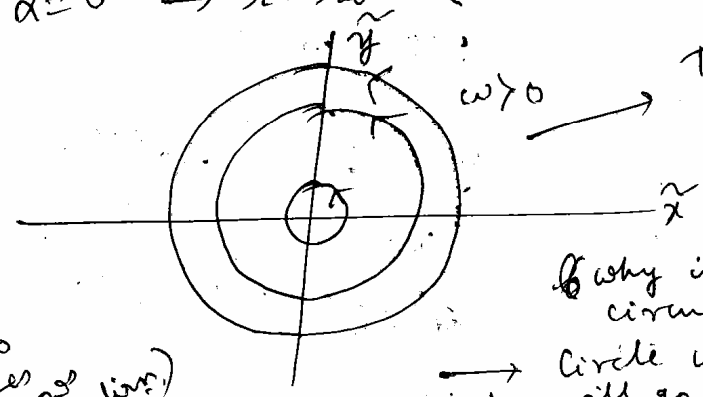
$\alpha > 0$
$\omega < 0$



changing the arrow dirn, we'll get $\alpha < 0$ & $\omega > 0$
↳ stable spiral

The radial part determines stability — θ only changes dirn, while r det. whether it's going out or in

c) $\alpha = 0 \rightarrow r = r_0$ & $\theta = \omega t + \theta_0$



$\omega > 0$ → These fixed pts. are known as elliptic fixed pts.

why is it not called circular fixed pts.?

($\omega < 0$ gives arrows in opp. dirn)

→ Circle under linear transfs. will go into an ellipse

$\tilde{x}$ & $\tilde{y}$ need to be transformed ... plane, where they'll give

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H.W.

$$A^{-1} = \begin{pmatrix} c & \alpha - a \\ 0 & \omega \end{pmatrix}$$

Now, $A^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & \alpha - a \\ 0 & \omega \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= \begin{pmatrix} \cancel{ca} + c\alpha - \cancel{ca} & bc + \alpha d - ad \\ 0 + \omega c & 0 + \omega d \end{pmatrix}$$

$$= \begin{pmatrix} c\alpha & bc + \alpha d - ad \\ \omega c & \omega d \end{pmatrix}$$

Again, $\begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix} A^{-1} = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}\begin{pmatrix} c & \alpha - a \\ 0 & \omega \end{pmatrix}$

$$= \begin{pmatrix} \alpha c & \alpha^2 - a\alpha - \omega^2 \\ \omega c & \omega\alpha - a\omega + \alpha\omega \end{pmatrix}$$

$$= \begin{pmatrix} c\alpha & (a+d)^2/4 - \frac{a}{2}(a+d) + bc + (a-d)^2/4 \\ \omega c & \cancel{2}\left(\frac{a+d}{\cancel{2}}\right)\omega - a\omega \end{pmatrix}$$

$$= \begin{pmatrix} c\alpha & \frac{2a^2 + \cancel{2d^2} + \cancel{2ad} - \cancel{2ad}}{4} - \frac{a^2 + ad}{2} + bc \\ \omega c & \omega d \end{pmatrix}$$

$$= \begin{pmatrix} c\alpha & \frac{\cancel{a^2} + d^2 - \cancel{a^2} - ad}{2} + bc \\ \omega c & \omega d \end{pmatrix}$$

$$\begin{pmatrix} c\alpha & \frac{d^2 - ad}{2} + bc \end{pmatrix}$$

Also, $bc + \alpha d - ad = bc + \left(\frac{a+d}{2}\right) d - ad$

$$= bc + \frac{ad}{2} + d^2/2 - ad$$

$$= d^2/2 - ad/2 + bc$$

$$\therefore A^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c\alpha & \cancel{\frac{d^2-ad}{2}} + bc + \alpha d - ad \\ \omega c & \omega d \end{pmatrix}$$

(proved)

$$d\tilde{x}/dt = \alpha\tilde{x} - \omega\tilde{y}$$
$$d\tilde{y}/dt = \omega\tilde{x} + \alpha\tilde{y}$$

Introducing
$\tilde{x} = r\cos\theta$
$\tilde{y} = r\sin\theta$

Here, $r = \sqrt{\tilde{x}^2 + \tilde{y}^2}$ & $\tan\theta = \tilde{y}/\tilde{x} \Rightarrow \theta = \tan^{-1}(\tilde{y}/\tilde{x})$

$$\therefore \partial\theta/\partial\tilde{y} = \frac{1}{1+(\tilde{y}/\tilde{x})^2}\cdot\frac{1}{\tilde{x}} = \frac{\tilde{x}^2/\tilde{x}}{\tilde{x}^2+\tilde{y}^2} = \frac{\tilde{x}}{r^2}$$

and $$\partial\theta/\partial\tilde{x} = \frac{1}{1+(\tilde{y}/\tilde{x})^2}\cdot\left(\frac{-\tilde{y}}{\tilde{x}^2}\right) = \frac{-\tilde{y}\,\cancel{\tilde{x}^2}/\cancel{\tilde{x}^2}}{\tilde{x}^2+\tilde{y}^2} = -\tilde{y}/r^2$$

$$\therefore d\theta/dt = \frac{\partial\theta}{\partial\tilde{x}}\, d\tilde{x}/dt + \partial\theta/\partial\tilde{y}\; d\tilde{y}/dt$$

$$= -\tilde{y}/r^2\,(\alpha\tilde{x} - \omega\tilde{y}) + \tilde{x}/r^2\,(\omega\tilde{x} + \alpha\tilde{y})$$

$$= \frac{-\cancel{\alpha\tilde{x}\tilde{y}} + \omega\tilde{y}^2 + \omega\tilde{x}^2 + \cancel{\alpha\tilde{x}\tilde{y}}}{r^2}$$

$$= \omega$$

Poincaré construction

Suppose we have a curve in 3 or more dim. How do we see its behaviour? We plot it only when it intersects a given 2-dim. plane & don't care what it does elsewhere.

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12/8/05

(x_F, y_F)

$$\frac{d}{dt}\begin{pmatrix} x-x_F \\ y-y_F \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x-x_F \\ y-y_F \end{pmatrix} \quad \text{—(1)}$$

$\tau = a+d$, $\delta = ad-bc$

$\lambda_1, \lambda_2 = \tau/2 \pm \sqrt{\tau^2/4 - \delta}$

1) $\tau^2 > 4\delta$

2) $\tau^2 < 4\delta$

3) $\tau^2 = 4\delta \Rightarrow \lambda_1 = \lambda_2 = \tau/2$

$\because$ 2 e. values are equal, we can't find 2 indep. e. vectors & can't diagonalise the matrix by similarity transfn. in general. But it's possible for a Hermitian matrix.

Jordan-Canonical form — upper triangular or lower " form

Standard method to diagonalise $\Rightarrow \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

Define $A^{-1} = \begin{pmatrix} a-d & 2b \\ 2c & 0 \end{pmatrix}$ Valid when b & c are non-zero [A is non-singular]

Show that $A^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix} A^{-1}$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A\begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix} A^{-1}$$

Using this, we'll simplify (1) -

Define : $\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = A^{-1}\begin{pmatrix} x-x_F \\ y-y_F \end{pmatrix}$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix}\lambda & 0\\ c & \lambda\end{pmatrix} A^{-1}\begin{pmatrix}x-x_F\\ y-y_F\end{pmatrix} = \begin{pmatrix}\lambda & 0\\ c & \lambda\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}$$

Even if we can't diagonalise the matrix, we can write the Diff. Eqns in the form

$$d\tilde{x}/dt = \lambda\tilde{x} \Rightarrow \tilde{x} = \tilde{x}_0 e^{\lambda t}$$

$$d\tilde{y}/dt = c\tilde{x} + \lambda\tilde{y} \Rightarrow \tilde{y} = \tilde{y}_0 e^{\lambda t} + c\tilde{x}_0 t e^{\lambda t}$$

Case 3 (a) $c = 0$ (Here, both b & c are zero)

Then the matrix is diagonal.

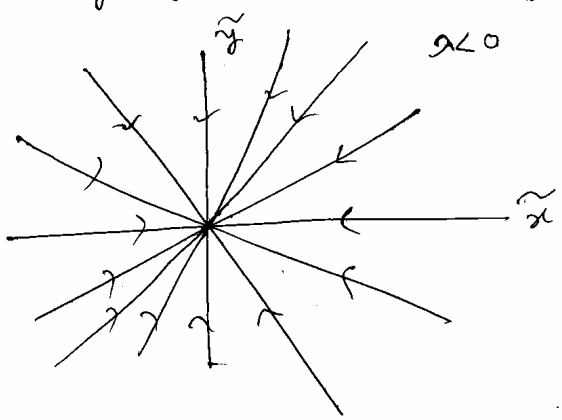
[In certain cases, the Jordan-Canonical gives a diagonal matrix]

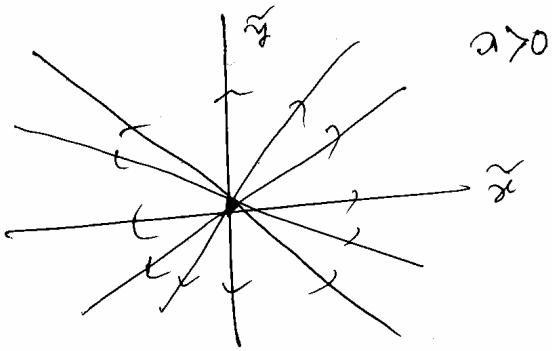
$$\tilde{x} = \tilde{x}_0 e^{\lambda t}$$

$$\tilde{y} = \tilde{y}_0 e^{\lambda t}$$

(The upper triangular form can be transformed into lower triangular form by similarity transform.)

$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ can do the transform





Arrows are drawn to tell us what happens when $t \uparrow$ — but nature of curves is insensitive to λ

Case 3 (b) $\quad c \neq 0$

$$e^{\lambda t} = \tilde{x}/\tilde{x}_0$$

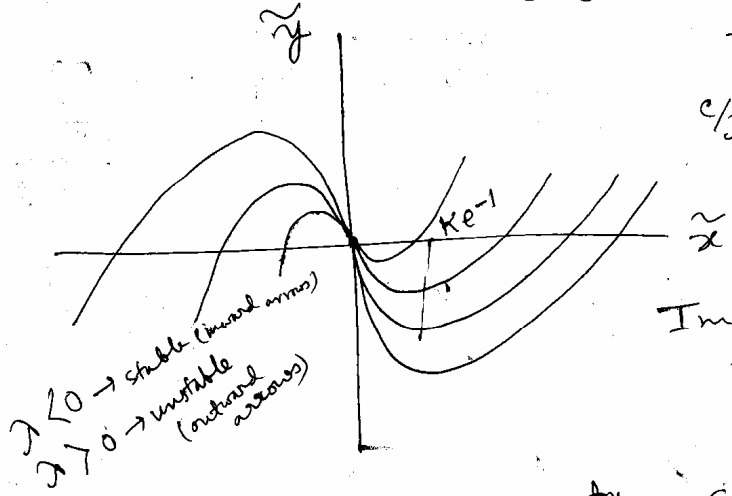
$$\tilde{y} = \tilde{y}_0 \, \tilde{x}/\tilde{x}_0 + c\tilde{x} \, \frac{1}{\lambda} \ln \tilde{x}/\tilde{x}_0$$

$$= c/\lambda \, \tilde{x} \left[\ln \tilde{x}/\tilde{x}_0 + \underbrace{\tilde{y}_0/\tilde{x}_0 \; \lambda/c}_{\text{const.}} \right]$$

$$= c/\lambda \, \tilde{x} \ln\left(\tilde{x}/K\right)$$

& we define it as $-\ln K$.

$c/\lambda > 0$



Improper nodes.

$$d\tilde{y}/d\tilde{x} = c/\lambda \ln(\tilde{x}/K) + \frac{c}{\lambda} \cancel{\tilde{x}} \, K/\cancel{\tilde{x}} \, \frac{1}{K} = c/\lambda \left[\ln(\tilde{x}/K) + 1\right]$$

For $\tilde{x} = 0$, $d\tilde{y}/d\tilde{x}$ is infinite

For $\tilde{x} = Ke^{-1}$, it turns $\Rightarrow$ $d\tilde{y}/d\tilde{x} = 0 = -1 + \ln(\tilde{x}/K)$ $\Rightarrow \tilde{x}/K = e^{-1}$

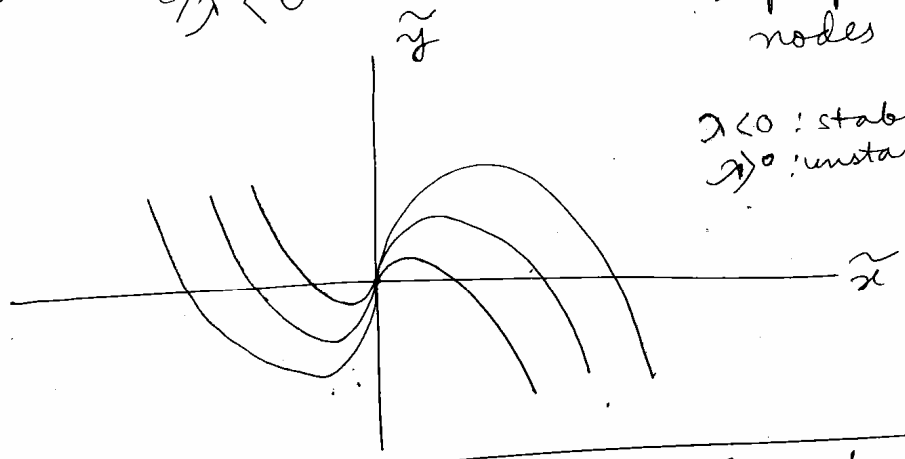
It can be seen that $-\tilde{x}$ & $-\tilde{y}$ will satisfy the same set of eqns & will be the mirror image of the R.H.S.

Here, the singularity is logarithmic + a power law singularity

$c/\lambda < 0$

Improper nodes

$\lambda < 0$: stable
$\lambda > 0$: unstable



($\tilde{x}$ & $\tilde{y}$ are related to x & y by some linear transfn.)

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = A^{-1} \begin{pmatrix} x - x_F \\ y - y_F \end{pmatrix}$$

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

$$\begin{pmatrix} x - x_F \\ y - y_F \end{pmatrix} = A \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

$$= \begin{pmatrix} p\tilde{x} + q\tilde{y} \\ r\tilde{x} + s\tilde{y} \end{pmatrix}$$ $\rightsquigarrow$ not an orthonormal matrix / transfn.

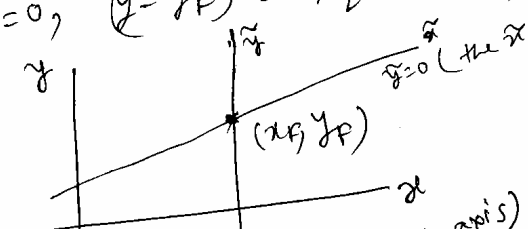
$x - x_F = p\tilde{x} + q\tilde{y}$
$y - y_F = r\tilde{x} + s\tilde{y}$

For $\tilde{y} = 0$, $x - x_F = p\tilde{x}$
$\therefore y - y_F = r/p\,(x - x_F)$

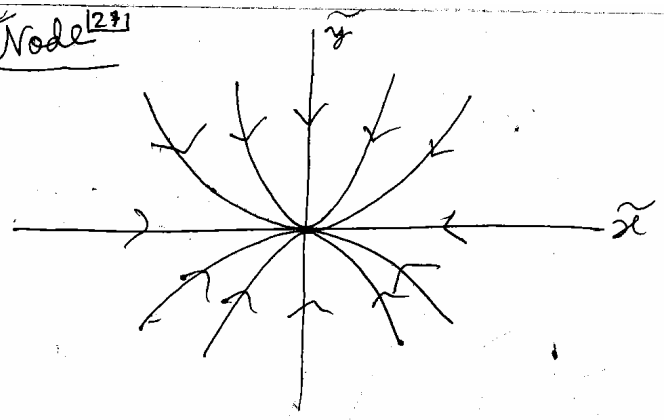
For $\tilde{x} = 0$, $x - x_F = q\tilde{y}$
$\therefore y - y_F = s/q\,(x - x_F)$

If $\tilde{y} = 0$, then $(y - y_F) = r/p\,(x - x_F)$

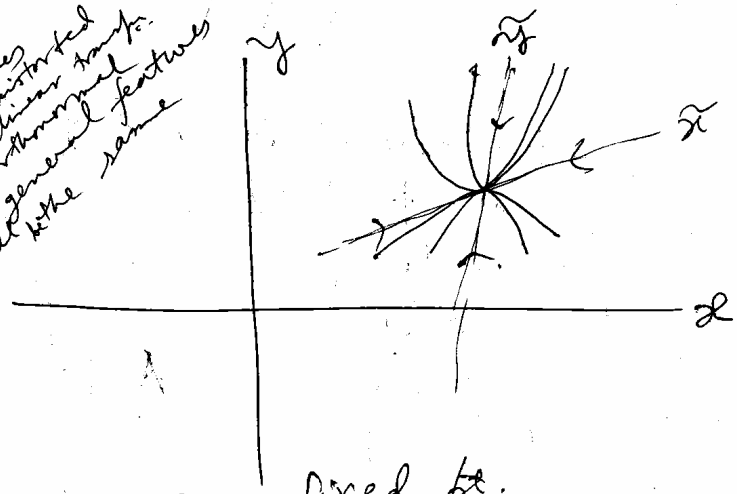
If $\tilde{x} = 0$, $(y - y_F) = s/q\,(x - x_F)$



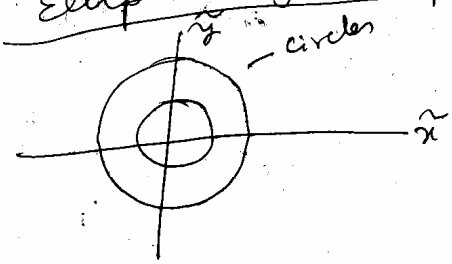
Stable Node [27]

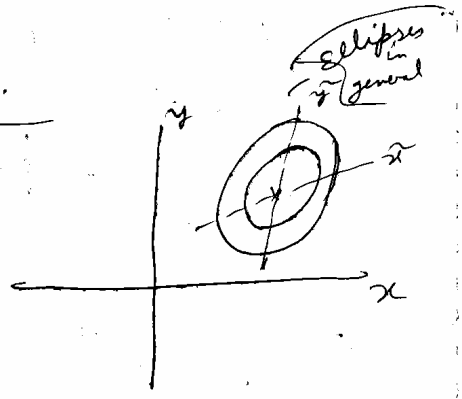


Here the curves will be distorted ∴ this linear transf. isn't orthonormal — but general features will be the same

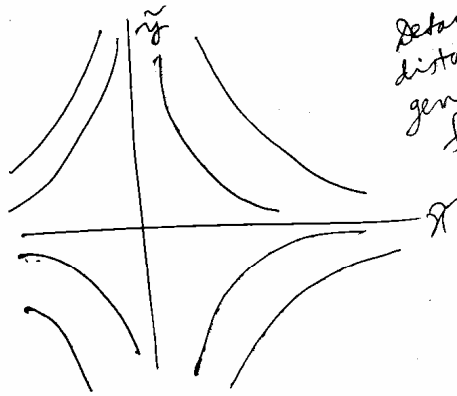


~~Ell~~ Elliptic fixed pt.

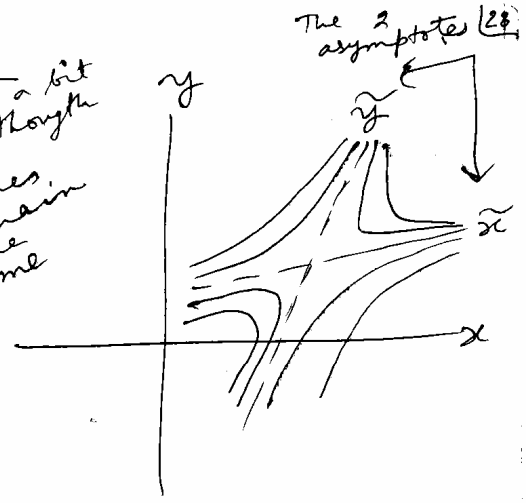




Hyperbolic fixed pt.



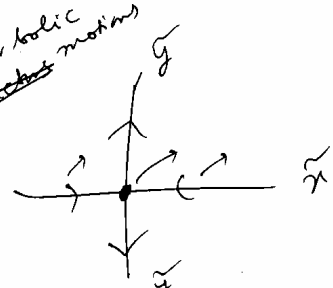
Details get a bit distorted, though general features remain the same



Suppose $\vec{r}_F = (x_F, y_F)$ is a fixed point
Suppose $\vec{r}(t)$ describes a motion with some specific initial condition.

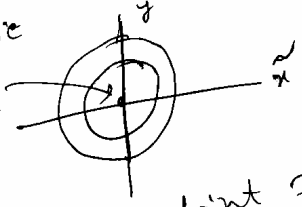
Defn :- $\vec{r}_F$ is an attractor of the motion if $\vec{r}(t)$ is $\vec{r}(t) \to \vec{r}_F$ as $t \to \infty$

For hyperbolic ~~attractors~~ motions



For stable spiral → infinite (by changing parameters diff. spirals can be drawn, which won't touch each other)

For elliptic → 1



Defn → Any point P in (x, y) plane is in the basin of attraction of $\vec{r}_F$ if $\vec{r}_F$ is the [illegible] of the motion beginning at P.

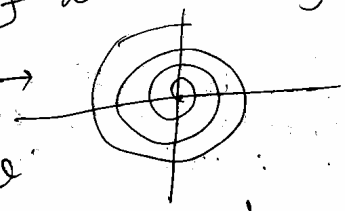
(29) Suppose D_R is a disk of radius R around $\vec{r}_F$:

$$(x-x_F)^2+(y-y_F)^2 < R^2$$

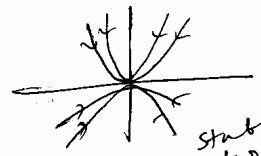
Doubt:- sir:- generally D_R must be an open set. Proof??

Defn. A fixed pt. $\vec{r}_F$ is strongly stable if there exists δ such that $\vec{r}_F$ is the attractor of every motion that begins inside D_δ, i.e., D_δ is in the basin of attraction of $\vec{r}_F$.

e.g. →



spiral

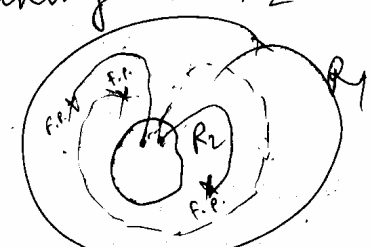


stable star

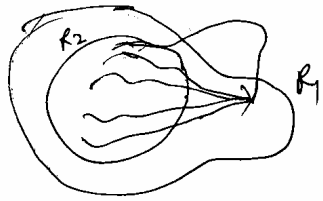
Lowest pt. of undamped pendulum isn't an e.g.
→ because $x=0$ & $v=0$ can't be reached for $x\neq 0$ &/or $v\neq 0$ initially

elliptic f.p. – stable in some sense as u hover around the fixed pt. & don't go far from the f.p.

Defn. :- A region R_1 in phase space (x-y space) contains the motion of a region R_2 if all motions beginning in R_2 lie inside R_1 for $t \geq 0$.



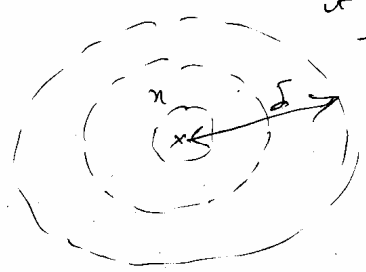
[R_1 will have a minimum size]
(R_1 ~~may~~ is in general. i.e. full phase space)



All Attractors should be contained in R_1 is a necessary, but not a sufficient condn.

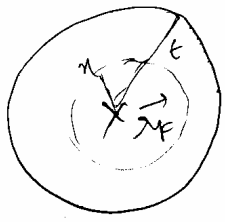
<u>Defn</u> :- A fixed pt. $\vec{x}_F$ is stable if there exists a δ such that for every $\epsilon < \delta$, there exists an η such that D_ϵ contains the motion of D_η.

We must be able to draw ~~some~~ disc of radius δ such that it will contain the motion for any $\epsilon < \delta$



Fix a no. δ (u decide the δ)
Now give anything $\epsilon < \delta$
If ~~to~~ I can give another no. $\eta < \epsilon$ such that any motion won't go out of ϵ

A fixed point $\vec{x}_F$ is stable if given any positive no. δ, for every $\epsilon < \delta$ there exists an η such that D_ϵ contains the motion of D_η.



[Given an ϵ, I'll have to find out a circle of radius η, u never wander out of the radius ϵ]

(This defn includes elliptic fixed pts. as e.g.) stable but not strongly stable.

We have to consider a generalised sphere (instead of a disk) for n-dim.

Then $(x^1 - x_F^1)^2 + (x^2 - x_F^2)^2 + \cdots + (x^n - x_F^n)^2 < R^2$ defines D_R.

E.g. → suppose. $dx/dt = x f_1(x, y)$ $\quad x = 0$ as a fixed pt.

$dy/dt = x f_2(x, y)$

$x = 0 \Rightarrow dx/dt = 0 = dy/dt$

So we have a continuous line of fixed pts. (along the y-axis).

Doubt 1 Why D_R is an open set?

⇒ If the boundary pts. are included ⇒ we can go backwards in time & then the trajectory would go beyond the boundary — then those outside pts. would have to be considered in D_R.

So, an open set — on going backwards in time, the trajectory would ~~tend~~ tend to reach the boundary asymptotically.

17/8/05 In a 2nd order autonomous system,

Possibilities:- as $t \to \infty$

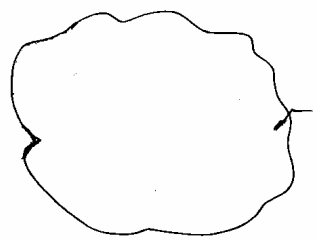
1) $\vec{r}(t) \xrightarrow{\text{will approach}}$ an absolutely stable fixed point

2) $\vec{r}(t)$ could be a periodic motion (e.g. — orbits around elliptic fixed point)

3) $\vec{r}(t) \to \infty$

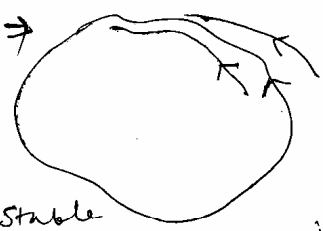
4) $\vec{r}(t)$ approaches a limit cycle.

Limit cycle is typically a closed periodic orbit



[Elliptic f.p. comes in whole families; but here, if u move away from the orbit, u either go away or approach the orbit]

neighbouring trajectories asymp. approach this approach (attracted from both sides)



Stable

neighbouring go away from (repulsive from both sides)

Unstable trajectories

In the generic case (fn. itself is zero & not its deriv.)

In special cases, it can be attractive on one side & rep. from another side in double zero.

Example

$$dx/dt = \alpha x(x^2+y^2-R^2) - \omega y \quad \text{—} (1)$$

$$dy/dt = \alpha y(x^2+y^2-R^2) + \omega x \quad \text{—} (2)$$

It's easier to survey the dynamics by going to the polar coord.:—

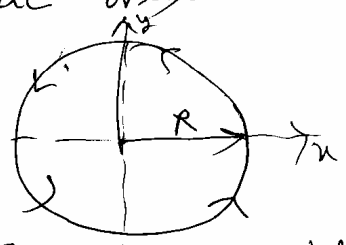
Polar coordinates: $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1} y/x$

Check that $dr/dt = \alpha r(r^2-R^2)$

$d\theta/dt = \omega \xrightarrow{\text{soln.}} \theta = \omega t + \theta_0$

From this, it follows that for $r=R$ & $\theta = \omega t$, $dr/dt = 0$

So there is at least one periodic orbit (with radius R) $\Rightarrow$



What if we displace it a little bit (i.e., $r = R+\rho$)?

Define: $\rho = r - R$

$\therefore$ (1) gives $d\rho/dt = \alpha(\rho+R)((\rho+R)^2-R^2)$

$\simeq 2\alpha R^2 \rho$ + terms of order ρ^2

[Taylor expansion]

$\Rightarrow \rho \simeq \rho_0 e^{2\alpha R^2 t}$

$\Rightarrow$ For $\alpha > 0$, ρ grows $\Big\}$ as $t \to \infty$

$\alpha < 0$ $\rho \to 0$

For ρ (+ve or -ve), ρ ~~diverges~~ from the $r=R$ trajectory for $\alpha>0$

for ρ (+ve or -ve) $\downarrow$ " " $r=R$ --- for $\alpha<0$.

Ex. → Check what eventually happens to the trajectories if $\alpha>0$, or $\alpha<0$.

[analyse the given eqns]

---

n-th order autonomous system

$$\frac{dx^i}{dt} = F^i(\vec{x}), \quad i=1, \ldots, n$$

Suppose we have a fixed point at $\vec{x}_F$.

This means that $F^i(\vec{x}_F)=0$ for $i=1, \ldots, n$.

We want to study the motions starting a little away from the f.p.

Define:- $y^i = x^i - x^i_F$

$$\therefore \frac{dy^i}{dt} = F^i(\vec{x}_F + \vec{y})$$

$$= \underbrace{F^i(\vec{x}_F)}_{=0} + \underbrace{\frac{\partial F^i}{\partial x^j}\Big|_{\vec{x}_F}}_{\text{matrix } M^i_j} y^j + \ldots \quad \text{[Taylor exp.]}$$

[To keep track that F-index is in numerator & x-index is in denominator, we write one index up & " down]

$$\Rightarrow \frac{dy^i}{dt} = M^i_j y^j$$

Generically, M has n eigenvalues:-

$$\det(M - \lambda \underbrace{I}_{\text{identity matrix}}) = 0$$

" + M's eigenvalues are different.

$$M v_{(k)} = \Lambda_{(k)} v_{(k)} \rightarrow \text{an } n\text{-dim. column vector}$$

$\Lambda_{(k)} \rightarrow$ k-th eigenvalue

The matrix M is a real matrix.

$\Rightarrow$ There can be complex eigenvalues, but they always come in complex conjugate pairs.

In a generic case, we have $2p$ complex eigenvalues & $n-2p$ real eigenvalues.

$$\alpha_{(1)} + i\omega_{(1)},\ \alpha_{(1)} - i\omega_{(1)},\ \alpha_{(2)} + i\omega_{(2)},\ \alpha_{(2)} - i\omega_{(2)},$$
$$\ldots,\ \alpha_{(p)} + i\omega_{(p)},\ \alpha_{(p)} - i\omega_{(p)}$$

$$\lambda_{(1)},\ \lambda_{(2)},\ \ldots,\ \lambda_{(n-2p)} \quad \left[\alpha_{(i)},\ \omega_{(i)},\ \lambda_{(i)} \text{ are real}\right]$$

[Linear eqn $M v_{(k)} = \Lambda_{(k)} v_{(k)}$ can't introduce complex eigenvalues — so, real e.values will give real e.vectors]

Eigenvector corresponding to $\lambda_{(R)}$:

$$\begin{pmatrix} a^1_{(k)} \\ \vdots \\ a^n_{(k)} \end{pmatrix}$$

$\therefore$ the e.value eqn is

$$\boxed{M^i_j\, a^j_{(k)} = \lambda_{(k)}\, a^i_{(k)}} \quad (2)$$

[Complex e.values will have complex e.vectors]

Eigenvector corresponding to $\alpha_{(k)} + i\omega_{(k)}$:

$$\begin{pmatrix} b_{(k)} + i\, c_{(k)} \\ \vdots \end{pmatrix}$$

$$M^i_j \left(b^j_{(k)} + i\, c^j_{(k)} \right) = \left(\alpha_{(k)} + i\, \omega_{(k)} \right) \left(b^i_{(k)} + i\, c^i_{(k)} \right)$$

$\longrightarrow$ eigenvalue eqn.

[Complex conj. e. value will have complex conj. e. vector.]

Separating into real & complex parts:-

$$\boxed{\begin{aligned} M^i_j \; b^j_{(k)} &= \alpha_{(k)}\, b^i_{(k)} - \omega_{(k)}\, c^i_{(k)} \\ M^i_j \; c^j_{(k)} &= \omega_{(k)}\, b^i_{(k)} + \alpha_{(k)}\, c^i_{(k)} \end{aligned}}$$

$\rightarrow$ Real part

$\rightarrow$ Im part

(b)

$\therefore$ (a) & (b) are one matrix eqns.

$$M\, a_{(k)} = \lambda_{(k)}\, a_{(k)}$$

$$M\, b_{(k)} = \alpha_{(k)}\, b_{(k)} - \omega_{(k)}\, c_{(k)}$$

$$M\, c_{(k)} = \omega_{(k)}\, b_{(k)} + \alpha_{(k)}\, c_{(k)}$$

These 3 sets of eqns can be combined into an $n \times n$ matrix eqn :-

$$M \begin{pmatrix} a^1_{(1)} & a^1_{(2)} & \cdots & a^1_{(n-2p)} & b^1_{(1)} & c^1_{(1)} & \cdots & b^1_{(p)} & c^1_{(p)} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ a^n_{(1)} & a^n_{(2)} & & a^n_{(n-2p)} & b^n_{(1)} & c^n_{(1)} & \cdots & b^n_{(p)} & c^n_{(p)} \end{pmatrix}$$

$$= \begin{pmatrix} a^1_{(1)} & \cdots & c^1_{(p)} \\ \vdots & & \vdots \\ n & & n \end{pmatrix} \begin{pmatrix} \lambda_{(1)} & 0 & & \\ 0 & \lambda_{(2)} & & \\ & & \ddots & \\ & & & \lambda_{(n-2p)} \\ & & & & \alpha_{(1)} \; \omega_{(1)} \\ & & & & \omega_{(1)} \; \alpha_{(1)} \end{pmatrix}$$

$MA = AK \Rightarrow A^{-1}M = KA^{-1}$ (~~assume that~~ we use the fact A is non-singular)

Recall the motion near the fixed pt.

$$\frac{dy^i}{dt} = M^i_j \, y^j \quad \left(y^i = x^i - x^i_F\right)$$

where

due to linearly indep. e. vectors

The corr. matrix eqn. is

$$\frac{d\vec{y}}{dt} = \underbrace{M}_{A \text{ matrix prdt.}} \vec{y}$$

$$\frac{d}{dt}\left(A^{-1}\vec{y}\right) = A^{-1}M\vec{y} = K\left(A^{-1}\vec{y}\right)$$

~~Define~~

$$\cancel{\vec{Z} = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^{n-2p} \\ \eta^1 \\ \tau^1 \\ \eta^2 \\ \tau^2 \\ \vdots \\ \eta^p \\ \tau^p \end{pmatrix} = A^{-1}\vec{y}}$$

Define :-

$$\vec{Z} = \begin{pmatrix} \hat{z}^1 \\ \vdots \\ \hat{z}^{n-2p} \\ \tilde{z}^1 \\ \check{z}^1 \\ \tilde{z}^2 \\ \check{z}^2 \\ \vdots \\ \tilde{z}^p \\ \check{z}^p \end{pmatrix} = A^{-1}\vec{y}$$

$$\Rightarrow \frac{d\hat{z}^i}{dt} = \lambda_{(i)}\hat{z}^i \quad \text{for } i = 1, \ldots, n-2p$$

[$\because$ the first $(n-2p)$ elements in K is diagonal]

$$\frac{d\tilde{z}^1}{dt} = \alpha_{(1)}\tilde{z}^1 + \omega_{(1)}\check{z}^1$$

$$\frac{d\check{z}^1}{dt} = -\omega_{(1)}\tilde{z}^1 + \alpha_{(1)}\check{z}^1$$

In general, $\frac{d\tilde{z}^i}{dt} = \alpha_{(i)} \tilde{z}^i + \omega_{(i)} \check{z}^i$

$$\frac{d\check{z}^i}{dt} = -\omega_{(i)} \tilde{z}^i + \alpha_{(i)} \check{z}^i$$

[∵ the rest of K is block-diagonal]

Solutions

$\hat{z}^i = \hat{z}^i_0 e^{\lambda_{(i)} t}$

The rest can be solved used polar coord.

Define $\left.\begin{array}{l} \tilde{z}^i = r^i_0 \cos\theta_i \\ \check{z}^i = r^i_0 \sin\theta_i \end{array}\right\}$ $\begin{array}{l} dr^i/dt = \alpha_{(i)} r^i \\ d\theta^i/dt = -\omega_{(i)} \end{array}$

soln. → $\boxed{\begin{array}{l} r^i = r^i_0 e^{\alpha_{(i)} t} \\ \theta^i = -\omega_{(i)} t + \theta^i_0 \end{array}}$

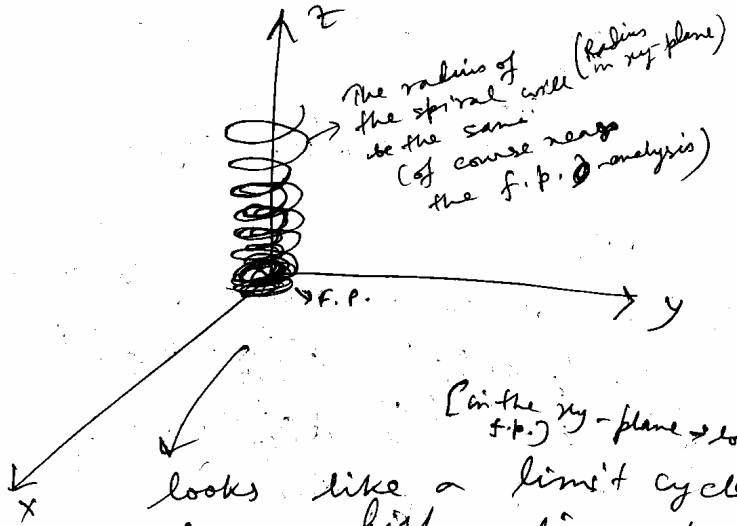
Complete solns of the eqn of motion near the f.p.

[The solns split up into many 2-dim systems]
↳ (We don't need any new input to analyse the n-dim. aut. system.)

$\boxed{\text{The f.p. is absolutely stable for } \lambda_i < 0, \ \alpha_{(i)} < 0 \text{ for every } i}$

In general, the approach to the f.p. will be very complicated — if we project on diff. planes, we can get spirals, ~~et~~ elliptic — — —, etc.

e.g. → Take a 3-dim. aut. system with 2 complex & 1 real e. values.



$\lambda \to -ve$
$\alpha = 0$
Take λ for z

z is tending to zero, but x & y are never going to be zero

[in the xy-plane (f.p.) → looks like elliptic

looks like a limit cycle from a higher-dim perspective

eventually approach a periodic orbit in the xy-plane as $t \to \infty$

($x^i \Rightarrow$ space of coords & vel. for classical systems)

$$dx^i/dt = F^i(\vec{x})$$

Coord. change

Define new coordinates

$$y^i = g^i(\vec{x})$$

↳ known fns

$$dy^i/dt = \partial g^i/\partial x^j \; dx^j/dt = \frac{\partial g^i(\vec{x})}{\partial x^j} F^j(\vec{x})$$

y^i's satisfy: $\frac{dy^i}{dt} = \tilde{F}^i(\vec{y})$,

$$\boxed{\tilde{F}^i(\vec{y}) = \frac{\partial g^i(\vec{x})}{\partial x^j} F^j(\vec{x})}$$

Sometimes, by a coord. change, we may simplify the initial set of eqns.

[In analysing f.p. also, we made a linear change of variables — but we can as well make a non-linear change of coord.]

H.W.

$$dr/dt = (\alpha r^3 - \alpha R^2 r) \quad \text{——— (1)}$$

Let $I = \int \frac{dr}{r(r-R)(r+R)}$

$$\therefore I = \int \left[\frac{-1/R^2}{r} + \frac{1/2R^2}{r-R} + \frac{1/2R^2}{r+R} \right] dr$$

$$= -1/R^2 \ln r + \frac{1}{2R^2} \ln |r-R| + 1/2R^2 \ln (r+R) + c$$

~~$\therefore$ (1) can be solved by~~

for $r<R$

$$I = -1/R^2 \ln r + \frac{1}{2R^2} \ln (R-r) + \frac{1}{2R^2} \ln (R+r) + c$$

$$= \frac{1}{R^2} \ln \left(\frac{\sqrt{R^2 - r^2}}{r} \right) + c$$

$\therefore$ (1) has the soln.

$$\ln \left(\sqrt{R^2/r^2 - 1} \right) = \alpha R^2 t + \ln r_0$$

$$\Rightarrow \quad R^2/r^2 - 1 = r_0^2 e^{2\alpha R^2 t}$$

$$\Rightarrow \quad \frac{1}{r^2} = \frac{1 + r_0^2 e^{2\alpha R^2 t}}{R^2}$$

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$$\Rightarrow r = \frac{R}{\sqrt{1+r_0^2 e^{2\alpha R^2 t}}}$$

(i) <u>For $\alpha > 0$</u> :— As $t \to \infty$, $r \to 0$

For large t,

$$r \approx \frac{R}{r_0} e^{-|\alpha| R^2 t}$$

So, we find a stable f.p. at $r = 0$.

(ii) <u>For $\alpha < 0$</u> :— As $t \to \infty$, $r \to R$

---

for $r > R$

$$I = \frac{1}{R^2} \ln\left(\frac{\sqrt{r^2 - R^2}}{r}\right) + c$$

$\therefore$ (1) has the soln

$$\ln\left(\sqrt{1 - R^2/r^2}\right) = \ln r_0 + \alpha R^2 t$$

$$\Rightarrow 1 - R^2/r^2 = r_0^2 e^{2\alpha R^2 t}$$

$$\Rightarrow R^2/r^2 = 1 - r_0^2 e^{2\alpha R^2 t}$$

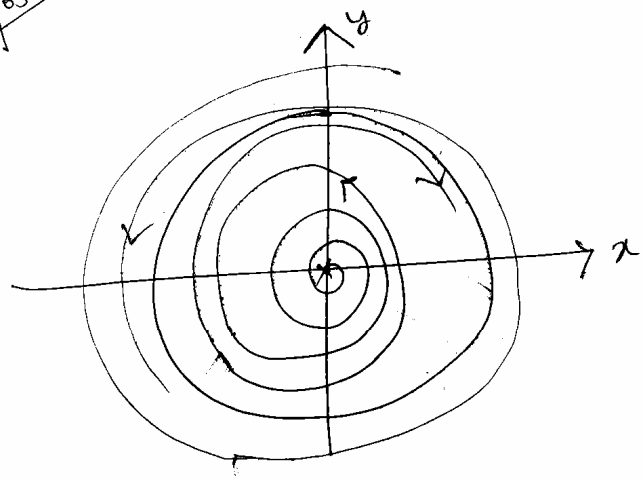
$$\Rightarrow r = \frac{R}{\sqrt{1 - r_0^2 e^{2\alpha R^2 t}}}$$

(i) <u>For $\alpha < 0$</u> :— As $t \to \infty$, $r \to R$

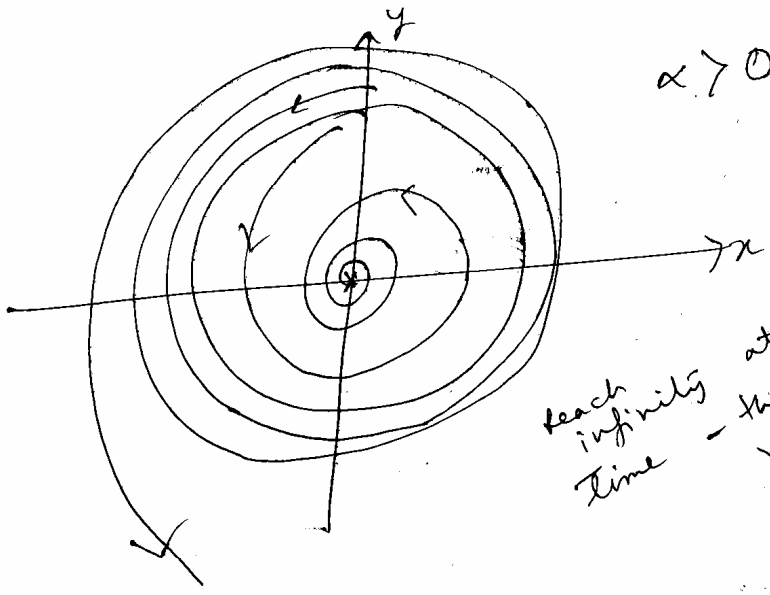
(ii) For $\alpha > 0$:— $r \to \infty$ as $1 - r_0^2 e^{2|\alpha| R^2 t} \to 0$

$\to -\frac{\ln r_0}{|\alpha| R^2}$

18/8/05



$\alpha < 0$



$\alpha > 0$

reach infinity at a finite time - the velocity ↑ very fast

$$r = \frac{R}{\sqrt{1 - r_0^2 e^{2\alpha R^2 t}}}$$

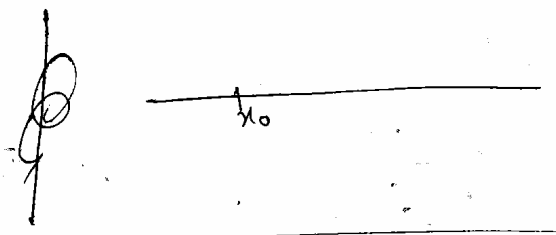
r_0 is small

$$r = \frac{R}{\sqrt{1 - r_0^2}} \simeq R$$

once it reaches infinity, it stays ...

$$dx/dt = x^3$$
$$x(t=0) = x_0 > 0$$
$$dx/x^3 = dt \Rightarrow -\tfrac{1}{2}\frac{1}{x^2} = t + c$$
$$-\tfrac{1}{2}\left(\frac{1}{x^2} - \frac{1}{x_0^2}\right) = t$$
$$\frac{1}{x^2} = \frac{1}{x_0^2} - 2t = \frac{1 - 2t + x_0^2}{x_0^2}$$
$$x^2 = \frac{x_0^2}{1 - 2t x_0^2}$$

vel. ↑ so as it rapidly goes out



$$dx/dt = \sqrt{x}$$

Here, u need infinite time to reach infinity, though vel. is infinite at infinity

Hamiltonian system (autonomous)

A system is called Hamiltonian if it has even number ($2n$) of phase space degrees of freedom and if there exists some coordinates system in which the eqns of motion take the form

$$\frac{dx^i}{dt} = \Omega^{ij}\frac{\partial H(\vec{x})}{\partial x^j} \quad \text{for some fn } H(\vec{x})$$

(repeated indices are summed over)

For an autonomous Hamiltonian.

Ω is a $2n \times 2n$ matrix & is a block diagonal matrix.

$$\Omega = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & 0 & 1 & \\ & & & & -1 & 0 & \\ & & & & & & \ddots \end{pmatrix}$$

$$x^1 = q^1,\ x^2 = p_1,\ x^3 = q^2,\ x^4 = p_2,\ \ldots,\ x^{2n-1} = q^n,\ x^{2n} = p_n$$

$$\frac{d}{dt}\begin{pmatrix} q^1 \\ p_1 \\ q^2 \\ p_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} \partial H/\partial q^1 \\ \partial H/\partial p_1 \\ \partial H/\partial q^2 \\ \partial H/\partial p_2 \\ \vdots \end{pmatrix}$$

In components,

$$\frac{dq^1}{dt} = \frac{\partial H}{\partial p_1}$$

$$dp_1/dt = -\partial H/\partial q^1$$

$$dq^2/dt = \frac{\partial H}{\partial p_2}$$

$$dp_2/dt = \frac{\partial H}{\partial q^2}$$

$\vdots$

upper index in a denominator is like a lower index & lower index ---- upper index

$d\vec{x}/dt = \Omega \vec{\nabla}_x H$ is used to write the matrix eqn.

All u r saying is that f_i has a specific str., viz, $\Omega^{ij} \frac{\partial H(\vec{x})}{\partial x^j}$

— special case of 2n order autonomous system

$$\therefore \quad \left.\begin{aligned} dq^i/dt &= \partial H/\partial p_i \\ dp_i/dt &= -\partial H/\partial q^i \end{aligned}\right\} \quad i = 1, \ldots, n.$$

Suppose we are given f_i's — how to check whether it is a Hamiltonian system?

— f_i's will take the specific form $\Omega^{ij} \frac{\partial H}{\partial x^j}$ only after making the appropriate transfn. $\Longrightarrow$ Have to find some general procedure / prop. to convert it into this specific form?

[We will see the transfn. to a generic coord. system, not in a sp. coord. system where this special form is ~~seen~~ seen]

<u>Introduce new coordinates</u>

$$y^i = g^i(\vec{x})$$

[These are invertible maps — fns can be inverted to give $x^i = g^i(\vec{y})$]

$$y^i = g^i(\vec{x}) \qquad x^i = \tilde{g}^i(\vec{y})$$

Inverse functions

<u>Definition</u>:

ω_{ij} is the matrix inverse of Ω^{ij}

$$\Omega^{ij}\omega_{jk} = \delta^i_k$$

$$\omega = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}$$

$$\therefore \quad dy^i/dt = \frac{\partial g^i(\vec{x})}{\partial x^k} \frac{dx^k}{dt}$$

$$= \frac{\partial g^i(\vec{x})}{\partial x^k} \Omega^{kl} \frac{\partial H(\vec{x})}{\partial x^l}$$

g^i is passive coord. transf.

another set of coord. on the same manifold

The space need not be Euclidean space

Define: $\tilde{H}(\vec{y}) = H(\vec{x})$

$$\frac{\partial \tilde{H}(\vec{y})}{\partial y^j} \frac{\partial g^j(\vec{x})}{\partial x^l} \quad \text{(Chain rule)}$$

Here we are not interested in the global topology. But if the coord. chart breaks down in a particular (like $r \to \infty$), then the global topology has to be considered

The topology has to be specified separately — it does not follow from the given set of eqns. ∵ the eqns have been constructed in a given coord. system

$$\Rightarrow \boxed{dy^i/dt = \tilde{\Omega}^{ij} \frac{\partial \tilde{H}(\vec{y})}{\partial y^j}} \qquad \boxed{\tilde{\Omega}^{ij} = \Omega^{kl} \frac{\partial g^i(\vec{x})}{\partial x^k} \frac{\partial g^j(\vec{x})}{\partial x^l}} \quad (a)$$

$$\tilde{\Omega}^{ji} = \Omega^{kl} \frac{\partial g^j}{\partial x^k} \frac{\partial g^i}{\partial x^l}$$ [Interchanging indices in (a)]

But k & l are repeated indices & can be relabelled :-

$$\tilde{\Omega}^{ji} = \Omega^{lk} \frac{\partial g^j}{\partial x^l} \frac{\partial g^i}{\partial x^k}$$

$$= -\Omega^{kl} \frac{\partial g^j}{\partial x^l} \frac{\partial g^i}{\partial x^k}$$ ($\because \Omega^{kl} = -\Omega^{lk}$, $\because \Omega$ was an antisym. matrix)

$$= -\tilde{\Omega}^{ij}$$

So, firstly we better make sure that $\tilde{\Omega}$ is an antisym. matrix.

This antisym. prop. is somewhat trivial.

The next prop. " " non- ".

Define $\tilde{\omega}(\vec{y})$ = matrix inverse of $\tilde{\Omega}$

$$\therefore \tilde{\Omega}^{ij}\tilde{\omega}_{jk} = \delta^{i}{}_{k}$$

We can think $\tilde{\Omega} = A(\vec{x})\,\Omega\,A^{T}(\vec{x})$,

where $A^{i}{}_{j}(\vec{x}) = \partial g^{i}/\partial x^{j}$

We have, $\tilde{\omega} = (\tilde{\Omega})^{-1} = (A^{T})^{-1}\underbrace{\Omega^{-1}}_{\omega}A^{-1}$

Now, $(A^{-1})^{i}{}_{j} = \partial x^{i}/\partial y^{j}$

$$= \frac{\partial g^{i}(\vec{y})}{\partial y^{j}}$$

& $(A^{T})^{-1} = (A^{-1})^{T}$

$$\therefore \tilde{\omega}_{ij} = \underbrace{\frac{\partial \tilde{g}^{k}(\vec{y})}{\partial y^{i}}}_{(A^{-1})^{T}{}_{i}{}^{k}}\,\omega_{kl}\,\frac{\partial g^{l}(\vec{y})}{\partial y^{j}} \quad \text{—} (1)$$

$$\Rightarrow \tilde{\omega}_{ij} = (A^{-1})^{T}{}_{i}{}^{k}\,\omega_{kl}\,(A^{-1})^{l}{}_{j}$$

$$\boxed{\frac{\partial \tilde{g}^{k}(\vec{y})}{\partial y^{j}}\,\frac{\partial y^{j}(\vec{x})}{\partial x^{l}} = \delta^{k}{}_{l}}$$

Claim :- that it is true

Condn. II

$$\frac{\partial \tilde{\omega}_{ij}(\vec{y})}{\partial y^m} + \frac{\partial \tilde{\omega}_{jm}}{\partial y^i} + \frac{\partial \tilde{\omega}_{mi}}{\partial y^j} = 0 \quad \text{— (b)}$$

Sometimes, we write this as

$$\left(\partial_m \tilde{\omega}_{ij} + \partial_i \tilde{\omega}_{jm} + \partial_j \tilde{\omega}_{mi} = 0 \right)$$

where $\partial_i = \partial/\partial y^i$

The significance of this is the following :-
suppose u find that LHS of (b) isn't zero — then the system isn't Hamiltonian.
$\Rightarrow$ Test for " being ".

Proof of (b) :-

$$\frac{\partial \tilde{\omega}_{ij}}{\partial y^m} = \omega_{kl} \overset{(1)}{\frac{\partial^2 \tilde{g}^k}{\partial y^m \partial y^i}} \frac{\partial \tilde{g}^l}{\partial y^j} + \omega_{kl} \frac{\partial \tilde{g}^k}{\partial y^i} \overset{(2)}{\frac{\partial^2 \tilde{g}^l}{\partial y^m \partial y^j}}$$

[Differentiating (1)]

cancel

By Cyclic permutation :-

$i \to j,\ j \to m,\ m \to i$

$$\frac{\partial \tilde{\omega}_{jm}}{\partial y^i} = \omega_{kl} \overset{(3)}{\frac{\partial^2 \tilde{g}^k}{\partial y^i \partial y^j}} \frac{\partial \tilde{g}^l}{\partial y^m} + \omega_{kl} \frac{\partial \tilde{g}^k}{\partial y^j} \overset{(1)}{\frac{\partial^2 \tilde{g}^l}{\partial y^i \partial y^m}}$$

cancel

$$\frac{\partial \tilde{\omega}_{mi}}{\partial y^j} = \omega_{kl} \overset{(2)}{\frac{\partial^2 \tilde{g}^k}{\partial y^j \partial y^m}} \frac{\partial \tilde{g}^l}{\partial y^i} + \omega_{kl} \frac{\partial \tilde{g}^k}{\partial y^m} \overset{(3)}{\frac{\partial^2 \tilde{g}^l}{\partial y^j \partial y^i}}$$

But, even after checking these 2 condns, it doesn't prove that the system is Hamiltonian. ??
These 2 condns are necessary but not sufficiently. [Refer Arnold]

~~Result~~ Result (Arnold)

These 2 condns are sufficient for the system to be Hamiltonian. [The proof in the same spirit as if the curl vanishes, we can write it as a gradient of a scalar]

What is the f.p. str. of a Hamiltonian system?

Fixed point structure

$$\frac{dx^i}{dt} = \underbrace{\Omega^{ij}\,\partial H/\partial x^j}_{F^i(\vec{x})}$$

fixed point: $f^i(\vec{x}_f) = 0$

$$\Rightarrow \Omega^{ij}\frac{\partial H}{\partial x^j}\Big|_{\vec{x}_f} = 0$$

But Ω^{ij} is a non-singular matrix

$\therefore$ we must have $\frac{\partial H}{\partial x^j}\Big|_{\vec{x}_f} = 0$

Introduce $(x^i - x^i_f)$

$$\frac{d}{dt}(x^i - x^i_f) = \Omega^{ij}\left[\underbrace{\frac{\partial H}{\partial x^j}\Big|_{\vec{x}_f}}_{=0} + \frac{\partial^2 H}{\partial x^j \partial x^k}\Big|_{\vec{x}_f}(x^k - x^k_f) + \cdots\right]$$

$$\frac{d}{dt}(x^i - x_F^i) = M^i{}_k (x^k - x_F^k)$$

$$M^i{}_k = \Omega^{ij} \frac{\partial^2 H}{\partial x^j \partial x^k}\Big|_{\vec{x}_F}$$

sym. matrix

sym. matrix

its e. values control the nature of the f.p.

An absolutely stable f.p. will be found if the real parts of all e. values are -ve.

My proof

$$M^i{}_j = -\Omega^{ji} \frac{\partial^2 H}{\partial x^i \partial x^j}\Big|_{\vec{x}_F} = -M^j{}_i$$

$$= -\mathrm{Tr}\, M$$

$$\Rightarrow 2\,\mathrm{Tr}\, M = 0$$

$$\Rightarrow \mathrm{Tr}\, M = 0$$

$$\mathrm{Tr}\, M = M^i{}_i = \underset{\text{anti sym}}{\Omega^{ij}} \underset{\text{sym.}}{\frac{\partial^2 H}{\partial x^j \partial x^i}}\Big|_{\vec{x}_F} = 0$$

$\Rightarrow$ sum of the e.values of $M = 0$

Hence, it is impossible to have all e.values to have real part < 0.

So no strongly stable f.p.

The best we can hope for is a f.p. which is stable, but not strongly stable — analog of elliptic f.p.

19/8/05

Hamiltonian system

In special coordinate system the eqns for time evolution take the form

$$\frac{dx^i}{dt} = \Omega^{ij} \frac{\partial H(\vec{x})}{\partial x^j}$$

$H(\vec{x})$: Hamiltonian

Behaviour near a fixed point $\vec{x}_F$:

$$\frac{d}{dt}(x^i - x_F^i) = M^i{}_j (x^j - x_F^j)$$

$$M^i{}_j = \partial^2 H$$

51)

We have $M^i_i = 0$

$\Rightarrow$ sum of all eigenvalues $= 0$.

Eigenvalues :-

$\alpha_{(1)} + i\omega_{(1)}$, $\alpha_{(1)} - i\omega_{(1)}$, ----,

$\alpha_{(p)} + i\omega_{(p)}$, $\alpha_{(p)} - i\omega_{(p)}$, & the rest of the e. values are real.

$\lambda_{(1)}, \lambda_{(2)}, \ldots \lambda_{2(n-p)}$

$2n$: Total dimension of the phase space.

$$\sum_{i=1}^{2(n-p)} \lambda_{(i)} + 2\sum_{i=1}^{p} \alpha_{(i)} = 0$$

$\Rightarrow$ It is impossible to have all $\lambda_{(i)} < 0$, & all $\alpha_{(i)} < 0$.

$\Rightarrow$ we cannot have strongly stable fixed point.

Suppose all the $\alpha_{(i)} = 0$, $\lambda_{(i)} = 0$.

Suppose ~~[illegible]~~ we have two real roots $\lambda_{(1)} = 0$, $\lambda_{(2)} = 0$ & other complex roots with $\alpha_{(1)} = \alpha_{(2)} = \cdots = 0$

What will be the generic form of the matrix?

In the general, we will not have closed orbits because ω's will not be commensurate in general

Stability will be det. by λ & α only — not by ω, $\because$ ω plays the role of $\dot{\theta}$ in polar coord.

$$\Rightarrow \begin{pmatrix} 0 & 0 & & & & & & \\ c & 0 & & & & & & \\ & & \alpha_{(1)} & \omega_{(1)} & & & & \\ & & -\omega_{(1)} & \alpha_{(1)} & & & & \\ & & & & \alpha_{(2)} & \omega_{(2)} & & \\ & & & & -\omega_{(2)} & \alpha_{(2)} & & \\ & & & & & & & \end{pmatrix}$$

It'll split into 2×2 blocks

The rest of the elements are zero

The eqn. is $\begin{pmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ \vdots \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2n} \end{pmatrix}$

How will the orbit look in the $z_3 - z_4$ plane?

$$dz_3/dt = \omega_{(1)} z_4$$

$$\frac{dz_4}{dt} = -\omega_{(1)} z_3$$

The orbits will be circles in this coord. system.

$$z_3 = r_0 \sin(\omega t + \theta_0)$$

$$z_4 = r_0 \cos(\omega t + \theta_0)$$

Similarly, z_5 & z_6 will show a similar behaviour.

If we go to some other coord. system, we'll get ellipses.

$$dz_1/dt = 0 \Rightarrow z_1 = z_1^{(0)}$$

$$dz_2/dt = c z_1 \Rightarrow \frac{dz_2}{dt} = c z_1^{(0)}$$

$$\Rightarrow z_2 = z_2^{(0)} + c z_1^{(0)} t$$

starts going linearly

z_2 won't reach the origin & will go away from the f.p.

$\Rightarrow$ Unstable fixed point.

[Of course, z_2 is fixed for $z_1^{(0)}$ equal to zero, but then u are not considering an arbitrary pt. in the neighbourhood of the f.p. to analyse the behaviour.

z_2-axis is a line of f.p.'s - but a little disp. will make it go away from the f.p.

So, if u have zero e.values, generically, the coord. will start growing - so it's not a good thing to have zero e.values.]

In order to get stable fixed point, all e.values should be imaginary (generic situation).

E.g. of zero eigenvalues:-

free particle: $H = p^2/2m$

$$dq/dt = p/m, \quad dp/dt = 0$$

Fixed point: $(p = 0, q = q_0)$

$F^q = p/m$ & $F^p = 0$ [by comparison]

So the matrix M has the form

$$M = \begin{pmatrix} \partial F^q/\partial q & \partial F^q/\partial p \\ \partial F^p/\partial q & \partial F^p/\partial p \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1/m \\ 0 & 0 \end{pmatrix}$$

this is in the upper triangular form (can be converted to lower —)

So, it shows how 2 e. values gives a f. p. generically unstable.

[If the total matrix is 0, then we'll have to consider 2nd order terms — in first order analysis — there will be no motion]

$$dx^i/dt = \Omega^{ij} \frac{\partial H(\vec{x})}{\partial x^j}$$

$$\frac{dH(\vec{x})}{dt} = \frac{\partial H(\vec{x})}{\partial x^i} dx^i/dt = \frac{\partial H}{\partial x^i} \Omega^{ij} \partial H/\partial x^j = 0$$

This is zero due to $\Omega^{ij} = -\Omega^{ji}$

This tells us that $H(\vec{x})$ remains constant along a phase curve.

We can restate this as :—

All phase curves must lie on constant $H(\vec{x})$ ~~subspaces~~ subspaces.

This means the following :—

$H(\vec{x}) = K$

→ one eqn in $2n$ variables & this gives us a $(2n-1)$ dimensional subspace.

another $(2n-1)$ dim

(These subspaces fill up the entire manifold & they don't intersect as long as $H(\vec{x})$ is a single-valued fn. of $\vec{x}$.)

A phase curve is a one-dim. space.

This result tells us that this (the phase curve) must lie completely inside on one of the $(2n-1)$ dimensional constant H surfaces.

[e.g. – Consider an ellipsoid – a 2-dim surface in 3-D space – a one-dim. ~~phase~~ curve will in general cut the ellipsoid – but here we know that the phase curve must lie on the ellipsoid]

(All variables on phase curve must change continuously – can't jump from one to another

e.g. – $H = p^2/2m$ for each H, there are 2 p's – but u can't jump from $+p$ to $-p$)

If $n=1$, then $2n-1=1$

In this case the phase curves are constant H surfaces

$\Rightarrow$ The problem is solvable.

(in the sense that it's a purely algebraic problem & don't have to solve diff. eqns)

$H(p,q)$ = const. will give an algebraic eqn. in p & q.

$$\frac{dq}{dt} = \partial H/\partial p$$

$$dp/dt = -\partial H/\partial q$$

But phase curves don't tell us about how fast the motion takes place.

Then, we write $dt = \dfrac{dq}{(\partial H/\partial p)}$

(for knowing phase curves, don't have to integrate anything – But to know t-dependence ...)

$\hookrightarrow$ known fn. of q, p

$\downarrow$

known fn. of q on a phase curve.

We basically solve $t + t_0 = \displaystyle\int^{q} \frac{dq}{(\partial H/\partial p)}$

$\downarrow$ known fn. of q

[illegible] a given problem is

Suppose $\vec{x}_F$ is a fixed point

$$\Omega^{ij} \frac{\partial H}{\partial x^j}\Big|_{\vec{x}_F} = 0 \Rightarrow \frac{\partial H}{\partial x^j}\Big|_{\vec{x}_F} = 0$$

So, $\vec{x}_F$ is an extremum of H.

$$H(\vec{x}) = H(\vec{x}_F) + \overbrace{\frac{\partial H}{\partial x^i}\Big|_{\vec{x}_F}}^{=0} (x^i - x^i_F)$$

$$+ \frac{1}{2} \frac{\partial^2 H}{\partial x^i \partial x^j}\Big|_{\vec{x}_F} (x^i - x^i_F)(x^j - x^j_F) + \cdots$$

So, $$H(\vec{x}) = H(\vec{x}_F) + \frac{1}{2} \underbrace{\frac{\partial^2 H}{\partial x^i \partial x^j}\Big|_{\vec{x}_F}}_{Q_{ij}} (x^i - x^i_F)(x^j - x^j_F)$$

$Q_{ij} \rightarrow$ a real symmetric matrix & can be diagonalized by a similarity transfn.

$$Q_{ij} = R \begin{pmatrix} \overset{Q_d}{\overset{\shortparallel}{K_1}} & & & \\ & K_2 & & \\ & & \ddots & \\ & & & K_{2n} \end{pmatrix} R^T$$

Define $y^i = (R^T)^i{}_j (x^j - x^j_F)$

my proof –

$Q_{ij} = R_i{}^l Q_{lm} (R^T)^m{}_j$

$= R_i{}^l K_l \delta_{lm} (R^T)^m{}_j$

$= R_i{}^m K_m (R^T)^m{}_j$

$= (R^T)^m{}_i K_m (R^T)^m{}_j$

$\therefore Q_{ij}(x^i - x^i_F)(x^j - x^j_F) = K_m y_m y_m = K_m y_m^2$

$$\Rightarrow H(\vec{x}) = H(\vec{x}_F) + \frac{1}{2} \sum_{i=1}^{2n} K_i y_i^2$$

Constant H surfaces have simple form near the fixed point.

$$H(\vec{x}) = K$$

$$\Rightarrow \boxed{\sum_{i=1}^{2n} K_i y_i^2 = \underbrace{K - H(\vec{x}_F)}_{\text{small, near}}}$$

$$H(\vec{x}) = H(\vec{x}_F) + \tfrac{1}{2}(x - x_F)^T \mathcal{G} (x - x_F) = H(\vec{x}_F) + \tfrac{1}{2} y^T \mathcal{G}_d y$$

$$\mathcal{G} \Downarrow R\,\mathcal{G}_d\,R^T \qquad \tfrac{1}{2} y^T \mathcal{G}_d y = \sum_i K_i y_i^2$$

(For positive Kapp i's (meaning H has a local minimum at $\vec{x}_F$), & we have ellipsoids around the f.p.)

If all the $K_i > 0$, then H has a local minimum at $\vec{x}_F$ & constant H surfaces are ellipsoids centred around $\vec{x}_F$.

This tells us that the fixed point is stable.
(on this long., we'll get all λ_i's & α_i's zero)

If any $K_i < 0$, then the surface is hyperboloid & hence not bounded near the f.p.
[at least near the f.p. – however if u analyse higher order terms, the motion may become bounded]

Typically we expect the fixed point to be unstable.

If $K_i = 0$ for some i

e.g. – All other K_i's > 0
(All these y_i's will be bounded
But, in that particular y_i, the phase curve isn't bounded – there's no restriction on y_i)

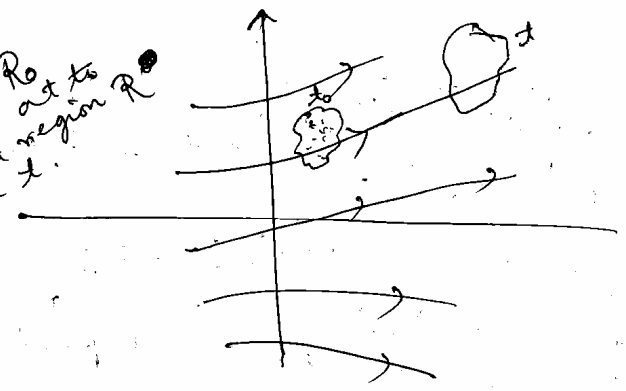
y_i coordinate can become large
$\Rightarrow$ generically the fixed point is unstable.

that stability will be near the min

Consider the eqns in special coord. system.

$$dx^i/dt = -\Omega^{ij}\,\partial H/\partial x^j$$

A region R_0 at t_0 beginning at t_0 goes to a region R at time t.



The flow lines remain frozen for an autonomous system — doesn't change with time

(Consider a family of motions where at $t = t_0$ we start inside a given volume, & observe what happens to it at time t — an open set gets mapped into some other " set — neighbouring pts. move to neighbouring pts.)

Is $\mathrm{Vol}(R) = \mathrm{Vol}(R_0)$ always?

$\Rightarrow$ If it's true, then incompressible fluid. otherwise it's compressible.

[The very fact that $\exists$ strongly stable f.p., shows that generically we have compressible systems — all the neighbouring pts. — get mapped to the f.p. — whole surface collapses to a pt.]

<u>Liouville theorem</u> :— For any Hamiltonian system, the flow is incompressible in the special coordinates system. [vol. changes by the Jacobian factor — hence the vol. may be diff. in another coord. system]

$\Rightarrow \mathrm{Vol}(R) = \mathrm{Vol}(R_0)$

[Liouville theorem also shows

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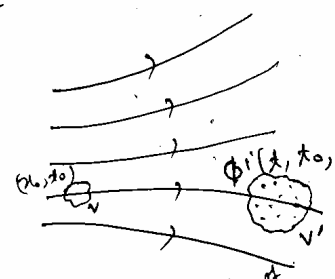


→ fn. to describe what vol. it will flow to
→ parametric eqn of the phase curve

The vol. can be thought of as a collection of pts. Evolve x_0 at t_0 & evolve it & see where the pt. goes.

$\boxed{\text{Is } V = V' \text{ ?}}$

Liouville's theorem says that ~~a Ham~~ "Flow in a Hamiltonian system is incompressible".

If we can show $\frac{dV}{dt}\Big|_{t=t_0} = 0$, then in general V won't change, because (x_0, t_0) isn't a special pt.

Trajectory:

$$x^i(t) = \phi^i(t, t_0, \vec{x}_0)$$ → shows how x^i will evolve as a fn of time

Define :-

$$S^i{}_j(t) = \frac{\partial \phi^i(t, t_0, \vec{x}_0)}{\partial x_0^j}$$

$$\delta x^i(t) = \frac{\partial \phi^i(t, t_0, \vec{x}_0)}{\partial x_0^j}\, \delta x_0^j$$ → (time fixed, but changing the location)

under a variation of the initial point

(The det. of the matrix S will tell us how much the vol. has changed.)

$$J(t, t_0, \vec{x}_0) = \det S(t, t_0, \vec{x}_0)$$

Vol. in one cord. system = (Jacobian factor) × $\delta x_1 \delta x_2$ ------

... show that $\frac{\partial}{\partial t} J(t, t_0, \vec{x}_0) = 0$ for Hamiltonian system.

$$\frac{\partial}{\partial t} S^i{}_j(t, t_0, \vec{x}_0) = \frac{\partial}{\partial t} \frac{\partial \phi^i(t, t_0, \vec{x}_0)}{\partial x_0^j}$$

[At $t = t_0$, $\phi^i(t, t_0, \vec{x}_0) = x_0^i$ & $S^i{}_j = \delta_{ij}$]

Doubt: ↑ deriv can't change as no long as these deriv. are continuous

← as they are indep. variables, we can change the order of the deriv.

$$= \frac{\partial}{\partial x_0^j} \frac{\partial \phi^i(t, t_0, \vec{x}_0)}{\partial t}$$

$$= \underbrace{\Omega^{ik}}_{\text{constant}} \frac{\partial}{\partial x_0^j} \frac{\partial H}{\partial x^k}\Big|_{\vec{x} = \vec{\phi}(t, t_0, \vec{x}_0)}$$

$x^i(t) = \phi^i(t, t_0, \vec{x}_0)$

$$\frac{dx^i}{dt} = \Omega^{ik} \frac{\partial H}{\partial x^k}$$

↓ understood for a fixed initial condition – so $\frac{d}{dt}$ & not $\frac{\partial}{\partial t}$

∴ we may write

$$\frac{\partial \phi^i(t, t_0, \vec{x}_0)}{\partial t} = \Omega^{ik} \frac{\partial H}{\partial x^k}\Big|_{\vec{x} = \vec{\phi}(t, t_0, \vec{x}_0)}$$

$$\left[\frac{\partial}{\partial t} S^i{}_j(t, t_0, \vec{x}_0)\right]\Big|_{t = t_0}$$

$$= \Omega^{ik} \frac{\partial}{\partial x_0^j} \frac{\partial H(\vec{x}_0)}{\partial x_0^k} \quad [\because \text{ at } t = t_0,\ \phi = x_0]$$

& $S^i{}_j(t_0, t_0, \vec{x}_0) = \delta^i{}_j$

[$\because \phi^i(t_0, t_0, \vec{x}_0) = x_0^i$]

[S isn't identically const. – the's matrix changes with time – If it were const. ≡ trivial – nothing evolves & $x(t) = x_0$ for all t]

$$S^i{}_j(t_0 + \Delta t, \vec{x}_0) = \underbrace{\delta^i{}_j}_{\text{Identity matrix}} + \Delta t\, \Omega^{ik} \frac{\partial^2 H(\vec{x}_0)}{\partial x_0^j \partial x_0^k}$$

[location fixed, but changing time t]

$J(t_0+\Delta t, t_0, \vec{x}_0) \equiv$ Det. of the R.H.S. matrix

$$\det(I + \underset{\text{small}}{\epsilon} A)$$

$$= \det \begin{pmatrix} 1+\epsilon A_{11} & \epsilon A_{12} & \epsilon A_{13} & \cdots \\ \epsilon A_{21} & 1+\epsilon A_{22} & \epsilon A_{23} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$n \times n$ matrix

$$= \underbrace{(1+\epsilon A_{11})(1+\epsilon A_{22})(1+\epsilon A_{33}) \cdots (1+\epsilon A_{nn})}_{+\, O(\epsilon^2)}$$

→ product of all diagonal terms

[Every time u pick an off-diagonal element & calculate its cofactor, it will at least of the order of ϵ^2]

$$= 1 + \epsilon(A_{11} + A_{22} + \cdots + A_{nn}) + O(\epsilon^2)$$

$$= 1 + \epsilon(\mathrm{Tr}\, A)$$

$$J(t_0+\Delta t, t_0, \vec{x}_0) = 1 + \Delta t \underbrace{\underset{\text{anti sym}}{\Omega^{ik}} \underset{\text{sym}}{\frac{\partial^2 H(\vec{x}_0)}{\partial x_0^i \partial x_0^k}}}_{=0}$$

$$= 1$$

$\Rightarrow \frac{\partial J(t, t_0, \vec{x}_0)}{\partial t} = 0$ at $t = t_0$

Hence it is zero in general $\Rightarrow$ Proves Liouville's theorem.

[Int. this over the whole curve, we can show that the volume isn't change anywhere]

Applications of Liouville's theorem :-

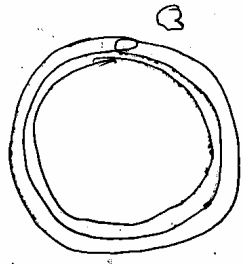
① Absence of strongly stable fixed points.

If V gets trapped to a small pt. at the origin, this is not volume preserving.

② Absence of strongly stable limit cycles.

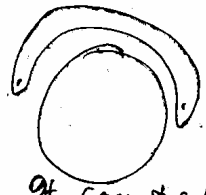
[A limit cycle stands in isolation in contrast to circles ~~ellipses~~ around elliptic f.p.'s]

If we can show that there is a limit of stretching out, we can " the above is true.



overlap of volumes

But these curves can't intersect for non-singular flow



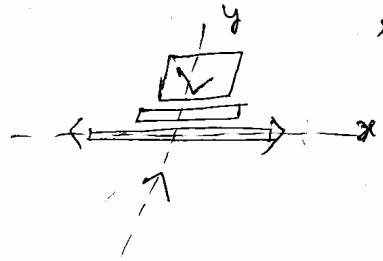
It can stretch at most to the length of the limit cycle

The volume can't spiral - can be seen clearly in 2-D

- In higher-dim, we have to see if a vol. can be taken to block either way.

Motion around hyperbolic fixed pt.

has the prop. that attractive in some dirn. & rep. in some other

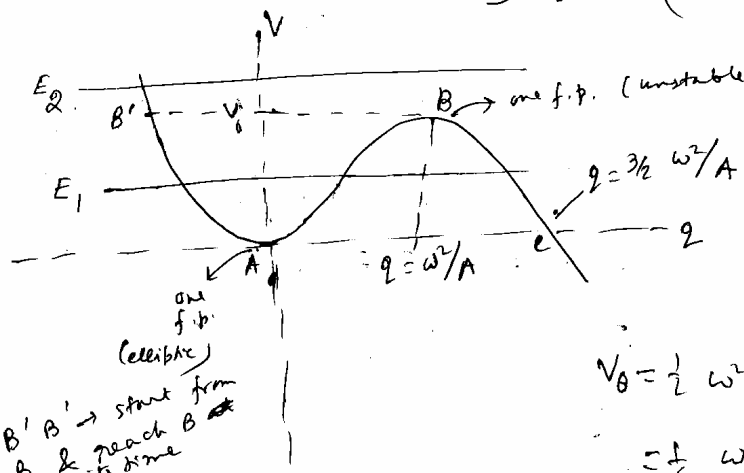


amt. of shrinking in one dirn
= " " stretching in the other dirn
(in 2-D)
In this way, vol. will be preserved.

Example:-

$$H = \frac{p^2}{2m} + V(q)$$

$$V(q) = \frac{1}{2}\omega^2 q^2 - \frac{1}{3}A q^3 \quad (A > 0)$$



$$V_0 = \frac{1}{2}\omega^2 \cdot (\omega^2/A)^2 - \frac{1}{3}A(\omega^2/A)^3$$
$$= \frac{1}{6}\omega^6/A^2$$

$$\frac{dq}{dt} = \partial H/\partial p = p/m$$
$$dp/dt = -\partial H/\partial q = -(\omega^2 q + A q^2).$$

Fixed pts:- $p = 0$ & $-\omega^2 q + A q^2 = 0 \Rightarrow q = 0$ & $q = \omega^2/A$

$\therefore$ the 2 f.p.'s are $(q = 0, p = 0)$, $(q = \omega^2/A, p = 0)$
elliptic hyperbolic

$$M \equiv \begin{pmatrix} \partial F^q/\partial q & \partial F^q/\partial p \\ \partial F^p/\partial q & \partial F^p/\partial p \end{pmatrix} = \begin{pmatrix} 0 & 1/m \\ -\omega^2 + 2Aq & 0 \end{pmatrix}$$

<u>At $(q=0, p=0)$</u>, $M = \begin{pmatrix} 0 & 1/m \\ -\omega^2 & 0 \end{pmatrix}$

e. values :- $\lambda^2 + \omega^2/m = 0 \Rightarrow \lambda = \pm i\,\omega/\sqrt{m}$

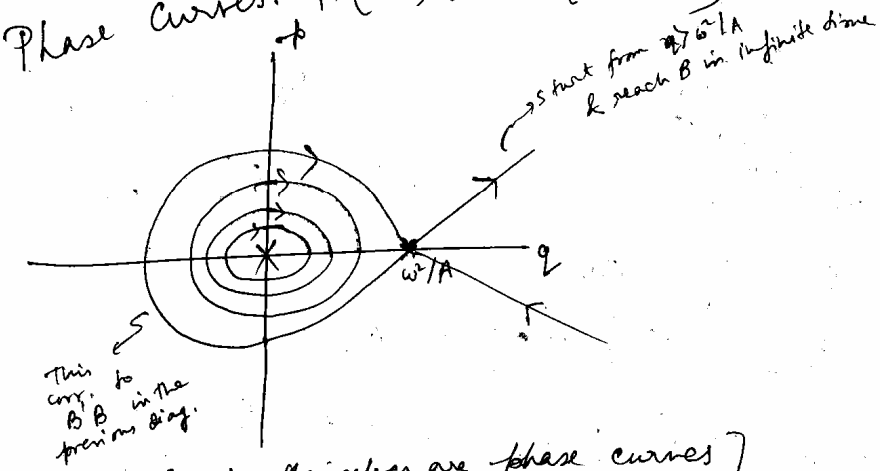
imag. e. values — so elliptic f.p.'s

<u>At $q = \omega^2/A$, $p = 0$</u>, $M = \begin{pmatrix} 0 & 1/m \\ \omega^2 & 0 \end{pmatrix}$

e. values :- $\lambda - \omega^2/m = 0 \Rightarrow \lambda = \pm \dfrac{\omega}{\sqrt{m}}$

hyperbolic f.p.

Phase curves: $H(q, p) = E$ (constant)



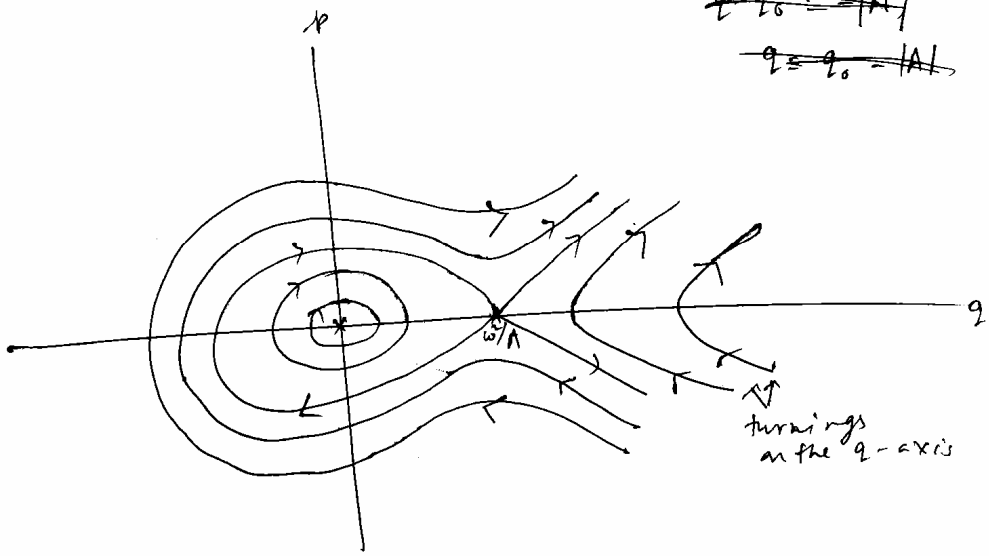
[Remember :- f.p.'s themselves are phase curves]

at $(q=0, p=0)$

Take $E > V_0$

$\cancel{q - q_0 = -|A|}$

$\cancel{q = q_0 - |A|}$



$$dq/dt = p/m \qquad \frac{p^2}{2m} + \frac{1}{2}\omega^2 q^2 - \frac{1}{3}Aq^3 = E$$

$$\Rightarrow p = \sqrt{2m\left(E - \frac{1}{2}\omega^2 q^2 + \frac{1}{3}Aq^3\right)}$$

$$dt = m\frac{dq}{p} = \frac{m\,dq}{\sqrt{2m\left(E - \frac{1}{2}\omega^2 q^2 + \frac{1}{3}Aq^3\right)}}$$

$$\therefore t - t_0 = \int_{q_0}^{q} \frac{m\,dq}{\sqrt{2m\left(E - \frac{1}{2}\omega^2 q^2 + \frac{1}{3}Aq^3\right)}}$$

In this sense, a 2nd order Ham. system is always solvable.

Under what condn. is a higher order autonomous system is solvable?

show $d^n x = \sqrt{\ldots}\, d^n y$ ← this factor is precisely the one the Jacobian gives

Provided the vol. is defined like

65) Doubts

① S^i_j not const. – But det. const. ? → yes

② Stretching out – limit cycle

③ phase curve int. ??! → only for hyperbolic fixed pts., stable star, etc.

As long as the motion is non-singular, it is reaching the f.p. in infinite time – so no problem.

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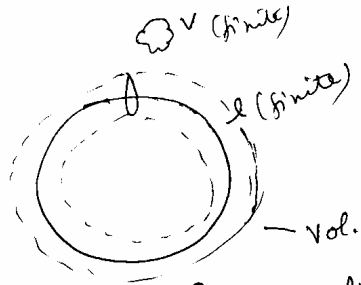


$V \to$ dim. of l^n in n-dim.

— vol. of the tube $< V$

$\Rightarrow$ Cross-section A $< V/l$ (uniform)

Part of that (V) will leak out of the tube; so, Liouville's theorem isn't satisfied.

Higher dimensional Hamiltonian system :-

Existence of Hamiltonian does not make the theory solvable.

$H = \text{constant}$ gives us a ~~2n-dim~~ $(2n-1)$-dim. subspace, whereas the phase curves are 1-dimensional.

Q) Under what condition is a higher dimensional autonomous Hamiltonian system solvable?

We'll show that we need $(n-1)$ independent conserved charges other than H satisfying certain conditions in order to solve the theory.

For $(2n-1)$-dim subspace $(2n-2)$ indep. conserved charges, i.e., ~~constants~~ give conserved quantities to give us a 1-dim. phase curve

Define: Given two fns $F(\vec{x})$, $G(\vec{x})$ we define the Poisson bracket:

$$\{F, G\} = \frac{\partial F}{\partial y^i} \Omega^{ij} \frac{\partial G}{\partial x^j} \underset{\text{show}}{=} \frac{\partial F}{\partial y^i} \tilde{\Omega}^{ij} \frac{\partial G}{\partial y^j}$$

in a general

where, $\tilde{\Omega}^{ij} = \frac{\partial y^i}{\partial x^k} \Omega^{kl} \frac{\partial y^j}{\partial x^l}$

Suppose $Q_1(\vec{x}), \ldots, Q_{n-1}(\vec{x})$ are the $(n-1)$ independent conserved charges other than H.

Then we need

$$\left.\begin{array}{l} \{Q_1, H\} = 0, \ \{Q_i, Q_j\} = 0 \text{ for solvability of the theory.} \end{array}\right\} \text{To be ~~proved~~ proven}$$

this is indeed an additional condn.

this isn't really an additional condition [follows from the fact that we need (n-1) indep. conserved charges]

$$x^1 = q^1, \ x^2 = p_1, \ x^3 = q^2, \ x^4 = p_2, \ \ldots$$

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i} \right)$$

$$\Omega = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$\boxed{\{F,G\} = \Omega^{ij} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j}}$$

① $\{F, F\} = 0$ [$\because \Omega^{ij}$ is an antisym. matrix while the rest is ~~anti~~ sym.]

② $\frac{dF}{dt} = \frac{\partial F}{\partial x^i} \frac{dx^i}{dt} = \frac{\partial F}{\partial x^i} \Omega^{ij} \frac{\partial H}{\partial x^j} = \{F, H\}$

[$\because \frac{dQ_i}{dt} = 0$, $\therefore$ it follows that $\{Q_i, H\} = 0$]

$$\frac{\partial F}{\partial x^i} \Omega^{ij} \frac{\partial G}{\partial x^j} = \frac{\partial G}{\partial x^j} (-\Omega^{ji}) \frac{\partial F}{\partial x^i} = -\frac{\partial G}{\partial x^j} \Omega^{ji} \frac{\partial F}{\partial x^i} = -\{G, F\}$$

Canonical transfns

If $y^i = f^i(\vec{x})$, then

$$\Omega^{ij} = \frac{\partial f}{\partial x^k} \Omega^{kl} \frac{\partial f^j}{\partial x^l}$$

The transfn from $\vec{x}$ coordinate system to $\vec{y}$ coordinate system is called canonical if

$$\tilde{\Omega}^{ij} = \Omega^{ij}$$

Properties of canonical transfns:-

Form of the Poisson bracket remains unchanged under canonical transfn

$$\{F, G\} = \frac{\partial F}{\partial x^i} \Omega^{ij} \frac{\partial G}{\partial x^j}$$

$$= \frac{\partial F}{\partial y^k} \underbrace{\frac{\partial y^k}{\partial x^i} \Omega^{ij} \frac{\partial y^l}{\partial x^j}}_{\tilde{\Omega}^{kl}} \frac{\partial G}{\partial y^l}$$

$$= \tilde{\Omega}^{kl} \frac{\partial F}{\partial y^k} \frac{\partial G}{\partial y^l}$$

$$= \Omega^{kl} \frac{\partial F}{\partial y^k} \frac{\partial G}{\partial y^l} \quad \text{if the transfn is canonical.}$$

The splitting of y^k into coord. & mom. will not occur for any arbitrary $\tilde{\Omega}^{kl}$

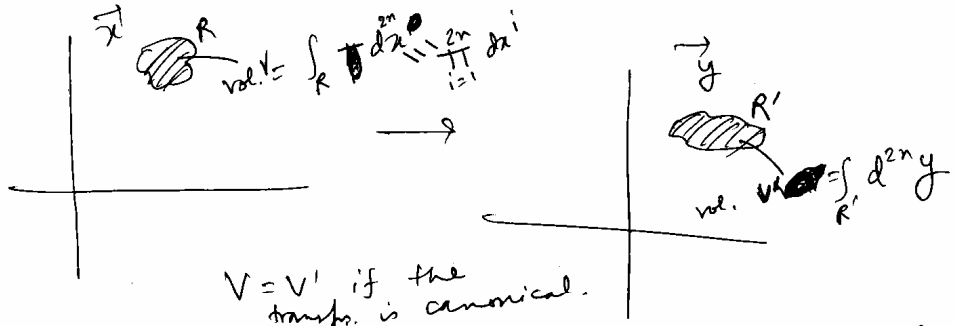
Then we can define

$$y^1 = q^1,\ y^2 = p_1,\ y^3 = q^2,\ y^4 = p_2, \ldots$$

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i} \right)$$

[of course F (or G) won't have the same functional form when expressed in q, p as was when expressed in x, t]

Canonical transfn. preserves phase space volume.



$V = V'$ if the transfn. is canonical.

The open set in $\vec{x}$-coord. sys. will be mapped into some open set in $\vec{y}$-coord. system.

For a general coord. transfn.

$$\int_{R'} d^{2n}y = \int_R J(\vec{x})\, d^{2n}x$$

$$J(\vec{x}) = \left| \det\left(\frac{\partial y^i}{\partial x^j} \right) \right|$$

(const. for a linear coord. transfn.; otherwise it will be a fn. of $\vec{x}$ & $\vec{y}$)

$$\tilde{\Omega}^{ij} = \frac{\partial y^i}{\partial x^k}\, \Omega^{kl}\, \frac{\partial y^j}{\partial x^l}$$

Now, $S^i{}_k = \dfrac{\partial y^i}{\partial x^k}$ (defn.)

$$\therefore \tilde{\Omega}^{ij} = S^i{}_k\, \Omega^{kl}\, S^j{}_l$$

$\tilde{\Omega} = S\,\Omega\,S^T$ [$\because$ k-index gets contracted correctly, while the l index is contracted on taking transpose]

$$\det \tilde{\Omega} = (\det \Omega)\,(\det S)^2$$

If $\tilde{\Omega} = \Omega$, then $|\det S| = 1$ [Note:- $\det \Omega = 1$]

$\therefore$ for a canonical transfn, $|\det S| = 1$ [$\because \tilde{\Omega} = \Omega$]

$$\Rightarrow \int_{R'} d^{2n} y = \int_{R} d^{2n} x .$$

[We have seen that flow preserves the phase space vol. (time evolution of the system) in Ham. system. Here also, phase space vol. is preserved in can. transfn. Is there any relation betn the 2? — There is indeed a relation, because this flow can be taken as a canon. transfn] → when we see from the active pt. of view

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(Now we give a notion of vol. which is coord.-indep.)

Previously, this notion of vol. was coord.-dep.

Return to the general coord. transfn

$$\int_{R} d^{2n} x = \int_{R'} \underbrace{\left(J(\vec{x}')\right)^{-1}}_{\text{Inverse Jacobian}} d^{2n} y$$

This isn't something written entirely in $\vec{y}$-coord. system $\because$ it depends on $\vec{x}'$. We write $J(\vec{x}')$ as sth. that is naturally given by $\vec{y}$-coord. sys.

$$|\det \tilde{\Omega}| = \underbrace{|\det \Omega|}_{1 = \det \Omega} \cdot \underbrace{|(\det S)^2|}_{(J(\vec{x}'))^2} = 1 \times \left(J(\vec{x}')\right)^2$$

$$\Rightarrow J(\vec{x}') = |\det \tilde{\Omega}|^{1/2}$$

$$\therefore \int_{R'} \left(J(\vec{x})\right)^{-1} d^{2n}y = \int_{R'} |\det \tilde{\Omega}|^{-1/2} d^{2n}y$$

Suppose we define vol. as ↑ & then we'll find Liouville's th. will hold as long as we use this vol. defn. in y-coord. That can be taken as the standard vol. in the $\vec{x}$-coord. sys.

Exercise:- Check the validity of Liouville's theorem for the volume $\int d^{2n}y\, |\det \tilde{\Omega}|^{-1/2}$ by working directly with the eqns of motion in the $\vec{y}$ coord. system, viz.,

$$\frac{dy^i}{dt} = \tilde{\Omega}^{ij} \frac{\partial \tilde{H}}{\partial y^j}$$

Symmetries

A canonical transfn. $y^i = f^i(\vec{x})$ is a symmetry of the equations of motion if the Hamiltonian in the $\vec{y}$ coordinate system has the same form as the Hamiltonian in the $\vec{x}$ coord. sys.

Under a general coord. transfn from $\vec{x} \to \vec{y}$: $y^i = f^i(\vec{x})$ eqns of motion become:

$$\frac{dy^i}{dt} = \tilde{\Omega}^{ij} \frac{\partial \tilde{H}(\vec{y})}{\partial y^i}$$

$[\tilde{H}(\vec{y}) = H(\vec{x})]$

fn. obtd. by expressing $\vec{x}$ in terms of $\vec{y}$ in $H(\vec{x})$

& $\tilde{\Omega}^{ij} \partial y^i$ obtd. x^i

For Canonical transfn. :-

$$\tilde{\Omega}^{ij} = \Omega^{ij}$$

Symmetry transfn.

$$\tilde{\Omega}^{ij} = \Omega^{ij}, \quad \tilde{H}(\vec{y}) = H(\vec{y})$$

[i.e., ~~the~~ $\tilde{H}$ has the same fn. form as H when we express $\vec{x}$ in terms of $\vec{y}$ i.e. $\tilde{H}(\vec{x})$]

$$\therefore \frac{dy^i}{dt} = \Omega^{ij} \frac{\partial H(\vec{y})}{\partial y^j}$$ → same form as the $\vec{x}$ eqn.

(this is → known as the) Passive viewpoint : Eqns of motion look the same in the new coordinate system

[The form of the eqn. of motion is the same in either coord. sys.]

Suppose we have a soln. of the eqns of motion → $x^i = \xi^i(t)$ → Given fns

In the $\vec{y}$ coord. system, they will appear as $y^i = f^i(\vec{\xi}(t))$ [∵ the transfn. law is $y^i = f^i(\vec{x})$]

But y^i's satisfy the same eqns of motion as the x^i's (for a sym. transfn.).

$\Rightarrow$ $x^i = f^i(\vec{\xi}(t))$ is a soln of the eqns of motion in the $\vec{x}$-coord. system.

(Given $x^i = \xi^i(t)$ as the solns of eqns. of motion, it is guaranteed that $x^i = f^i(\vec{\xi}(t))$ are also the eqns of motion) [of course, $x^i = f^i(\vec{\xi}(t))$ aren't going to respect the same old initial conds.]

hint.

Active viewpt. → transfn. which allows us to go from one soln. to another soln. (in the same coord. sys.)

Doubt
① (n-1) charges?
② Phase. flow analogous to can. transfn
③ Problem given

t_0, t → fixed
Take $\vec{x}$ pt.

$$y^i = f^i(\vec{x}, t, t_0)$$

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Canonical transfn :-

$$(f^1(\vec{x}), \ldots, f^n(\vec{x}))$$

$$\frac{\partial f^i}{\partial x^k}\frac{\partial f^j}{\partial x^l}\Omega^{kl} = \Omega^{ij} \qquad \Omega = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

Infinitesimal canonical transfns :-

$$f^i(x) = x^i + \epsilon\,\phi^i(\vec{x}) + O(\epsilon^2)$$

ϵ ↓ infinitesimal parameter

$$y^i = x^i + \epsilon\,\phi^i(\vec{x})$$

$$\frac{\partial f^i(\vec{x})}{\partial x^j} = \delta^i{}_j + \epsilon\frac{\partial \phi^i(\vec{x})}{\partial x^j}$$

$$\left(\delta^i{}_k + \epsilon\frac{\partial \phi^i}{\partial x^k}\right)\left(\delta^j{}_l + \epsilon\frac{\partial \phi^j}{\partial x^l}\right)\Omega^{kl} = \Omega^{ij} + O(\epsilon^2)$$

$$\Rightarrow \cancel{\Omega^{ij}} + \epsilon\frac{\partial \phi^i}{\partial x^k}\Omega^{kj} + \epsilon\frac{\partial \phi^j}{\partial x^l}\Omega^{il} = \cancel{\Omega^{ij}} + O(\epsilon^2)$$

(we are neglecting the ϵ^2 term)

$$\Rightarrow \frac{\partial \phi^i}{\partial x^k}\Omega^{kj} + \frac{\partial \phi^j}{\partial x^l}\Omega^{il} = 0$$

$$\Downarrow$$

$$\Rightarrow \omega_{jn}\,\omega_{mi} \times (\ldots) = 0$$

Note → $\Omega^{kj}\omega_{jn} = \delta^k{}_n$

$\Omega^{il}\omega_{mi} = \omega_{mi}\Omega^{il} = \delta_m{}^l$

$$\Rightarrow \frac{\partial \phi^i}{\partial x^k}\delta^k{}_n\,\omega_{mi} + \frac{\partial \phi^j}{\partial x^l}\delta_m{}^l\,\omega_{jn} = 0$$

$$\Rightarrow \boxed{\frac{\partial \phi^i}{\partial x^n}\omega_{mi} - \frac{\partial \phi^j}{\partial x^m}\omega_{nj} = 0}$$

[comes because we write ω_{nj} instead of ω_{jn}]

Define $A_m = \omega_{mi}\,\phi^i$

Hence the eqn. $\boxed{\dfrac{\partial \phi^i}{\partial x^n}\,\omega_{mi} - \dfrac{\partial \phi^j}{\partial x^m}\,\omega_{nj} = 0}$

gives $\dfrac{\partial A_m}{\partial x^n} - \dfrac{\partial A_n}{\partial x^m} = 0$

(This is like the curl of a vector is zero in 3-dim.)

Hence, we have, $A_m = \partial_m \psi(\vec{x})$ $\left[\!\left[\partial_m = \dfrac{\partial}{\partial x^m} \right]\!\right]$ for some fn. $\psi(\vec{x})$.

We find that,

$$\omega_{mi}\,\phi^i = \frac{\partial \psi}{\partial x^m} \Rightarrow \Omega^{nm}\omega_{mi}\phi^i = \Omega^{nm}\frac{\partial \psi}{\partial x^m}$$

$$\Rightarrow \delta^n{}_i\,\phi^i = \Omega^{nm}\frac{\partial \psi}{\partial x^m}$$

$$\Rightarrow \boxed{\phi^i = \Omega^{in}\frac{\partial \psi}{\partial x^n}} \qquad \Rightarrow \phi^n = \Omega^{nm}\frac{\partial \psi}{\partial x^m}$$

$$\Rightarrow \phi^i = \Omega^{in}\frac{\partial \psi}{\partial x^n}$$

Hence $y^i = x^i + \epsilon\,\phi^i(\vec{x})$ gives us

$$y^i = x^i + \epsilon\,\Omega^{in}\frac{\partial \psi}{\partial x^n}$$

(Writing in this manner has an adv. than writing in terms of $\phi^i(\vec{x})$. Because previously, ϕ^i had to be chosen such that they satisfy certain conds., but here ψ can be chosen as any fn. we like)

ψ:- generator of infinitesimal canonical transfn.

$$\delta x^i = y^i - x^i \xrightarrow{\text{defn}} \delta x^i = \epsilon\,\Omega^{in}\frac{\partial \psi}{\partial x^n} = \epsilon\{x^i, \psi\}$$

$\therefore$ [illegible] where

Defn. ~~then~~ $\delta F = F(\vec{y}) - F(\vec{x})$

$\downarrow$ final config. $\quad \downarrow$ initial configuration ($\because$ we are talking about active transfn.)

$$\Rightarrow \delta F = F(\vec{x} + \delta\vec{x}) - F(\vec{x}) = \frac{\partial F}{\partial x^i}\delta x^i = \epsilon\,\Omega^{in}\frac{\partial F}{\partial x^i}\frac{\partial \psi}{\partial x^n}$$

If we have a can. transfn. generated by ψ, then we have $\delta F = \epsilon\,\Omega^{in}\frac{\partial F}{\partial x^i}\frac{\partial \psi}{\partial x^n} = \epsilon\{F, \psi\}$

Note :- $\delta\psi = 0$
i.e., the change in generator itself is zero

Suppose we take $F = H \rightsquigarrow$ the Hamiltonian

So, $\delta H \left(= H(\vec{y}) - H(\vec{x})\right) = \epsilon\{H, \psi\}$

If ψ generates a symmetry of H, then

$\delta H = 0 \Rightarrow \{H, \psi\} = 0.$

$\Rightarrow \frac{d\psi}{dt} = 0 \Rightarrow \psi$ is conserved.

(ψ is a const. of motion)

(The converse is also true.)

Converse :- Suppose there is some conserved quantity $Q(\vec{x})$.

This implies that $\{H, Q\} = 0$

This in turn implies that $\delta x^i = \epsilon\,\Omega^{in}\frac{\partial Q}{\partial x^n}$ is a symmetry of H. [using $\delta x^i = \epsilon\,\Omega^{in}\frac{\partial \psi}{\partial x^n} = \epsilon\{x^i, \psi\}$]

$\{H, Q\} = 0$

$\Rightarrow \frac{\partial H}{\partial x^i}\Omega^{ij}\frac{\partial Q}{\partial x^j} = 0$

Now consider infinitesimal canonical transfn. $\delta x^i = \epsilon\,\Omega^{ij}\,\partial Q/\partial x^j$ (I can choose the ψ to be Q)

Hence :- (If there is a conservation law, then it implies symmetry, the generator of sym. being the conserved charge.)
Cons. charges are related to existence of symmetries

The more conserved charges we have the better is our chance of solving the problem.

Passive picture

Start with $H(\vec{x})$ & go to the $\vec{y}$-coord. system.
For a can. trans., $H(\vec{x}) = \tilde{H}(\vec{y})$
$= \tilde{H}(\vec{x}(\vec{y}))$

Change : $\tilde{H}(\vec{y}) - H(\vec{y})$
$= H(\vec{x}) - H(\vec{y})$

exactly opposite to that in active picture

(there is only a change of sign).

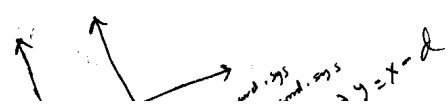
Every Ham. sys. has a conserd quantity, i.e., the Ham. itself,
So, we want to study the

Symmetry generated by H :-

$\delta x^i = \epsilon \Omega^{ij} \frac{\partial H}{\partial x^j}$ [we have just substituted H for g]

$= \epsilon \frac{dx^i}{dt} = x^i(t+\epsilon) - x^i(t)$

New coordinate $y^i = x^i + \delta x^i$ $= x^i(t+\epsilon)$



Consider an arbit. pt. in phase space :—

$\underset{\vec{x}}{\times}$ $\times\, \vec{y}$

$\vec{y}$ is the pt. that $\vec{x}$ goes to in time t

We can integrate it for a finite time & clearly find that :—

H generates time translation symmetry.

Examples :-

(1) free-particle : $H = \sum_{\alpha=1}^{3} p_\alpha^2/2m$

So, free particle Ham. depends only on p_α's & not on q_α's.

let $q^\alpha \longrightarrow q^\alpha + \epsilon n^\alpha$ arbitrary constants

$p^\alpha \to p^\alpha$

H doesn't change under this transfn. & this is a symmetry of H.

1) This is a symmetry of H

2) Is this canonical?

Let's try to find a generator to check (2).

We are done if we can find the generator.

suggestion :- take $\psi = \sum_\alpha n^\alpha p_\alpha$

Recall $\{F, G\} = \Omega^{ij} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j}$

Using q, p notation, we have $\{F, G\} = \frac{\partial F}{\partial q^\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q^\alpha}$

$$\delta q^\alpha = \epsilon\{q^\alpha, \psi\} = \epsilon\left[(1)\frac{\partial \psi}{\partial p_\alpha} - 0\right] = \epsilon n^\alpha$$

$$\delta p_\alpha = \epsilon\{p_\alpha, \psi\} = \epsilon\left[0 - \underbrace{\frac{\partial \psi}{\partial q^\alpha}}_{0}\right] = 0 - 0 = 0$$

[$\because \psi$ doesn't depend on q^α]

$\because$ n^α's are arbit. constants,
we have 3 conserved charges, viz,
p_1, p_2, p_3.

② Particle moving under a central force:-

$$H = \sum_{\alpha=1}^{3} p_\alpha^2/2m + V(r)$$

$$r = \sqrt{(q^1)^2 + (q^2)^2 + (q^3)^2}$$

The Ham. has a rotational invariance:-

$$p_\alpha \to R_{\alpha\beta} p_\beta \quad \text{with } RR^T = \mathbb{1}$$

Let's say $q^\alpha \to S_{\alpha\beta} q^\beta$ $\quad SS^T = \mathbb{1}$

R & S can be indep. matrices for inv. of H.

$\therefore$ H is invariant under indep. rot. of p & q.

Does it ~~generate~~ give a conserved charge?

→ Yes, we get 6 — but sth wrong is there

We haven't checked whether the transfns are canonical.

<u>checking</u>:- $P_\alpha = R_{\alpha\beta} p_\beta$ $\quad Q^\alpha = S_{\alpha\beta} q^\beta$

$\because$ $\tilde{\Omega} = \Omega$ for canonical transfn, the Poisson brackets for P_α & p_α should be the same.

$P_\beta = R_{\beta\gamma} p_\gamma$
$\frac{\partial P_\beta}{\partial p_i} = R_{\beta\gamma} \frac{\partial p_\gamma}{\partial p_i}$
$= R_{\beta\gamma} \delta_{\gamma i}$
$= R_{\beta i}$

We have, $\{q^\alpha, q^\beta\} = 0$, $\{p^\alpha, p^\beta\} 0$,

$$\{q^\alpha, p_\beta\} = \delta^\alpha{}_\beta$$

$$\{Q^\alpha, Q^\beta\} = \frac{\partial Q^\alpha}{\partial q^i}\frac{\partial Q^\beta}{\partial p_i} - \frac{\partial Q^\alpha}{\partial p^i}\frac{\partial Q^\alpha}{\partial q^i}$$

$P_\alpha P_\alpha$
$= R_{\alpha\beta} p_\beta R_{\alpha\gamma} p_\gamma$
$= \delta_{\beta\gamma} p_\beta p_\gamma$
$= p_\beta^2$

$$\{P_\alpha, P_\beta\} = \frac{\partial P_\alpha}{\partial q^i}\frac{\partial P_\beta}{\partial p_i} - \frac{\partial P_\alpha}{\partial p_i}\frac{\partial P_\beta}{\partial q^i} = 0$$

$$\{Q^\alpha, P_\beta\} = \frac{\partial Q^\alpha}{\partial q^i}\frac{\partial P_\beta}{\partial p_i} - \underbrace{\frac{\partial Q^\alpha}{\partial p_i}}_{=0}\underbrace{\frac{\partial P_\beta}{\partial q^i}}_{=0}$$

$$= S_{\alpha\beta}\,\delta^\beta{}_i\, R_{\beta\gamma}\,\delta^\gamma{}_i$$

$$= S_{\alpha i} R_{\beta i} \overset{?}{=} \delta^\alpha{}_\beta$$ (we must put this restriction)

So, we can't have arbit. S & R.

$\therefore\ SR^T = \mathbb{1} \Rightarrow S = (R^T)^{-1}$

But we already had the restriction $RR^T = \mathbb{1}$

$\therefore\ S = R$

$\therefore$ to get conserved charges, we must take $S = R$, i.e.,

$$\boxed{P_\alpha = R_{\alpha\beta}\, p_m\ ,\ Q^\alpha = R_{\alpha\beta}\, q^\beta}$$

Hence, we must have 3 conserved charges.

Infinitesimal transfn.

$$R_{\alpha\beta} = \delta_{\alpha\beta} + \epsilon\,\omega_{\alpha\beta}$$

$\because\ RR^T = \mathbb{1} \Rightarrow \omega_{\alpha\beta} = -\omega_{\beta\alpha}$

$$\omega = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} = a_1 t_1 + a_2 t_2 + a_3 t_3$$

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},\quad T_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

&

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$(RR^T)_{ik}$ ... $(R^T)_{jk}$ ... $(\delta_{kj} + \epsilon\omega_{kj})$ $(\delta_{ij}\omega_{kj} + \delta_{kj}\omega_{ij})$ ... $\omega_{ik})$

$\epsilon_{\alpha\beta\gamma}$

$$P_\alpha = R_{\alpha\beta} p_\beta = p_\alpha + \epsilon \omega_{\alpha\beta} p_\beta$$

$$Q^\alpha = R_{\alpha\beta} q^\beta = q^\alpha + \epsilon \omega_{\alpha\beta} q^\beta$$

$$\therefore \delta p_\alpha = \epsilon \omega_{\alpha\beta} p_\beta$$

$$\& \quad \delta q^\alpha = \epsilon \omega_{\alpha\beta} q^\beta$$

We are now looking for the generator.

We are ~~suppose~~ looking for a ψ such that

$$\delta p_\alpha = -\epsilon \frac{\partial \psi}{\partial q^\alpha}$$

$$\delta q^\alpha = \epsilon \frac{\partial \psi}{\partial p_\alpha}$$

$\Rightarrow$ ψ must be such that

$-\cancel{\epsilon} \frac{\partial \psi}{\partial q^\alpha} = \cancel{\epsilon} \omega_{\alpha\beta} p_\beta$

$\Rightarrow \frac{\partial \psi}{\partial q^\alpha} = \omega_{\beta\alpha} p_\beta$

$\& \frac{\partial \psi}{\partial p_\alpha} = \omega_{\alpha\beta} q^\beta$

$\delta p^\alpha = \epsilon \{p_\alpha, \psi\} = -\epsilon \left(\frac{\partial p_\alpha}{\partial q^i}\frac{\partial \psi}{\partial p_i} - \frac{\partial p_\alpha}{\partial p_i}\frac{\partial \psi}{\partial q^i}\right)$

$= \epsilon \left[\frac{\partial p_\alpha}{\partial q^i}\frac{\partial \psi}{\partial p_i}\right]$

$= \epsilon \left[0 - \delta^\alpha_i \frac{\partial \psi}{\partial q^i}\right]$

$= -\epsilon \frac{\partial \psi}{\partial q^\alpha}$

Guess :- $\boxed{\psi = \omega_{\alpha'\beta'} q^{\beta'} p_{\alpha'}}$

$$\therefore \delta p_\alpha = -\epsilon\, \delta^{\beta'}_{\alpha} \omega_{\alpha'\beta'} p_{\alpha'}$$

$$= -\epsilon\, \omega_{\alpha'\alpha} p_{\alpha'}$$

$$= \epsilon\, \omega_{\alpha\alpha'} p_{\alpha'}$$

$$\delta q^\alpha = \epsilon \omega_{\alpha\beta} q^\beta$$

$\therefore$ ψ is found to be the generator of this canonical transfn.

$$\psi = a_1 (T_1)_{\alpha\beta} q^\beta p_\alpha + a_2 (T_2)_{\alpha\beta} q^\beta p_\alpha + a_3 (T_3)_{\alpha\beta} q^\beta p_\alpha$$

[where a_1, a_2 & a_3 are arbitr. constants]

$\therefore$ there are 3 conserved charges :- [we can choose $a_1 = 1$ & $a_2 = a_3 = 0$]

~~$L_1 = (T_1)_{\alpha\beta} q^\beta p_\alpha$~~

$$L_1 = (T_1)_{\alpha\beta} q^\beta p_\alpha = q^3 p_2 - q^2 p_3$$

$$L_2 = (T_2)_{\alpha\beta} q^\beta p_\alpha = p_3 q^1 - p_1 q^3$$

1/9/05

Finite Canonical transfn.

$\boxed{\text{Motivation}}$ (for studying finite Can. transfn.) :-

in fact one of the motivation

Whereas infinitesimal canonical transfns are useful for finding conserved charges, finite canonical transfns are useful for solving the system using these conserved charges.

(means we have to have a set of algebraic eqns &/or a set of integrals) may not be explicitly solvable - an implicit relation

Suppose we have a $2N$'th order autonomous Hamiltonian system

$$q^1, \ldots, q^N, p_1, \ldots, p_N, \quad H(\vec{q}, \vec{p}).$$

Suppose we have N independent conserved charges $I_1(\vec{q}, \vec{p}), I_2(\vec{q}, \vec{p}), \ldots, I_N(\vec{q}, \vec{p})$ satisfying $\{I_\alpha, I_\beta\} = 0$.

(one of this can be the Hamiltonian)

We'll show that there is a canonical transfn. to new coordinates $Q_1, \ldots, Q_N, P_1, \ldots, P_N$ such that

$$P_\alpha = I_\alpha(\vec{q}, \vec{p}) \quad \text{for } \alpha = 1, \ldots, N$$

$$Q_\alpha = g_\alpha(\vec{q}, \vec{p})$$

$\hookrightarrow$ fns which can be found by knowing $I_\alpha(\vec{q}, \vec{p})$

[there is a definite algorithm of getting g_α's from I_α's — algebraic eqn or integrals can be obtained]

Consistency condn. $\rightarrow$ $\{I_\alpha, I_\beta\} = 0$ $\therefore$ we must have $\{P_\alpha, P_\beta\} = 0$

We have, $\frac{dP_\alpha}{dt} = 0 \Rightarrow \frac{\partial \tilde{H}}{\partial Q_\alpha} = 0$ $\left[\frac{dP_\alpha}{dt} = 0 \Rightarrow \{P_\alpha, \tilde{H}\} = 0 \Rightarrow \frac{\partial P_\alpha}{\partial Q_i}\frac{\partial \tilde{H}}{\partial P_i} - \frac{\partial P_\alpha}{\partial P_i}\frac{\partial \tilde{H}}{\partial Q_i} = 0\right]$

This means that H is indep. of all Q_α's.

$\tilde{H}(\vec{P}, \vec{Q}) = H(\vec{p}, \vec{q})$

Hence H is a fn. of $(I_1, \ldots, I_N)$.

... also a coord. quantity

$$\therefore \quad \frac{dQ_\alpha}{dt} = \frac{\partial H}{\partial P_\alpha} = \omega_\alpha(\vec{P})$$

General soln :—

$$P_\alpha = P_\alpha^0 \longrightarrow \text{constants}$$

(These fix the initial condns)

$$Q^\alpha = \omega_\alpha(\vec{P})\,t + Q_\alpha^0$$

$\rightarrow$ constant

(This result is trivial. But the nontrivial part is the construction of the fns g_α's)

This is the general method of solving a $2N^{th}$ order aut. Ham. sys. with N indep. consvd charges.

$Q_\alpha = g_\alpha(\vec{q}, \vec{p})$ is clearly a finite canonical transfn. & so we must study " "

We must be able to express $\vec{q}, \vec{p}$ in terms of Q, P (invertible) — bcos H is given in terms of q's & p's & we may want to go back to this original coord system.

This procedure requires us to study finite canonical transfn.

Study properties of ~~fin~~ finite canonical transfn.

$(x^1, \ldots, x^{2N}) \rightarrow$ a set of special coordinates

so that
$$\Omega_{ij} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

$$\omega = \Omega^{-1} = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

Consider a canonical transfn to $\tilde{x}^i = f^i(\vec{x})$

$$\frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial \tilde{x}^j}{\partial x^l}\Omega^{kl} = \Omega^{ij}$$

$$\frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial \tilde{x}^j}{\partial x^l}\Omega^{kl}\times\omega_{li}\omega_{kj} = \Omega^{ij}\times\omega_{li}\omega_{kj}$$

$$\Rightarrow \frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial \tilde{x}^j}{\partial x^l}\delta^k_i \times\omega_{kj} = \delta_l{}^j\,\omega_{kj}$$

$\Rightarrow$ - - - -

equivalent to $\Leftrightarrow$ $$\frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial \tilde{x}^j}{\partial x^l}\omega_{ij} = \omega_{kl}$$

Define: $x^1 = q^1,\ x^2 = p_1,\ x^3 = q^2,\ x^4 = p_2, \ldots$

$\tilde{x}^1 = \tilde{q}^1,\ \tilde{x}^2 = \tilde{p}_1,\ \tilde{x}^3 = \tilde{q}^2,\ \tilde{x}^4 = \tilde{p}_2, \ldots$

Define:-

$$A = \begin{pmatrix} p_1 \\ 0 \\ p_2 \\ 0 \\ p_3 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} x^2 \\ 0 \\ x^4 \\ 0 \\ x^6 \\ 0 \\ \vdots \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} \tilde{p}_1 \\ 0 \\ \tilde{p}_2 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} \tilde{x}^2 \\ 0 \\ \tilde{x}^4 \\ 0 \\ \tilde{x}^6 \\ 0 \\ \vdots \end{pmatrix}$$

~~Define~~ Notation $\Rightarrow \partial_i = \frac{\partial}{\partial x^i}$ & $\tilde{\partial}_i = \frac{\partial}{\partial \tilde{x}^i}$

$\therefore$ we have $\boxed{\partial_i A_j - \partial_j A_i = \omega_{ij}}$

Similarly, $\boxed{\tilde{\partial}_i \tilde{B}_j - \tilde{\partial}_j \tilde{B}_i = \omega_{ij}}$

Our eventual goal is with the help of a single unrestricted fn we want to write all the g_x's

Define :-

$$B_i = \frac{\partial \tilde{x}^k}{\partial x^i}\tilde{B}_k \quad \text{[transforms as a covariant vector]}$$

$$\partial_i B_j - \partial_j B_i = \frac{\partial}{\partial x^i}\left(\frac{\partial \tilde{x}^k}{\partial x^j}\tilde{B}_k\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial \tilde{x}^k}{\partial x^i}\tilde{B}_k\right)$$

$$= \cancel{\frac{\partial^2 \tilde{x}^k}{\partial x^i \partial x^j}\tilde{B}_k} + \frac{\partial \tilde{x}^k}{\partial x^j}\frac{\partial \tilde{B}_k}{\partial x^i} - \cancel{\frac{\partial^2 \tilde{x}^k}{\partial x^j \partial x^i}\tilde{B}_k} - \frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{B}_k}{\partial x^j}$$

($\tilde{B}_k$'s are fns of $\tilde{x}^i$'s chain rule)

$$\Rightarrow \partial_i B_j - \partial_j B_i = \frac{\partial \tilde{x}^l}{\partial x^j}\frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{B}_l}{\partial \tilde{x}^k} - \frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^j}\frac{\partial \tilde{B}_k}{\partial \tilde{x}^l}$$

[interchanging the labels k & l]

$$= \frac{\partial \tilde{x}^l}{\partial x^j}\frac{\partial \tilde{x}^k}{\partial x^i}\left(\tilde{\partial}_k \tilde{B}_l - \tilde{\partial}_l \tilde{B}_k\right)$$

$$= \frac{\partial \tilde{x}^l}{\partial x^j}\frac{\partial \tilde{x}^k}{\partial x^i}\,\omega_{kl}$$

$$\Rightarrow \boxed{\partial_i B_i - \partial_j B_i = \omega_{ij}}$$

So, we have 2 relations :—

$$\partial_i A_j - \partial_j A_i = \omega_{ij} \quad \text{——— (1)}$$

$$\partial_i B_j - \partial_i B_i = \omega_{ij} \quad \text{——— (2)}$$

Define :— $C_i = A_i - B_i$ ——— (3)

$$\Rightarrow \partial_i C_j - \partial_j C_i = 0 \quad \text{—(3)}$$ [(1) − (2) gives this]

This results from the fact that the transfn is canonical (we have encoded the can. transfn. in this ~~result~~ eqn)

$$\Rightarrow C_i = \frac{\partial F_1}{\partial x^i}$$

[Conversely, if $C_i = \frac{\partial F_1}{\partial x^i}$, then we'll get (3) & we can work backwards to show that the transfn. is canonical. It guarantees that the " " ".]

Let's consider ~~a change $\delta\vec{x}$ in $\vec{x}$~~ 2 points $\vec{x}$ & $\vec{x} + \delta\vec{x}$ close to each other.

$\vec{x} \xrightarrow{\text{corr. image}} \vec{\tilde{x}}$ $\qquad$ $\vec{x} + \delta\vec{x} \xrightarrow{\text{corr. image}} \vec{\tilde{x}} + \vec{\delta\tilde{x}}$

$$\delta F_1 = \frac{\partial F_1}{\partial x_i}\,\delta x^i = C_i\,\delta x^i = A_i\,\delta x^i - B_i\,\delta x^i$$

$$= A_i\,\delta x^i - \frac{\partial \tilde{x}^k}{\partial x^i}\,\tilde{B}_k\,\delta x^i$$

$$= A_i\,\delta x^i - \tilde{B}_k\,\delta \tilde{x}^k$$

(B_i isn't a simple object, but $\tilde{B}_k$ is, ∴ we write in terms of $\tilde{B}_k$)

Writing F_1 in terms of p_α & q_α ⇒

$$\boxed{\delta F_1 = \sum_{\alpha=1}^{N} \left(p_\alpha\,\delta q^\alpha - \tilde{p}_\alpha\,\delta \tilde{q}^\alpha \right)}$$

Suppose we regard F_1 as a fn. of $q^1, \ldots, q^N$, $\tilde{q}^1, \ldots, \tilde{q}^N$.

[F_1 is a fn. of any $2N$ indep. variables. ∴ In principle we can choose q^α's & $\tilde{q}^\alpha$'s as the indep. variables]

$$\Rightarrow p_\alpha = \frac{\partial F_1(\vec{q}, \vec{\tilde{q}})}{\partial q^\alpha}, \quad \tilde{p}_\alpha = -\frac{\partial F_1(\vec{q}, \vec{\tilde{q}})}{\partial \tilde{q}^\alpha}$$

[These eqns can be used to express $\tilde{q}$ & $\tilde{p}$ as fn. of q & p]

Use these to determine $\tilde{q}^\alpha$ in terms of $\vec{q}, \vec{p}$

[Given F_1, we need to solve a set of ordinary eqns to solve for $\tilde{q}^\alpha$'s]

not diff. eqns

$\tilde{p}_\alpha$ as fn. of $\vec{q}, \vec{p}$.

Summary :-

① For every generic canonical transfn. from $(\vec{q}, \vec{p})$ to $(\vec{\tilde{q}}, \vec{\tilde{p}})$ we have ~~a fn.~~ a fn. $F_1(\vec{q}, \vec{\tilde{q}})$ such that $p_\alpha = \frac{\partial F_1}{\partial q^\alpha}$, $\tilde{p}_\alpha = -\frac{\partial F_1}{\partial \tilde{q}^\alpha}$

exceptions → $\vec{q}$ & $\vec{\tilde{q}}$ aren't indep. variables

Exception: If $\vec{q}$ & $\vec{\tilde{q}}$ are not independent variables.

$\ldots \vec{\tilde{q}})$ we have a $\ldots \vec{\tilde{q}})$

1] given by:

$$p_\alpha = \frac{\partial f_1}{\partial q^\alpha}, \quad \tilde{p}_\alpha = -\frac{\partial f_1}{\partial \tilde{q}^\alpha}$$

($\therefore$ we can generate a can. transfn given a f_1)

Example

$$f_1(\vec{q}, \vec{\tilde{q}}) = \sum_\alpha q^\alpha \tilde{q}^\alpha$$

(Let's see what can. transfn it generates.)

$$\therefore p_\alpha = \frac{\partial f_1}{\partial q^\alpha} = \tilde{q}^\alpha$$

$$\Rightarrow \tilde{q}^\alpha = p_\alpha \quad \text{—— (i)}$$

Again $\tilde{p}_\alpha = -\frac{\partial f_1}{\partial \tilde{q}^\alpha} = -q^\alpha$

$$\Rightarrow \tilde{p}_\alpha = -q^\alpha \quad \text{—— (ii)}$$

This basically interchanges the coordinates & momenta.

Example (where q^α & $\tilde{q}_\alpha$ aren't indep.)

Suppose $\tilde{q}^\alpha = q^\alpha$, $\tilde{p}_\alpha = p_\alpha$ (Identity transfn)

This transfn can't be generated by f_1 because f_1 requires that $\tilde{q}^\alpha$ & q^α can be treated as indep. variables.

Here, we get $f_1 = \text{const.}$

(Of course, it's a special case, & not a generic case.)

Try: If $A_i = \tilde{A}_i$, $B_i = \tilde{B}_i = A_i$, $\therefore C_i = 0$ $\Rightarrow f_1 = \text{const.}$

Recall the starting pt.

$$A = \begin{pmatrix} p_1 \\ 0 \\ p_2 \\ 0 \\ \vdots \end{pmatrix} \quad \& \quad \tilde{B} = \begin{pmatrix} \tilde{p}_1 \\ 0 \\ \tilde{p}_2 \\ 0 \\ \vdots \end{pmatrix}$$

& $\partial_i A_j - \partial_j A_i = \omega_{ij}$ —(a)

$\tilde{\partial}_i \tilde{B}_j - \tilde{\partial}_j \tilde{B}_i = \omega_{ij}$ —(b)

Suppose instead of choosing A & $\tilde{B}$ this way, we choose A & $\tilde{B}$ such that (a) & (b) are satisfied nevertheless →

Choose $A = \begin{pmatrix} 0 \\ -q^1 \\ 0 \\ -q^2 \\ \vdots \end{pmatrix}$ & $\tilde{B} = \begin{pmatrix} \tilde{p}_1 \\ 0 \\ \tilde{p}_2 \\ 0 \\ \vdots \end{pmatrix}$

This choice again ~~satisfy~~ satisfies (a) & (b).

Steps

(1) Define B_i in the same way.

(2) $C_i = A_i - B_i \Rightarrow \partial_i C_j - \partial_j C_i = 0$ —(c)

$\Rightarrow C_i = \dfrac{\partial f_2}{\partial x^i}$ for some fn. f_2

[∵ the defn. of A has changed, f_2 will be diff. from f_1, although C_i's obey the same diff. eqns.]

In many books, $C_i = \frac{\partial f_2}{\partial x^i}$ is given so that we get $q^\alpha = \frac{\partial f_2(\vec{\tilde{q}}, \vec{p})}{\partial p_\alpha}$

$$\delta f_2 = A_i \delta x^i - \tilde{B}_i \delta \tilde{x}^i$$
$$= -\sum_{\alpha=1}^{N} q^\alpha \delta p_\alpha - \sum_{\alpha=1}^{N} \tilde{p}_\alpha \delta \tilde{q}^\alpha$$

(We can have a simple interpretation if we treat p_α & $\tilde{q}^\alpha$ as indep. variables)

Suppose $\vec{p}$ & $\vec{\tilde{q}}$ can be taken as indep. variables

Then, $q^\alpha = -\dfrac{\partial f_2(\vec{\tilde{q}}, \vec{p})}{\partial p_\alpha}$, $\tilde{p}_\alpha = -\dfrac{\partial f_2(\vec{\tilde{q}}, \vec{p})}{\partial \tilde{q}^\alpha}$

⇓

N eqns. & $\vec{\tilde{q}}$ is

Example:-

$$f_2(\vec{\tilde{q}}, \vec{p}) = -\sum_\alpha \tilde{q}^\alpha p_\alpha$$

$$q^\alpha = -\frac{\partial f_2}{\partial p_\alpha} = \tilde{q}^\alpha$$

$$\tilde{p}_\alpha = -\frac{\partial f_2}{\partial \tilde{q}^\alpha} = p_\alpha$$

(Clearly, an identity transfn)

So, we can do things here, which we could not do with f_1.
But, f_2 can't generate transfns in which q^α & $\tilde{q}^\alpha$ are indep.
So, every generic f will not give all transfns (will miss some).

If we choose $A = \begin{pmatrix} p_1 \\ 0 \\ p_2 \end{pmatrix}$ $A = \begin{pmatrix} p_1 \\ 0 \\ p_2 \\ 0 \\ \vdots \end{pmatrix}$

& $B = \begin{pmatrix} 0 \\ -\tilde{q}^1 \\ 0 \\ -\tilde{q}^2 \\ \vdots \end{pmatrix}$,

then $\tilde{q}^\alpha = \dfrac{\partial f_3(\vec{p}, \vec{\tilde{q}})}{\partial \tilde{p}_\alpha}$, $p_\alpha = \dfrac{\partial f_3(\vec{p}, \vec{\tilde{q}})}{\partial q^\alpha}$

If we choose $A = \begin{pmatrix} 0 \\ -q^1 \\ 0 \\ -q^2 \\ 0 \\ \vdots \end{pmatrix}$ & $B = \begin{pmatrix} 0 \\ -\tilde{q}^1 \\ 0 \\ -\tilde{q}^2 \\ 0 \\ \vdots \end{pmatrix}$,

then $q^\alpha = -\dfrac{\partial f_4(\vec{p}, \vec{\tilde{p}})}{\partial p_\alpha}$

$\partial f_4(\vec{p}, \vec{\tilde{p}})$

All canonical transfn, however, can't be generated by one of these 4 possibilities. U can cook up can. transfn which can't be generated by one of these 4. But, u can use a fcntn of these possibilities to generate the total transfn.

There can be $2^N \times 2^N = 2^{2N}$ choices for A & B (even if u don't take linear combns of q & p).

Suppose $A = \begin{pmatrix} p_1 & \\ 0 & \\ \vdots & \\ 0 & \\ & -q^1 \\ & \vdots \end{pmatrix}$ — 1 block, 1 block

There are N such blocks, each of which can be chosen in 2 ways.

8/9/05 $\quad F_3(\vec{q}, \vec{\tilde{p}}) \qquad \tilde{q}^\alpha = \dfrac{\partial F_3}{\partial \tilde{p}_\alpha}, \; p_\alpha = \dfrac{\partial F_3}{\partial q^\alpha}$

This shows that if we take

$$F_3 = \sum_\alpha q^\alpha \tilde{p}_\alpha,$$

then we get $\tilde{q}^\alpha = q^\alpha$ & $\tilde{p}_\alpha = p_\alpha$ (Identity transfn.)

For $F_3 = \sum_\alpha q^\alpha \tilde{p}_\alpha + \epsilon \psi(\vec{q}, \vec{\tilde{p}})$

$$\Rightarrow \tilde{q}^\alpha = q^\alpha + \epsilon \frac{\partial \psi(\vec{q}, \vec{\tilde{p}})}{\partial \tilde{p}_\alpha}$$

$$p_\alpha = \tilde{p}_\alpha + \epsilon \frac{\partial \psi}{\partial q^\alpha}(\vec{q}, \vec{\tilde{p}})$$

$$\epsilon\left[\frac{\partial \psi(\vec{q}, \vec{\tilde{p}})}{\partial p_\alpha} + (\tilde{p}_\alpha - p_\alpha)\frac{\partial^2 \psi(\vec{q}, \vec{\tilde{p}})}{\partial^2 p_\alpha} + \cdots\right] = \epsilon \frac{\partial \psi(\vec{q}, \vec{\tilde{p}})}{\partial p_\alpha} - \epsilon^2 \frac{\partial^2 \psi(\vec{q}, \vec{\tilde{p}})}{\partial^2 p_\alpha} + \cdots$$

$$\therefore \tilde{q}^\alpha = q^\alpha + \epsilon \frac{\partial \psi(\vec{q}, \vec{\tilde{p}})}{\partial p_\alpha} + O(\epsilon^2)$$

$$\& \; p_\alpha = \tilde{p}_\alpha + \epsilon \frac{\partial \psi}{\partial q^\alpha}(\vec{q}, \vec{\tilde{p}}) = \tilde{p}_\alpha + \epsilon \frac{\partial \psi(\vec{q}, \vec{\tilde{p}})}{\partial q^\alpha} + O(\epsilon^2)$$

Hence, $\tilde{q}^\alpha = q^\alpha + \epsilon \dfrac{\partial \psi(\vec{q}, \vec{\tilde{p}})}{\partial p_\alpha} + O(\epsilon^2)$ → agrees with the general form of infinite-

(We can do this with f_2.
But this can't be done with f_1 & f_4 because in that case identity transf. isn't allowed & hence sth close to identity transf. isn't also allowed.)

Suppose a Hamiltonian system has N independent conserved charges $I_1(\vec{q}, \vec{p}), \ldots, I_N(\vec{q}, \vec{p})$.

$$\therefore \{I_\alpha, I_\beta\} = 0$$

<u>Problem</u>: Solve the system

<u>Strategy</u>:- Find new coordinate system $(\vec{\tilde{q}}, \vec{\tilde{p}})$ such that the Hamiltonian is a function of $\vec{\tilde{p}}$ only, i.e.,

$$H(\vec{q}, \vec{p}) = \tilde{H}(\vec{\tilde{p}})$$

The eqns of motion in $(\vec{\tilde{q}}, \vec{\tilde{p}})$ coord. system are

$$\frac{d\tilde{q}^\alpha}{dt} = \frac{\partial \tilde{H}}{\partial \tilde{p}_\alpha} = \omega_\alpha(\vec{\tilde{p}})$$

$$\frac{d\tilde{p}_\alpha}{dt} = -\frac{\partial \tilde{H}}{\partial \tilde{q}^\alpha} = 0 \Rightarrow \tilde{p}_\alpha = \underbrace{\tilde{p}_\alpha(0)}_{\text{constants}}$$

Also $\tilde{q}^\alpha = \omega_\alpha(\vec{\tilde{p}})t + \underbrace{\tilde{q}^\alpha(0)}_{\text{constants}}$

This gives $\vec{\tilde{q}}$ & $\vec{\tilde{p}}$ as fns of t

Since we know the relationship betn $(\vec{q}, \vec{p})$ & $(\vec{\tilde{q}}, \vec{\tilde{p}})$.

[In a generic system, we can use any of the f_i's $(i=1,2,3,4)$]
Here we look for an $f_3(\vec{q}, \vec{\tilde{p}})$ which generates such a transformation.

$$\tilde{q}^\alpha = \frac{\partial f_3}{\partial \tilde{p}_\alpha}, \quad p_\alpha = \frac{\partial f_3}{\partial q^\alpha}$$

(our goal is to construct the f_3)

Choice of $\tilde{p}_\alpha$: $\tilde{p}_\alpha = f_\alpha(\vec{I})$

arbitrary indep. fns (our choice)

It's natural ~~that~~ to assume that $\tilde{p}_\alpha$'s are some fns of the const. charges, ∵ they themselves are constants

Consistency check:

$$\{\tilde{p}_\alpha, \tilde{p}_\beta\} = \frac{\partial \tilde{p}_\alpha}{\partial q^\gamma}\frac{\partial \tilde{p}_\beta}{\partial p_\gamma} - (\alpha \leftrightarrow \beta)$$

$$= \frac{\partial \tilde{p}_\alpha}{\partial I_\mu}\frac{\partial I_\mu}{\partial q^\gamma}\frac{\partial \tilde{p}_\beta}{\partial I_\nu}\frac{\partial I_\nu}{\partial p_\gamma} - \frac{\partial \tilde{p}_\alpha}{\partial I_\mu}\frac{\partial I_\mu}{\partial p_\gamma}\frac{\partial \tilde{p}_\beta}{\partial I_\nu}\frac{\partial I_\nu}{\partial q^\gamma}$$

$$= \frac{\partial \tilde{p}_\alpha}{\partial I_\mu}\frac{\partial \tilde{p}_\beta}{\partial I_\nu}\underbrace{\{I_\mu, I_\nu\}}_{=0}$$

$$= 0$$

(If f_α's aren't indep., we will get a singular matrix k: $\{\tilde{p}_\alpha, \tilde{p}_\beta\} = 0$ even if $\{I_\mu, I_\nu\} \neq 0$)

(This is a necessary condn for us to be able to write $\tilde{p}_\alpha = f_\alpha(\vec{I})$. But it doesn't prove that we can always do it — because first we have to find the f_3.)

Let's find what kind of f_3 will achieve this

(All we can say that there is no immediate contradiction in making an ansatz. For total consistency, we must have $\{\tilde{q}^\alpha, \tilde{p}_\beta\} = \delta^\alpha{}_\beta$, $\{\tilde{q}^\alpha, \tilde{q}^\beta\} = 0$)

$$\tilde{p}_\alpha = f_\alpha\left(\vec{I}(\vec{q}, \vec{p})\right) = h_\alpha(\vec{q}, \vec{p})$$

known fns

$$\Downarrow$$

$$p_\alpha = g_\alpha(\vec{q}, \tilde{\vec{p}})$$

known functions

(Think p_α's as unknown variables)

f_α's should be indep. otherwise it can't have been possible to invert them

$$p_\alpha = g_\alpha(\vec{q}, \vec{p})$$

$\hookrightarrow$ known fns

$\vec{p}$ can be thought as constant parameters — can be det. for any $\vec{p}$

$$\Downarrow$$

$$\boxed{\frac{\partial F_3(\vec{q}, \vec{p})}{\partial q^\alpha} = g_\alpha(\vec{q}, \vec{p})} \quad (\alpha = 1, 2, \ldots N)$$

[If we can solve these eqns to find F_3, we are done. All we need to show that we ~~do~~ are able to find F_3 in this way]

irrespective of what $f_\alpha(t)$ we have chosen

Q. (a) Do these eqns have a soln?

(b) If yes, can we express the soln in terms of integrals of known fns?

Consistency condn (for the eqns to be solvable) :-

$$\boxed{\frac{\partial g_\alpha(\vec{q}, \vec{p})}{\partial q^\beta} = \frac{\partial g_\beta(\vec{q}, \vec{p})}{\partial q^\alpha}} \quad (A)$$

(similar to curl of a vector is zero)

~~Given this~~ If this is satisfied, then there is some $F_3(\vec{q}, \vec{p})$ for which the eqns $\frac{\partial F_3}{\partial q^\alpha} = g_\alpha$ hold.

[Note :- If $\frac{\partial A_m}{\partial x^n} - \frac{\partial A_n}{\partial x^m} = 0$, then $A_m = \frac{\partial}{\partial x^m}\phi(\vec{x})$ for some $\vec{x}$]

(Let us first try to see whether (A) is satisfied)

$$p_\alpha = g_\alpha(\vec{q}, \vec{p})$$

[Clearly g_α's can't be arbit. fns of $\vec{q}, \vec{p}$.

we know h_α's, though not the g_α's

Property of $h_\alpha \longrightarrow$ ① Indep. fns
② Poisson bracket with themselves is zero ?

$\Delta p_\alpha = \left(\frac{\partial g_\alpha}{\ldots}\right) \Delta q^\beta$

$$dp_\alpha = \left(\frac{\partial g_\alpha}{\partial \tilde{p}_\beta}\right)_{\vec{q},\vec{\tilde{p}}} d\tilde{p}_\beta + \left(\frac{\partial g_\alpha(\vec{q},\vec{\tilde{p}})}{\partial q^\beta}\right)_{\vec{q},\vec{\tilde{p}}} dq^\beta$$

$\delta_\alpha^{\ \gamma} dp_\gamma$

$$= \left(\frac{\partial g_\alpha}{\partial \tilde{p}_\beta}\right)_{\vec{q},\vec{\tilde{p}}} \left\{ \left(\frac{\partial h_\beta(\vec{q},\vec{\tilde{p}})}{\partial q^\gamma}\right)_{\vec{q},\vec{\tilde{p}}} dq^\gamma + \left(\frac{\partial h_\beta(\vec{q},\vec{\tilde{p}})}{\partial p_\gamma}\right)_{\vec{q},\vec{\tilde{p}}} dp_\gamma \right\}$$

using $\tilde{p}_\alpha = h_\alpha(\vec{q},\vec{\tilde{p}})$

$$+ \left(\frac{\partial g_\alpha(\vec{q},\vec{\tilde{p}})}{\partial q^\beta}\right)_{\vec{q},\vec{\tilde{p}}} dq^\beta$$

(β index can be changed to γ)

[$\because$ this is a general equ, we can take p_α & q^α as the indep. variables & the coeff. of dp_α & dq^α on both sides should match)

Matching coefficients of dq^γ & dp_γ on both sides :-

$$\boxed{\begin{aligned} &\left(\frac{\partial g_\alpha}{\partial \tilde{p}_\beta}\right)_{\vec{q},\vec{\tilde{p}}} \left(\frac{\partial h_\beta}{\partial q^\gamma}\right)_{\vec{q},\vec{\tilde{p}}} = -\left(\frac{\partial g_\alpha}{\partial q^\gamma}\right)_{\vec{q},\vec{\tilde{p}}} \quad \text{--- (2)} \\ &\& \left(\frac{\partial g_\alpha}{\partial \tilde{p}_\beta}\right)_{\vec{q},\vec{\tilde{p}}} \left(\frac{\partial h_\beta}{\partial p_\gamma}\right)_{\vec{q},\vec{\tilde{p}}} = \delta_\alpha^{\ \gamma} \quad \text{--- (1)} \end{aligned}}$$

$$\left(\frac{\partial h_\beta}{\partial q^\gamma}\right)_{\vec{q},\vec{\tilde{p}}} \left(\frac{\partial h_\delta}{\partial p_\gamma}\right)_{\vec{q},\vec{\tilde{p}}} - \left(\frac{\partial h_\beta}{\partial p_\gamma}\right)_{\vec{q},\vec{\tilde{p}}} \left(\frac{\partial h_\delta}{\partial q^\gamma}\right)_{\vec{q},\vec{\tilde{p}}} = 0 \quad \text{--- (3)}$$

(from the vanishing of the Poisson

Multiply (3) by

$$\left(\frac{\partial g_\mu}{\partial p_\delta}\right)_{\vec{q},\vec{p}} \left(\frac{\partial g_\nu}{\partial p_\beta}\right)_{\vec{q},\vec{p}}$$

& we get,

$$\delta_\mu{}^\gamma \left[-\left(\frac{\partial g_\nu}{\partial q^\gamma}\right)_{\vec{q},\vec{p}}\right] + \delta^\gamma_\nu \left(\frac{\partial g_\mu}{\partial q^\gamma}\right)_{\vec{q},\vec{p}} = 0$$

[using ②]

from the cancel

$$\left(\frac{\partial g_\mu}{\partial p_\delta}\right)_{\vec{q},\vec{p}} \left(\frac{\partial h_\delta}{\partial p_\gamma}\right)_{\vec{q},\vec{p}} \equiv \delta_\alpha{}^\gamma \text{ [using ①]}$$

$$\Rightarrow \boxed{-\left(\frac{\partial g_\nu}{\partial q^\mu}\right)_{\vec{q},\vec{p}} + \left(\frac{\partial g_\mu}{\partial q^\nu}\right)_{\vec{q},\vec{p}} = 0}$$

(This proves that $\exists$ some F_3 such that $\vec{g}$ is the gradient of the F_3.)

We have shown that there is some $F_3(\vec{q},\vec{p})$ such that

$$\frac{\partial F_3(\vec{q},\vec{p})}{\partial q^\alpha} = g_\alpha(\vec{q},\vec{p})$$

$\boxed{\text{Q. How do we find } F_3\text{?}}$

$\vec{E} = -\vec{\nabla}\phi \longrightarrow$ In electrostatics

& $\phi = -\int_{\vec{x}} \vec{E}(\vec{x}) \cdot d\vec{x}$

Analogously, we have

$$F_3(\vec{q},\vec{p}) = \int_{\vec{q}(0)}^{\vec{q}} g_\alpha(\vec{q}',\vec{p})\, dq'^\alpha + F(\vec{p})$$

(The $f(\vec{\tilde{p}})$ introduces a further ambiguity in our choice [firstly, f_α's were arbit.] So, these canonical transfn's choice isn't unique - there can be many such can. transfn.)

Example of contour

Straight line from $\vec{q}^{(0)}$ to $\vec{q}$.

$$\vec{q}' = \vec{q}^{(0)} + u\left(\vec{q} - \vec{q}^{(0)}\right), \quad 0 \le u \le 1$$

$$\therefore \; dq'^{\alpha} = \left(q^{\alpha} - q^{(0)\alpha}\right) du$$

$$F_3(\vec{q}, \vec{\tilde{p}}) = \int_0^1 \Big[du \, \overbrace{f_\alpha\left(\vec{q}^{(0)} + u\left(\vec{q} - \vec{q}^{(0)}\right), \vec{\tilde{p}}\right)}^{\text{known fn of } u \; (\because f_\alpha\text{'s are known fns.})} \left(q^{\alpha} - q^{(0)\alpha}\right) \Big] + f(\vec{\tilde{p}})$$

$\Rightarrow F_3$ is known.

Since F_3 is known, the canonical transfn is known.

Until now, given any set of charges I_α's such that $\{I_\alpha, I_\beta\} = 0$, we can find this type of can. transfn where $\tilde{p}_\alpha$'s are fns of I_β's. They are conserved or not not didn't matter in this derivation.

If we further assume that the charges are conserved then H must be a fn. of $I_1, \ldots, I_N$.

$$\tilde{H} = H(I_1, \ldots, I_N)$$

known fns of $\tilde{p}_1, \ldots, \tilde{p}_N$

$$\tilde{H} = \tilde{H}(\vec{\tilde{p}}) \rightarrow \text{No } \vec{\tilde{q}} \text{ dependence of } \tilde{H}.$$

$\Rightarrow$ Eqns are solvable in $\tilde{q}, \tilde{p}$ coordinate system.

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Generators of Canonical trs.

$$F_3(\vec{q}, \vec{\tilde{p}})$$

$$\tilde{q}^\alpha = \frac{\partial F_3}{\partial \tilde{p}_\alpha}, \quad p_\alpha = \frac{\partial F_3}{\partial q^\alpha}$$

Supposed to solve $\tilde{q}^\alpha$ in terms of $\vec{q}$ & $\vec{p}$, know $\vec{\tilde{p}}$ from

supposed to solve this to find $\vec{\tilde{p}}$ as a fn. of $\vec{p}$ & $\vec{q}$

$\longrightarrow$ guaranteed to be a canonical transfn.

If a $2N$'th order Hamiltonian system has N independent conserved charges $I_1, \ldots, I_N$ satisfying $\{I_\alpha, I_\beta\} = 0$, then the system is solvable.

The way we proved it is as follows:-

1. choose new momentum variables $\tilde{p}_\alpha$ as

$$\tilde{p}_\alpha = \phi_\alpha(\vec{I}(\vec{q}, \vec{p})) \qquad \alpha = 1, 2, \ldots, N$$

$\downarrow$ N independent fns (arbitrary)

2. ~~Define~~ Solve these eqns to write

$$\phi_\alpha = g_\alpha(\vec{\tilde{p}}; \vec{q})$$

3. Define: $F_3(\vec{q}, \vec{\tilde{p}}) = \int^{\vec{q}} g_\alpha(\vec{\tilde{p}}, \vec{q}')\, dq'^\alpha + F(\vec{\tilde{p}})$

integration in q' space along an arbit. contour

$\downarrow$ arbitrary fn

From this it follows that

$$\frac{\partial F_3(\vec{\tilde{p}}, \vec{q})}{\partial q^\alpha} = g_\alpha(\vec{\tilde{p}}, \vec{q})$$

[comes from the above integral $-\vec{q}$ comes from the limit]

... generated by

Once we have determined f_3 by this procedure, we have $\tilde{q}^\alpha = \frac{\partial f_3}{\partial \tilde{p}_\alpha}$ = known ["solvable" ⇒ " upto ~~doing~~ writing integrals]

4. $\tilde{H}(\vec{\tilde{q}}, \vec{\tilde{p}}) = H(\vec{q}, \vec{p})$ (This is a canonical transform)

(∵ $\tilde{p}_\alpha$'s are fns of I_β's)

$$0 = \frac{d\tilde{p}_\alpha}{dt} = -\frac{\partial \tilde{H}}{\partial \tilde{q}^\alpha}$$

$$\Rightarrow \frac{\partial \tilde{H}}{\partial \tilde{q}^\alpha} = 0$$

$$\Rightarrow \tilde{H} = \tilde{H}(\vec{\tilde{p}})$$

Hence, $\frac{d\tilde{q}^\alpha}{dt} = \frac{\partial \tilde{H}}{\partial \tilde{p}_\alpha} = \omega_\alpha(\vec{\tilde{p}})$ → derivative of $\tilde{H}$

Soln. ⇒ $\tilde{p}_\alpha = \tilde{p}_\alpha^{(0)}$ (= constant)

$\therefore \tilde{q}^\alpha = \omega_\alpha(\vec{\tilde{p}}^{(0)})\, t + \tilde{q}^{\alpha(0)}$ (constant)

So, once we have solved for $\tilde{p}_\alpha$ & $\tilde{q}^\alpha$, we have also solved for p_β & q^β, ∵ we know $p_\alpha = g_\alpha(\vec{\tilde{p}}, \vec{q})$ & ---

In this sense, the system is solvable.

Example Simple harmonic oscillator

$$H = p^2/2m + \frac{1}{2} k q^2$$

This is a 2-dim. Ham. sys. → H itself is the conserved charge

$$I = H = p^2/2m + \frac{1}{2} k q^2$$

Step 1 Introduce $\tilde{p} = \Phi(H) \Rightarrow H = \Psi(\tilde{p})$ (inverse fn)

$$\Psi(\tilde{p}) = \frac{p^2}{2m} + \frac{1}{2} k q^2$$ [Analog of $\tilde{p}_\alpha = \Phi_\alpha(\vec{I}(\vec{q}, \vec{p}))$]

Step 2 $p = \sqrt{2m\left(\Psi(\tilde{p}) - \frac{1}{2} k q^2\right)} \longrightarrow g(\tilde{p}, q)$

Step 3

$$\cancel{F_3(q,\tilde{p}) = \int^{\tilde{q}} \sqrt{2m(\psi(\tilde{p}) - \tfrac{1}{2}kq'^2)}\, dq' + F(\tilde{p})}$$

$$F_3(q,\tilde{p}) = \int^{q} \sqrt{2m\left(\psi(\tilde{p}) - \tfrac{1}{2}kq'^2\right)}\, dq' + F(\tilde{p})$$

Consider the canonical transfn generated by $F_3(\tilde{p}, q)$.

$$\boxed{p = \frac{\partial F_3}{\partial q} = \sqrt{2m\left(\psi(\tilde{p}) - \tfrac{1}{2}Kq^2\right)}} \quad (1)$$

$$\tilde{q} = \frac{\partial F_3}{\partial \tilde{p}} = \int^{q} \frac{\partial}{\partial \tilde{p}} \sqrt{2m\left(\psi(\tilde{p}) - \tfrac{1}{2}kq'^2\right)}\, dq' + \frac{dF}{d\tilde{p}}$$

we can do the integral & then get $\frac{\partial F_3}{\partial \tilde{p}}$.

$$= \sqrt{\frac{m}{2}}\, \psi'(\tilde{p}) \int^{q} \frac{dq'}{\sqrt{2m\left(\psi(\tilde{p}) - \tfrac{1}{2}kq'^2\right)}} + F'(\tilde{p})$$

$$= \sqrt{\frac{m}{K}}\, \psi'(\tilde{p}) \sin^{-1} \frac{q}{\sqrt{2\psi(\tilde{p})/k}} + F'(\tilde{p})$$

We get,

$$\boxed{q = \sqrt{\frac{2\psi(\tilde{p})}{K}} \sin\left\{\frac{\sqrt{K/m}\,(\tilde{q} - F'(\tilde{p}))}{\psi'(\tilde{p})}\right\}} \quad (2)$$

Step 4

$$\tilde{H}(\tilde{q},\tilde{p}) = H(q,p) = \psi(\tilde{p})$$

$$\frac{d\tilde{p}}{dt} = 0 \Rightarrow \tilde{p} = \tilde{p}^{(0)}$$

$$\frac{d\tilde{q}}{dt} = \frac{\partial \tilde{H}}{\partial \tilde{p}} = \psi'(\tilde{p}) \Rightarrow \tilde{q} = \psi'(\tilde{p}^{(0)})t + \tilde{q}^{(0)}$$

From (2), we get,

$$q = \underbrace{\sqrt{\frac{2\psi(\tilde{p}^{(0)})}{K}}}_{\text{a constant}} \sin\left\{\sqrt{K/m}\left(t + \underbrace{\frac{\tilde{q}^{(0)}}{\psi'(\tilde{p}^{(0)})} - \frac{F'(\tilde{p}^{(0)})}{\psi(\tilde{p}^{(0)})}}_{\text{a constant}}\right)\right\}$$

$\because$ $\tilde{p}^{(0)}$ is fixed by us

$\therefore$ the solution is

$$q = C_1 \sin\left[\sqrt{\frac{K}{m}}\,(t + C_2)\right]$$

$C_1, C_2 \rightarrow$ constants

standard soln of a H.O.

But, we know $\psi(\tilde{p}^{(0)}) = H(q, p) = \text{constant} = E$

$$\therefore \boxed{C_1 = \sqrt{\frac{2E}{K}}}$$

We find that our final soln doesn't depend on our choice of ψ, which is physical bcos soln shouldn't depend on some arbit. choice of some fn

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$H = \frac{\vec{p}^2}{2m} + V(\lambda)$ $\quad \lambda^2 = x^2 + y^2 + z^2$

If we look for the symmetries of the Hamiltonian, we find:

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \rightarrow R \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow S \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$R^T R = \mathbb{1}$ $\qquad S^T S = \mathbb{1}$

This isn't a sym. of eqns of motion, bcos in general that isn't a can. transfn.

In the case $R = S$ only it is a " "

... things $\rightarrow$ | To have sym. transfn

In Newtonian sys., it isn't easy to find the Hamiltonian, except in simple cases

Suppose we have N particles of mass $m_1, \ldots, m_N$ moving under forces

$$F^1(\vec{q}), \ldots, F^N(\vec{q}).$$

$$m_\alpha \frac{d^2 q^\alpha}{dt^2} = F^\alpha(\vec{q})$$

[assuming forces depend only on the position, not on the momentum & here we take q^α's to be Cartesian coordinates]

Suppose further that

$$F^\alpha(\vec{q}) = -\frac{\partial V(\vec{q})}{\partial q^\alpha}$$

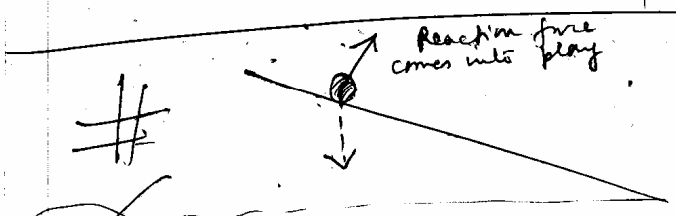
$$m_\alpha \frac{d^2 q^\alpha}{dt^2} = -\frac{\partial V}{\partial q^\alpha}$$

Define $p_\alpha = m_\alpha \dfrac{dq^\alpha}{dt}$.

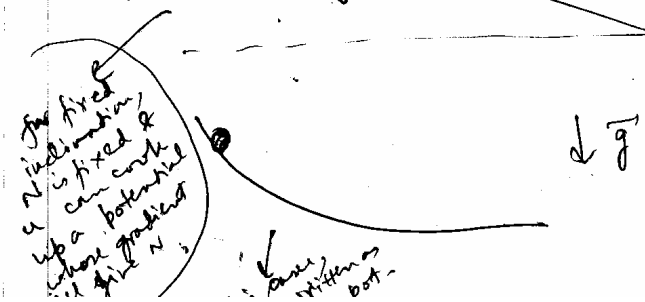
$$\frac{dq^\alpha}{dt} = \frac{p_\alpha}{m_\alpha} = \frac{\partial H}{\partial p_\alpha}$$

$$\frac{dp_\alpha}{dt} = -\frac{\partial V}{\partial q^\alpha} = -\frac{\partial H}{\partial q^\alpha}$$

if $H = \sum_\alpha p_\alpha^2 / 2m_\alpha + V(\vec{q})$



The simplicity is lost



The force due to reason will be very complicated – the tangent changes & the Normal force changes force also

These systems are easily studied in the Lagrangian formulation.

Hamiltonian itself is a const. of motion, for higher dim. however, this doesn't imply the system is solvable

Definition :- Lagrangian is the Legendre transform of the Hamiltonian with respect to the momentum variables.

Given $H(q, p)$, we define

$$u^\alpha = \frac{\partial H}{\partial p_\alpha}$$

and $$L(\vec{q}, \vec{u}) = \sum_\alpha u^\alpha p_\alpha - H(\vec{q}, \vec{p})$$

(To begin with, these are fns of q^α's & p_α's. But, using $u^\alpha = \frac{\partial H}{\partial p_\alpha}$, we write L as a fn. of q^α & p_α)

(We must know how to go from p_α variables to the u^α variables)

Consider small variations in $\vec{q}$ & $\vec{p}$:

$$\delta L = \sum_\alpha \delta u^\alpha p_\alpha + \cancel{\sum_\alpha u^\alpha \delta p_\alpha} - \sum_\alpha \frac{\partial H}{\partial q^\alpha} \delta q^\alpha - \cancel{\sum_\alpha \frac{\partial H}{\partial p_\alpha} \delta p_\alpha}$$

$$\Rightarrow \delta L = \sum_\alpha \left(\delta u^\alpha p_\alpha - \frac{\partial H}{\partial q^\alpha} \delta q^\alpha \right)$$

(But, we should take $\vec{u}_\alpha$, $\vec{q}_\alpha$ to be indep. variables while writing L.)

$$\Rightarrow \left.\frac{\partial L}{\partial u^\alpha}\right|_{\vec{q}, \vec{u}} = p_\alpha, \quad \left.\frac{\partial L}{\partial q^\alpha}\right|_{\vec{q}, \vec{u}} = -\left.\frac{\partial H}{\partial q^\alpha}\right|_{\vec{q}, \vec{p}}$$

(These eqns tell how to go from L to H)

Given $L(\vec{q}, \vec{u})$, we define

$$p_\alpha = \left.\frac{\partial L}{\partial u^\alpha}\right|_{\vec{q}, \vec{u}}$$

$$H(\vec{q}, \vec{p}) = \sum_\alpha u^\alpha p_\alpha - L(\vec{q}, \vec{u})$$

Equations of motion :-

$$\frac{dq^\alpha}{dt} = \frac{\partial H}{\partial p_\alpha} \Rightarrow \boxed{\frac{dq^\alpha}{dt} = u^\alpha}$$

$$\frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial q^\alpha} \Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial u^\alpha}\right) = \frac{\partial L}{\partial q^\alpha}}$$

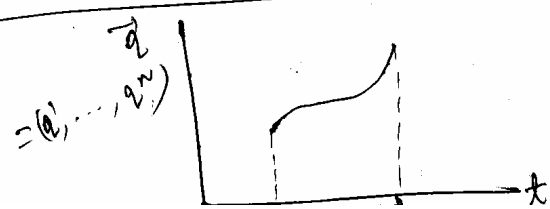
Eqns of motion in the Lagrangian formulation

[One adv. of using L is the simplicity of the 1st eqn. : $\frac{dq^\alpha}{dt} = u^\alpha$
In Lag. formulation, irrespective of what L is, we always have $dq^\alpha/dt = u^\alpha$ (doesn't change).
In Ham. formulation, both eqns change if H changes]

u^α is always the velocity
(keeping this in mind, we don't even have to write always $dq^\alpha/dt = u^\alpha$)

[In Lag. formulation, we don't think u^α's as variables. We take only q^α's to be variables & have N 2nd order ODE's]

[u, q indep. from the first order viewpt. & solve for them separately
From the 2nd order viewpt., not indep — u's are derivatives of q's]



Trajectory : Set of N fns

$$q^1 = f^1(t)$$
$$q^2 = f^2(t)$$
$$\vdots$$
$$q^N = f^N(t)$$

N Eqns in (N+1) variables will give you a one-dimensional curve.

[Don't think t as ~~an parameter~~ arbitrary parameter. t is a variable — It is the time]

A functional is an algorithm that generates a no. given some fns. e.g. → $S[\vec{f}(t)]$

A fn. is an algorithm " " a no. given some nos.

For given $L(\vec{q}, \vec{u})$, define

$$S[\vec{f}(t)] = \int_{t_a}^{t_b} dt \; L\left(\vec{q} = \vec{f}(t),\ \vec{u} = \frac{d}{dt}\vec{f}(t)\right)$$

→ Action associated with the trajectory $\vec{q} = \vec{f}(t)$.

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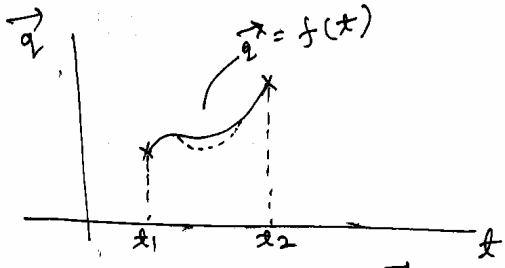
$$L = \sum_\alpha u^\alpha p_\alpha - H \qquad u^\alpha = \frac{\partial H}{\partial p_\alpha}$$

$$\boxed{\begin{aligned} \frac{dq^\alpha}{dt} &= u^\alpha \\ \frac{d}{dt}\left(\frac{\partial L}{\partial u^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} &= 0 \end{aligned}}$$

$$S[f(t)] = \int_{t_1}^{t_2} dt \; L\left(\vec{q} = \vec{f}(t),\ \vec{u} = \frac{\partial \vec{f}(t)}{dt}\right)$$



Take a new curve $\vec{q} = \vec{f}(t) + \delta\vec{f}(t)$ [Slightly deviated from the original dirn.]

$$S[\vec{f}(t) + \delta\vec{f}(t)] = \int_{t_1}^{t_2} dt\, L\left(\vec{f}(t) + \delta\vec{f}(t), \frac{d\vec{f}}{dt} + \frac{d}{dt}\delta\vec{f}(t)\right)$$

$$= \int_{t_1}^{t_2} dt \left[L\left(\vec{f}(t), \frac{d\vec{f}(t)}{dt}\right) + \frac{\partial L}{\partial q^\alpha}\delta f^\alpha(t) + \frac{\partial L}{\partial \dot{q}^\alpha}\frac{d}{dt}\delta f^\alpha(t) + O(\delta f^2)\right]$$

$$S[\vec{f}(t) + \delta\vec{f}(t)]$$

$$= S[\vec{f}(t)] + \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q^\alpha}\delta f^\alpha(t) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha}\delta f^\alpha(t)\right] + \frac{\partial L}{\partial \dot{q}^\alpha}\delta f^\alpha(t)\Big|_{t_1}^{t_2} + O(\delta f^2)$$

(Integrating the 3rd term by parts)

If $\vec{f}(t)$ is a solution of the classical eqns of motion then the second term $= 0$.

If $\delta\vec{f}(t) = 0$ at t_1, t_2 then the third term $= 0$.

$\Rightarrow S[f + \delta f] = S(f)$ to first order in $\delta\vec{f}$.

$\Rightarrow$ The action has an extremum at the solution of classical eqs. of motion.

Reverse is also true.

If $\delta S = 0$ to first order under an arbitrary δf with $\delta f(t_1) = 0 = \delta f(t_2)$, then $\vec{f}(t)$ satisfies classical eqs. of motion.

Take $\delta f^\alpha(t) = \epsilon\, \delta^\alpha_\beta\, \delta(t-t')$ [that is our choice]

$\beta \to$ fixed index, $t' \to$ fixed time

$$\delta S \equiv S[f+\delta f] - S[f] = \epsilon\left(\frac{\partial L}{\partial q^\beta} - \frac{d}{dt}\frac{\partial L}{\partial u^\beta}\right)$$

$q = t'$, $\vec{q} = \vec{f}(t)$, $\vec{u} = \frac{d\vec{f}}{dt}$

But this is supposed to be zero [$\because$ β can be chosen & t' can be chosen arbitrarily]

$$\frac{\partial L}{\partial q^\beta} - \frac{d}{dt}\frac{\partial L}{\partial u^\beta} = 0$$

Doubt $\to$ δf well behaved fn.
$\to$ Not really whether we must see whether δS is well-behaved for our choice of δf

$L(\vec{q}, \vec{u})$

$\tilde{q}^\alpha = \phi^\alpha(\vec{q}, \vec{u})$

$\tilde{u}^\alpha = \psi^\alpha(\vec{q}, \vec{u})$

Under what condition the eqs. of motion for $\vec{\tilde{q}}$ and $\vec{\tilde{u}}$ look the same as that of $\vec{q}, \vec{u}$?

In Ham. formulation, we have to have
① canonical transf.
② Hamo unchanged

$\longrightarrow u^\alpha = \frac{dq^\alpha}{dt}$

Let's see under what condn this is going to be preserved.

$$\frac{d\tilde{q}^\alpha}{dt} \overset{?}{=} \tilde{u}^\alpha \Rightarrow \text{Need}: \frac{\partial \phi^\alpha}{\partial q^\beta}\frac{dq^\beta}{dt} + \frac{\partial \phi^\alpha}{\partial u^\beta}\frac{du^\beta}{dt} = \psi^\alpha(\vec{q}, \vec{u})$$

$$\Rightarrow \frac{\partial \phi^\alpha}{\partial q^\beta} u^\beta + \frac{\partial \phi^\alpha}{\partial u^\beta}\frac{du^\beta}{dt} = \psi^\alpha(\vec{q}, \vec{u})$$

As it stands, this eqn. doesn't make sense.
RHS $\to$ fn. of $\vec{q}$ & $\vec{u}$ only
LHS $\to$ " " $\vec{q}$, $\vec{u}$ & $\dot{\vec{u}}$.

Solve for $\frac{du^\beta}{dt}$ in terms of $\vec{q}, \vec{u}$ using the second eqs. of motion:

$$\frac{d}{dt}\underbrace{\frac{\partial L}{\partial u^\alpha}}_{g_\alpha(\vec{q}, \vec{u})} - \frac{\partial L}{\partial q^\alpha} = 0$$

Considering the generic case when $\frac{\partial g_\alpha}{\partial u^\beta}$ has non-zero eigenvalue

otherwise we can't go back & forth betn. Lag. & Ham.

$$\rightarrow \frac{\partial g_\alpha}{\partial q^\beta}\frac{dq^\beta}{dt} + \frac{\partial g_\alpha}{\partial u^\beta}\frac{du^\beta}{dt} - \frac{\partial L}{\partial q^\alpha} = 0$$

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The above eqn. gives $\frac{du_\beta}{dt}$ as fn. of $\vec{q}$ & $\vec{u}$.

[This procedure doesn't impose any condn. on ϕ^α — it is completely arbitrary — but then ψ^α is det. in terms of ϕ^α.]

$\phi^\alpha(\vec{q},\vec{u})$ is arbitrary at this stage. $\psi^\alpha(\vec{q},\vec{u})$ is determined in terms of $\phi^\alpha(\vec{q},\vec{u})$:—

$$\psi^\alpha(\vec{q},\vec{u}) = \frac{\partial \phi^\alpha}{\partial q^\beta} u^\beta + \frac{\partial \phi^\alpha}{\partial u^\beta}\frac{du^\beta}{dt}$$

↳ needs to be expressed in terms of $\vec{q}$, $\vec{u}$.

Still now we have considered half of the eqns remain unchanged. Now we try to see what condns the other set remains unchanged.

Under what condn. on $\phi^\alpha(\vec{q},\vec{u})$

$$\frac{d}{dt}\frac{\partial L}{\partial \tilde{u}_\alpha} - \frac{\partial L}{\partial \tilde{q}^\alpha} = 0 \ ?$$

→ (Only under some special choice of ϕ^α, this should be possible.)

There are many approaches:—

① Brute force

$$\tilde{q}^\alpha = \phi^\alpha(\vec{q},\vec{u})$$

$$\tilde{u}^\alpha = \psi^\alpha(\vec{q},\vec{u})$$

Convert $\frac{\partial}{\partial \tilde{q}_\alpha}$, $\frac{\partial}{\partial \tilde{u}_\alpha}$ to $\frac{\partial}{\partial q^\alpha}$, $\frac{\partial}{\partial u^\alpha}$ using chain rule & see under what condn.

$$\frac{d}{dt}\frac{\partial L}{\partial \tilde{u}} - \frac{\partial L}{\partial \tilde{q}} = 0 .$$

~~We want to have $L(\vec{q},\vec{u})$ to have the same form as~~

$L(\vec{q},\vec{u}) = L(\tilde{\vec{q}},\tilde{\vec{u}})$ wrong ✗

We are demanding form of eqns will be unchanged

② We already know from ~~the~~ the Hamiltonian formulation that

$$H(\vec{q}, \vec{p}) = H(\tilde{\vec{q}}, \tilde{\vec{p}}).$$

$$\sum_\alpha p_\alpha \frac{\partial H}{\partial p_\alpha} - L(\vec{q}, \vec{u}) = \sum_\alpha \tilde{p}_\alpha \frac{\partial H}{\partial \tilde{p}_\alpha} - L(\tilde{\vec{q}}, \tilde{\vec{u}})$$

Approaches ① & ② aren't very convenient - ① is cumbersome, while in ② we have to switch to Ham. formulation & we aren't gaining anything.

If $L(\vec{q}, \vec{u}) = L(\tilde{\vec{q}}, \tilde{\vec{u}})$ is true, then

$$\frac{d}{dt}\frac{\partial L(\tilde{\vec{q}}, \tilde{\vec{u}})}{\partial u^\alpha} - \frac{\partial L(\tilde{\vec{q}}, \tilde{\vec{u}})}{\partial q^\alpha} = 0,$$

but this is not what we want

③ <u>Simplest approach: Action formalism</u>

Result: Given $\phi^\alpha(\vec{q}, \vec{u})$ consider

$$L\left(\Phi(\vec{q}, \vec{u}), \frac{d}{dt}\Phi(\vec{q}, \vec{u})\right)$$

a fn. of $\vec{q}, \vec{u}$ & $\frac{d\vec{u}}{dt}$

$$\frac{d\phi^\alpha(\vec{q}, \vec{u})}{dt} = \frac{\partial \phi_\alpha}{\partial q^\beta} u^\beta + \frac{\partial \phi^\beta}{\partial u^\beta}\frac{du^\beta}{dt}$$

<u>Claim</u>: $\tilde{q}^\alpha = \phi^\alpha(\vec{q}, \vec{u})$ is a symmetry of the eqns of motion if

$$L\left(\vec{\phi}(\vec{q}, \vec{u}), \frac{d\vec{\phi}}{dt}(\vec{q}, \vec{u})\right) = L(\vec{q}, \vec{u}) + \frac{d}{dt}K(\vec{q}, \vec{u})$$

for any fn. $K(\vec{q}, \vec{u})$

$\boxed{\text{Proof}}$:-

$S[\vec{q} = \vec{f}(t)]$: Action associated with ~~a~~ some path.

~~Suppose $\vec{f}(t)$ is a classical solution of the eq~~

The idea is all the $\dot{\vec{u}}$ dependent part in LHS can be written in the derivative of $K(\vec{q}, \vec{u})$

ϕ^α has to be chosen in order to satisfy this

Suppose $\left(\vec{q} = \vec{f}(t), \vec{u} = \frac{d\vec{f}}{dt}\right)$ is a solution of the eqns of motion.

In the Active sense

((Given $f(t)$, we can find the action involving $\phi(f(t))$.))

$$S[\vec{\phi}(\vec{f}(t))] = \int_{t_1}^{t_2} dt\; L\left(\vec{\phi}(\vec{f}(t)), \frac{d}{dt}\vec{\phi}(\vec{f}(t))\right)$$

action evaluated for the path $\vec{\phi}(\vec{f}(t))$

$$= \int_{t_1}^{t_2} dt\; L\left(\vec{f}(t), \frac{d}{dt}\vec{f}(t)\right) + K\left(\vec{f}(t), \frac{d\vec{f}}{dt}\right)\Big|_{t_1}^{t_2}$$

action evaluated for the path $\vec{f}(t)$

$$\Rightarrow S[\vec{\phi}(\vec{f}(t))] = S[\vec{f}(t)] + K\left(\vec{f}(t), \frac{d\vec{f}}{dt}\right)\Big|_{t_1}^{t_2}$$

$\rightarrow$ true for any $\vec{f}(t)$.

Now suppose $\vec{q} = \vec{f}(t)$, $\vec{u} = \frac{d\vec{f}}{dt}$ is a soln of eqns of motion.

$\Rightarrow \delta S[\vec{f}(t)] = 0$ to first order under any variation of the path which vanishes at t_1, t_2

We see that

$\delta S[\vec{\phi}(\vec{f}(t))] = \delta S[\vec{f}(t)]$

for any $\vec{f}(t)$ [doesn't have to be a soln of eqns of motion]

Doubt

What is the purity that the trajectory will pass through the initial & final pts.

[If $\vec{f}(t)$ is the soln of an eqn of motion, then RHS $= 0$ & hence LHS $= 0$]

If $\vec{q} = \vec{f}(t)$, $\vec{u} = \frac{d\vec{f}}{dt}$ is a soln of the eqns of motion then $\delta S[f] = 0$ to first order in δf

$\Rightarrow \delta S[\vec{\phi}(\vec{f})] = 0$ to first order in δf.

non-vanishing Jacobian

$\Rightarrow \vec{q} = \vec{\phi}(\vec{f}(t)),\ \vec{u} = \frac{d}{dt}\vec{\phi}(\vec{f}(t))$ is a soln of the eqns of motion.

$$\delta\phi^\alpha(f) = \frac{\partial \phi^\alpha(\vec{q},\vec{u})}{\partial q^\beta}\Big|_{\vec{q}=\vec{f}} \delta f^\beta + \frac{\partial \phi^\alpha}{\partial u^\beta}\frac{d}{dt}(\delta f^\beta)$$

[As long as the transfn is ~~non-sings~~ non-singular, variation of $\delta\phi^\alpha$ can expressed as variation of δf^β]

If we can extremise S for the given 2 points, then the system passes through them — if it isn't possible, then the system doesn't pass through them

At $x=0$ $\sin x$ & $\sin x + e^{-1/x}$ has all deriv. same (in $x>0$ domain)

H.W.

Check the validity of Liouville's theorem for the volume $\int d^{2n}y\, |\det \tilde{\Omega}|^{-1/2}$ by working directly with the equations of motion in the $\vec{y}$-coordinate system, viz.,

$$\frac{dy^i}{dt} = \tilde{\Omega}^{ij} \frac{\partial \tilde{H}}{\partial y^j}$$

Soln. $\Rightarrow$ The trajectory is given by

$$y^i(t) = \tilde{\phi}^i(t, t_0, \vec{y}_0) \quad \text{—— (1)}$$

Let $\tilde{S}^i{}_j = \dfrac{\partial \tilde{\phi}^i(t, t_0, \vec{y}_0)}{\partial y_0^j}$ $\quad \boxed{y_0^j = \tilde{\phi}^i(t_0, t_0, \vec{y}_0)}$

Hence, $\delta y^i(t) = \dfrac{\partial \tilde{\phi}^i(t, t_0, \vec{x}_0)}{\partial y_0^j}\, \delta y_0^j$

Let $\tilde{J}(t, t_0, \vec{y}_0) = |\det \tilde{S}(t, t_0, \vec{y}_0)|$

Now, $\dfrac{\partial}{\partial t}\left[\tilde{S}^i{}_j(t, t_0, \vec{y}_0)\right] = \dfrac{\partial}{\partial t}\left(\dfrac{\partial \tilde{\phi}^i(t, t_0, \vec{y}_0)}{\partial y_0^j}\right)$

$$= \frac{\partial}{\partial y_0^j} \frac{\partial \tilde{\phi}^i(t, t_0, \vec{y}_0)}{\partial t}$$

$$= \frac{\partial}{\partial y_0^j} \frac{\partial y^i(t)}{\partial t}$$

$$= \frac{\partial}{\partial y_0^j}\left(\tilde{\Omega}^{ik}_{(t, t_0, \vec{y}_0)} \frac{\partial \tilde{H}}{\partial y^k}\right)\Bigg|_{\vec{y} = \tilde{\phi}(t, t_0, \vec{y}_0)}$$

$$\Rightarrow \left[\frac{\partial}{\partial t} \tilde{S}^i{}_j(t, t_0, \vec{y}_0)\right]_{t = t_0} = \frac{\partial}{\partial y_0^j}\left[\tilde{\Omega}^{ik}(t_0, t_0, \vec{y}_0) \frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^k}\right]$$

Now, $\tilde{S}^i{}_j(t_0, t_0, \vec{y}_0) = \delta^i{}_j$

$$\therefore \tilde{S}^i{}_j(t_0 + \Delta t, t_0, \vec{y}_0) = \delta^i{}_j + \Delta t \underbrace{\frac{\partial}{\partial y_0^j}\left[\tilde{\Omega}^{ik}(t_0, t_0, \vec{y}_0) \frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^k}\right]}_{N^i{}_j}$$

$$\Rightarrow \det\left[\tilde{S}^i{}_j(t_0 + \Delta t, t_0, \vec{y}_0)\right] = 1 + \Delta t\, \mathrm{tr}\,(N)$$

$$\Rightarrow \boxed{d^{2n} y = \left[1 + \Delta t \frac{\partial}{\partial y_0^i}\left[\tilde{\Omega}^{ik}(t_0, t_0, \vec{y}_0)\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^k}\right]\right] d^{2n} y_0}$$

Again, $\tilde{\Omega}^{ik}_{(t)} = \frac{\partial y^i(t)}{\partial x^r}\,\Omega^{rl}\,\frac{\partial y^k}{\partial x^l}$

$$\therefore \frac{\partial \tilde{\Omega}^{ik}_{(t, t_0, \vec{y}_0)}}{\partial t} = \Omega^{rl}\left[\frac{\partial}{\partial x^r}\left(\frac{\partial y^i}{\partial t}\right)\frac{\partial y^k}{\partial x^l} + \frac{\partial y^i}{\partial x^r}\frac{\partial}{\partial x^l}\left(\frac{\partial y^k}{\partial t}\right)\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\tilde{\Omega}^{ik}_{(t, t_0, \vec{y}_0)}\Big|_{t=t_0} = \Omega^{rl}\left[\frac{\partial}{\partial x_0^r}\left\{\tilde{\Omega}^{i\alpha}(t_0, t_0, \vec{y}_0)\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right\}\frac{\partial y_0^k}{\partial x_0^l} + \frac{\partial y_0^i}{\partial x_0^r}\frac{\partial}{\partial x_0^l}\left\{\tilde{\Omega}^{k\alpha}(t_0, t_0, \vec{y}_0)\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right\}\right]$$

$$= \left[\frac{\partial y_0^k}{\partial x_0^l}\Omega^{rl}\frac{\partial y_0^\beta}{\partial x_0^r}\frac{\partial}{\partial y_0^\beta}\left\{\tilde{\Omega}^{i\alpha}(t_0, t_0, \vec{y}_0)\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right\}\right] + \left[\frac{\partial y_0^i}{\partial x_0^r}\Omega^{rl}\frac{\partial y_0^\beta}{\partial x_0^l}\frac{\partial}{\partial y_0^\beta}\left\{\tilde{\Omega}^{k\alpha}(t_0, t_0, \vec{y}_0)\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right\}\right]$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\tilde{\Omega}^{ik}_{(t, t_0, \vec{y}_0)}\Big|_{t=t_0} = \tilde{\Omega}^{\beta k}_{(t_0, t_0, \vec{y}_0)}\frac{\partial}{\partial y_0^\beta}\left[\tilde{\Omega}^{i\alpha}(t_0, t_0, \vec{y}_0)\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right] + \tilde{\Omega}^{i\beta}(t_0, t_0, \vec{y}_0)\frac{\partial}{\partial y_0^\beta}\left[\tilde{\Omega}^{k\alpha}(t_0, t_0, \vec{y}_0)\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right]}$$

Now, $\tilde{\Omega}(t_0 + \Delta t, t_0, \vec{y}_0) = \tilde{\Omega}(t_0, t_0, \vec{y}_0) + \Delta t \frac{\partial \tilde{\Omega}(t, t_0, \vec{y}_0)}{\partial t}\Big|_{t=t_0}$

$$= \tilde{\Omega}(t_0, t_0, \vec{y}_0)\left[1 + \Delta t\, \overset{=\tilde{\omega}}{\tilde{\Omega}^{-1}(t_0, t_0, \vec{y}_0)}\frac{\partial \tilde{\Omega}(t, t_0, \vec{y}_0)}{\partial t}\Big|_{t=t_0}\right]$$

$$\Rightarrow \tilde{\Omega}^{ik}(t_0 + \Delta t, t_0, \vec{y}_0) = \tilde{\Omega}^{i\alpha}(t_0, t_0, \vec{y}_0)\left[\delta^k_\alpha + \Delta t\, \tilde{\omega}_{\alpha l}(t_0, t_0, \vec{y}_0)\,\tilde{\Omega}^{\beta k}(t_0, t_0, \vec{y}_0)\frac{\partial}{\partial y_0^\beta}\left(\tilde{\Omega}^{i\alpha}(t_0, t_0, \vec{y}_0)\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right)\right]$$

NB

$$\Rightarrow \tilde{\Omega}^{ik}(t_0+\Delta t, t_0, \vec{y}_0)$$

$$= \tilde{\Omega}^{ij}(t_0, t_0, \vec{y}_0)\left[\delta_j^{\ k} + \Delta t\ \tilde{\omega}_{jl}(t_0, t_0, \vec{y}_0)\frac{\partial \tilde{\Omega}^{lk}(t, t_0, \vec{y}_0)}{\partial t}\Big|_{t=t_0}\right]$$

$$= \tilde{\Omega}^{ij}(t_0, t_0, \vec{y}_0)\left[\delta_j^{\ k} + \Delta t\ \tilde{\omega}_{jl}(t_0, t_0, \vec{y}_0)\tilde{\Omega}^{\beta k}(t_0, t_0, \vec{y}_0)\frac{\partial}{\partial y_0^\beta}\left(\tilde{\Omega}^{l\alpha}(t_0, t_0, \vec{y}_0)\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right)\right.$$

$$\left. + \Delta t\ \underbrace{\tilde{\omega}_{jl}(t_0, t_0, \vec{y}_0)\tilde{\Omega}^{l\beta}(t_0, t_0, \vec{y}_0)}_{\delta_j^{\ \beta}}\frac{\partial}{\partial y_0^\beta}\left(\tilde{\Omega}^{k\alpha}(t_0, t_0, \vec{y}_0)\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right)\right]$$

$$= \tilde{\Omega}_0^{ij}\left[\delta_j^{\ k} + \Delta t\left\{\tilde{\omega}_{0\,jl}\tilde{\Omega}_0^{\beta k}\frac{\partial}{\partial y_0^\beta}\left(\tilde{\Omega}_0^{l\alpha}\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right) + \frac{\partial}{\partial y_0^j}\left(\tilde{\Omega}_0^{k\alpha}\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right)\right\}\right]$$

$$M_j^{\ k}$$

$$\therefore \det \tilde{\Omega}(t_0+\Delta t, t_0, \vec{y}_0)$$

$$= \det \tilde{\Omega}(t_0, t_0, \vec{y}_0)\left[1 + \Delta t\ \underbrace{\mathrm{tr}(M)}_{M_j^{\ j}}\right]$$

$$= \det \tilde{\Omega}(t_0, t_0, \vec{y}_0)\left[1 + \Delta t\left\{\underbrace{\tilde{\omega}_{0\,jl}\tilde{\Omega}_0^{\beta j}}_{\delta^\beta_l}\frac{\partial}{\partial y_0^\beta}\left(\tilde{\Omega}_0^{l\alpha}\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right) + \frac{\partial}{\partial y_0^j}\left(\tilde{\Omega}_0^{k\alpha}\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right)\right\}\right]$$

$$= \det \tilde{\Omega}(t_0, t_0, \vec{y}_0)\left[1 + \Delta t\left\{\frac{\partial}{\partial y_0^k}\left(\tilde{\Omega}_0^{l\alpha}\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right) + \frac{\partial}{\partial y_0^j}\left(\tilde{\Omega}_0^{k\alpha}\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right)\right\}\right]$$

$$= \det \tilde{\Omega}(t_0, t_0, \vec{y}_0)\left[1 + 2\Delta t\frac{\partial}{\partial y_0^l}\left(\tilde{\Omega}_0^{l\alpha}\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right)\right]$$

$$\therefore \left|\det \tilde{\Omega}(t_0+\Delta t, t_0, \vec{y}_0)\right|^{-1/2} = \left|\det \tilde{\Omega}(t_0, t_0, \vec{y}_0)\right|^{-1/2}\left[1 - \Delta t\frac{\partial}{\partial y_0^l}\left(\tilde{\Omega}_0^{l\alpha}\frac{\partial \tilde{H}(\vec{y}_0)}{\partial y_0^\alpha}\right)\right]$$

Hence,

$$\int d^{2n} y \, |\det \tilde{\Omega}(t_0+\Delta t, t_0, \bar{y}_0)|^{-1/2}$$

$$= \int d^{2n} y_0 \left[1 + \Delta t \frac{\partial}{\partial y_0^i} \left\{ \tilde{\Omega}_0^{ik} \frac{\partial \tilde{H}(\bar{y}_0)}{\partial y_0^k} \right\}\right] |\det \tilde{\Omega}(t_0, t_0, y_0)|^{-1/2} \times \left[1 - \Delta t \frac{\partial}{\partial y_0^\alpha} \left(\tilde{\Omega}_0^{l\alpha} \frac{\partial \tilde{H}(\bar{y}_0)}{\partial y_0^\alpha} \right)\right]$$

$$= \int d^{2n} y_0 \, |\det \tilde{\Omega}(t_0, t_0, y_0)|^{-1/2} \left[1 - (\Delta t)^2 \frac{\partial}{\partial y_0^i} \left\{ \tilde{\Omega}_0^{ik} \frac{\partial \tilde{H}(\bar{y}_0)}{\partial y_0^k} \right\} \times \frac{\partial}{\partial y_0^l} \left(\tilde{\Omega}_0^{l\alpha} \frac{\partial \tilde{H}(\bar{y}_0)}{\partial y_0^\alpha} \right)\right]$$

$$\simeq \int d^{2n} y_0 \, |\det \tilde{\Omega}(t_0, t_0, y_0)|^{-1/2}$$

[Neglecting terms containing $(\Delta t)^2$ & higher powers of Δt]

Hence, Liouville's th. is proved.

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16/9/05

$L(\vec{q}, \vec{u})$

A generic transformation

$$\tilde{q}^\alpha = \phi^\alpha(\vec{q}, \vec{u})$$

$$\tilde{u}^\alpha = \psi^\alpha(\vec{q}, \vec{u})$$

Under what condition do $\vec{\tilde{q}}$ and $\vec{\tilde{u}}$ satisfy the same eqns of motion as $\vec{q}$ and $\vec{u}$?

$$\underbrace{\frac{dq^\alpha}{dt} = u^\alpha}, \quad \frac{d}{dt}\frac{\partial L}{\partial u^\alpha} = \frac{\partial L}{\partial q^\alpha}$$

Want $\frac{d\tilde{q}^\alpha}{dt} = \tilde{u}^\alpha \Rightarrow \psi^\alpha(\vec{q}, \vec{u}) = \frac{d}{dt}\phi^\alpha(\vec{q}, \vec{u})$

$$= \frac{\partial \phi^\alpha}{\partial q^\beta} u_\beta + \frac{\partial \phi^\alpha}{\partial u^\beta} \cdot \frac{du^\beta}{dt}.$$

Express $\frac{du^\beta}{dt}$ in terms of $\vec{q}$ & $\vec{u}$ using

$$\frac{d}{dt}\frac{\partial L}{\partial u^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0$$

$\Rightarrow$ determines $\psi^\alpha(\vec{q}, \vec{u})$ for given $\vec{\phi}(\vec{q}, \vec{u})$ and $L(\vec{q}, \vec{u})$. [ψ^α isn't indep. & is det. in terms of ϕ^α & L]

What condn. do we need on $\phi^\alpha(\vec{q}, \vec{u})$ for the Euler-Lagrange eqn. to have the same form?

$$\frac{d}{dt}\frac{\partial L(\vec{\tilde{q}}, \vec{\tilde{u}})}{\partial \tilde{u}^\alpha} - \frac{\partial L(\vec{\tilde{q}}, \vec{\tilde{u}})}{\partial \tilde{q}^\alpha} = 0$$

A sufficient condn. for that to happen is:

$$L\left(\vec{\phi}(\vec{q}, \vec{u}), \frac{d}{dt}\vec{\phi}(\vec{q}, \vec{u})\right) = L(\vec{q}, \vec{u}) + \frac{d}{dt}K(\vec{q}, \vec{u})$$

for some $K(\vec{q}, \vec{u})$

$\hookrightarrow$ must hold without using Euler-Lagrange eqn.

(... $dq^\alpha = u^\alpha$ eqn.)

Action is a functional. We use $f(t)$ & then $\dot{q}(t)$ is known & so only q's are indep.

In Lag. formulation, the set ... α is made ...

Example

$$H = \sum_\alpha p_\alpha^2/2m + V(|\vec{q}|)$$

invariant under orthogonal rotation

$$|\vec{q}| = \sqrt{(q^1)^2 + (q^2)^2 + (q^3)^2}$$

$$u_\alpha = \frac{\partial H}{\partial p_\alpha} = p_\alpha/m$$

$$L = \sum_\alpha u^\alpha p_\alpha - H = \frac{1}{2} m \sum_\alpha (u_\alpha)^2 - V(|\vec{q}|)$$

Rotational symmetry: $\tilde{q}^\alpha = R^\alpha{}_\beta q^\beta$, $\boxed{R R^T = \mathbb{1}}$

(Adv. in Lag. framework — transfn. law of u^α is no longer indep., unlike in Ham. framework)

$$\tilde{u}^\alpha = R^\alpha{}_\beta \, dq^\beta/dt = R^\alpha{}_\beta u^\beta$$

$$L(\vec{\tilde{q}}, \vec{\tilde{u}}) = L(\vec{q}, \vec{u})$$

In this case $\vec{K}(\vec{u}, \vec{q}) = 0$

this has been possible bcos ϕ is a fn. of q^α only

Example 2 Infinitesimal time translation

$$\tilde{q}^\alpha = \underbrace{q^\alpha + \epsilon u^\alpha}_{\phi^\alpha(\vec{q}, \vec{u})} + O(\epsilon^2)$$

If ϕ has a non-trivial u^α dependence, there will be $\dot{u}^\alpha$ dependence, which can't be eliminated bcos u aren't allowed to use Euler-Lag. eqn. & $K \neq 0$ in general

$$\tilde{u}^\alpha = \frac{d\tilde{q}^\alpha}{dt} = \frac{dq^\alpha}{dt} + \epsilon \frac{du^\alpha}{dt} + O(\epsilon^2)$$

(for this u are to use eqn. of motion)

$$= u^\alpha - \epsilon/m \, \frac{\partial V}{\partial q^\alpha} \Rightarrow \psi^\alpha(\vec{q}, \vec{u})$$

Euler Lagrange eqn.

$$\frac{d}{dt}(m u^\alpha) + \frac{\partial V}{\partial q^\alpha} = 0$$

$$\frac{du^\alpha}{dt} = -\frac{1}{m}\frac{\partial V}{\partial q^\alpha}$$

$$L(\vec{\Phi}(\vec{q}, \vec{u}), \tfrac{d}{dt}\vec{\Phi}(\vec{q}, \vec{u}))$$

$$= L(\vec{q} + \epsilon \vec{u}, \, d\vec{q}/dt + \epsilon \, d\vec{u}/dt)$$

$$= L(\vec{q} + \epsilon \vec{u}, \, \vec{u} + \epsilon \, d\vec{u}/dt)$$

$$= \frac{1}{2} m \left(\vec{u} + \epsilon \frac{d\vec{u}}{dt}\right)^2 - V(|\vec{q} + \epsilon \vec{u}|)$$

$$= \frac{1}{2} m \vec{u}^2 + \epsilon m \vec{u} \cdot \frac{d\vec{u}}{dt} - V(|\vec{q}|) - \epsilon u^\alpha \frac{\partial V(|\vec{q}|)}{\partial q^\alpha} + O(\epsilon^2)$$

$$= L(\vec{q}, \vec{u}) + \epsilon \frac{d}{dt}\left[\frac{1}{2} m \vec{u}^2 - V(|\vec{q}|)\right]$$

In this case, $K(\vec{q}, \vec{u}) = \epsilon\left(\frac{1}{2} m \vec{u}^2 - V(|\vec{q}|)\right)$

$\vec{q} = \vec{f}(t)$

$S[\vec{f} + \delta\vec{f}] - S(\vec{f})$?

$\vec{f} + \delta\vec{f}$ doesn't in general satisfy Euler-Lag. eqn. replaced $S[\vec{\Phi} + \delta\vec{\Phi}]$

Consider an infinitesimal symmetry trs.

$$\tilde{q}^\alpha = q^\alpha + \epsilon\, \chi^\alpha(\vec{q}, \vec{u}) + O(\epsilon^2)$$

Suppose

$$L\left(\vec{q} + \epsilon \vec{\chi}(\vec{q}, \vec{u}),\ \frac{d\vec{q}}{dt} + \epsilon \frac{d}{dt}\chi(\vec{q}, \vec{u})\right)$$
$$= L(\vec{q}, \vec{u}) - \epsilon \frac{d}{dt} N(\vec{q}, \vec{u})$$

Then this is a symmetry transformation.

[In the limit $\epsilon \to 0$, the diff. must vanish]

<u>Claim</u> :- $Q = \frac{\partial L}{\partial u^\alpha} \chi^\alpha(\vec{q}, \vec{u}) + N(\vec{q}, \vec{u})$ is conserved. $\longrightarrow$ $\boxed{\text{Noether method}}$

$$\epsilon \left[\frac{\partial L}{\partial q^\alpha} \chi^\alpha(\vec{q}, \vec{u}) + \frac{\partial L}{\partial u^\alpha} \frac{d}{dt} \chi^\alpha(\vec{q}, \vec{u}) + \frac{d}{dt} N(\vec{q}, \vec{u})\right] = 0$$

$$\Rightarrow \left[\qquad '' \qquad\right] = 0 \quad \text{—(A)}$$

Now,
$$\frac{dQ}{dt} = \frac{d}{dt}\frac{\partial L}{\partial u^\alpha} \chi^\alpha + \underbrace{\frac{\partial L}{\partial u^\alpha}\frac{d\chi^\alpha}{dt} + \frac{dN}{dt}}_{-\frac{\partial L}{\partial q^\alpha}\chi^\alpha \ \text{(using (A))}}$$

$$= \left(\frac{d}{dt}\frac{\partial L}{\partial u^\alpha} - \frac{\partial L}{\partial q^\alpha}\right)\chi^\alpha$$

$= 0$, when eqns of motion are satisfied.

(While testing whether Q is conserved, u must use eqns of motion. Sym. transfn. has to be tested without using eqns of motion)

Example. Time translation:

$$X^\alpha(\vec{q}, \vec{u}) = u^\alpha, \quad N = -\frac{1}{2} m \vec{u}^2 + v(|\vec{q}|)$$

$$\therefore \quad Q = \frac{\partial L}{\partial u^\alpha} X^\alpha(\vec{q}, \vec{u}) + N(\vec{q}, \vec{u})$$

$$\left(= m u^\alpha \delta_{\beta\alpha} u^\alpha - \frac{1}{2} m \vec{u}^2 + v(|\vec{q}|)\right)$$

$$L = \frac{1}{2} m \vec{u}^2 - v(|\vec{q}|)$$

$$= m \vec{u}^2 + N$$

$$= \frac{1}{2} m \vec{u}^2 + v(|\vec{q}|) \rightarrow \text{Hamiltonian}$$

$$L = \frac{1}{2} m \vec{u}^2 - V(|\vec{q}|)$$

Rotation: $\tilde{q}^\alpha = q^\alpha + R^\alpha{}_\beta q^\beta$

For infinitesimal rotation, it will take the form $\tilde{q}^\alpha = q^\alpha + \epsilon \omega_{\alpha\beta} q^\beta$

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}$$

linear combn. of 3 antisymm. matrices

$$\omega_{\alpha\beta} = a_1 (T_1)_{\alpha\beta} + a_2 (T_2)_{\alpha\beta} + a_3 (T_3)_{\alpha\beta},$$

where $T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, $T_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

& $T_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$a_1, a_2 a_3$ are arbitrary.

$$X^\alpha(\vec{q}, \vec{u}) = \omega_{\alpha\beta} q^\beta, \quad N(\vec{q}, \vec{u}) = 0$$

[∵ it is true for finite transfs. that $N=0$, then it must be true for infinitesimal transfs.]

$$Q = m u^\alpha \omega_{\alpha\beta} q^\beta = a_1 L_1 + a_2 L_2 + a_3 L_3$$

conserved charges $\boxed{L_i = m (T_i)_{\alpha\beta} u^\alpha q^\beta}$

Angular momentum

119) What we have achieved is:—

Suppose we are given $L(\vec{q}, \vec{u})$ & an infinitesimal symmetry trs. $\vec{q} \to \vec{q} + \phi(\vec{q}, \vec{u})$

going via the Ham. framework

Construct ① $H(\vec{q}, \vec{p})$

use Noether method to construct a conserved quantity Q.

② Symmetry trs:—

$$\tilde{q}^\alpha = q^\alpha + \epsilon\, \phi^\alpha(\vec{q}, \vec{u})$$

$$\tilde{u}^\alpha = u^\alpha + \epsilon \frac{\partial \phi^\alpha}{\partial q^\beta} u^\beta + \epsilon \frac{\partial \phi^\alpha}{\partial u^\beta} \, du^\beta/dt$$

$$\Rightarrow \psi^\alpha(\vec{q}, \vec{u})$$

③ Knowing the relation betn $\vec{u}$ & $\vec{p}$, write these as

$$\tilde{q}^\alpha = q^\alpha + \epsilon F^\alpha(\vec{q}, \vec{p})$$

$$\tilde{p}_\alpha = p_\alpha + \epsilon G_\alpha(\vec{q}, \vec{p})$$

(we get a) conserved charge Q' such that

$$F^\alpha = \{q^\alpha, Q'\}$$

$$G_\alpha = \{p_\alpha, Q'\}$$

Are Q & Q' ~~are~~ the same?

→ There must be some relation betn. Q & Q'. (In the e.g. studied, they were the same).

We would like to show that Q & Q' are the same (upto a sign or a normalisation const.)

We want to show that

$$\{q^\alpha, Q\} = F^\alpha$$

$$\{p_\alpha, Q\} = G_\alpha$$

Who tells u there is the same conserved charge associated with the same sym. transfs, (as we use 2 diff methods)?

→ Need to prove it.

Begin with :-

$$L\left(\vec{q}+\epsilon\vec{X}, \frac{d\vec{q}}{dt}+\epsilon\frac{d\vec{X}}{dt}\right) = L(\vec{q},\vec{u}) - \epsilon\frac{dN}{dt}$$

$$\Rightarrow \epsilon\left[\frac{\partial L}{\partial q^\alpha}X^\alpha + \epsilon\frac{\partial L}{\partial u^\alpha}\frac{dX^\alpha}{dt} + \frac{dN}{dt}\right] = 0$$

$$\Rightarrow \frac{\partial L}{\partial q^\alpha}X^\alpha(\vec{q},\vec{u}) + \frac{\partial L}{\partial u^\alpha}\frac{\partial X^\alpha}{\partial q^\beta}u^\beta + \frac{\partial L}{\partial u^\alpha}\frac{\partial X^\alpha}{\partial u^\beta}\frac{du^\beta}{dt} + \frac{\partial N}{\partial q^\alpha}u^\alpha + \frac{\partial N}{\partial u^\beta}\frac{du^\beta}{dt} = 0$$

(This eqn has to hold without using Euler-Lag. eqns & we then can't manipulate du^β/dt & so the coeff. of u^β should be 0)

$$\Rightarrow \boxed{\frac{\partial L}{\partial u^\alpha}\frac{\partial X^\alpha}{\partial u^\beta} + \frac{\partial N}{\partial u^\beta} = \quad \text{for every } \beta}$$

$$\& \quad \boxed{\frac{\partial L}{\partial q^\alpha}X^\alpha(\vec{q},\vec{u}) + \frac{\partial L}{\partial u^\alpha}\frac{\partial X^\alpha}{\partial q^\beta}u^\beta = 0}$$

[It's a constraint on X^α]

(Given a X^α, we see if we can find a N such that the above eqns are satisfied — if satisfied, X^α is the generator of a sym. transfn)

(Check that these eqns are satisfied for our e.g. on time translation & rotation)

$$Q = \frac{\partial L}{\partial u^\alpha}X^\alpha + N$$
$$= p_\alpha X^\alpha + N$$

Consider an arbitrary variation in the phase space

$$q^\alpha \to q^\alpha + \delta q^\alpha, \quad u^\alpha \to u^\alpha + \delta u^\alpha, \quad p_\alpha \to p_\alpha + \delta p_\alpha$$

using chain rule

$$\therefore \delta Q = \delta p_\alpha X^\alpha + p_\alpha \delta X^\alpha + \delta N$$
$$= \delta p_\alpha X^\alpha + p_\alpha\left[\frac{\partial X^\alpha}{\partial q^\beta}\delta q^\beta + \frac{\partial X^\alpha}{\partial u^\beta}\delta u^\beta\right] + \frac{\partial N}{\partial q^\beta}\delta q^\beta + \frac{\partial N}{\partial u^\beta}\delta u^\beta$$

121) Now, $\frac{\partial N}{\partial u^\beta} = -\frac{\partial L}{\partial u^\alpha}\frac{\partial X^\alpha}{\partial u^\beta} = -p_\alpha \frac{\partial X^\alpha}{\partial u^\beta}$

$$\therefore \quad \delta\varphi = \delta p_\alpha X^\alpha + p_\alpha \frac{\partial X^\alpha}{\partial q^\beta}\delta q^\beta + \frac{\partial N}{\partial q^\beta}\delta q^\beta$$

$$= X^\alpha \delta p_\alpha + \left[p_\alpha \frac{\partial X^\alpha}{\partial q^\beta} + \frac{\partial N}{\partial q^\beta}\right]\delta q^\beta$$

Now, considering φ as fn. of q^α & p_α, we get,

$$\left.\frac{\partial \varphi}{\partial p_\alpha}\right|_{\vec{q},\vec{p}} = X^\alpha \quad \& \quad \frac{\partial \varphi}{\partial q^\beta} = \left(p_\alpha \frac{\partial X^\alpha}{\partial q^\beta} + \frac{\partial N}{\partial q^\beta}\right)$$

Consider infinitesimal transfn generated by φ:-

$$\delta q^\alpha = \epsilon\{q^\alpha, \varphi\} = \epsilon \frac{\partial \varphi}{\partial p_\alpha} = \epsilon X^\alpha$$ get this correctly

$$\delta p_\beta = \epsilon\{p_\beta, \varphi\} = -\epsilon \frac{\partial \varphi}{\partial q^\beta}$$

$$= -\epsilon\left\{p_\alpha \frac{\partial X^\alpha}{\partial q^\beta} + \frac{\partial N}{\partial q^\beta}\right\}$$

(For the coordinate variable, φ & φ' generate the same transfn.)

Ex: (Check u get the same thing)
Check that this is correct.

[Actually, if the first is correct, we are guaranteed that the 2nd must also be correct — otherwise it won't preserve $u^\alpha = \frac{dq^\alpha}{dt}$, as req'd. in Lag. formulation]

Doubt

① H.O. example. $p = \pm\sqrt{---}$ → relation betn. p & q is still there.

But we have taken only $p = +\sqrt{---}$ (consider local behaviour)

② Non-zero e.value of $\frac{\partial g_\alpha}{\partial u^\beta}$ → Brute force not in generic case

(after variation)

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③ Passive picture → $H(\vec{x}) - H(\vec{y})$ → $\tilde{H}(\vec{y}) - H(\vec{y})$ → what does it mean

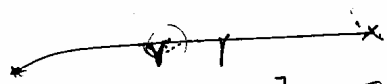
④ $S(\phi(t)) = S(f(t)) + K(f(t), \dot{f}(t))\Big|_{t_1}^{t_2}$

& $\delta S(\phi(t)) = \delta S(f(t))$

Does this imply that $L(\phi(t), \dot{\phi}(t)) = L(f(t), \dot{f}(t)) + \frac{dK}{dt}$?

$S[\phi(t)] = S[f(t)] + \cdots$

$S[\phi(t) + \delta\phi(t)] = S[f(\phi(t) + \delta\phi(t))] + \cdots$



$\Delta S[\phi] = S[f(\phi)] - S[\phi]$

$\frac{d}{dt}\frac{\partial L}{\partial u^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0$

$g_\alpha(\vec{q}, \vec{u})$

$\Rightarrow \frac{\partial g_\alpha}{\partial q^\beta} u^\beta + \frac{\partial g_\alpha}{\partial u^\beta}\cdot\frac{du^\beta}{dt} - \frac{\partial L}{\partial q^\alpha} = 0$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\frac{\partial g_\alpha}{\partial u^\beta}\frac{du^\beta}{dt} = -\frac{\partial g_\alpha}{\partial q^\beta} u^\beta + \frac{\partial L}{\partial q^\alpha}$

$M^{-1} = \frac{1}{\det M}\begin{pmatrix} \cdot \end{pmatrix}$

$\det M = 0$

$M^\alpha{}_\beta(\vec{q}, \vec{u})\frac{du^\beta}{dt} = F_\alpha(\vec{q}, \vec{u})$

$M\cdot\frac{d\vec{u}}{dt} = \vec{F}$

$\frac{d\vec{u}}{dt} = M^{-1}\vec{F}$

$M = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \; \lambda_n \end{pmatrix}$

$M^{-1} = \begin{pmatrix} \lambda_1^{-1} & & \\ & \lambda_2^{-1} & \\ & & \ddots \; \lambda_n^{-1} \end{pmatrix}$

$$\{q^\alpha, Q\} = \frac{\partial Q}{\partial p_\alpha} = x^\alpha$$

$$\{p_\alpha, Q\} = -\frac{\partial Q}{\partial q^\alpha} = \ldots$$

$$\delta u^\beta =$$

$$\downarrow$$

$$\delta p_\beta$$

$$\delta f(\ldots) = \epsilon \; F^{\circ}(\ldots)$$

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$L(\vec{q}, \vec{u})$

Imagine we define some new coord. $\tilde{q}^\alpha = f^\alpha(\vec{q})$ (not a fn of u^α)

What is the Lagrangian that describes the eqns for $\vec{\tilde{q}}$ & $\vec{\tilde{u}} = \frac{d\vec{\tilde{q}}}{dt}$?

$$\frac{dq^\alpha}{dt} = u^\alpha, \quad \frac{d}{dt}\frac{\partial L}{\partial u^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0$$

We want to look for some fn. $\tilde{L}(\vec{\tilde{q}}, \vec{\tilde{u}})$ such that $\vec{\tilde{q}}$ & $\vec{\tilde{u}}$ eqns are

$$\frac{d\tilde{q}^\alpha}{dt} = \tilde{u}^\alpha, \quad \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \tilde{u}^\alpha} - \frac{\partial \tilde{L}}{\partial \tilde{q}^\alpha} = 0$$

We are not looking for symmetries in this case.

First let us see how $\frac{d\tilde{q}^\alpha}{dt} = \tilde{u}^\alpha$ can be satisfied :-

$$\tilde{u}^\alpha = \frac{d\tilde{q}^\alpha}{dt} = \frac{d}{dt} f^\alpha(\vec{q}) = \frac{\partial f^\alpha}{\partial q^\beta}\frac{dq^\beta}{dt} = \frac{\partial f^\alpha}{\partial q^\beta} u^\beta$$

$$\Rightarrow \tilde{u}^\alpha = \frac{\partial f^\alpha}{\partial q^\beta} u^\beta$$

These eqns guarantee that $\frac{dq^\alpha}{dt} = u^\alpha$ & $\frac{d\tilde{q}^\alpha}{dt} = \tilde{u}^\alpha$ look the same.

$$\frac{d}{dt}\frac{\partial L}{\partial u^\alpha} - \frac{\partial L}{\partial q^\alpha} = \frac{d}{dt}\left(\frac{\partial L}{\partial \tilde{q}^\beta}\frac{\partial \tilde{q}^\beta}{\partial u^\alpha} + \frac{\partial L}{\partial \tilde{u}^\beta}\frac{\partial \tilde{u}^\beta}{\partial u^\alpha}\right) - \frac{\partial L}{\partial \tilde{q}^\beta}\frac{\partial \tilde{q}^\beta}{\partial q^\alpha} - \frac{\partial L}{\partial \tilde{u}^\beta}\frac{\partial \tilde{u}^\beta}{\partial q^\alpha}$$

$L(\vec{q}, \vec{u})$ fns of $\tilde{q}^\alpha$ & $\tilde{u}^\alpha$

$$= \frac{d}{dt}\left(0 + \frac{\partial L}{\partial \tilde{u}^\beta}\frac{\partial f^\beta}{\partial q^\alpha}\right) - \frac{\partial L}{\partial \tilde{q}^\beta}\frac{\partial f^\beta}{\partial q^\alpha} - \frac{\partial L}{\partial \tilde{u}^\beta}\frac{\partial^2 f^\beta}{\partial q^\gamma \partial q^\alpha}u^\gamma$$

$$= \frac{d}{dt}\frac{\partial L}{\partial \tilde{u}^\beta}\frac{\partial f^\beta}{\partial q^\alpha} + \cancel{\frac{\partial L}{\partial \tilde{u}^\beta}\frac{\partial^2 f^\beta}{\partial q^\alpha \partial q^\gamma}\frac{dq^\gamma}{dt}} - \frac{\partial L}{\partial \tilde{q}^\beta}\frac{\partial f^\beta}{\partial q^\alpha} - \cancel{\frac{\partial L}{\partial \tilde{u}^\beta}\frac{\partial^2 f^\beta}{\partial q^\gamma \partial q^\alpha}u^\gamma}$$

$$= \frac{\partial f^\beta}{\partial q^\alpha}\left(\frac{d}{dt}\frac{\partial L}{\partial \tilde{u}^\beta} - \frac{\partial L}{\partial \tilde{q}^\beta}\right)$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial u^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0$$

[$\frac{\partial f^\beta}{\partial q^\alpha}$ must be a non-singular matrix]

$$\longleftrightarrow \frac{d}{dt}\frac{\partial L}{\partial \tilde{u}^\alpha} - \frac{\partial L}{\partial \tilde{q}^\alpha} = 0$$

$$\therefore \tilde{L}(\vec{\tilde{q}}, \vec{\tilde{u}}) = L(\vec{q}, \vec{u}) \Rightarrow \tilde{L}\left(\vec{f}(\vec{q}), \frac{\partial \vec{f}}{\partial q^\beta}u^\beta\right) = L(\vec{q}, \vec{u})$$

functional forms of L & $\tilde{L}$ are different

So, when we change coordinates we just take the same lagrangian & write ~~replace~~ $\vec{q}$ & $\vec{u}$ in terms of $\vec{\tilde{q}}$ & $\vec{\tilde{u}}$.

$\tilde{q}^\alpha = f^\alpha(\vec{q})$

Suppose $q^\alpha(t) = F^\alpha(t)$ and $u^\alpha(t) = \frac{d}{dt}F^\alpha(t)$ is a soln

$$0 = \int^{t_2} L\left(\vec{F}(t), \frac{d\vec{F}}{dt}\right) dt \text{ is extremized}$$

$$S = \int_{t_1}^{t_2} \tilde{L}\left(\vec{f}(\vec{F}(t)), \frac{d}{dt}\vec{f}(\vec{F}(t))\right) dt$$

$\Rightarrow$ Extremization of first form of S implies extremization of second form of S. ($\because$ they are identical).

follows from the defn of $\tilde{L}(\vec{q}, \vec{\dot{q}}) = L(\vec{q}, \vec{\dot{q}})$

If we consider $S = \int_{t_1}^{t_2} \tilde{L}(\vec{q}(t), \frac{d\vec{q}(t)}{dt})$

then if $\vec{q} = \vec{F}(t)$ is a solution of eqns of motion derived from $L(\vec{q}, \vec{\dot{q}})$, then $\vec{\tilde{q}} = \vec{f}(\vec{F}(t))$ is a solution of the eqns of motion derived from $\tilde{L}(\vec{\tilde{q}}, \vec{\dot{\tilde{q}}})$.

Define $\phi(\vec{\tilde{q}}, \vec{\dot{\tilde{q}}})$ &

take $S = \int_{t_1}^{t_2} \phi\left(\vec{f}(\vec{F}(t)), \frac{d}{dt}\vec{f}(\vec{F}(t))\right)$

Then we can can identify $\phi(\vec{\tilde{q}}, \vec{\dot{\tilde{q}}})$ with $\tilde{L}(\vec{\tilde{q}}, \vec{\dot{\tilde{q}}})$, becos ϕ acts as the Lagrangian for the arguments $\vec{f}(\vec{F}(t))$ & $\frac{d}{dt}\vec{f}(\vec{F}(t))$

E.g. $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$

Introduce polar coordinates

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

In this case, $u_x = \frac{dx}{dt} = \frac{dr}{dt} \sin\theta \cos\phi + r \cos\theta \frac{d\theta}{dt} \cos\phi - r \sin\theta \sin\phi \frac{d\phi}{dt}$

$= u_r \sin\theta \cos\phi + u_\theta r \cos\theta \cos\phi - u_\phi r \sin\theta \sin\phi$

Similarly, $u_y = \ldots$, $u_z = \ldots$

$$\tilde{L}(r, \theta, \phi, u_r, u_\theta, u_\phi) = L(x, y, z, u_x, u_y, u_z)$$
$$= \frac{1}{2} m \left(u_r^2 + r^2 u_\theta^2 + r^2 \sin^2\theta\, u_\phi^2\right) - V(r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$$

Once we have got the Lagrangian, it is easy to change coordinates

Suppose we have $L(\vec{q}, \vec{u})$.
Suppose q^i's are subject to certain constraints:-

$\phi_1 = 0, \phi_2 = 0, \ldots$

$$\phi_1(\vec{q}) = 0$$
$$\phi_2(\vec{q}) = 0$$
$$\vdots$$
$$\phi_k(\vec{q}) = 0$$

ϕ's are continuous well defined fns of q^i's.

Q.) What are the modified eqns of motion?

Physical realization: Add to L a term $-\frac{1}{\epsilon} \sum_{i=1}^{k} (\phi_i(\vec{q}))^2$ & take $\epsilon \to 0$ limit

($\frac{1}{\epsilon}$ is taken because we are putting an infinite potential in a dirn we try to go away from the surface — But these potential is zero as long as we remain on the surface — a particle can't have infinite energy)

(In Ham., this will correspond to adding a term $+\frac{1}{\epsilon} \sum_i (\phi_i)^2$ to the Hamiltonian)

127 Consider a more general type of constraints

Assuming these to be smooth fns. – no singularities etc.

$$\phi_1(\vec{q}) = \hat{\theta}_1$$
$$\phi_2(\vec{q}) = \hat{\theta}_2$$
$$\vdots$$
$$\phi_k(\vec{q}) = \hat{\theta}_k$$

fixed numbers

This subspace is parametrized by $N-k$ coordinates $\theta_1, \theta_2, \ldots, \theta_{N-k}$.

We need $n-k$ coord. to describe the particle motion in $(n-k)$-dim ~~subspace~~ of n-dim. space

~~$q^\alpha = f^\alpha(\theta_1, \ldots, \theta_{N-k})$~~

$$q^\alpha = f^\alpha(\theta_1, \ldots, \theta_{N-k}; \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k)$$

Changing the ~~[illegible]~~ $\hat{\theta}_i$'s will change the subspace we are considering – ~~[illegible]~~ The ~~[illegible]~~ diff. ~~values~~ sets of values of $\hat{\theta}_i$'s give a family of surfaces.

Consider $\theta_1, \ldots, \theta_{N-k}$ & $\hat{\theta}_1, \ldots, \hat{\theta}_k$ as the new coordinates.

Original $(N-k)$ dimensional subspace corresponds to

$$\hat{\theta}_1 = 0, \ \hat{\theta}_2 = 0, \ldots, \hat{\theta}_k = 0.$$

Now, as we change the θ_α's, we are moving on this subspace.

Changing $\hat{\theta}_i$ moves you away from the subspace.

$$\tilde{L}(\vec{\theta}, \vec{\hat{\theta}}, \vec{U}, \vec{\hat{U}}) = L(\vec{q}, \vec{u})$$

not necessarily orthogonal

$\tilde{L}$: One set of variables are along the subspace, while the other set is transverse

$$\Rightarrow \hat{L}(\vec{\theta}, \vec{\hat{\theta}}, \vec{U}, \vec{\hat{U}}) - \frac{1}{\epsilon} \sum_{i=1}^{k} (\hat{\theta}_i)^2$$

Eqns of motion

$$dq^\alpha/dt = v^\alpha, \quad \frac{d\hat{q}^\alpha}{dt} = \hat{v}^\alpha \quad \text{—— (1)}$$

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial v^\alpha} - \frac{\partial \tilde{L}}{\partial q^\alpha} = 0$$

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \hat{v}^i} - \frac{\partial \tilde{L}}{\partial \hat{q}^i} + \frac{2}{\epsilon}\hat{q}_i = 0 \quad \text{—— (2)}$$

$\tilde{L}$ is a smooth fn. locally ϕ_i is a smooth fns, until any cond. or velocity doesn't become infinite

$$\Downarrow$$

$$\hat{q}_i = -\epsilon_2\left[\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \hat{v}^i} - \frac{\partial \tilde{L}}{\partial \hat{q}^i}\right]$$

$$\Rightarrow \hat{q}_i \to 0 \quad \text{as } \epsilon \to 0$$

[So, these eqns are equiv. to setting $\hat{q}_i = 0$]

$$\hat{v}^i = \frac{d\hat{q}^i}{dt} \to 0 \quad \text{as } \epsilon \to 0$$

So, we forget about the eqns (1) & (2).

$\therefore$ we have, $\frac{d}{dt}\frac{\partial \tilde{L}}{\partial v^\alpha} - \frac{\partial \tilde{L}}{\partial q^\alpha} = 0$, $\frac{dq^\alpha}{dt} = v^\alpha$

& set $\hat{q}_i = 0$, $\hat{v}_i = 0$.

If we define

$$\bar{L}(\vec{q}, \vec{v}) = \tilde{L}(\vec{q}, \vec{\hat{q}} = 0, \vec{v}, \vec{\hat{v}} = 0),$$

then the Euler-Lagrange eqns takes the form

$$\frac{d}{dt}\frac{\partial \bar{L}}{\partial v^\alpha} - \frac{\partial \bar{L}}{\partial q^\alpha} = 0.$$

While calculating $\frac{\partial \tilde{L}}{\partial v^\alpha}$ & $\frac{\partial \tilde{L}}{\partial q^\alpha}$, we are supposed to keep $\hat{q}_i$ & $\hat{v}_i$ fixed & at the end put $\hat{q}_i$ & $\hat{v}_i$ equal to zero — this is equiv. to putting $\hat{q}_i$ & $\hat{v}_i$ zero from the beginning

129) $\tilde{L}(\vec{Q}, \hat{\vec{Q}}, \vec{U}, \hat{\vec{U}})$

$= L\left(\vec{q} = \vec{f}(Q_1, \ldots Q_{N-k}, \hat{Q}_1, \ldots, \hat{Q}_k), \vec{u} = \ldots\right)$

$\bar{L} = \tilde{L}(\vec{Q}, \hat{\vec{Q}} = 0, \vec{U}, \hat{\vec{U}} = 0)$

$= L\left(\vec{q} = \vec{f}(Q_1, \ldots, Q_{N-k}, \hat{Q}_1 = 0, \ldots, \hat{Q}_k = 0), \vec{u} = \ldots\right)$

Q_i's were the coordinates of the subspace

~~$\Rightarrow Q_1, Q_{N-k}$~~

$\Rightarrow Q_1, \ldots, Q_{N-k}$ are parameters labelling the subspace

$\phi_1(\vec{q}) = 0, \ \phi_2(\vec{q}) = 0, \ldots, \phi_k(\vec{q}) = 0$

Prescription

Solve the constraints

$\phi_i(\vec{q}) = 0, \quad i = 1, \ldots, k$

parametrically.

$\Rightarrow q^\alpha = f^\alpha(Q^1, \ldots, Q^{N-k})$

$\because \hat{Q}_i$'s have been put to zero anyway

In $L(\vec{q}, \vec{u})$, put $q^\alpha = f^\alpha(\vec{Q})$, $u^\alpha = \frac{\partial f^\alpha}{\partial Q^\beta} U^\beta$

to get $\bar{L}(\vec{Q}, \vec{U})$.

$\bar{L}$ describes the dynamics of the constrained system.

E.g. Consider a system with coordinates (x, y) and $L(x, y, u_x, u_y) = \frac{1}{2} m(u_x^2 + u_y^2) + mgy$

subject to a constraint $y = ax^3 + bx^2 + c$.

↳ nic.

(Let us choose the parameter such that)

(130)

$x = \varphi$

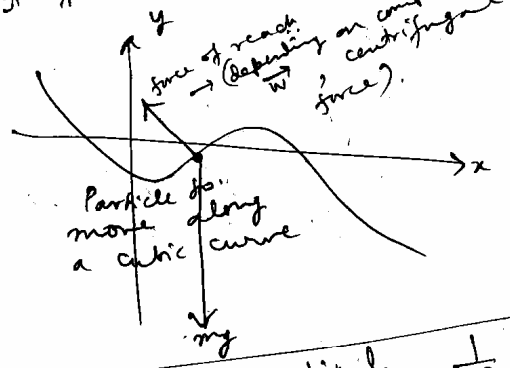
$y = a\varphi^3 + b\varphi^2 + c$

(x itself can be chosen as the parameter, as an algebraic eqn is given connecting x & y)

$u_x = \frac{dx}{dt} = u_\varphi$

$u_y = dy/dt = (3a\varphi^2 + 2b\varphi)\, u_\varphi$

$\tilde{L}(\varphi, u_\varphi) = \frac{1}{2} m u_\varphi^2 \{1 + (3a\varphi^2 + 2b\varphi^2)\} + mg(a\varphi^3 + b\varphi^2 + c)$



We have used the additional force due to constraint by taking an extra potential

Extra potential $\frac{1}{\epsilon} \sum_i (\phi_i(\vec{q}))^2$

Calculating the additional force due to constraint $\xRightarrow{\text{this implies}}$ the extra force $F_\alpha = -\frac{2}{\epsilon} \sum_i \phi_i(\vec{q}) \frac{\partial \phi_i(\vec{q})}{\partial q^\alpha}$

$\hat{\varphi}_i$ eq. $\Rightarrow \hat{\varphi}_i = -\frac{\epsilon}{2}\left(\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \hat{U}_i} - \frac{\partial \tilde{L}}{\partial \hat{\varphi}_i}\right)$ $(\because \phi_i = \hat{\varphi}_i)$

$$\therefore \boxed{F_\alpha = \sum_i \left(\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \hat{U}_i} - \frac{\partial \tilde{L}}{\partial \hat{\varphi}_i}\right)\frac{\partial \phi_i}{\partial q^\alpha}}$$

Suppose u take $(\phi_i)^4$ → not linear in ϵ It is $\epsilon^{1/3}$

$[\phi_i(\vec{q})]^2$ has the adv. ~~$\phi_i(\vec{q})$~~ that the force is prop. to $\phi_i(\vec{q})$

though we have taken an infinite potential, we are getting a finite force — an actual force

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① Doubt → $(\phi_i)^q$ → singular

② In $L(\vec{q}, \vec{u}) + \frac{1}{\epsilon}\sum \phi_i^2(q)$, q indep.

$$\delta u^\beta = \epsilon \frac{\partial X^\alpha}{\partial q^\beta} u^\beta + \epsilon \frac{\partial X^\alpha}{\partial u^\beta} \frac{du^\beta}{dt}$$

$$= \epsilon \left(\frac{\partial X^\alpha}{\partial q^\beta} u^\beta + \frac{\partial X^\beta}{\partial u^\beta} \underbrace{\frac{d}{dt} \frac{\partial H}{\partial p^\beta}} \right)$$

$$\frac{\partial^2 H}{\partial q^\gamma \partial p^\beta} \frac{\partial H}{\partial p^\gamma} + \frac{\partial^2 H}{\partial p^\gamma \partial p^\beta} \left(\frac{\partial H}{\partial q^\gamma} \right)$$

$$\delta u^\beta = \delta \left(\frac{\partial H}{\partial p^\beta} \right) = \frac{\partial^2 H}{\partial q^\alpha \partial p^\beta} \delta q^\alpha + \frac{\partial^2 H}{\partial p^\alpha \partial p^\beta} \delta p^\beta$$

Now $H = \sum p_\alpha q^\alpha - L$

$$\therefore \frac{\partial H}{\partial p_\beta} = q^\beta - \frac{\partial L}{\partial p_\beta}$$

$$\frac{\partial^2 H}{\partial p_\gamma \partial p_\beta} = - \frac{\partial^2 L}{\partial p_\gamma \partial p_\beta} \quad \& \quad \frac{\partial^2 H}{\partial q^\gamma \partial p_\beta} = \delta_\gamma{}^\beta - \frac{\partial^2 L}{\partial q^\gamma \partial p_\beta}$$

$$\frac{\partial L}{\partial q^\gamma} = \frac{dp_\gamma}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial q^\gamma}\right)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{\partial}{\partial p_\gamma} \frac{\partial H}{\partial p_\beta} = \frac{\partial u^\beta}{\partial p_\gamma} \qquad \frac{\partial}{\partial q^\alpha}(u^\beta)$$

$$\delta u^\beta = \epsilon X^\alpha \frac{\partial^2 H}{\partial q^\alpha \partial p_\beta} + \frac{\partial^2 H}{\partial p^\alpha \partial p^\beta} \delta \phi_\beta$$

$$\Rightarrow \epsilon \frac{\partial X^\alpha}{\partial q^\beta} u^\beta + \epsilon \frac{\partial X^\beta}{\partial u^\beta} \frac{\partial^2 H}{\partial q^\gamma \partial p^\beta} \frac{\partial H}{\partial p^\gamma} - \frac{\partial H}{\partial q^\gamma} \frac{\partial^2 H}{\partial p^\gamma \partial p_\beta}$$

$$= \epsilon X^\alpha \frac{\partial^2 H}{\partial q^\alpha \partial p_\beta} + \frac{\partial^2 H}{\partial p^\alpha \partial p_\beta} \delta p_\beta$$

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Problem

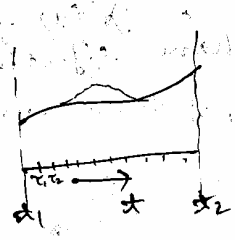
$L(\vec{q}, \vec{u})$ $\quad q^\alpha : \alpha = 1, 2, \ldots, N$

Constraints $\phi_i(\vec{q}) = 0$

Q.) How do we derive the eqns of motion?

$$\Rightarrow S = \int_{t_1}^{t_2} L(\vec{q}(t), \vec{u}_1 = d\vec{q}/dt)\, dt$$

& extremize S with respect to $\vec{q}(t)$ (subject to $\vec{q}(t)$ doesn't change at boundaries)



Discretize time:

$\vec{q}(t) \overset{\text{equiv.}}{\underset{\text{to specifying}}{\longleftrightarrow}} \vec{q}(\tau_1), \vec{q}(\tau_2), \ldots$

Consider S as $S \rightarrow$ ordinary fn. of $\vec{q}(\tau_1), \vec{q}(\tau_2), \ldots$

(approx. functional S by an ordinary fn. given these variables)

(not a functional)

(U know the fn. $\vec{q}(t)$ reasonably accurately. As the intervals are smaller & smaller, u know better & better)

Extremization of S

$\longleftrightarrow$ Extremization w.r.t. $\vec{q}(\tau_1), \vec{q}(\tau_2), \ldots$

In this picture, S is a multivariable fn. for N q's & k intervals we have Nk variables.

Consider a fn. of n variables $f(x_1, x_2, \ldots, x_n)$.

Extremize f subject to a set of constraints :-

k eqns. $\leftarrow$ $\boxed{g_1(\vec{x}) = 0,\ g_2(\vec{x}) = 0, \ldots}$ — ①

Standard way to solve this problem is using Lagrange multiplier

δf + some arbitrary linear combn. of δg_i's $= 0$

$$\delta f + \sum_s \lambda_s \delta g_s = 0$$

$$\Rightarrow \sum_i \frac{\partial f}{\partial x^i} \delta x^i + \sum \lambda_s \frac{\partial g_s}{\partial x^i} \delta x^i = 0$$

$\delta f = 0$ will give too many condns, all of which can't be subject to the constraints

$$\Rightarrow \boxed{\frac{\partial f}{\partial x^i} + \sum_s \lambda_s \frac{\partial g_s}{\partial x^i} = 0} \quad (2)$$

$\rightarrow n$ eqns.

At this stage λ_s is arbitrary — not actually

But λ_s's are fixed due to

We regard (1) & (2) as $(n+k)$ eqns in $n+k$ variables $x_1, x_2, \ldots, x_n, \lambda_1, \ldots, \lambda_k$ & solve them.

We have to insert/introduce a $\lambda^i(t)$ for each interval (bcos we have effectively Mk variables & constraints, if there were M constraint eqns)

$$\boxed{\delta S + \sum_{i=1}^{K} \int dt\, \lambda^i(t)\, \delta\phi^i(\vec{q}(t)) = 0 \qquad \phi^i(\vec{q}(t)) = 0}$$

this corresponds to summing over all time intervals

In the discrete version, we have to introduce $\lambda^i(\tau_1), \lambda^i(\tau_2), \ldots$

Suppose there are M intervals.

$\therefore$ no. of q^α's : MN

M intervals, N q^α originally

no. of constraints : MK for each $q^\alpha(\tau_i)$

\# of variational eqns : NM

\# of λ's $= MK$

this is equal to the no. of variables

Everything gets multiplied by M

Consider a new Lagrangian

$$L'(\vec{q}, \vec{u}, \vec{\lambda}) = L(\vec{q}, \vec{u}) + \sum_{i=1}^{k} \lambda_i \phi^i(\vec{q})$$

think of these as new coordinates $\lambda^1, \ldots, \lambda^k$

For the λ variables, L' depends only on coordinates, but not on the corresponding ~~variables~~ velocities — so L' is a bit strange

$$S' = \int_{t_1}^{t_2} dt\, L' = \underbrace{\int_{t_1}^{t_2} dt\, L\left(\vec{q}, \frac{d\vec{q}}{dt}\right)}_{= S} + \sum_{i=1}^{k} \int dt\, \lambda_i(t)\, \phi_i(\vec{q}(t))$$

$$\therefore \delta S' = 0$$

$$\Rightarrow 0 = \delta S + \sum_{i=1}^{k} \int dt \left[\delta\lambda_i(t)\, \phi_i(\vec{q}(t)) + \lambda_i\, \delta\phi_i(t)\right]$$

λ_i eqn. of motion $\Rightarrow \phi_i(\vec{q}(t)) = 0$

q^α eqn. of motion :-

$$\delta S + \sum_{i=1}^{k} \int dt\, \lambda_i(t)\, \delta\phi_i(\vec{q}(t)) = 0$$

[ϕ_i depends only on q^α, & so $\delta\phi_i$ doesn't contain any $\delta\lambda$]

More explicitly :-

$$\phi_i(\vec{q}(t)) = 0$$

$$-\frac{d}{dt}\frac{\partial L}{\partial u^\alpha} + \frac{\partial L}{\partial q^\alpha} + \sum_{i=1}^{k} \lambda_i \frac{\partial \phi_i(\vec{q})}{\partial q^\alpha} = 0$$

Solving this system with constraints $\Leftrightarrow$ mapping this problem writing a new Lagrangian with some additional variables

Relationship with earlier analysis :-

Choose coordinates $\vec{\varphi}$ & $\vec{\tilde{\varphi}}$

along the constrained surface, transverse

(135) $\tilde{Q}_i = \phi_i(\vec{q})$

First look at the Q-eqns :-

these eqns really give the time evolution of the system

$$-\frac{d}{dt}\frac{\partial L}{\partial U^\alpha} + \frac{\partial L}{\partial Q^\alpha} + 0 = 0$$

($\because$ Q's & $\tilde{Q}$'s are indep.)

Then the $\tilde{Q}$-eqns :-

$$-\frac{d}{dt}\frac{\partial L}{\partial \tilde{U}^i} + \frac{\partial L}{\partial \tilde{Q}^i} + \lambda_i = 0$$

(the eqns transverse to the constrained surface trivial bcos we consider that) $\rightarrow$ these eqns determine λ_i in terms of $\vec{Q}, \vec{U}$.

$$\tilde{Q}^i = 0 \Rightarrow \tilde{U}^i = 0$$

($\because$ we already know $\tilde{Q}^i = 0$ & $\tilde{U}^i = 0$ so, the $\tilde{Q}$ eqns are algebraic eqns in terms of $\vec{Q}, \vec{U}$ & λ_i)

[$\lambda_i \rightarrow$ interpretation of generalized force transverse to the constrained surface - force of reacn needed to maintain the constraint]

Non-holonomic constraints :-

Suppose the constraints involve $\vec{q}$ and $\vec{u}$.

$$\phi_i(\vec{q}, \vec{u}) = 0 \qquad i = 1, \ldots, k$$

Naive guess:

Constraints restricting the velocities also

Use $L'(\vec{q}, \vec{u}, \vec{\lambda}) = L(\vec{q}, \vec{u}) + \sum_{i=1}^{k} \lambda_i \phi_i(\vec{q}, \vec{u})$

& extremize the action.

$\Longrightarrow$ Turns out not to be correct.

[this is because $u = dq/dt$ is built-in. The non-trivial part of (Euler-Lag. eqn) can be got by variational ...]

If ϕ_i's are integrable such that u's can be expressed as fns ...

Reference :-
M. R. Flannery, Am. J. Phys. V 73 p 265

Try to find other ways of looking for conservation laws & solve the system.

Hamilton-Jacobi method

Symmetries may not be apparent. But once u have found conserved charges, u have a symmetry

Strategy: Look for a canonical trs. of the form:

$$p_\alpha = \frac{\partial F_3(\vec{\tilde{p}}, \vec{q})}{\partial q^\alpha}, \quad \tilde{q}^\alpha = \frac{\partial F_3(\vec{\tilde{p}}, \vec{q})}{\partial \tilde{p}_\alpha}$$

Such that: $H(\vec{q}, \vec{p}) = \tilde{H}(\vec{\tilde{p}})$

[Now we are assuming that we don't know the conservation laws. So if we are able to do this :- $H(\vec{q}, \vec{p}) = \tilde{H}(\tilde{p})$ ---]

If we are able to do this then eqns of motion simplify, because

$$\frac{d\tilde{p}_\alpha}{dt} = -\frac{\partial \tilde{H}}{\partial \tilde{q}_\alpha} = 0 \Rightarrow \tilde{p}_\alpha = \text{constant}$$

$$\frac{\partial q^\alpha}{dt} = \frac{\partial \tilde{H}}{\partial \tilde{p}_\alpha} = \omega_\alpha(\vec{\tilde{p}})$$

$$q^\alpha = \omega_\alpha(\vec{\tilde{p}})\, t + \kappa^\alpha$$

Applying canonical transformation from $\vec{q}, \vec{p} \to \vec{\tilde{q}}, \vec{\tilde{p}}$ we can find $\vec{q}(t), \vec{p}(t)$.

How do we go about finding F_3?

Need to solve

$$H\left(\vec{q}, \frac{\partial F_3(\vec{\tilde{p}}, \vec{q})}{\partial q^\alpha}\right) = \tilde{H}(\vec{\tilde{p}})$$

... differential

F_3, $\tilde{H}$ → unknown

(137) Regard this as a partial differential equ for $F_3(\vec{q})$ at fixed $\vec{p}$.

So that the whole thing is indep. of q. ?!

Treat $\vec{p}$'s as constants & drop all reference to these.

$$H\left(\vec{q}, \frac{\partial F_3}{\partial \vec{q}}\right) = C_0.$$

e.g. for harmonic oscillator $\frac{1}{2} k \vec{q}^{\,2} + \frac{1}{2} m \left(\frac{\partial F_3}{\partial q}\right)^2 = C_0$

(try to manipulate this complicated eqn) manipulate & try to write it as

$$\phi_1\left(q^1, \frac{\partial F_3}{\partial q^1}, C_0\right) = \psi_1\left(q^2, \frac{\partial F_3}{\partial q^2}, q^3, \frac{\partial F_3}{\partial q^3}, \ldots, C_0\right)$$

Look for a solution for F_3 of the form

$$F_3(\vec{q}) = F_3^{(1)}(q^1, C_0) + \tilde{F}_3^{(1)}(q^2, q^3, \ldots, C_0)$$

$$\phi_1\left(q^1, \frac{\partial F_3^{(1)}}{\partial q^1}, C_0\right) = \psi_1\left(q^2, \frac{\partial \tilde{F}_3^{(1)}}{\partial q^2}, q^3, \frac{\partial \tilde{F}_3^{(1)}}{\partial q^3}, \ldots, C_0\right) = C_1$$

$$\Rightarrow \phi_1\left(q^1, \frac{\partial F_3^{(1)}}{\partial q^1}, C_0\right) = C_1 \Rightarrow \frac{\partial F_3^{(1)}}{\partial q^1} = g_1(q^1, C_0, C_1)$$

$$\Rightarrow F_3^{(1)}(q^1) = \int^{q^1} g_1(x)\, dx$$

Each side should be equal to a constant

$$\psi_1\left(q^2, \frac{\partial \tilde{F}_3^{(1)}}{\partial q^2}, q^3, \frac{\partial \tilde{F}_3^{(1)}}{\partial q^3}, \ldots, C_0\right) = C_1$$

manipulate

$$\phi_2\left(q^2, \frac{\partial \tilde{F}_3^{(1)}}{\partial q^2}\right) = \psi_2\left(q^3, \frac{\partial \tilde{F}_3^{(1)}}{\partial q^3}, q^4, \frac{\partial \tilde{F}_3^{(1)}}{\partial q^4}, \ldots\right)$$

We are looking for some F_3 which has the form $F_3 = F_3^{(1)}(\cdot) + \tilde{F}_3^{(1)}(\cdot)$

Look for a soln of the form

$$\Rightarrow \phi_2\left(q^2, \frac{\partial F_3^{(2)}(q^2)}{\partial q^2}\right) = \psi_2\left(q^3, \frac{\partial F_3^{(2)}}{\partial q^3}, \ldots\right)$$

$$\Rightarrow F_3^{(2)}(q^2) = \ldots = c_2$$

If we succeed till the end then!

$$F_3(q^1, \ldots, q^N) = F_3^{(1)}(q^1, c_0, c_1)$$
$$+ F_3^{(2)}(q^2, c_0, c_1, c_2)$$
$$+ \ldots + F_3^{(N-1)}(q^{N-1}, c_0, c_1, \ldots, c_{N-1})$$
$$+ \tilde{F}_3^{(N-1)}(q^N, c_0, c_1, \ldots, c_{N-1})$$

we can call it $F_3^{(N)}$.

[We can set the constants $c_0, c_1, \ldots, c_{N-1}$ as our $\tilde{p}_\alpha$'s or for our convenience some arbit. fn. $\tilde{p}_\alpha$'s]

[No need to repeat the process for the last one – $F_3^{(N)}$ ~~because~~ bcos it is det. in terms of c_{N-1}]

Example :- Central force in 3-dim :-

$$L = \frac{1}{2} m\left(u_x^2 + u_y^2 + u_z^2\right) - V\left(\sqrt{x^2+y^2+z^2}\right)$$

(r, θ, ϕ) coordinate

$$L = \frac{1}{2} m\left(u_r^2 + r^2 u_\theta^2 + r^2 \sin^2\theta\, u_\phi^2\right) - V(r)$$

$$p_r = \frac{\partial L}{\partial u_r} = m u_r, \quad p_\theta = \frac{\partial L}{\partial u_\theta} = m r^2 u_\theta,$$

$$p_\phi = \frac{\partial L}{\partial u_\phi} = m r^2 \sin^2\phi\, u_\phi$$

$$\ldots + u_\theta p_\theta + u_\phi p_\phi - L = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right)$$

(139)

$$H(\vec{q}, \partial F_3/\partial \vec{q}) = C_0$$

$$\frac{1}{2m}\left[\left(\frac{\partial F_3}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial F_3}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial F_3}{\partial \phi}\right)^2\right] + V(r) = C_0$$

Multiply by $2m r^2$ (to separate r part from θ part) $\Rightarrow$

$$r^2\left(\frac{\partial F_3}{\partial r}\right)^2 + 2m r^2 V(r) - 2m C_0 r^2 = -\left(\frac{\partial F_3}{\partial \theta}\right)^2 - \frac{1}{\sin^2\theta}\left(\frac{\partial F_3}{\partial \phi}\right)^2$$

We look for $F_3 = F_3^{(1)}(r) + \tilde{F}_3^{(1)}(\theta, \phi)$

$$\boxed{r^2\left(\frac{\partial F_3^{(1)}}{\partial r}\right)^2 + 2m r^2 V(r) - 2m C_0 r^2 = -C_1}$$

$$\| $$

$$-\left(\frac{\partial \tilde{F}_3^{(1)}}{\partial \theta}\right)^2 - \frac{1}{\sin^2\theta}\left(\frac{\partial \tilde{F}_3^{(1)}}{\partial \phi}\right)^2 = -C_1$$

Can write $F_3^{(1)}$ as an integral

Multiply by $\sin^2\theta$ for separation

$$-\sin^2\theta\left(\frac{\partial \tilde{F}_3^{(1)}}{\partial \theta}\right)^2 + C_1 \sin^2\theta = \left(\frac{\partial \tilde{F}_3^{(1)}}{\partial \phi}\right)^2$$

$$\tilde{F}_3^{(1)}(\theta, \phi) = F_3^{(2)}(\theta) + \tilde{F}_3^{(2)}(\phi)$$

$$-\sin^2\theta\left(\frac{\partial F_3^{(2)}}{\partial \theta}\right)^2 + C_1 \sin^2\theta = C_2; \quad \left(\frac{\partial \tilde{F}_3^{(2)}}{\partial \phi}\right)^2 = C_2$$

Use elliptic coordinates:

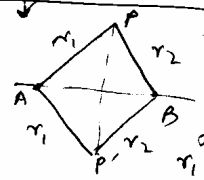
$$\xi = r_1 + r_2, \quad \eta = r_1 - r_2$$

It's not good globally — but locally it's okay — bcos we are trying to solve the problem locally

Ex: check that in these coordinates

$$H = 2 p_\xi^2 \frac{\xi^2 - 4c^2}{\xi^2 - \eta^2} + 2 p_\eta^2 \frac{4c^2 - \eta^2}{\xi^2 - \eta^2} - \frac{4k\xi}{\xi^2 - \eta^2}$$

[$2c \equiv$ distance betn A & B]



Both P & P' possible for same r_1 & r_2.

Look for $F_3(\xi, \eta)$ such that

$$2\left(\frac{\partial F_3}{\partial \xi}\right)^2 \frac{\xi^2 - 4c^2}{\xi^2 - \eta^2} + 2\left(\frac{\partial F_3}{\partial \eta}\right)^2 \frac{4c^2 - \eta^2}{\xi^2 - \eta^2} - 4\frac{k\xi}{\xi^2 - \eta^2} = \underbrace{C_0}_{\text{constant}}$$

Multiply by $(\xi^2 - \eta^2)$ to get

$$2\left(\frac{\partial F_3}{\partial \xi}\right)^2 (\xi^2 - 4c^2) + 2\left(\frac{\partial F_3}{\partial \eta}\right)^2 (4c^2 - \eta^2) - 4k\xi = C_0(\xi^2 - \eta^2)$$

$$\Rightarrow 2\left(\frac{\partial F_3}{\partial \xi}\right)^2 (\xi^2 - 4c^2) - C_0\xi^2 - 4k\xi = -2\left(\frac{\partial F_3}{\partial \eta}\right)^2 (4c^2 - \eta^2) - C_0\eta^2$$

Parabola limit of hyperbola & ellipse with focus going to infinity

Look for a solution of the form

$$F_3(\xi, \eta) = F_3^{(1)}(\xi) + \tilde{F}_3^{(1)}(\eta)$$

Now, we have,

$$2\left(\frac{dF_3^{(1)}}{\ }\right)^2 (\xi^2 - 4c^2) - C_0\xi^2 - 4k\xi$$

LHS → fn. of ξ only
RHS → " " η "

We can now express $\tilde{F}_3^{(1)}$ & $\tilde{F}_3^{(2)}$ as integrals :—

$$F_3^{(1)}(\xi, C_0, C_1) = \frac{1}{\sqrt{2}} \int^{\xi} d\xi' \frac{1}{\sqrt{\xi'^2 - 4c^2}} \sqrt{C_1^2 + C_0 \xi'^2 + 4k\xi'}$$

$$\tilde{F}_3^{(1)}(\eta, C_0, C_1) = \frac{1}{\sqrt{2}} \int^{\eta} d\eta' \frac{1}{\sqrt{4c^2 - \eta^2}} \sqrt{-C_1 - C_0 \eta'^2}$$

Lower limits not specified so that we can get the const. of integration

$$F_3(\xi, \eta, C_0, C_1) = F_3^{(1)}(\xi, C_0; C_1) + \tilde{F}_3^{(1)}(\eta; C_0, C_1)$$

$\Rightarrow F_3(\xi, \eta, \tilde{p}_1, \tilde{p}_2)$

Defining $\tilde{p}_1 = C_0$ & $\tilde{p}_2 = C_1$

$C_0, C_1 \to$ parameters

$$\tilde{q}^{\alpha} = \frac{\partial F_3}{\partial \tilde{p}_\alpha}, \quad p_\xi = \frac{\partial F_3}{\partial \xi}, \quad p_\eta = \frac{\partial F_3}{\partial \eta}$$

Get ξ & η in terms of $\tilde{p}$ & $\tilde{q}$

$\tilde{p}$ in terms of p & q

Hamiltonian is just C_0, i.e., $\tilde{p}_1$, in $\tilde{q}, \tilde{p}$ coord. — so it's easier in this coord.

Do a coord. system where the Hamiltn – Jacobi method can be applied.

Symmetry → not a geometrical sym., but a sym. in phase space

First check numerically whether a problem is integrable. If it is, then apply all sorts of analytical methods – otherwise don't even try.

143) Another method for finding conserved charges:

$\boxed{\text{LAX PAIR}}$

A & B are called Lax pair

Suppose we are given $H(\vec{q}, \vec{p})$.

$$\frac{dq^\alpha}{dt} = \frac{\partial H}{\partial p_\alpha}, \quad \frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial q^\alpha}. \quad (\alpha = 1, 2, \ldots, N)$$

We try to construct a pair of $N \times N$ matrices $A_{\alpha\beta}(\vec{q}, \vec{p})$ & $B_{\alpha\beta}(\vec{q}, \vec{p})$ such that eqns of motion

$$\Rightarrow \frac{dA}{dt} = BA - AB$$

this situation holds

$$\frac{dA_{\alpha\beta}}{dt} = B_{\alpha\gamma} A_{\gamma\beta} - A_{\alpha\gamma} B_{\gamma\beta}$$

Construct $2N^2$ fns of $\vec{q}$ & $\vec{p}$ in the form of $A_{\alpha\beta}$ & $B_{\alpha\beta}$

(B eqn. is arbitrary, we can have $\frac{dB}{dt}$ as anything)

Can't be applied to any problem — some " has some permutation symmetry betn. $\vec{q}$ & $\vec{p}$ & then we use this method — otherwise here we are to find $2N^2$ fns instead of a conserved quantity $dQ/dt = 0$

$$\frac{d}{dt} \mathrm{Tr}(A^n) = \frac{d}{dt} \mathrm{Tr}(\overbrace{AA \cdots A}^{n\text{-times}})$$

$$= \mathrm{Tr}\left(\frac{dA}{dt} A^{n-1}\right) + \mathrm{Tr}\left(A \frac{dA}{dt} A^{n-2}\right) + \cdots + \mathrm{Tr}\left(A^{n-1} \frac{dA}{dt}\right)$$

$\frac{dA}{dt}$ won't necessarily commute with A, so u don't change the order while differentiating

$$\frac{d}{dt}\left(A_{\alpha_1\alpha_2} A_{\alpha_2\alpha_3} \cdots A_{\alpha_n\alpha_1}\right)$$

Trace is a summation over indices & diff. can be taken within summation over indices

$$\Rightarrow \frac{d}{dt}\left(\mathrm{Tr}\, A^n\right) = n\,\mathrm{Tr}\left(\frac{dA}{dt} A^{n-1}\right)$$

$$= n\,\mathrm{Tr}\left((BA - AB)\, A^{n-1}\right)$$

$$= n\,\mathrm{Tr}\left(BA^n\right) - n\,\mathrm{Tr}\left(BA^n\right)$$

[again using cyclic prop. of trace]

$$= 0$$

Using (144) cyclic ~~permutation~~ property of trace

(This is true for any value of n) — (so we have constructed a conserv. charge.)

$\Rightarrow$ $\mathrm{Tr}(A^n)$ is conserved for any n.

(We have $2N$ indep. variables & we can at most construct $2N$ indep. fns out of them — so conserv. charge can't be more than $2N$)

$$\mathrm{Tr}(A^n) = \lambda_1^n + \lambda_2^n + \cdots + \lambda_N^n$$

each of λ_i is a fn of $\vec{q}$ & $\vec{p}$

You can get at most N indep. quantities.

$\Rightarrow$ N independent conserved charges.

$\mathrm{Tr}(A^n)$ $\quad n = 1, 2, \ldots, N$

(We can apply this to problems where A & B are easier to find than $\mathrm{Tr}(A^n) \rightarrow$ the conserv. charge))

If we can't find $N \times N$ matrix, you can find a $m \times m$ matrix ($m < N$) which will give less no. of conserv. charges.

e.g. $\rightarrow$ Ham. is a sum of an integrable & a non- " system Hamiltonian. In the complex theory, all conserv. charges of int. sys. are still conserv. charges, but less than that need to solve the new problem.

(145) Example $\Rightarrow$ 3-body Toda system.

$$H = \frac{1}{2}\sum_{\alpha=1}^{3} p_\alpha^2 + V(\vec{q})$$

We have set mass $m=1$ for convenience

$$V(\vec{q}) = e^{-(q^1-q^2)} + e^{-(q^2-q^3)} + e^{-(q^3-q^1)}$$

$$\frac{dq^\alpha}{dt} = p_\alpha, \quad \frac{dp_1}{dt} = -e^{-(q^3-q^1)} + e^{-(q^1-q^2)}$$

$$\frac{dp_2}{dt} = e^{-(q^2-q^3)} - e^{-(q^1-q^2)}$$

$$\frac{dp_3}{dt} = e^{-(q^3-q^1)} - e^{-(q^2-q^3)}$$

Not worrying about dimensions — otherwise can multiply by some constants

[p_1 generates q^1 goes to $q^1+\epsilon$
p_2 " q^2 " " $q^2+\epsilon$
p_3 " q^3 " " $q^3+\epsilon$
H is consvd.]

$\rightarrow$ Basically shifting the centre of mass-coord. & $V(\vec{q})$ doesn't depend on centre of mass coord.

& for this, H is consvd.

So we find $\Rightarrow$ ① H is conserved
② $p_1+p_2+p_3$ is conserved.

p_1, p_2, p_3 can be considered as the sum of mom. of 3 one-dim. particle motion

looks like the trace & then p_1, p_2, p_3 must be the diagonal elements

$$dA/dt = BA - AB$$ [linear in A on both sides]

(It looks natural that A & B should be built of exponentials.)

Define $a_1 = \frac{1}{2} e^{-\ln(q^1-q^2)}$

$a_2 = \frac{1}{2} e^{-\frac{1}{2}(q^2-q^3)}$

$$A = \begin{pmatrix} p_1/2 & a_1 & a_3 \\ a_1 & p_2/2 & a_2 \\ a_3 & a_2 & p_3/2 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & a_1 & -a_3 \\ -a_1 & 0 & a_2 \\ a_3 & -a_2 & 0 \end{pmatrix}$$

Ex. :- Check that $\frac{dA}{dt} = BA - AB$

Ex. Check that $\mathrm{Tr}\, A^2 = H$

Conserved charges :-
① H ② $p_1+p_2+p_3$
③ $\mathrm{Tr}(A^3)$

We have to make sure that $p_1+p_2+p_3$ has vanishing Poisson bracket with $\mathrm{Tr}(A^3)$

New charge
$\mathrm{Tr}(A^3) =$ Complicated fn of $\vec{q}$, $\vec{p}$

$\Downarrow$

New conserved charge and a hidden symmetry.

It is manifestly invariant under a sim. trans. of q^1, q^2, q^3 by same amt. & so has vanishing Poisson bracket with $p_1+p_2+p_3$

$p_1+p_2+p_3$ generates simultaneous translations of q^1, q^2, q^3 by same amt. So, the sym. generated by $p_1+p_2+p_3$ leaves a_1, a_2, a_3 unchanged & so $p_1+p_2+p_3$ has vanishing Poisson bracket with each element of A

Don't need to check vanishing Poisson bracket with H bcos a cons. charge is by defn. has vanishing P. brack. with H

(147) Epilogue

Given conserved charges $Q_1, Q_2, \ldots$, there are symmetries.
The correspondence goes both ways:—

$$Q_1, Q_2 \ldots \Longleftrightarrow \text{symmetries}$$

~~Now we recognise~~

Q) If $\{Q_1, Q_2\} = 0$, what does this imply for the symmetries generated by Q_1 & Q_2?

Suppose $F(\vec{q}, \vec{p})$ is some fn. of $\vec{q}, \vec{p}$.
Under infinitesimal transfn. generated by Q_1,

$$F(\vec{q}, \vec{p}) \xrightarrow[\text{some new fn.}]{\text{goes to}} G(\vec{q}, \vec{p}) = F(\vec{q}, \vec{p}) + \epsilon\{F, Q_1\}$$

Now apply infinitesimal trs. generated by Q_2 on this new fn. $G(\vec{q}, \vec{p})$:—

$$G(\vec{q}, \vec{p}) \longrightarrow G(\vec{q}, \vec{p}) + \eta\{G(\vec{q}, \vec{p}), Q_2\}$$
$$= F(\vec{q}, \vec{p}) + \epsilon\{F, Q_1\}$$
$$+ \eta\left(\{F(\vec{q}, \vec{p}), Q_2\} + \epsilon\{\{F, Q_1\}, Q_2\}\right)$$

We are thinking ϵ & η as indep. & we'll neglect ϵ^2, η^2 terms though we'll keep the term

the ϵ^2 & η^2 would cancel out anyway

Do this in reverse order:—

$$F(\vec{q}, \vec{p}) \rightarrow H(\vec{q}, \vec{p}) = F(\vec{q}, \vec{p}) + \eta\{F, Q_2\}$$
$$H(\vec{q}, \vec{p}) \rightarrow H(\vec{q}, \vec{p}) + \epsilon\{H, Q_1\}$$
$$= F(\vec{q}, \vec{p}) + \eta\{F, Q_2\}$$
$$+ \epsilon\left(\{F(\vec{q}, \vec{p}), Q_1\} + \eta\{\{F, Q_2\}, Q_1\}\right)$$

ϵ, η interchanged & Q_1, Q_2 interch-

Now we want to study betn these 2 :-

Difference

$$= \epsilon \eta \left(\{\{F, Q_1\}, Q_2\} - \{\{F, Q_2\}, Q_1\} \right)$$

Ex: Prove that given 3-fns $A(\vec{q}, \vec{p})$, $B(\vec{q}, \vec{p})$ an $C(\vec{q}, \vec{p})$, the following is true :-

$$\{\{A, B\}, C\} - \{\{A, C\}, B\}$$

$$= \{A, \{B, C\}\} \Rightarrow \text{Jacobi identity}$$

$$= \epsilon \eta \{F, \{Q_1, Q_2\}\}$$

If $\{Q_1, Q_2\} = 0$ (which is what we need for using them for solving the problem), then the difference $= 0$ & we saw that the two symmetries commute

(It doesn't depend on the order in which we apply the symmetries)

Conversely, if two symmetries commute, then the associated charges will have zero Poisson bracket.

In rare cases this may fail.

$\{Q_1, Q_2\}$ can at the most be a const., but it happens in rare cases — most cases it is zero.

This shows why

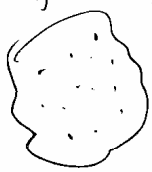
$\}=0$ for any ... at

(149)

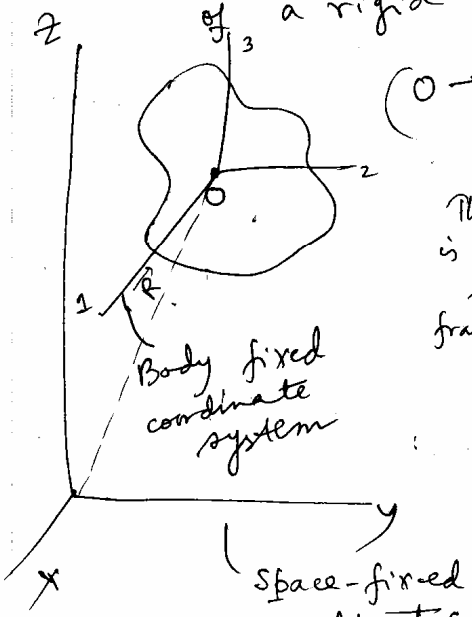
28/9/05

Dynamics of Rigid Body

Rigid body → A collection of point masses at fixed position relative to each other (All relative distances are fixed).



Q.) How many independent coordinates do we need to specify the configuration of a rigid body?



(O → fixed pt. relative to the body

The set of axes is fixed on the body — some frame erected inside the body)

Treat the system as a constrained system. Parametrically solve all the constraint eqns in terms of the indep. coord. & substitute in the Lag.

If we specify the coordinates of O & the orientation of 1, 2 & 3 in the Space-fixed coordinate system then the rigid body is completely fixed.

$\vec{R} = (R^x, R^y, R^z)$ coordinates of O.

$\hat{n}_1, \hat{n}_2, \hat{n}_3$: unit vectors along 1, 2 & 3 all measured in the x, y, z coordinate

... Dynamics of the

This gives us $4 \times 3 = 12$ parameters

(But there are constraints, $\because$ these are unit vectors. And also we have taken an orthogonal coord. sys. to begin with, it'll always remain orthogonal)

$\hat{n}_1^2 = 1,\ \hat{n}_2^2 = 1,\ \hat{n}_3^2 = 1,\ \hat{n}_1 \cdot \hat{n}_2 = 0,\ \hat{n}_1 \cdot \hat{n}_3 = 0,\ \hat{n}_2 \cdot \hat{n}_3 = 0$

$\Rightarrow 12 - 6 = 6$ independent coordinates

3 vectors $\hat{n}_1, \hat{n}_2, \hat{n}_3$ & another one $\vec{R}$. $\therefore 4 \times 3$ components $= 12$

A unit vector has 2 indep. parameters.

u can also say that to fix x, u need 2 param. & then u can rotate it about 0 → so another 1 param.

$\therefore 3+3 = 6$ parameters

$$\hat{n}_1 = \begin{pmatrix} n_1^x \\ n_1^y \\ n_1^z \end{pmatrix}, \quad \hat{n}_2 = \begin{pmatrix} n_2^x \\ n_2^y \\ n_2^z \end{pmatrix}, \quad \hat{n}_3 = \begin{pmatrix} n_3^x \\ n_3^y \\ n_3^z \end{pmatrix}$$

(Introduce the matrix S having the following form) →

$$S = \begin{pmatrix} n_1^x & n_2^x & n_3^x \\ n_1^y & n_2^y & n_3^y \\ n_1^z & n_2^z & n_3^z \end{pmatrix}$$

There is a constraint on S that

$$S^T S = \mathbb{1} \xrightarrow[\text{follows from}]{\text{This}} \hat{n}_i \cdot \hat{n}_j = \delta_{ij}$$

These 2 constraints are equivalent

$$\det S = 1 \Leftrightarrow \hat{n}_3 = \hat{n}_1 \times \hat{n}_2$$

[So we'll restrict to $\det S = 1$. Note for $\hat{n}_3 = -\hat{n}_1 \times \hat{n}_2$, we'll get $\det S = -1$]

$\Rightarrow S$ is an $SO(3)$ matrix

This shows that the independent coordinate degrees of freedom of a rigid body are

1) A vector $\vec{R} = (R^x, R^y, R^z)$

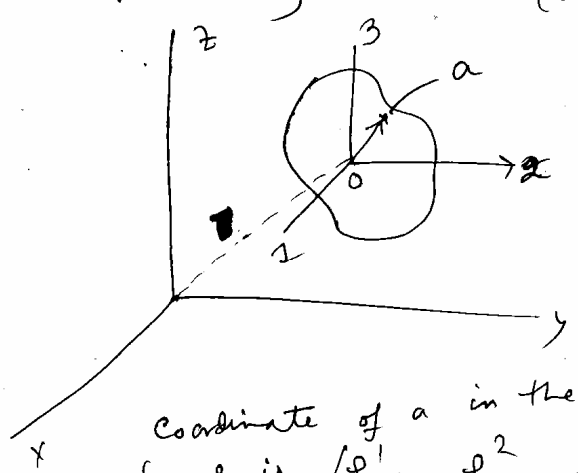
$\begin{pmatrix} \vec{n}_1 \cdot \vec{n}_1 & \vec{n}_1 \cdot \vec{n}_2 & \vec{n}_1 \cdot \vec{n}_3 \\ \vec{n}_2 \cdot \vec{n}_1 & \vec{n}_2 \cdot \vec{n}_2 & \vec{n}_2 \cdot \vec{n}_3 \\ \vec{n}_3 \cdot \vec{n}_1 & \vec{n}_3 \cdot \vec{n}_2 & \vec{n}_3 \cdot \vec{n}_3 \end{pmatrix}$

151 [$SO(N)$ is an $N \times N$ matrix satisfying $S^T S = \mathbb{1}$ & $\det S = 1 \Rightarrow$ specifies rotation in N-dim.]

($SO(3)$ is the rotation matrix in 3-dim.)

[We'll write down the Lag. in terms of 3×3 matrix S but at the end we'll eliminate the unwanted variables & consider only the indep. variables]

(we take a point on the rigid body)



Coordinate of a in the body fixed ~~frame~~ frame is $\underbrace{\left(\rho^1_{(a)}, \rho^2_{(a)}, \rho^3_{(a)}\right)}_{\text{fixed numbers}}$ (they don't change as the body is moving)

Coordinate of a in the space-fixed system is $\vec{r}_{(a)} = \vec{R} + \rho^1_{(a)}\,\hat{n}_1 + \rho^2_{(a)}\,\hat{n}_2 + \rho^3_{(a)}\,\hat{n}_3$

~~Total kinetic energy~~

$$= \begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix} + S \begin{pmatrix} \rho^1_{(a)} \\ \rho^2_{(a)} \\ \rho^3_{(a)} \end{pmatrix}$$

(In a more compact notation)

$$\vec{r}_{(a)} = \vec{R} + S\vec{\rho}_{(a)} \quad \text{where} \quad \vec{\rho}_{(a)} = \begin{pmatrix} \rho^1_{(a)} \\ \rho^2_{(a)} \\ \rho^3_{(a)} \end{pmatrix}$$

(The velocity of a is)

$$\frac{d\vec{r}_{(a)}}{dt} = \frac{d\vec{R}}{dt} + \frac{dS}{dt}\vec{\rho}_{(a)} \quad \left[\because \frac{d\vec{\rho}_{(a)}}{dt} \text{ is zero, } \vec{\rho}_{(a)} \text{ being fixed}\right]$$

Total kinetic energy

$$T = \frac{1}{2}\sum_a m_{(a)}\left(\frac{d\vec{r}_{(a)}}{dt}\right)^2$$

$$\Rightarrow T = \frac{1}{2}\sum_a m_{(a)}\left(\left(\frac{d\vec{R}}{dt}\right)^2 + 2\frac{d\vec{R}}{dt}\cdot\frac{dS}{dt}\vec{\rho}_a + \left(\frac{dS}{dt}\vec{\rho}_{(a)}\right)\cdot\left(\frac{dS}{dt}\vec{\rho}_{(a)}\right)\right)$$

$$\Rightarrow T = \frac{1}{2}M\left(\frac{d\vec{R}}{dt}\right)^2 + \frac{d\vec{R}}{dt}\cdot\frac{dS}{dt}\underbrace{\sum_a m_{(a)}\vec{\rho}_{(a)}}_{\text{Time independent}} + \frac{1}{2}\frac{dS_{ij}}{dt}\frac{dS_{ik}}{dt}\underbrace{\sum_a m_{(a)}\rho^j_a\rho^k_{(a)}}_{\text{Time independent}}$$

[where $M = \sum_a m_{(a)}$]

Now, $\sum_a m_{(a)}\vec{\rho}_a = M\vec{\rho}_{(c.m.)}$

coordinate of centre of mass in the body fixed system

Define : $K_{ij} = \sum_a m_{(a)}\rho^i_{(a)}\rho^j_{(a)}$ [constant when you fix the body fixed cond.]

$$\boxed{T = \frac{1}{2}M\left(\frac{d\vec{R}}{dt}\right)^2 + M\frac{d\vec{R}}{dt}\cdot\frac{dS}{dt}\vec{\rho}_{(c.m.)} + \frac{1}{2}\,\mathrm{Tr}\left(\frac{dS}{dt}K\frac{dS^T}{dt}\right)}$$

$\left(\frac{dS}{dt}\right)_{ij}K_{jk}\left(\frac{dS^T}{dt}\right)_{ki}$

$\mathrm{Tr}\left(\frac{dS}{dt}K\frac{dS^T}{dt}\right)$

$K_{ij} \to$ matrix ??

158)

(We'll try to decouple the 2 coord. systems — easier to solve the problem)

$$\left(\frac{d\vec{R}}{dt} \cdot \frac{dS}{dt} \longrightarrow \text{coupling terms} \right)$$

1) If the rigid body has no other constraint we choose O at the centre of mass

$$\xrightarrow[\text{fixes}]{\text{this}} \quad \vec{\rho}^{(cm)} = 0$$

Hence, $T = \frac{1}{2} M \left(\frac{d\vec{R}}{dt} \right)^2 + \frac{1}{2} \mathrm{Tr} \left(\frac{dS}{dt} K \frac{dS^T}{dt} \right)$

[Doesn't mean we decouple the problem completely bcos the potential term may involve coupling.
However in a uniform grav. field, pot. depends only on the centre of mass location — not so in other cases]

2) Suppose the rigid body is allowed to rotate with some point P fixed in space.

[e.g. → If we fix the tip of a rotating top]

Take O to be at the point which is fixed in space.

(In this case, the constraint takes a very simple form)

Constraint : $\vec{R} = \vec{R_0} \longrightarrow$ fixed

Hence, $T = \frac{1}{2} \mathrm{Tr} \left(\frac{dS}{dt} K \frac{dS^T}{dt} \right)$

[Top moving in a 2-D plane. The z-coord. of tip can't move up & down. that is a suitable choice of

$$K_{ij} = \sum_a m_{(a)} \rho^i_{(a)} \rho^j_{(a)}$$

If it is a continuous solid, we can write $K_{ij} = \int d^3\rho \; D(\vec{\rho}) \, \rho_i \rho_j$

$D(\vec{\rho})$ → density of the solid at $\vec{\rho}$

Inertia matrix I_{ij}:

Definition: $I_{ij} = \sum_a m_{(a)} \left(\vec{\rho}^{\,2}_{(a)} \delta_{ij} - \rho_{(a)i} \rho_{(a)j} \right)$ summing over all blobs

$$\left[\text{or } I_{ij} = \int d^3\rho \; D(\vec{\rho}) \left(\rho^2 \delta_{ij} - \rho_i \rho_j \right) \right]$$

Note: $\mathrm{Tr}\, K = \sum_a m_{(a)} \vec{\rho}^{\,2}_{(a)}$

$$I = (\mathrm{Tr}\, K) \mathbb{1} - K$$

$[\because \mathrm{Tr}\, \mathbb{1} = 3]$

$$\Rightarrow \begin{cases} \mathrm{Tr}\, I = 2\, \mathrm{Tr}\, K \\ K = \frac{1}{2} \mathrm{Tr}(I) - I \end{cases}$$

$\mathrm{Tr}\, I = (\mathrm{Tr}\, K) \cdot 3 - \mathrm{Tr}\, K = 2\, \mathrm{Tr}\, K$

[We'll try to develop techniques aimed at solving the part $\frac{1}{2} \mathrm{Tr}\left(\frac{dS}{dt} K \frac{dS^T}{dt} \right)$. At the end, we'll add the part $\frac{1}{2} M \left(\frac{d\vec{R}}{dt} \right)^2$. We assume its possible to decouple the potential term also.]

gives the symmetry for 3-D point particle

Part of L that is of interest is $\frac{1}{2} \mathrm{tr} \left(\frac{dS}{dt} K \frac{dS^T}{dt} \right)$ + Possible potential term

Since K is symmetric, we can find an orthogonal matrix W such that $[W^T W = \mathbb{1}]$

$$K = W K_d W^T$$

W → fixed orthogonal matrix of det $= 1$

156)

Here K_d is diagonal : $\begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix}$

$$\therefore \quad T = \frac{1}{2}\left(\mathrm{Tr}\, \frac{dS}{dt}\, W K_d W^T \frac{dS^T}{dt}\right)$$

Define $\tilde{S} = \overset{\to SO(3)}{S}\, \overset{\to SO(3)}{W}$

[2 notations ↓ product of 2 SO(3) matrix is also an SO(3) matrix]

also an SO(3) matrix [3×3 orthogonal matrix of det = 1]

We define $\tilde{S}$ as our new dynamical variable bcos if we can solve in terms of $\tilde{S}$, we are done

My - - - [$\tilde{S}$ is SO(3) : bcos]

$\tilde{S}^T = W^T S^T$

$\therefore \tilde{S}^T S = W^T S^T S W = W^T W = \mathbb{1}$

~~Det~~ $\mathrm{Det}(S) = (\mathrm{Det}\, S)(\mathrm{Det}\, W) = 1 \times 1 = 1$

$$\therefore \quad T = \frac{1}{2}\, \mathrm{Tr}\left(\frac{d\tilde{S}}{dt}\, K_d\, \frac{d\tilde{S}^T}{dt}\right)$$

$$S = \begin{pmatrix} \vec{n}_1 & \vec{n}_2 & \vec{n}_3 \end{pmatrix}$$

$$\tilde{S} = SW = \begin{pmatrix} \vec{n}_1 & \vec{n}_2 & \vec{n}_3 \end{pmatrix} \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \vec{\tilde{n}}_1 & \vec{\tilde{n}}_2 & \vec{\tilde{n}}_3 \end{pmatrix}$$

As if we are multiplying a row matrix with a sq. matrix

where $\vec{\tilde{n}}_1 = W_{11}\vec{n}_1 + W_{21}\vec{n}_2 + W_{31}\vec{n}_3$

$\vec{\tilde{n}}_2 = W_{12}\vec{n}_1 + W_{22}\vec{n}_2 + W_{32}\vec{n}_3$

$\vec{\tilde{n}}_3 = W_{13}\vec{n}_1 + W_{23}\vec{n}_2 + W_{33}\vec{n}_3$

$\Rightarrow$ We are making a different choice of the body fixed frame.

Actually we have consider each $\vec{n}_i$ as a column matrix

$\vec{\tilde{n}}_1, \vec{\tilde{n}}_2$ & $\vec{\tilde{n}}_3$ are fixed linear combn. of $\vec{n}_1, \vec{n}_2, \vec{n}_3$ ↓ effectively we have chosen a body fixed acc. to convenience fixed remains

of course, $\vec{n}_1, \vec{n}_2, \vec{n}_3$ aren't ...

Summary

We need to analyse the Lagrangian

$$L = \frac{1}{2} \mathrm{Tr}\left(\frac{d\tilde{S}}{dt} K_d \frac{d\tilde{S}^T}{dt} \right) - \text{Potential terms}$$

$$K_d = \begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix}$$

(S is still a constrained variable ∴ S has $3 \times 3 = 9$ elements, but there are 3 indep. degrees of freedom or parameters)

It's not necessary that all the symmetries of the kinetic term will survive when we add the potential term

158)

$$H(\vec{p},\vec{q}) = \tilde{H}(\tilde{\vec{p}},\tilde{\vec{q}})$$

$$\Rightarrow p_\alpha \frac{\partial H}{\partial p_\alpha} - L(\vec{q},\vec{u}) = \tilde{p}_\alpha \frac{\partial H}{\partial \tilde{p}_\alpha} - L(\tilde{\vec{q}},\vec{u})$$

$$\Rightarrow \cancel{p_\alpha q^\alpha - L(\vec{q},\vec{u})} = \tilde{p}_\alpha \tilde{q}^\alpha \cancel{- L(\tilde{q},\vec{u})} + \epsilon \frac{dN}{dt}$$

$$\Rightarrow -p_\alpha q^\alpha + \tilde{p}_\alpha \tilde{q}^\alpha + \epsilon \frac{dN}{dt} = 0$$

Now, $\frac{dN}{dt} = \frac{\partial N}{\partial q^\alpha} u^\alpha + \frac{\partial N}{\partial u^\beta} du^\beta/dt$

$\therefore$ we have,

$$-p_\alpha q^\alpha + \tilde{p}_\alpha \tilde{q}^\alpha + \epsilon \frac{\partial N}{\partial q^\alpha} u^\alpha + \epsilon \frac{\partial N}{\partial u^\beta} du^\beta/dt = 0$$

Using $\frac{\partial N}{\partial u^\beta} = -\epsilon \frac{\partial L}{\partial u^\alpha} \frac{\partial X^\alpha}{\partial u^\beta}$,

we get,

$$-p_\alpha q^\alpha + \tilde{p}_\alpha \tilde{q}^\alpha + \epsilon \frac{\partial N}{\partial q^\alpha} u^\alpha - \epsilon \frac{\partial L}{\partial u^\alpha} \frac{\partial X^\alpha}{\partial u^\beta} du^\beta/dt = 0$$

$$\Rightarrow -p_\alpha q^\alpha + \tilde{p}_\alpha \tilde{q}^\alpha + \epsilon \frac{\partial N}{\partial q^\alpha} u^\alpha - \epsilon p_\alpha \frac{\partial X^\alpha}{\partial u^\beta} du^\beta/dt = 0$$

Now, $\delta u^\alpha = \epsilon \frac{\partial X^\alpha}{\partial q^\beta} u^\beta + \epsilon \frac{\partial X^\alpha}{\partial u^\beta} du^\beta/dt$

$$\Rightarrow \boxed{-\epsilon \frac{\partial X^\alpha}{\partial u^\beta} du^\beta/dt = \epsilon \frac{\partial X^\alpha}{\partial q^\beta} u^\beta - \delta u^\alpha}$$

$\therefore$ we get, $-p_\alpha q^\alpha + \tilde{p}_\alpha \tilde{q}^\alpha + \epsilon \frac{\partial N}{\partial q^\alpha} u^\alpha + \epsilon p_\alpha \frac{\partial X^\alpha}{\partial q^\beta} u^\beta - p_\alpha \delta u^\alpha = 0$

$$\Rightarrow \cancel{-p_\alpha q^\alpha} + \cancel{p_\alpha q^\alpha} + \delta p_\alpha \delta q^\alpha + q^\alpha \delta p_\alpha + p_\alpha \delta q^\alpha - p_\alpha \delta u^\alpha + \epsilon \underbrace{\left[\frac{\partial N}{\partial q^\alpha} u^\alpha + p_\alpha \frac{\partial X^\alpha}{\partial q^\beta} u^\beta\right]}_{= -\delta p_{\alpha,\alpha}} = 0$$

$$\tilde{p}_\alpha \tilde{q}^\alpha = (p_\alpha + \delta p_\alpha)(q^\alpha + \delta q^\alpha) = p_\alpha q^\alpha + \delta p_\alpha \delta q^\alpha + q^\alpha \delta p_\alpha + p_\alpha \delta q^\alpha$$

If we substitute

we get,

$$\delta p_\alpha \, \delta q^\alpha + q^\alpha \delta p_\alpha + p_\alpha \delta q^\alpha - p_\alpha \delta u^\alpha - \delta p_\beta u^\beta = 0$$

$$-p_\alpha q^\alpha + \tilde{p}_\alpha \tilde{q}^\alpha - p_\alpha \delta u^\alpha - \delta p_\alpha u^\alpha = 0$$

LHS $\delta p_\alpha \chi^\alpha + q^\alpha \delta p_\alpha + p_\alpha \chi^\alpha - p_\alpha \tilde{u}^\alpha + p_\alpha \tilde{u}^\alpha - \tilde{p}_\alpha u^\alpha + p_\alpha u^\alpha$

LHS $\delta p_\alpha \, \delta q^\alpha + q^\alpha \delta p_\alpha + p_\alpha \delta q^\alpha - p_\alpha \frac{\delta \dot{q}^\alpha}{dt} - \delta p_\alpha \dot{q}^\alpha = 0$

LHS $-p_\alpha q^\alpha + \tilde{p}_\alpha \tilde{q}^\alpha - p_\alpha \delta \dot{q}^\alpha - \delta p_\alpha \dot{q}^\alpha = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} = 0$$

$$\frac{d}{dt} p_\alpha - \frac{\partial L}{\partial q^\alpha} = 0$$

$$\begin{pmatrix} T^a_{11} & T^a_{12} & T^a_{13} \end{pmatrix} \begin{pmatrix} \dot{s}_{13} \\ \dot{s}_{23} \\ \dot{s}_3 \end{pmatrix}_{3\times 1}$$

1×3

$$= \begin{pmatrix} T^a_{11} \dot{s}_{13} + T^a_{12} \dot{s}_{23} + \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}_{1\times 3} \begin{pmatrix} T^a_{11} & T^a_{12} & T^a_{13} \end{pmatrix}_{3\times 3}$$

$\begin{matrix} 23 \\ 2(13) \end{matrix}$

$$\begin{pmatrix} T^a & T^a_{12} & T^a_{13} \end{pmatrix}$$

160) $Q_3 = -\mathrm{Tr}\left(-\Omega K_d \tilde{S}^{-1} T^3 \tilde{S}\right)$

$$\Rightarrow dQ_3/dt = -\mathrm{Tr}\left(\frac{d\Omega}{dt} K_d \tilde{S}^{-1} T^3 \tilde{S}\right) - \mathrm{Tr}\left(\Omega K_d \tilde{S}^{-1} T^3 \frac{d\tilde{S}}{dt}\right) - \mathrm{Tr}\left(\Omega K_d \frac{d\tilde{S}^{-1}}{dt} T^3 \tilde{S}\right)$$

$\Omega = \tilde{S}^{-1} d\tilde{S}/dt$

$\therefore d\tilde{S}/dt = \tilde{S}\Omega$

$\Omega^T = \tilde{S}^T d\tilde{S}/dt$

$\therefore \Omega^T = \frac{d\tilde{S}^T}{dt}\tilde{S}$

$\Rightarrow \Omega^T = \frac{d\tilde{S}^{-1}}{dt}\tilde{S}$

$\Rightarrow d\tilde{S}^{-1}/dt = \Omega^T \tilde{S}^{-1}$

$$= -\mathrm{Tr}\left(\frac{d\Omega}{dt} K_d \tilde{S}^{-1} T^3 \tilde{S}\right) - \mathrm{Tr}\left(\Omega K_d \tilde{S}^{-1} T^3 \tilde{S}\Omega\right) - \mathrm{Tr}\left(\Omega K_d \Omega^T \tilde{S}^{-1} T^3 \tilde{S}\right)$$

$$= -\mathrm{Tr}\left[\left(\frac{d\Omega}{dt} K_d + \Omega^2 K_d + \Omega K_d \Omega^T\right)\left(\tilde{S}^{-1} T^3 \tilde{S}\right)\right]$$

Now, $\Omega^2 K_d = \begin{pmatrix} -\lambda_1(\omega_2^2+\omega_3^2) & \lambda_2\omega_1\omega_2 & \lambda_3\omega_1\omega_3 \\ \lambda_1\omega_1\omega_2 & -\lambda_2(\omega_1^2+\omega_3^2) & \lambda_3\omega_2\omega_3 \\ \lambda_1\omega_1\omega_3 & \lambda_2\omega_2\omega_3 & -\lambda_3(\omega_1^2+\omega_2^2) \end{pmatrix}$ — ①

$\frac{d\Omega}{dt} K_d = \begin{pmatrix} 0 & \lambda_2\dot{\omega}_3 & -\lambda_3\dot{\omega}_2 \\ -\lambda_1\dot{\omega}_3 & 0 & \lambda_3\dot{\omega}_1 \\ \lambda_1\dot{\omega}_2 & -\lambda_2\dot{\omega}_1 & 0 \end{pmatrix}$ — ②

$\Omega K_d = \begin{pmatrix} 0 & \lambda_2\omega_3 & -\lambda_3\omega_2 \\ -\lambda_1\omega_3 & 0 & \lambda_3\omega_1 \\ \lambda_1\omega_2 & -\lambda_2\omega_1 & 0 \end{pmatrix}$

$\therefore \Omega K_d \Omega^T = \begin{pmatrix} 0 & \lambda_2\omega_3 & -\lambda_3\omega_2 \\ -\lambda_1\omega_3 & 0 & \lambda_3\omega_1 \\ \lambda_1\omega_2 & -\lambda_2\omega_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$

$= \begin{pmatrix} \lambda_2\omega_3^2+\lambda_3\omega_2^2 & -\lambda_3\omega_1\omega_2 & -\lambda_2\omega_1\omega_3 \\ -\lambda_3\omega_1\omega_2 & \lambda_1\omega_3^2+\lambda_3\omega_1^2 & -\lambda_1\omega_2\omega_3 \\ -\lambda_2\omega_1\omega_3 & -\lambda_1\omega_2\omega_3 & +\lambda_1\omega_2^2+\lambda_2\omega_1^2 \end{pmatrix}$ — ③

$\lambda_2\dot{\omega}_3 - (\lambda_3-\lambda_2)\omega_1\omega_2$

① + ② + ③ ⇒