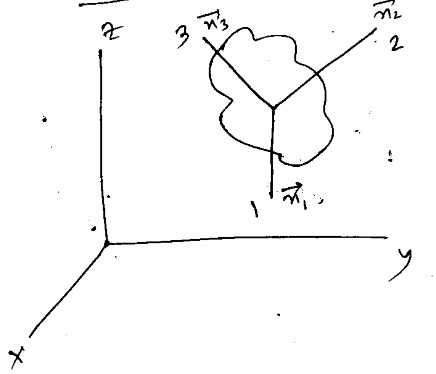
29/9/05

Analyze the Dynamics of a rigid body. (1)



Take the rigid body & erect a coord. frame in that

$S = (\vec{n}_1 \quad \vec{n}_3 \quad \vec{n}_3)$

Think ~~each~~ $\vec{n}_1$, $\vec{n}_2$ & $\vec{n}_3$ as column vectors.

$S^T S = \mathbb{1}$

$$T = \frac{1}{2} \underbrace{M}_{\text{Total mass}} \left(\frac{d\vec{R}}{dt}\right)^2 + \frac{1}{2} \mathrm{Tr}\left(\frac{dS}{dt} K \frac{dS^T}{dt}\right)$$

K : 3×3 matrix associated with the mass distribution inside the body

[Assuming the potential part also decouple, we will focus on $\frac{1}{2}\left(\mathrm{tr}\left(\frac{dS}{dt} K \frac{dS^T}{dt}\right)\right)$]

Focus on the part $T = \frac{1}{2} \mathrm{Tr}\left(\frac{dS}{dt} K \frac{dS^T}{dt}\right)$

[By choosing the body-fixed coord. judiciously we ensure K is diagonal]

Make a judicious ~~gaice~~ choice of the body fixed frame $\tilde{1}, \tilde{2}, \tilde{3}$ & the corresponding matrix $\tilde{S}$.

$$T = \frac{1}{2} \mathrm{Tr}\left(\frac{d\tilde{S}}{dt} K_d \frac{d\tilde{S}^T}{dt}\right) \qquad K_d = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

[we will study the possible symmetries of this T]

$L = T - V$ (V is in general a complicated fn. We will check whether a particular sym. is also a sym. of V. If not, we just leave out that sym.)

2) (for the time being)

Look for symmetries of T.

[Here, there are 3 indep. variables. S is a 3×3 unitary matrix parametrized by $\theta_1, \theta_2, \theta_3$]

$$S(\theta_1, \theta_2, \theta_3)$$
$$\to S(\underbrace{\tilde\theta_1, \tilde\theta_2, \tilde\theta_3}_{\text{Transformed } \theta\text{'s}})$$

[suppose a transfn. takes S to a new orthogonal matrix $S'(\theta_1', \theta_2', \theta_3')$ — That is effectively transfn. of $\theta_1, \theta_2, \theta_3$ to $\theta_1', \theta_2', \theta_3'$]

[Then we have to consider whether the allowed orth. transfn. is sym.]

A transfn. of coord. is a rule that generates a new orthogonal matrix from any given orthogonal matrix.

[Any transfn. of $\theta_1, \theta_2, \theta_3$ to give $\tilde\theta_1, \tilde\theta_2, \tilde\theta_3$ — that is necessarily an orthogonal transfn. of S. Instead of transforming $\theta_1, \theta_2, \theta_3$, we talk of an orth. transfn. of S]

[If S is an orthogonal transfn., S^2 is also an orth. transfn.]

[S goes to S^T is not a sym. transfn. as long as K_d is non-trivial]

[$S \to -S$ is not possible bcos then it is not an $SO(3)$ matrix as the det. is becoming -1]

(1) $\tilde S \longrightarrow \tilde S V$

We must have $V^T V = \mathbb{1}$

$$T \to \frac{1}{2}\mathrm{Tr}\left(\frac{d\tilde S}{dt}\, V K_d V^T \frac{d\tilde S^T}{dt}\right).$$

looks as if K_d has changed

(But a transfn. sym. can it act on constants. Here K_d is a const. only for passive interpretation it is as if we are changing body fixed axes)

is a symmetry if

$$[V, K_d] = 0$$

[If the e.values of K_d are diff., there is no V which commutes. Of course $V = \mathbb{1}$ is trivial]

(2) $\tilde S \longrightarrow U\tilde S, \quad U^T U = \mathbb{1}$

$$T \to \frac{1}{2}\mathrm{Tr}\left(U\frac{d\tilde S}{dt} K_d \frac{d\tilde S^T}{dt} U^T\right)$$

$$= \frac{1}{2}\mathrm{Tr}\left(U^T U \frac{d\tilde S}{dt} K_d \frac{d\tilde S^T}{dt}\right) \quad \text{[By cyclicity of trace]}$$

$$= \frac{1}{2}\mathrm{Tr}\left(\frac{d\tilde S}{dt} K_d \frac{d\tilde S^T}{dt}\right)$$

[This is always a sym. of kinetic term irrespective of K_d]

[$\because$ a general orth. matrix has 3 parameters, there are 3 such symmetries] $\boxed{3}$

[V can always be characterised by rotation about x, y, z axis. But rot. about diff. axes don't commute. So, the 3 charges won't have vanishing Poisson brackets]

[If K_d is $\lambda \mathbb{1}$, then there are 3 consrd. charges for $[V, K_d] = 0$. But still they don't commute & so no vanishing Poisson bracket]

Take $\tilde{S}$ & $\tilde{S} \xrightarrow[\text{of } \tilde{S}]{U \text{ transformation}} U\tilde{S} \xrightarrow[\text{transformation}]{V} U\tilde{S}V$

$$\tilde{S} \xrightarrow[\text{transfn.}]{V} \tilde{S}V \xrightarrow[\text{transfn.}]{U} U\tilde{S}V$$

Matrix prdt. is associative & so ~~$U(\tilde{S}V)$~~ $(U\tilde{S})V = U(\tilde{S}V)$

[If you make 2 U transfn. or 2 V transfns. successively, the ordering will make a diff. But, for 2 successive U, V transfn. ordering won't make a diff.]

[So, a consrd. charge for $[V, K_d] = [V, \mathbb{1}] = 0$ & a consrd. charge for $\tilde{S} \to U\tilde{S}$ will have a vanishing Poisson bracket with each other]

U transfn. is always a symmetry.

Under U trs. $\tilde{S} \xrightarrow{\text{goes to}} U\tilde{S}$

$$\tilde{S} = (\vec{\tilde{n}}_1, \vec{\tilde{n}}_2, \vec{\tilde{n}}_3) \longrightarrow U(\vec{\tilde{n}}_1, \vec{\tilde{n}}_2, \vec{\tilde{n}}_3) = (U\tilde{n}_1, U\tilde{n}_2, U\tilde{n}_3)$$

4) [So U is effectively making an overall ~~transf.~~ rotation of body.
In passive sense, ~~if U is rotat~~ you are rotating ~~the~~ space ~~body~~ fixed axes.]

U generates overall rotation of the body !
(active viewpoint)
or rotation of the space-fixed axes
[passive viewpoint).
The corresponding conserved charges are the angular momentum.

$$\tilde{S} \to \tilde{S} V$$

$$(\vec{n}_1\ \vec{n}_2\ \vec{n}_3) \to (\vec{n}_1\ \vec{n}_2\ \vec{n}_3) \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}$$

$$= \Big(\underbrace{V_{11}\vec{n}_1 + V_{21}\vec{n}_2 + V_{31}\vec{n}_3}_{\text{new } \vec{n}_1} \quad \underbrace{V_{12}\vec{n}_1 + V_{22}\vec{n}_2 + V_{32}\vec{n}_3}_{\text{new } \vec{n}_2} \quad \underbrace{V_{13}\vec{n}_1 + V_{23}\vec{n}_2 + V_{33}\vec{n}_3}_{\text{new } \vec{n}_3} \Big)$$

(Effectively, we are replacing $\vec{n}_i$ by a linear combn. of $\vec{n}_1$, $\vec{n}_2$ & $\vec{n}_3$)

[This has no. ~~ref~~ ref. to the space fixed axes.
So, this means choosing a new set of body fixed axes. New $\vec{n}_i$ is just another fixed vector in the body]
Remember that old $\vec{n}_i$'s were fixed vectors

Transformation induced by V corresponds to a new choice of the body fixed frame.
(Here you aren't rotating the body — so not rotation in the usual sense. Unless the body has some sym., V transfn. isn't a sym. bcos the body looks diff. for diff. ~~so~~ ~~to~~ sets of body fixed axes.)

Q) Under what conditions are V-transfs. symmetries?

Infinitesimal transfs. $V = I + \eta P + \mathcal{O}(\eta^2)$

(we'll work to first order in η)

$$V^T V = \mathbb{1} = (\mathbb{1} + \eta P^T)(\mathbb{1} + \eta P) = \mathbb{1} + \mathcal{O}(\eta^2)$$

$$\Rightarrow \eta (P + P^T) + \mathcal{O}(\eta^2) = 0$$

$$\Rightarrow P = -P^T$$

$$\Rightarrow V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \eta_3 & -\eta_2 \\ -\eta_3 & 0 & \eta_1 \\ \eta_2 & -\eta_1 & 0 \end{pmatrix} + \mathcal{O}(\eta^2)$$

η_1, η_2, η_3 small

$$V k_d = k_d V \qquad k_d = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\Downarrow$$

$$\left.\begin{aligned} (\lambda_1 - \lambda_2)\eta_3 &= 0 \\ (\lambda_1 - \lambda_3)\eta_2 &= 0 \\ (\lambda_2 - \lambda_3)\eta_1 &= 0 \end{aligned}\right\}$$

$\Rightarrow$ If $\lambda_1, \lambda_2, \lambda_3$ are different then all η_i's must vanish.

$\Rightarrow$ no infinitesimal symmetry

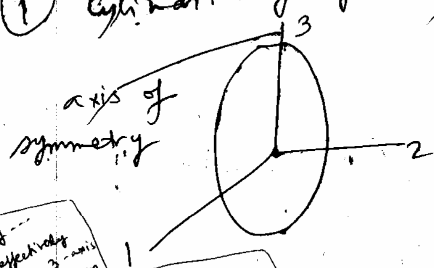
If $\lambda_1 = \lambda_2 \neq \lambda_3$, then $\eta_3 \neq 0$, $\eta_1, \eta_2 = 0 \Rightarrow$ one infinitesimal symmetry

If $\lambda_1 = \lambda_2 = \lambda_3$ then $\eta_1, \eta_2, \eta_3 \neq 0$ $\Rightarrow$ ~~there~~ three infinitesimal symmetry

6 Example :

For a sphere, the e. values are equal. But if e. values are equal, the body may not be sym. ∵ a rigid body has infinitely many pts. — A few condns on infinite pts.

(1) Cylindrically symmetric body



$\Rightarrow$ 1) K is diagonal

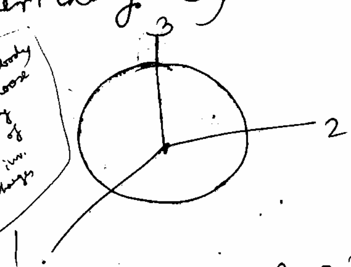
2) $\lambda_1 = \lambda_2$

(Body looks exactly the same from the pt. of view of 1 & 2)

Hy ... V effectively leaves the 3-axis inv. & makes us chose a new set of 1 & 2 axes — clearly this doesn't change the diagonal form of K w.r.t. new set of axes chosen — rotational inv. about 3-axis gives the coord. change

We may have $\lambda_1 = \lambda_2$ even if we don't have a symmetry.

(2) spherically symmetric body



$$K = \lambda \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3$

Here, K is always diagonal whatever body set of axes we choose & hence V effectively gives a new set of body axes — rot. inv. gives 3 coord. changes

We may have $\lambda_1 = \lambda_2 = \lambda_3$ even if the body is not spherically symmetric. (But it will behave as if it is sph. sym. — as far as its motion is concerned as λ_1, λ_2 & λ_3 only enter the eqns of motion)

$\tilde{S}(\theta_1, \theta_2, \theta_3)$

Both U & V transfns act as

$\dot{\theta}^i \rightarrow \phi^i(\vec{\theta})$

[The transfns don't involve $\dot{\theta}^i$'s & so don't depend on velocities]

Suppose $L \to L$ under the transfn. (True for the symmetries considered here)

Consider an infinitesimal transfn.:

$$\theta^i \longrightarrow \theta^i + \epsilon f^i(\vec{\theta})$$

Associated conserved charge (if this is a symmetry) is

$$\mathcal{Q} = \frac{\partial L}{\partial \dot{\theta}_i} f^i(\vec{\theta})$$

$$= \frac{\partial T}{\partial \dot{\theta}_i} f^i(\vec{\theta})$$

[We don't yet know the pot. But we assume that pot. is inv. under this transfn.]

[V doesn't depend on $\dot{\theta}$ & so it will be sufficient if we consider T only, assuming it is sym. for V also]

$$\delta \tilde{S}(\theta_1, \theta_2, \theta_3) = \frac{\partial \tilde{S}}{\partial \theta^i} \delta \theta^i$$

$$= \frac{\partial \tilde{S}}{\partial \theta^i} \epsilon f^i(\vec{\theta})$$

$$= \epsilon \Phi$$

Definition: $\Phi = \dfrac{\partial \tilde{S}}{\partial \theta^i} f^i(\vec{\theta})$

$\left[\because \Phi^T = \frac{\partial \tilde{S}^T}{\partial \theta^i} f^i \vec{\theta}\right]$

[$\because$ change in $\tilde{S}$ can be calculated, Φ is known]

$$T = \frac{1}{2} \mathrm{Tr}\left(\frac{d\tilde{S}}{dt} K_d \frac{d\tilde{S}^T}{dt}\right)$$

$$= \frac{1}{2} \mathrm{tr}\left(\frac{\partial \tilde{S}}{\partial \theta_k} \frac{d\theta_k}{dt} K_d \frac{\partial \tilde{S}^T}{\partial \theta_j} \frac{d\theta_j}{dt}\right)$$

$$\therefore \mathcal{Q} = \frac{\partial T}{\partial \dot{\theta}_i} f^i(\vec{\theta}) = \frac{1}{2} f^i(\vec{\theta}) \mathrm{Tr}\left(\frac{\partial \tilde{S}}{\partial \theta_i} K_d \frac{\partial \tilde{S}}{\partial \theta_j} \frac{d\theta_i}{dt}\right)$$

$$+ \frac{1}{2} f^i(\vec{\theta}) \mathrm{Tr}\left(\frac{\partial \tilde{S}}{\partial \theta_k} \frac{d\theta_k}{dt} K_d \frac{\partial \tilde{S}^T}{\partial \theta_i}\right)$$

$\mathcal{Q} = \frac{1}{2} f^i(\vec{\theta}) \mathrm{tr}\left(\frac{\partial \tilde{S}}{\partial \theta_k} \delta_{ik} K_d \frac{\partial \tilde{S}^T}{\partial \theta_j} \frac{d\theta_j}{dt}\right)$
$+ \frac{1}{2} f^i(\vec{\theta}) \mathrm{tr}\left(\frac{\partial \tilde{S}}{\partial \theta_k} K_d \delta_{ji} \frac{\partial \tilde{S}^T}{\partial \theta_i}\right)$
$=$

$$= \frac{1}{2} \mathrm{Tr}\left(\Phi K_d \frac{d\tilde{S}^T}{dt}\right) + \frac{1}{2} \mathrm{tr}\left(\frac{d\tilde{S}}{dt} K_d \Phi^T\right)$$

Transpose of one another

$$= \mathrm{Tr}\left(\frac{d\tilde{S}}{dt} K_d \Phi^T\right)$$

$$\boxed{\because f^i(\vec{\theta}) \frac{\partial \tilde{S}}{\partial \theta^i} = \Phi}$$

8] Define $\Omega = \tilde{S}^{-1} \frac{d\tilde{S}}{dt}$

$$\boxed{Q = \mathrm{Tr}\left(\tilde{S}\,\Omega\, K_d\, \Phi^T\right)}$$

Expression for a consrvd. charge for a given sym. transfn has been (~~it has~~ exp. for Φ) found without any ref. to how you parametrise S — so it is indep. of the parametrisation)

(The charge is really consrvd. or not depends on the problem — e.g. it'll depend on pot. & how many e. values are equal)

Physical interpretation of Ω:

$$\Omega = \tilde{S}^{-1}\frac{d\tilde{S}}{dt} = \tilde{S}^T \frac{d\tilde{S}}{dt}$$

$$\Omega + \Omega^T = \tilde{S}^T\frac{d\tilde{S}}{dt} + \frac{d\tilde{S}^T}{dt}\tilde{S}$$

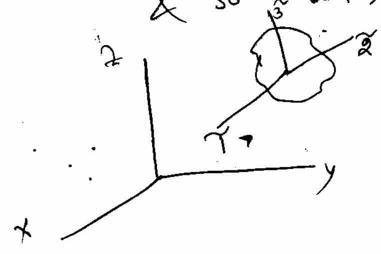
$$= \frac{d}{dt}\left(\underbrace{\tilde{S}^T\tilde{S}}_{\mathbb{1}}\right) = 0$$

$\Rightarrow \Omega$ is antisymmetric

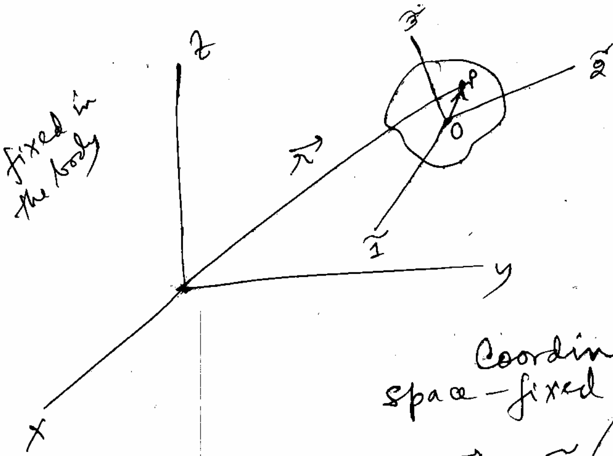
(we'll parametrise Ω as follows:-)

$$\Omega = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

[Ω is a time dep. matrix, $\because S$ is so & so $\omega_1, \omega_2, \omega_3$ are also time-dep.]



P: fixed in the body



Let us consider a fixed point on the rigid body having coordinates $(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3)$ in the body fixed frame.

Coordinates of P in the space-fixed frame:

$$\vec{r} = \vec{R} + \tilde{S}\underbrace{\begin{pmatrix}\tilde{\rho}_1\\ \tilde{\rho}_2\\ \tilde{\rho}_3\end{pmatrix}}_{\vec{\tilde{\rho}}}$$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\tilde{S}}{dt}\vec{\tilde{\rho}}$$

$$\frac{d}{dt}\left(\overrightarrow{OP}\right) = \frac{d}{dt}\left(\vec{r} - \vec{R}\right) = \frac{d\tilde{S}}{dt}\vec{\tilde{\rho}}$$

[Here, $\overrightarrow{OP} = \vec{r} - \vec{R}$]

In the space-fixed frame

Take an inertial frame that is instantaneously coincident with the body fixed frame & has the same velocity as O.

In that frame

$$\frac{d}{dt}\left(\overrightarrow{OP}\right) = \tilde{S}^{-1}\left(\frac{d\tilde{S}}{dt}\vec{\tilde{\rho}}\right)$$

$$= \Omega\,\vec{\tilde{\rho}}$$

$$= \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}\begin{pmatrix}\tilde{\rho}_1\\ \tilde{\rho}_2\\ \tilde{\rho}_3\end{pmatrix}$$

$$= \begin{pmatrix}\omega_3\tilde{\rho}_2 - \omega_2\tilde{\rho}_3\\ \omega_1\tilde{\rho}_3 - \omega_3\tilde{\rho}_1\\ \omega_2\tilde{\rho}_1 - \omega_1\tilde{\rho}_2\end{pmatrix} = \left(\vec{\omega} \times \vec{\tilde{\rho}}\right)$$

Think of 2 successive transfm. First transform from xyz to an inertial frame having same vel. as instantaneous vel. of O. Then reorient this new inertial frame along the body fixed axes

10

$\vec{\omega} \to$ instantaneous ang. vel. of the body in that inertial frame coincident -- -- & same vel. as O -- -- . . .

30/9/05

$$T = \frac{1}{2}\,\mathrm{Tr}\left(\frac{d\tilde{S}}{dt} K_d \frac{d\tilde{S}^T}{dt}\right)$$

$\tilde{S}$ is a 3×3 orthogonal matrix

$$K_d = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

[we also have some pot. terms which we aren't talking about right now]

Symmetries of T :-

① $\tilde{S} \to U\tilde{S}$ $\quad U^TU = \mathbb{1}$, $\det U = 1$ is a symmetry of T

② $\tilde{S} \to \tilde{S}V$ $\quad V^TV = \mathbb{1}$, $\det V = 1$ is a sym. of T if $K_d V = V K_d$.

Consider an infinitesimal transfn (of $\tilde{S}$) :-

$$\tilde{S} \to \tilde{S} + \epsilon\, \underbrace{\Phi}_{3\times 3 \text{ matrix}}$$

If this is a symmetry of L, then the corresponding conserved charge

$$Q = \mathrm{Tr}\left(\tilde{S}\,\Omega\, K_d\, \Phi^T\right)$$

where $\Omega = \tilde{S}^{-1}\dfrac{d\tilde{S}}{dt}$

We have constructed Q by Noether's method

~~The~~ If Q is a sym. of V also, the actual form of V doesn't enter into the expression for Q ∵ V is indep. of vel.

We'll try to construct conserved charges corresponding to U & V trs. assuming that they are symmetries.

① U-trs. :-

$$U = \mathbb{1} + \sum_{\alpha=1}^{3} \epsilon^\alpha T^\alpha$$

[where $(T^a)_{ij} = \epsilon^{aij}$ ← totally anti-symmetric matrix, $\epsilon^{123} = 1$]

U: expected to be an orthogonal matrix ∴ we add an antisym. matrix.

ϵ^α: infinitesimal parameters

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

[for any fixed a, $(T^a)_{ij} = \epsilon^{aij}$ is an antisymmetric matrix]

$$\tilde{S} \longrightarrow U\tilde{S} = \tilde{S} + \sum_{a=1}^{3} \epsilon^a \underbrace{T^a \tilde{S}}_{\Phi^a}$$

If you take 3 successive indep. transfns, then we have 3 candidates:–
$T^1\tilde{S}$, $T^2\tilde{S}$ & $T^3\tilde{S}$
$\downarrow \quad \downarrow \quad \downarrow$
$\phi_1 \quad \phi_2 \quad \phi_3$

Φ^1, Φ^2 and Φ^3 will give 3 diff. consvd. charges.

Conserved charges

$$Q^a = \mathrm{Tr}\left(\tilde{S}\,\Omega K_d\,(\Phi^a)^T\right)$$

$$\left(\Phi^a = T^a\tilde{S} \quad \therefore (\Phi^a)^T = \tilde{S}^T (T^a)^T = \tilde{S}^T(-T^a)\right.$$

[$\because T^a$ is an antisymm. matrix]

$$= -\mathrm{Tr}\left(\Omega K_d\,\tilde{S}^T T^a \tilde{S}\right)$$

$$\therefore \boxed{Q^a = -\mathrm{Tr}\left(\Omega K_d\,\tilde{S}^{-1} T^a \tilde{S}\right)}$$

($\because \tilde{S}^{-1} = S^T$, $\tilde{S}$ being an orthogonal matrix)

[It shows that it is ΩK_d times a similarity transfn $\tilde{S}^{-1} T^a \tilde{S}$]

② V – trs. :–

also an orthogonal matrix $\swarrow$ $V = \mathbb{1} + \sum_{a=1}^{3} \bar{\epsilon}^a T^a$

$$\tilde{S} \longrightarrow \tilde{S}V = \tilde{S} + \sum_{a=1}^{3} \bar{\epsilon}^a \underbrace{(\tilde{S}T^a)}_{\text{This is now our } \bar{\Phi}^a}$$

corresponding consvd. charge $\longrightarrow$

$$\bar{Q}^a = \mathrm{Tr}\left(\tilde{S}\,\Omega K_d (\bar{\Phi}^a)^T\right)$$

$$= -\mathrm{Tr}\left(\tilde{S}\,\Omega K_d T^a \tilde{S}^T\right)$$

$$= -\mathrm{Tr}\left(\Omega K_d T^a \tilde{S}^T\tilde{S}\right) \quad [\because \tilde{S}^T\tilde{S} = \mathbb{1}]$$

$$= -\mathrm{Tr}\left(\Omega K_d T^a\right)$$

$\therefore$ $$\boxed{\mathcal{P}^a = -\mathrm{Tr}\left(\Omega\, K_d\, T^a\right)}$$

[Here lowering or raising index doesn't really matter bcos we haven't told how to raise or lower index]

Calculation of $\{\mathcal{P}_a, \mathcal{P}_b\}$

$\mathcal{P}_a$'s generate
$$\delta\tilde{S} = \epsilon_a T^a \tilde{S}$$

This tells us that
$$\{\tilde{S}, \mathcal{P}^a\} = T^a \tilde{S}$$

→ This whole thing is in terms of 3×3 matrices ($\mathcal{P}^a$'s aren't !!)

Canonical transfn generated by $\mathcal{P}^a$ on anything; $\tilde{S}$ in particular

$$\{\{\tilde{S}, \mathcal{P}^a\}, \mathcal{P}^b\} = \{T^a\tilde{S}, \mathcal{P}^b\}$$

$$\Rightarrow \{\{\tilde{S}_{ij}, \mathcal{P}^a\}, \mathcal{P}^b\} = \{T^a_{ik}\tilde{S}_{kj}, \mathcal{P}^b\}$$

$$= T^a_{ik}\{\tilde{S}_{kj}, \mathcal{P}^b\}$$

$\because$ these are constants & the deriv. doesn't act on them

$$= T^a_{ik}\left(T^b\tilde{S}\right)_{kj}$$

$$= \left(T^a T^b \tilde{S}\right)_{ij}$$

[The matrix doesn't interfere with $\mathcal{P}^a$'s – $\mathcal{P}^a$'s are just single objects]

$$\boxed{\{\{\tilde{S}, \mathcal{P}^a\}, \mathcal{P}^b\} = T^a T^b \tilde{S}}$$

Still now we haven't introduced the Θ_i's & conjugate momenta & so we are trying to calculate $\{\mathcal{P}_a, \mathcal{P}_b\}$ in an indirect way.

$$\{\{\tilde{S}, \mathcal{P}^b\}, \mathcal{P}^a\} = T^b T^a \tilde{S}$$

Take the difference:

$$\{\{\tilde{S}, \mathcal{P}^a\}, \mathcal{P}^b\} - \{\{\tilde{S}, \mathcal{P}^b\}, \mathcal{P}^a\}$$

$$= \left(T^a T^b - T^b T^a\right)\tilde{S}$$

Now $(T^a T^b - T^b T^a)_{ij} = \epsilon^{aik}\epsilon^{bkj} - \epsilon^{bik}\epsilon^{akj} = \delta_{aj}\delta_{bi} - \delta_{ai}\delta_{bj}$

$\therefore\ = \epsilon^{cab}\epsilon^{cji} = \epsilon^{abc}(T^c)_{ji}$

$= -\epsilon^{abc}(T^c)_{ij}$

Now, $\boxed{T^a T^b - T^b T^a = -\sum_{c=1}^{3} \epsilon^{abc} T^c} \to$ Ex.

$\therefore \{\{\tilde{S}, \varphi^a\}, \varphi^b\} - \{\{\tilde{S}, \varphi^b\}, \varphi^a\} = -\sum_{c=1}^{3} \epsilon^{abc} \underbrace{T^c \tilde{S}}_{\varphi^c}$.

Ex: check using the defn of Poisson bracket that given any three functions $F(\vec{q},\vec{p})$, $G(\vec{q},\vec{p})$, $H(\vec{q},\vec{p})$

$\{\{F, G\}, H\} - \{\{F, H\}, G\} = \{F, \{G, H\}\}$

Jacobi's identity: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

$$\{\tilde{S}, \{\varphi^a, \varphi^b\}\} = -\sum \epsilon^{abc} \{\tilde{S}, \varphi^c\}$$

$$\Rightarrow \{\tilde{S}, \{\varphi^a, \varphi^b\} + \sum_c \epsilon^{abc} \varphi^c\} = 0$$

$$\cancel{\Rightarrow \frac{\partial \tilde{S}}{\partial \theta^i} \{\theta^i, \{\varphi^a, \varphi^b\} + \sum_c \epsilon^{abc} \varphi^c\} = 0}$$

[So this is actually 3 eqns]

[$\tilde{S}$ depends on the coord. & not on the conjugate momenta]

$$\Rightarrow \frac{\partial \tilde{S}}{\partial \theta^i} \underbrace{\frac{\partial}{\partial p_i} \left(\{\varphi^a, \varphi^b\} + \sum_c \epsilon^{abc} \varphi^c \right)}_{\text{must vanish}} = 0$$

$$\Rightarrow \frac{\partial}{\partial p_i} \left(\{\varphi^a, \varphi^b\} + \sum_c \epsilon^{abc} \varphi^c \right) = 0$$

So, $\left(\{\varphi^a, \varphi^b\} + \sum_c \epsilon^{abc} \varphi^c\right)$ should depend on the coord. & not on the momenta.

(Now, we want to show that it can't happen unless the quantity itself is zero)

You can also see how φ^a transforms under a transfn. generated by φ^b & can calculate $\{\varphi_a, \varphi_b\}$

$\mathcal{J}^a$ depends linearly on $\dot{\theta}^i$'s [due to Ω, $= S^{-1}\frac{dS}{dt}$]

Each term in $\mathcal{J}^a$ must have a $\dot{\theta}^i$ & there can't be any $\dot{\theta}^i$-indep. term

$$T = \frac{1}{2}\mathrm{Tr}\left(\frac{d\tilde{S}}{dt} K_d \frac{d\tilde{S}^T}{dt}\right)$$

another $\dot{\theta}_i$ comes from here

one $\dot{\theta}_i$ comes from here

quadratic in $\dot{\theta}_i$'s

$p_i = \frac{\partial T}{\partial \dot{\theta}^i}$ = linear in $\dot{\theta}^i$'s

$\mathcal{J}^a$ depends linearly on p_i's.

$$\{\mathcal{J}^a, \mathcal{J}^b\} = \frac{\partial \mathcal{J}^a}{\partial \theta^i}\frac{\partial \mathcal{J}^b}{\partial p_i} - \frac{\partial \mathcal{J}^a}{\partial p_i}\frac{\partial \mathcal{J}^b}{\partial \theta^i}$$

linear in p_i

doesn't depend on p_i

$\rightarrow$ linear in p_i

Hence, we get, $\{\mathcal{J}^a, \mathcal{J}^b\} + \sum_c \epsilon^{abc}\mathcal{J}^c = 0$

$$\Rightarrow \{\mathcal{J}^a, \mathcal{J}^b\} = -\sum_c \epsilon^{abc}\mathcal{J}^c$$

$\mathcal{J}^a$ is linear & homogeneous in $\dot{\theta}$'s

$\{\mathcal{J}^a, \mathcal{J}^b\} + \sum_c \epsilon^{abc}\mathcal{J}^c$ is also linear & homogeneous in p_i's & hence $---- = 0$

If the body is totally asymmetric, V can't be used

$$\vec{\mathcal{J}}^2 = (\mathcal{J}^1)^2 + (\mathcal{J}^2)^2 + (\mathcal{J}^3)^2 = \sum_a \mathcal{J}^a\mathcal{J}^a$$

$$\{\vec{\mathcal{J}}^2, \mathcal{J}^b\} = \sum_a \{\mathcal{J}^a\mathcal{J}^a, \mathcal{J}^b\}$$

Ex. Check that
$$\{FG, H\} = \{F, H\}G + \{G, H\}F$$

Using the above result, we get,
$$\{\vec{\wp}^2, \wp^b\} = 2\sum_a \wp^a \underbrace{\{\wp^a, \wp^b\}}_{-\sum_c \epsilon^{abc}\wp^c}$$ [with $F = G = \wp^a$]

$$= -2\sum_a \sum_c \underset{\substack{\text{anti-sym.}\\ \text{in } a \,\&\, c}}{\epsilon^{abc}} \underbrace{\wp^a \wp^c}_{\substack{\text{sym. in}\\ a \,\&\, c}}$$

$$= 0$$

[If $\wp^a$ has vanishing Poisson bracket with H, $\vec{\wp}^2$ ~~has~~ also ~~have~~ --- with H]

If all U trs. are symmetries, then $\wp^3$, $\vec{\wp}^2$ & H have vanishing Poisson bracket with each other.

[$\{\wp^3, H\} = 0$ becos ~~its~~ our assumption is that the model is such that $\wp^3$ is consvd.]

[It's a system with 3 coord. & we are getting 3 consvd. charges ($\wp^3$, $\vec{\wp}^2$ & H) & so its solvable]

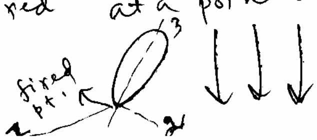
① Free totally asymmetric rigid body

No potential & obviously all U trs. are symmetries.

$\therefore$ $\wp^3$, $\vec{\wp}^2$, H are the consvd. charges.

For totality: $\wp^3$, $\vec{\wp}^2$, H, p_1, p_2, p_3 → 6 consvd. charges & 6 deg. of freedom

② Take a rigid body with axial symmetry fixed at a point on its axis (Rigid body with constraints).



subject to uniform gravitational field along 3-dirn.

16) $\bar{\mathcal{Q}}^3$ is consrvd. ($\because \lambda_1 = \lambda_2 \neq \lambda_3$)

[An arbit. rot. sym. is broken due to the grav. field (which is along z-dirn.)

$\because$ Pot. depends only on z, rotation about z is allowed.]

$\therefore \mathcal{Q}^3$ is another consrvd. charge.

$\therefore \bar{\mathcal{Q}}^3, \mathcal{Q}^3$ & $H \Longrightarrow$ 3 consrvd. charges.

[$\bar{\mathcal{Q}}^2$ is consrvd. only if all $\mathcal{Q}^i$'s are consrvd, $\because$ $\{\bar{\mathcal{Q}}^2, H\} \neq 0$ if any $\{\mathcal{Q}^i, H\} \neq 0$]

Free asymmetric rigid body

$$L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mathrm{Tr}\left(\frac{d\tilde{S}}{dt} K_d \frac{d\tilde{S}^T}{dt}\right)$$

work in the centre of mass frame $\vec{R} = 0$.

Totally asym. rigid body in uniform grav. field isn't solvable bcos we have only $\mathcal{Q}^3$ & H as the consrvd. charges

Symmetries: (1) $\tilde{S} \to U\tilde{S} \Rightarrow$ conserved charges $\mathcal{Q}^a = -\mathrm{Tr}(\Omega K_d \tilde{S}^{-1} t^a \tilde{S})$

(2) Time translation $\Rightarrow H = \frac{1}{2}\mathrm{Tr}\left(\frac{d\tilde{S}}{dt} K_d \frac{d\tilde{S}^T}{dt}\right)$

$$\Omega = \tilde{S}^{-1}\frac{d\tilde{S}}{dt} \Rightarrow \frac{d\tilde{S}}{dt} = \tilde{S}\Omega$$

$$H = \frac{1}{2}\mathrm{Tr}(\tilde{S}\Omega K_d \underset{-\Omega}{\underset{\shortparallel}{\Omega^T}} \tilde{S}^T)$$

$$= -\frac{1}{2}\mathrm{Tr}(\Omega K_d \Omega)$$

[H depends on the coord. velocities only through the combn. Ω — doesn't involve any other $\tilde{S}$].

$$\dot{\Omega} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

$$K_d = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\Rightarrow H = \frac{1}{2}(\lambda_1+\lambda_2)\omega_3^2 + \frac{1}{2}(\lambda_2+\lambda_3)\omega_1^2 + \frac{1}{2}(\lambda_1+\lambda_3)\omega_2^2$$

Remember the defn of inertial matrix:

$$I = (\mathrm{Tr}\,K)\,\mathbb{1} - K$$

In the basis where K is diagonal, I is also diagonal:

$$I = (\lambda_1+\lambda_2+\lambda_3)\,\mathbb{1} - K_d$$

$$= \begin{pmatrix} \underbrace{\lambda_2+\lambda_3}_{=I_1} & 0 & 0 \\ 0 & \underbrace{\lambda_1+\lambda_3}_{=I_2} & 0 \\ 0 & 0 & \underbrace{\lambda_1+\lambda_2}_{=I_3} \end{pmatrix}$$

M.I.'s ~~about~~ along the principal axes of the body

$I_1, I_2, I_3 \longrightarrow$ 3 e.values of the inertial matrix

$$\therefore\ H = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

[We would like to avoid writing explicitly the comp. of S bcos that needs parametrization in terms of Euler angles]

$$\vec{\mathcal{J}}^2 = \sum_a \mathcal{J}^a \mathcal{J}^a \qquad \mathrm{Tr}\left(\Omega K_d \tilde{S}^{-1} T^a \tilde{S}\right)\ \mathrm{Tr}\left(\Omega K_d \tilde{S}^{-1} T^a \tilde{S}\right)$$

$$= \sum_a \mathrm{Tr}\left(\Omega K_d \tilde{S}^{-1} T^a \tilde{S}\right) \ldots$$

$$= \sum_a \left[(\tilde{S}\Omega K_d \tilde{S}^{-1})_{ij} (T^a)_{ji} \, (\tilde{S}\Omega K_d \tilde{S}^{-1})_{kl} (T^a)_{lk} \right]$$

a dependent terms

[We don't want to get rid of Ω's — we want to get rid of $\tilde{S}$'s]

Now, $\sum_a (T^a)_{ji}(T^a)_{lk} = \sum_a \epsilon^{aji}\epsilon^{alk} = \delta^{jl}\delta^{ik} - \delta^{jk}\delta^{il}$ ~~$\delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il}$~~

18

$$\Rightarrow \vec{\mathcal{L}}^2 = \left(\tilde{S}\Omega K_d S^{-1}\right)_{ij}\left(\tilde{S}\Omega K_d \tilde{S}^{-1}\right)_{kl}\left(\delta^{ik}\delta^{jl} - \delta^{jk}\delta^{il}\right)$$

$$= \left(\tilde{S}\Omega K_d S^{-1}\right)_{ij}\left(\tilde{S}\Omega K_d \tilde{S}^{-1}\right)_{ij} - \left(\tilde{S}\Omega K_d S^{-1}\right)_{ij}\left(\tilde{S}\Omega K_d \tilde{S}^{-1}\right)_{ji}$$

We want to reinterpret this as matrix prdt.

transpose

$$= \left(\tilde{S}\Omega K_d \tilde{S}^{-1}\right)_{ij}\left(\tilde{S}\Omega K_d \tilde{S}^{-1}\right)^T_{ji} - \mathrm{Tr}\left(\tilde{S}\Omega K_d \underbrace{S^{-1}\tilde{S}}_{\mathbb{1}}\Omega K_d \tilde{S}^{-1}\right)$$

$$= \mathrm{Tr}\left(\left(\tilde{S}\Omega K_d \tilde{S}^{-1}\right)\left(\tilde{S}\Omega K_d \tilde{S}^{-1}\right)^T\right) - \mathrm{Tr}\left(\tilde{S}\Omega K_d \Omega K_d \tilde{S}^{-1}\right)$$

$$= \mathrm{Tr}\left(\tilde{S}\Omega K_d \underbrace{\tilde{S}^{-1}(\tilde{S}^{-1})^T}_{\mathbb{1}} K_d \Omega^+ \tilde{S}^T\right) - \mathrm{Tr}\left(\Omega K_d \Omega K_d \underbrace{\tilde{S}^{-1}\tilde{S}}_{\mathbb{1}}\right)$$

$$= \mathrm{Tr}\left(\tilde{S}^T\tilde{S}\Omega K_d K_d \Omega^+\right) - \mathrm{Tr}\left(\Omega K_d \Omega K_d\right)$$

$$= \mathrm{Tr}\left(\Omega K_d K_d \Omega^+\right) - \mathrm{Tr}\left(\Omega K_d \Omega K_d\right)$$

$$= -\mathrm{Tr}\left(\Omega K_d^2 \Omega\right) - \mathrm{Tr}\left(\Omega K_d \Omega K_d\right)$$

Ex: Show that $\vec{\mathcal{L}}^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$

3-dim. ellipse eqn. & so can be $\vec{\omega}$ such that its tip always lies on this ellipse

H.W.

$$\Omega = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

$$K_d = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\therefore \; K_d^2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}$$

$$K_d^2 \Omega = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \lambda_1^2 \omega_3 & -\lambda_1^2 \omega_2 \\ -\lambda_2^2 \omega_3 & 0 & \omega_1 \lambda_2^2 \\ \omega_2 \lambda_3^2 & -\omega_1 \lambda_3^2 & 0 \end{pmatrix}$$

$$\Omega K_d^2 \Omega = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda_1^2 \omega_3 & -\lambda_1^2 \omega_2 \\ -\omega_3 \lambda_2^2 & 0 & \omega_1 \lambda_2^2 \\ \omega_2 \lambda_3^2 & -\omega_1 \lambda_3^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\omega_3^2 \lambda_2^2 - \omega_2^2 \lambda_3^2 & -\omega_1 \omega_2 \lambda_3^2 & \omega_1 \omega_3 \lambda_2^2 \\ \omega_1 \omega_2 \lambda_3^2 & -\lambda_1^2 \omega_3^2 - \omega_1^2 \lambda_3^2 & -\lambda_1^2 \omega_2 \omega_3 \\ -\omega_1 \omega_3 \lambda_2^2 & \omega_2 \omega_3 \lambda_1^2 & -\omega_2^2 \lambda_1^2 - \omega_1^2 \lambda_2^2 \end{pmatrix}$$

$$\therefore -\mathrm{tr}\left(\Omega K_d^2 \Omega\right) = \omega_3^2 \lambda_2^2 + \omega_2^2 \lambda_3^2 + \omega_3^2 \lambda_1^2 + \lambda_3^2 \omega_1^2 + \lambda_1^2 \omega_2^2 + \lambda_2^2 \omega_1^2$$

$$= \omega_3^2 (\lambda_1^2 + \lambda_2^2) + \omega_2^2 (\lambda_1^2 + \lambda_3^2) + \omega_1^2 (\lambda_3^2 + \lambda_2^2)$$

29

$$\Omega K_d = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \omega_3\lambda_2 & -\omega_2\lambda_3 \\ -\omega_3\lambda_1 & 0 & \omega_1\lambda_3 \\ \omega_2\lambda_1 & -\omega_1\lambda_2 & 0 \end{pmatrix}$$

$$\therefore \Omega K_d \Omega K_d = \begin{pmatrix} 0 & \omega_3\lambda_2 & -\omega_2\lambda_3 \\ -\omega_3\lambda_1 & 0 & \omega_1\lambda_3 \\ \omega_2\lambda_1 & -\omega_1\lambda_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega_3\lambda_2 & -\omega_2\lambda_3 \\ -\omega_3\lambda_1 & 0 & \omega_1\lambda_3 \\ \omega_2\lambda_1 & -\omega_1\lambda_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\omega_3^2\lambda_1\lambda_2 - \omega_2^2\lambda_1\lambda_3 & \omega_2^2\lambda_2\lambda_3 & \omega_1\omega_3^2\lambda_2\lambda_3 \\ \omega_1\omega_2\lambda_1\lambda_3 & -\omega_3^2\lambda_1\lambda_2 - \omega_1^2\lambda_2\lambda_3 & \omega_2\lambda_3\lambda_1\lambda_3 \\ \omega_2\omega_3\lambda_1\lambda_2 & \omega_2\omega_3\lambda_1\lambda_2 & -\omega_2^2\lambda_1\lambda_3 - \omega_1^2\lambda_2\lambda_3 \end{pmatrix}$$

$$\therefore -\mathrm{Tr}(\Omega K_d \Omega K_d)$$

$$= \omega_3^2\lambda_1\lambda_2 + \omega_2^2\lambda_1\lambda_3 + \omega_3^2\lambda_1\lambda_2 + \omega_1\omega_2\lambda_2\lambda_3 + \omega_2^2\lambda_1\lambda_3 + \omega_1\omega_2\lambda_2\lambda_3$$

$$= 2\omega_3^2\lambda_1\lambda_2 + 2\omega_2^2\lambda_1\lambda_3 + 2\omega_1^2\lambda_2\lambda_3$$

g^2

$$\therefore \vec{g}^2 = \omega_3^2(\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2) + \omega_2^2(\lambda_1^2+\lambda_3^2+2\lambda_1\lambda_3) + \omega_1^2(\lambda_2^2+\lambda_3^2+2\lambda_2\lambda_3)$$

$$= I_3^2\omega_3^2 + I_2^2\omega_2^2 + I_1^2\omega_1^2$$

$$= \sum_{i=1}^{3} I_i^2\omega_i^2$$

5/10/05 Free asymmetric rigid body

Conserved charges Q_1, Q_2, Q_3, H.

$(Q_3, \vec{Q}^2, H)$ form a set of charges with vanishing Poisson bracket

$\Rightarrow$ system is solvable.

Q_3 depends on $\tilde{S}$ and $\Omega = \tilde{S}^{-1}\dfrac{d\tilde{S}}{dt}$.

$\vec{Q}^2$ and H depend only on Ω. rather than on the full details of $\tilde{S}$

Conserved charges

$$H = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

$$\vec{Q}^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

$$\Omega = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

H = constant $\Rightarrow$ ellipsoid

$\vec{Q}^2$ = constant $\Rightarrow$ ellipsoid

Tip of $\vec{\omega}$ lies on a one dimensional curve at the intersection of these two ellipsoids.

Study time evolution of $\vec{\omega}$

$\xrightarrow{\text{this requires}}$ must analyze the eqns of motion.

$$\text{Action} = \frac{1}{2}\int dt \; \mathrm{Tr}\left(\frac{d\tilde{S}}{dt} K_d \frac{d\tilde{S}^T}{dt}\right)$$

$$\delta(\text{Action}) = \frac{1}{2}\int dt\, \mathrm{Tr}\left(\frac{d\,\delta\tilde{S}}{dt} K_d \frac{d\tilde{S}^T}{dt} + \frac{d\tilde{S}}{dt} K_d \frac{d(\delta\tilde{S}^T)}{dt}\right)$$

Integrate by parts

$$= -\frac{1}{2}\int dt\, \mathrm{Tr}\left(\delta\tilde{S} K_d \frac{d^2\tilde{S}^T}{dt^2} + \frac{d^2\tilde{S}}{dt^2} K_d\, \delta\tilde{S}^T\right)$$

$$= -\int dt\, \mathrm{Tr}\left(\frac{d^2\tilde{S}}{dt^2} K_d\, \delta\tilde{S}^T\right).$$

[using cyclic prop. of trace]

U can do the integration component by component, though they are matrices.

($\tilde{S}$ isn't an indep. 3x3 matrix, its not an arbit 3x3 matrix — so an arbit matrix variation won't fully with the fact that $\tilde{S}$ is a 3x3 orthogonal matrix)

22) We can write,

$$\delta(\text{Action}) = -\int dt \, \mathrm{tr}\left(\tilde{S}^{-1} \frac{d^2 \tilde{S}}{dt^2} K_d \, \delta \tilde{S}^T \tilde{S}\right)$$

(we have inserted these)

We should expect 3 indep. eqns of motion. If we had set each ij-th comp. equal to 0, we would have got 9 eqns, which is obviously wrong

Now $\delta \tilde{S}^T \tilde{S} + \underbrace{\tilde{S}^T \delta \tilde{S}}_{(\delta \tilde{S}^T \tilde{S})^T} = \delta(\underbrace{\tilde{S}^T S}_{\mathbb{1}}) = 0$

$\therefore (\delta \tilde{S}^T \tilde{S})$ is an antisymmetric matrix.

($\because \tilde{S}$ has 3 indep. variations ---)

Consider an arbitrary 3×3 ~~matrix~~ anti-symmetric matrix A.

We can always find a $\delta \tilde{S}$ such that $\delta \tilde{S}^T \tilde{S} = \epsilon A$
↳ small number

3 indep. variation that you can think of ($\because$ there are 3 adjustable parameters)

(To prove the above statement,) → $\delta \tilde{S}^T \tilde{S} = \epsilon A$. (Now let's try to check if this is orthonormal)

Let us define $\delta \tilde{S}^T$ by the eqn.

$\delta \tilde{S}^T = \epsilon A \tilde{S}^{-1}$

Proof

[Q] $(\tilde{S}^T + \delta \tilde{S}^T)(\tilde{S} + \delta \tilde{S})$ is identity to the order ϵ, then ---

$$(\tilde{S}^T + \delta \tilde{S}^T)(\tilde{S} + \delta \tilde{S})$$
$$= \tilde{S}^T \tilde{S} + \delta \tilde{S}^T \tilde{S} + \tilde{S}^T \delta \tilde{S} + O(\epsilon^2)$$
$$= \mathbb{1} + \epsilon A + \epsilon A^T = \mathbb{1} \quad \text{if } A + A^T = 0$$

[So, $\delta \tilde{S}^T \tilde{S} = \epsilon A$ preserves the ~~id~~ condition that $\delta(\tilde{S}^T S) = \mathbb{1}$]

Choose $\delta \tilde{S}^T = \epsilon A \tilde{S}^{-1}$

$$\Rightarrow \delta(\text{Action}) = \int dt \, \mathrm{Tr}\Big(\overbrace{\tilde{S}^{-1} \frac{d^2\tilde{S}}{dt^2} K_d}^{B} \, \epsilon A\Big)$$

arbitrary anti-symmetric matrix

This should be zero for arbitrary A.
(This will be the eqn of motion — ∴ we get an eqn of motion for an arbit. variation of $\tilde{S}$)

$\mathrm{Tr}(BA) = 0$ for arbitrary antisymmetric matrix A.

$$\Rightarrow B_{ij} A_{ji} = 0. \quad (B \text{ must be symmetric})$$

$\xrightarrow[\text{implies}]{\text{this}}$ $B_{ij} = B_{ji}$

$$\Rightarrow B_{ij} - B_{ji} = 0$$

[The anti-sym. part of B must vanish — so B must be sym.. — Note any matrix can be written as sum of a sym. & an anti-sym. part]

(for $i = j$, it is trivial, ∴ no. of indep. eqn. is 3 — there are ~~3 ind~~ 3 indep. parameters other than the diagonal elements of B)

$$\Rightarrow \boxed{\tilde{S}^{-1} \frac{d^2\tilde{S}}{dt^2} K_d - \left(\tilde{S}^{-1} \frac{d^2\tilde{S}}{dt^2} K_d\right)^T = 0}$$

(It will be advantageous to rewrite in terms of Ω, because there is a nice algebraic eqn. betn. ω_1, ω_2 & ω_3)

Now $\Omega = \tilde{S}^{-1} \frac{d\tilde{S}}{dt} \Rightarrow \frac{d\tilde{S}}{dt} = \tilde{S}\Omega$

$$\Rightarrow \tilde{S}^{-1} \frac{d}{dt}(\tilde{S}\Omega) K_d - \left\{\tilde{S}^{-1} \frac{d}{dt}(\tilde{S}\Omega) K_d\right\}^T = 0$$

$$\Downarrow$$

$$\tilde{S}^{-1} \frac{d\tilde{S}}{dt} \Omega K_d + \tilde{S}^{-1} \tilde{S} \frac{d\Omega}{dt} K_d$$

$$= \Omega^2 K_d + \frac{d\Omega}{dt} K_d$$

24

$$\Rightarrow \underbrace{-\Omega^2 K_d}_{\substack{\text{symmetric matrix}\\ [\because\ \Omega \text{ is antisym.}]}} + \overbrace{\underbrace{\frac{d\Omega}{dt} K_d}_{\substack{\text{antisym.}\\ \text{matrix}}}}^{\text{sym. matrix}} - \underbrace{K_d \Omega^2}_{\substack{\text{symmetric}\\ \text{matrix}}} + K_d\left(-\frac{d\Omega}{dt}\right) = 0$$

$$K_d = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} &\frac{d\Omega}{dt} K_d + K_d \frac{d\Omega}{dt} = \begin{pmatrix} 0 & (\lambda_1+\lambda_2)\dot\omega_3 & (\lambda_1+\lambda_3)\dot\omega_2 \\ (\lambda_1+\lambda_2)\dot\omega_3 & 0 & (\lambda_2+\lambda_3)\dot\omega_1 \\ (\lambda_1+\lambda_3)\dot\omega_2 & (\lambda_2+\lambda_3)\dot\omega_1 & 0 \end{pmatrix} \leftarrow \text{sym. matrix} \\ &-\Omega^2 K_d - K_d \Omega^2 = \begin{pmatrix} 0 & (\lambda_2-\lambda_1)\omega_1\omega_2 & (\lambda_3-\lambda_1)\omega_1\omega_3 \\ -(\lambda_2-\lambda_1)\omega_1\omega_2 & 0 & (\lambda_3-\lambda_2)\omega_2\omega_3 \\ -(\lambda_3-\lambda_1)\omega_1\omega_3 & -(\lambda_3-\lambda_2)\omega_2\omega_3 & 0 \end{pmatrix} \leftarrow \text{antisym. matrix} \end{aligned}}$$

$$\Rightarrow (\lambda_1+\lambda_2)\frac{d\omega_3}{dt} = (\lambda_1-\lambda_2)\,\omega_1\omega_2$$

$$(\lambda_1+\lambda_3)\frac{d\omega_2}{dt} = (\lambda_3-\lambda_1)\,\omega_3\omega_1$$

$$(\lambda_2+\lambda_3)\frac{d\omega_1}{dt} = (\lambda_2-\lambda_3)\,\omega_2\omega_3$$

Putting $I_1 = \lambda_2+\lambda_3$; $I_2 = \lambda_1+\lambda_3$, $I_3 = \lambda_1+\lambda_2$, we get,

$$\boxed{\begin{aligned} I_3 \frac{d\omega_3}{dt} &= (I_2 - I_1)\,\omega_1\omega_3 \\ I_2 \frac{d\omega_2}{dt} &= (I_1 - I_3)\,\omega_1\omega_3 \\ I_1 \frac{d\omega_1}{dt} &= (I_3 - I_2)\,\omega_2\omega_3 \end{aligned}} \quad \text{— (A)}$$

(ω_i's can in turn be rewritten in terms of $\vec{S}$)

Eqns. of motion in terms of ω_i's

Note: a sign diff. from Gold.-201. Same sign if we take $\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ \omega_2 & \omega_1 & 0 \end{pmatrix}$

Ex. Verify the conservation laws of H & $\vec{g}^{\,2}$ directly using these eqns. Do the same for g_3. (g_3 is written in terms of $\vec{S}$ — so write $\frac{dg_3}{dt}$ which involves $d\vec{S}/dt$ in terms of ω_i's)

Try to manipulate...

Show $dH/dt = 0$, $d\vec{g}^{\,2}/dt = 0$, $dg_3/dt = 0$

$$\frac{dH}{dt} = \frac{1}{2}\frac{d}{dt}\left[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right]$$
$$= \omega_1 I_1 \, d\omega_1/dt + \omega_2 I_2 \, d\omega_2/dt + \omega_3 I_3 \, d\omega_3/dt$$
$$= \omega_1\omega_2\omega_3\left[I_2 - I_1 + I_1 - I_3 + I_3 - I_2\right] = 0$$

$$d\vec{g}^{\,2}/dt = \frac{d}{dt}\left[I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2\right]$$
$$= 2\omega_1\omega_2\omega_3\left[I_1(I_3 - I_2) + I_2(I_1 - I_3) + I_3(I_2 - I_1)\right]$$
$$= 0$$

To solve these eqns, find ω_3 and ω_2 as fn. of ω_1:

Using $H =$ constant
$\vec{J}^2 =$ Constant

We have 2 linear eqns in ω_i^2's

Substitute into any of the 3 eqns (A) $\xrightarrow[\text{give us}]{\text{this'll}}$ $\frac{d\omega_1}{dt}$ as a fn. of ω_1

$$\frac{d\omega_1}{dt} = F(\omega_1) \Rightarrow t = \int_{\omega_1(0)}^{\omega_1} \frac{d\omega_1'}{F(\omega_1')}$$

[This gives us t as a fn. of ω_1 & also ω_1 as a fn. of t. Then we can write ω_2 & ω_3 also in terms of t]

Qualitative study of the dynamics of ω_i's

The differential eqn. for $\vec{\omega}$ can be regarded as a dynamical system $\longrightarrow$ 3rd order autonomous dynamical system.

$\omega_1, \omega_2, \omega_3$ can be thought of as a dynamical system of 3rd order — ∵ we have 3 first order eqns in 3 variables.

[First thing we should look for is fixed pt. — ~~for~~ fixed pts., $\vec{\omega}$ doesn't change]

Fixed points

(1) $\omega_1 = \omega_2 = \omega_3 = 0$ (trivial fixed point — ∵ body isn't rotating)

(2) $\omega_1 = \omega_2 = 0$

(3) $\omega_1 = \omega_3 = 0$

(4) $\omega_2 = \omega_3 = 0$

[Here we are considering $I_1 \neq I_2 \neq I_3$]

$\vec{\omega} \to$ angular vel. in the body fixed coord.

Axis of rot. won't remain fixed for any arbit. choice — it will start ~~w~~obbling about the ~~body~~ fixed frame ~~But for I_1, I_2, I_3 fixed~~

Rigid body rotating about a fixed axis — for fixed point

But for axes which diagonalise the inertia-matrix → fixed

26 Stability of the fixed point $\omega_3 = \omega_2 = 0$

$\downarrow$ rotation about 1 axis

$\omega_3 = \omega_3^{(0)}$ $\rightsquigarrow$ (fixed by how much energy the body has — some kind of initial condn.)

$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = \begin{pmatrix}\omega_1^{(0)}\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}\frac{I_3 - I_2}{I_1}\, yz\\ \frac{I_1 - I_3}{I_2}\, \omega_1^{(0)} z\\ \frac{I_2 - I_1}{I_3}\, \omega_1^{(0)} y\end{pmatrix}$$

[we'll throw away terms which are quadratic or higher order in x, y, z]

$$I_3 \dot{\omega}_3 = (I_2 - I_1)\, \omega_2 \omega_1$$
$$I_2 \dot{\omega}_2 = (I_1 - I_3)\, \omega_1 \omega_3$$
$$I_1 \dot{\omega}_1 = (I_3 - I_2)\, \omega_2 \omega_3$$

(But we aren't ~~kept~~ supposed to keep yz, which is 2nd order)

$$\therefore \frac{d}{dt}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ \frac{I_1 - I_3}{I_2}\, \omega_1^{(0)} z\\ \frac{I_2 - I_1}{I_3}\, \omega_1^{(0)} y\end{pmatrix} + \text{quadratic in } x, y, z$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \frac{I_1 - I_3}{I_2}\, \omega_1^{(0)}\\ 0 & \frac{I_2 - I_1}{I_3}\, \omega_1^{(0)} & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

Eigenvalues:

$$\lambda = 0$$

$$\text{or } \lambda^2 = \frac{I_1 - I_3}{I_2} \cdot \frac{I_2 - I_1}{I_3} \left(\omega_1^{(0)}\right)^2$$

[The first row & column doesn't do anything — it's already in the block diagonal form — one eigenvalue must be zero]

[Both e-values −ve impossible ∵ trace is zero
The best we can have is that these 2
imaginary] — [also, λs will come in opposite pairs — one +ve & one −ve if λ^2 is +ve]

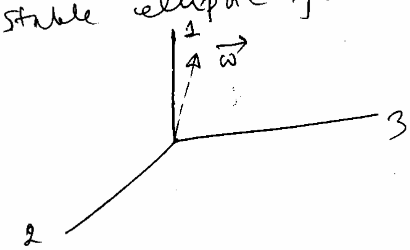
Two cases

(1) I_1 lies betn. I_2 and I_3
$\Rightarrow \lambda^2$ is positive $\Rightarrow$ 1 +ve & 1 −ve e-value.
$\Rightarrow$ hyperbolic fixed point — not stable
($\vec{\omega}$ will deviate from one of the axes as time goes on)

(2) Either $I_1 > I_2, I_3$
or $I_1 < I_2, I_3$

$\boxed{I_1 \text{ must be the max. or minimum of all the } I_\alpha\text{'s.}}$

$\Rightarrow \lambda^2 < 0$
$\Rightarrow \lambda = \pm i\omega$
$\Rightarrow$ stable elliptic fixed point.



[$\vec{\omega}$, slightly deviated from being along 1 axis, precesses about 1 axis.
y & 3 axes will precess with ang. freq. ω
∵ $\lambda = \pm i\omega$]

$$\frac{dy}{dt} = \frac{I_1 - I_3}{I_2}\,\omega_1^{(0)}\, z$$

$$\frac{d^2y}{dt^2} = \frac{I_1 - I_3}{I_2}\,\omega_1^{(0)}\left(\frac{I_2 - I_1}{I_3}\right)\omega_1^{(0)}\, y$$

$$= -\omega^2 y$$

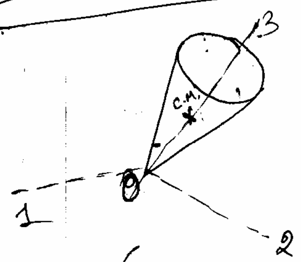
$\Rightarrow y = A\cos(\omega t + \phi)$

Also, $z = -\dfrac{A I_2}{I_1 - I_3}\,\dfrac{1}{\omega_1^{(0)}}\,\omega \sin(\omega t + \phi)$

[∴ y & z has ~~angular~~ periodic motion with ang. freq. ω]

Axially symmetric rigid body under uniform gravity, fixed at a point on the axis of symmetry.

28



$\downarrow g$

3 changes →
① More symmetrical body
② But subject to uniform gravity — so giving up some of the symmetries
③ Harder problem as C.M. coord. can't be ~~chosen to simplify~~ matters.

[This choice of axes diagonalises the matrix K — Check]

$$T = \frac{1}{2}\left(\frac{d\tilde{S}}{dt} K_d \frac{d\tilde{S}^{T}}{dt}\right)$$

$$K_d = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

[$\because \lambda_1 = \lambda_2$, due to complete sym. betn. the 1 & 2 axes]

The additional potential term would not depend on $\tilde{S}$ as the centre of mass moves independently — uniform grav. doesn't affect the CM coord.

[We choose the origin at the fixed pt. instead of the C.M. — the price we have to pay is that the location of C.M. will matter & grav. pot. will depend on $\tilde{S}$]

$$V = \sum_{a} m_{(a)}\, g\, z_{(a)}$$

a-th particle; mass of the a-th particle; z-coordinate of this particle

$$\vec{r}_{(a)} = \vec{R} + \tilde{S}\,\vec{\rho}_{(a)}$$

$\vec{r}_{(a)}$: coordinate vector in space fixed frame

$\vec{R}$: Coordinate of O in space-fixed frame

$\vec{\rho}_{(a)}$: Coordinate of a in body fixed frame

$$\cancel{V = g \sum_a (Z + \tilde{S}}$$

$$\sum_a m_{(a)} \vec{r}_{(a)} = \left(\sum_a m_{(a)}\right) \vec{r} + \tilde{S} \sum_a m_{(a)} \vec{\rho}_{(a)}$$

$$= \underset{\substack{\text{coord. of} \\ \text{the tip}}}{M\vec{R}} + \tilde{S} M \vec{\rho}_{c.m.}$$

fixed, the tip is fixed

$$\vec{\rho}_{c.m.} = \begin{pmatrix} 0 \\ 0 \\ \rho^{(0)} \end{pmatrix}$$

$$V = g \sum_a m_a z_{(a)}$$

$$= g M \underset{\substack{\downarrow \\ z \text{ comp. of } 0}}{z} + M \rho^{(0)} \left\{ \tilde{S} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}_3$$

(means 3rd comp. of this)

$$= g M z + M \rho^{(0)} g \tilde{S}_{33}$$

Note,

$$\sum m_{(a)} \vec{r}_{(a)} = M\vec{R} + M\rho_{(0)} \times \begin{pmatrix} \tilde{S}_{13} \\ \tilde{S}_{23} \\ \tilde{S}_{33} \end{pmatrix}$$

$$\therefore V = \text{const.} + M \rho^{(0)} g \tilde{S}_{33}$$

$$T = \frac{1}{2} \operatorname{Tr}\left(\frac{d\tilde{S}}{dt} K_d \frac{d\tilde{S}^T}{dt} \right)$$

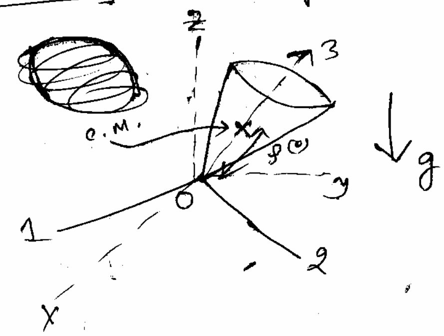
$$L = T - V$$

The pot. term breaks rotational sym. — only rotation about z-axis is a symmetry.

6/5/10

(30)

Axially symmetric rigid body moving under gravity, with a point on the symmetry axis fixed in space.



$$L = T - V$$

$$T = \frac{1}{2}\,\mathrm{Tr}\left(\frac{d\tilde{S}}{dt}\,\kappa_d\,\frac{d\tilde{S}^T}{dt}\right)$$

$$V = Mg\,\rho^{(0)}\,\tilde{S}_{33}$$

Symmetry: $\tilde{S} \longrightarrow U\tilde{S}$ is a symmetry of T for $U^T U = \mathbb{1}$.

$$V \to Mg\,\rho^{(0)}\,(U\tilde{S})_{33}$$

$$\Downarrow$$

$$U_{3i}\,\tilde{S}_{i3} = \tilde{S}_{33} \text{ for symmetry}$$

(This has to be true for any $\tilde{S}$, which is an orthogonal matrix)

$$\Downarrow$$

$$\begin{pmatrix} \tilde{S}_{13} \\ \tilde{S}_{23} \\ \tilde{S}_{33} \end{pmatrix}$$

[Any column of an orthogonal matrix should have unit norm — $\tilde{S}_{13}^2 + \tilde{S}_{23}^2 + \tilde{S}_{33}^2 = 1$]

$\Rightarrow U_{3i}\, \tilde{S}_{i3} = \tilde{S}_{33}$ $\quad \Rightarrow U_{3i} \tilde{S}_{i3} \tilde{S}_{j3} = \tilde{S}_{33} \tilde{S}_{j3} \Rightarrow U_{3i} \delta_{ij} = \delta_{3j}$

$\Rightarrow \quad U_{3i} = \delta_{i3}$

Choose $\tilde{S}_{3i} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow U_{33} = 1$

$\tilde{S}_{3i} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow U_{32} = 0$

$\tilde{S}_{3i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow U_{31} = 0$

$\therefore$ we must have $U_{3i} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Rotation about z-axis.

[This is the most general way to parametrize U_{3i}]

Infinitesimal version: $\theta = \epsilon$

$$V = \begin{pmatrix} 1 & \epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1} + \epsilon\, T^3$$

$\therefore$ Conserved charge $= \mathcal{Q}^3$

$\tilde{S} \longrightarrow \tilde{S} V$ is a symmetry of T if

$V K_d = K_d V$ $\quad (V^T V = \mathbb{1})$

$$K_d = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

(V has to be of this form to commute with K_d)

$$V K_d = K_d V \Rightarrow V = \begin{pmatrix} \cos\bar{\theta} & \sin\bar{\theta} & 0 \\ -\sin\bar{\theta} & \cos\bar{\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\longrightarrow$ rotation about body fixed sym. axis.

Infinitesimal version : $V = \mathbb{1} + \epsilon T^3$

32 Conserved charge $\bar{Q}_3$.

It is easiest check this in the infinitesimal form that V must be of this form

Potential term :-

$$\tilde{V} \Rightarrow M g . \rho^{(0)} (\tilde{S} V)_{33}$$
$$= M g \rho^{(0)} \tilde{S}_{3i} \underbrace{V_{i3}}_{\tilde{S}_{3i} \delta_{i3} = \tilde{S}_{33}}.$$

(This shows that $\tilde{V}$ doesn't change)

Thus $\tilde{V} \to \tilde{V}$

∴ this is indeed a symmetry.

Final list of conserved charges!

Q_3, $\bar{Q}_3$, H.

We know that $\{Q_3, \bar{Q}_3\} = 0$,

& $\{Q_3, H\} = 0$, $\{\bar{Q}_3, H\} = 0$

⇒ The system is solvable.

Choose convenient coordinate system :-

So far we haven't chosen any coords system & haven't parametrised $\tilde{S}$

ϕ^1, ϕ^2, ϕ^3 are the 3 coordinates.

Under Q_3 trs: $\phi^i \to f^i(\vec{\phi})$

$\bar{Q}_3$ trs: $\phi^i \to g^i(\vec{\phi})$

Some angles which parametrize $\tilde{S}$

Suppose we can choose coordinate system such that :

Under $\mathcal{G}_3$: $\phi_1 \to \underset{\substack{\downarrow \\ \text{parameter of transfn.}}}{\phi_1 + \theta}$, $\phi_2 \to \phi_2$, $\phi_3 \to \phi_3$

Under $\bar{\mathcal{G}}_3$: $\phi_1 \to \phi_1$, $\phi_2 \to \phi_2 + \theta$, $\phi_3 \to \phi_3$

H is independent of ~~θ_1 and θ_2~~ ϕ_1 and ϕ_2.

p_1 & p_2 are conserved

momentum conjugate to ϕ_1 and ϕ_2

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial \phi_1} = 0$$ (p_1 generates a translation of ϕ_1)

$$\frac{dp_2}{dt} = -\frac{\partial H}{\partial \phi_2} = 0$$ (p_2 generates a translation of ϕ_2)

$\therefore\ p_1 = \mathcal{G}_3$

& $p_2 = \bar{\mathcal{G}}_3$

~~The~~ only non-trivial dynamics ~~[illegible]~~ will come from $(\phi_3, p_3) \longrightarrow$ like a 1-dimensional problem.

Analog of ang. mom. for the space-fixed axes — that is fixed

Parametrization of $\tilde{S}$

Define:

$$R_3(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \text{rotation about 3-axis}$$

$$R_2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \to \text{rotation about 2-axis}$$

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -s\theta & \cos\theta \end{pmatrix} \to \text{rotation about 1-axis}$$

34) A generic $\tilde{S}$ can be parametrized as follows:

$$\tilde{S} = R_i(\alpha)\; R_j(\beta)\; R_k(\gamma) \qquad j \neq i \;\&\; j \neq k$$

(If $i = j$ or $j = k$ then we have 2 parameters — total rotation about the same axis)

U transfn:

$$\tilde{S} \to U\tilde{S} = R_3(\theta)\,\tilde{S}$$

choose $i = 3$, then $\alpha \to \alpha + \theta$,
$\beta \to \beta$, $\gamma \to \gamma$.

V transfn:

$$\tilde{S} \to \tilde{S}V = \tilde{S}\,R_3(\theta)$$

choose $k = 3$, then $\gamma \to \gamma + \theta$

Then $\alpha \to \alpha$, $\beta \to \beta$, $\gamma \to \gamma + \theta$

We shall use

$$\tilde{S} = R_3(\phi)\, R_1(\chi)\, R_3(\psi)$$

We will take $j = 2$ because $j \neq i$ & $j \neq k$

(ϕ, χ, ψ are now our parameters)

ϕ, χ, ψ are the coordinates.

L and H should be independent of ϕ and ψ.

Physical interpretation:-

Consider the symmetry axis: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in the body fixed frame.

In the space-fixed frame, it is $\tilde{S}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

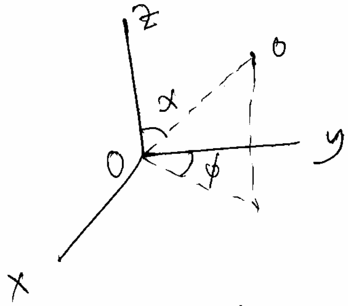
$$R_3(\psi)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$R_1(\chi)R_3(\psi)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sin\chi \\ \cos\chi \end{pmatrix}$$

$$R_3(\phi)R_1(\chi)R_3(\psi)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\phi\sin\chi \\ -\cos\phi\sin\chi \\ \cos\chi \end{pmatrix}$$

$$= \begin{pmatrix} \sin\phi \sin\chi \\ \cos\phi \sin\chi \\ \cos\chi \end{pmatrix}$$

Note
Wald → [155]
Here, ψ of Wald $= \phi$ of textbook
θ of Wald $= \chi$ of copy



$\phi \to$ azimuthal angle measured from y-axis

$\chi \to$ polar angle

$\psi \to$ angle of rotation about the sym. axis in the body fixed frame

$$T = \frac{1}{2} \mathrm{Tr}\left(\frac{d\tilde{S}}{dt} K_d \frac{d\tilde{S}^T}{dt} \right)$$

$$K_d = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Ex. Check that

$$T = \frac{1}{2} I_1 (\dot{\chi}^2 + \sin^2\chi\, \dot{\phi}^2) + \frac{1}{2} I_3 (\dot{\psi} + \cos\chi\, \dot{\phi})^2$$

$I_1 = \lambda_1 + \lambda_3$
$I_3 = 2\lambda_1$

Potential $V = Mg\, \rho^{(0)} \tilde{S}_{33} = Mg\, \underbrace{\rho^{(0)} \cos\chi}_{\text{Ht. of c.m.}}$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1 \sin^2\chi\, \dot{\phi} + I_3 \cos\chi\, (\dot{\psi} + \cos\chi\, \dot{\phi})$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 (\dot{\psi} + \dot{\phi} \cos\chi)$$

p_ϕ & p_ψ are conserved but $\dot{\phi}$ & $\dot{\psi}$ need not be consvd.

$H = T + V$

$$p_\phi = I_1 \sin^2\chi\, \dot{\phi} + I_3 \cos\chi \frac{p_\psi}{I_3}$$

$$\boxed{\dot{\phi} = \frac{p_\phi - p_\psi \cos\chi}{I_1 \sin^2\chi}}$$

$$\boxed{\dot{\psi} = \frac{p_\psi}{I_3} - \cos\chi \frac{p_\phi - p_\psi \cos\chi}{I_1 \sin^2\chi}}$$

(36) Hamiltonian $H = T + V$

$$= \frac{1}{2} I_1 \dot{\chi}^2 + \frac{1}{2 I_1 \sin^2\chi}\left(p_\phi - p_\psi \cos\chi\right)^2 + \frac{p_\psi^2}{2I_3} + Mg\,\rho^{(0)} \cos\chi = E$$

p_ϕ, p_ψ, E are constants.

$\therefore$ we can write $\frac{d\chi}{dt} = F(\chi)$

$$\Rightarrow \int_{\chi_0}^{\chi} \frac{d\chi'}{F(\chi')} = t$$

$\therefore$ the problem is solved in terms of integrals.

↑ Infinite energy

Define $u = \cos\chi$

$\therefore \dot{u} = -\sin\chi\, \dot{\chi} \Rightarrow \dot{\chi}^2 = \frac{\dot{u}^2}{1-u^2}$

$$\Rightarrow E = \frac{1}{2} I_1 \frac{\dot{u}^2}{1-u^2} + \frac{1}{2I_1(1-u^2)}\left(p_\phi - p_\psi u\right)^2 + \frac{p_\psi^2}{2I_3} + Mg\,\rho^{(0)} u.$$

$$\Rightarrow \dot{u}^2 = \frac{2(1-u^2)}{I_1}\left(E - \frac{p_\psi^2}{2I_3} - Mg\,\rho^{(0)} u\right) - \frac{1}{I_1}\left(p_\phi - p_\psi u\right)^2$$

$= f(u) \rightarrow$ a cubic polynomial

$\times$ Goldstein 213

defn.

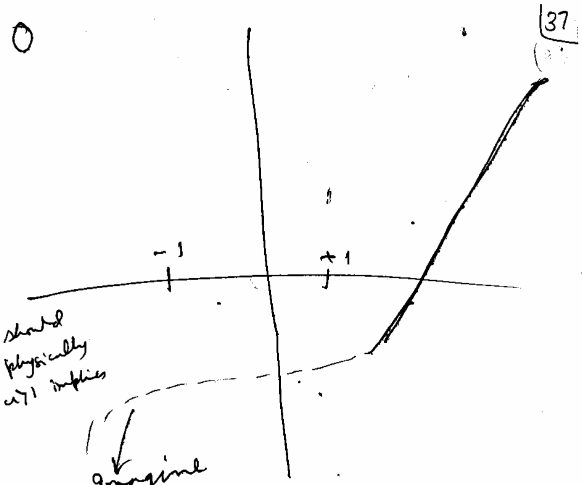
The behaviour of the system will be controlled by $f(u)$.

(This $f(u)$ is a cubic polynomial in u & is easier to solve)

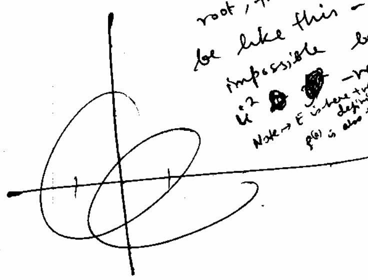
As $u \to \infty$, $f(u) \to \infty$

As $u \to -\infty$, $f(u) \to -\infty$

At $u = \pm 1$, $f(u) < 0$



[If there is only one root, then the figure should be like this — but physically impossible bcos $u > 1$ implies $\dot{u}^2$ -ve
Note → E is here +ve definite
$g(u)$ is also +ve]



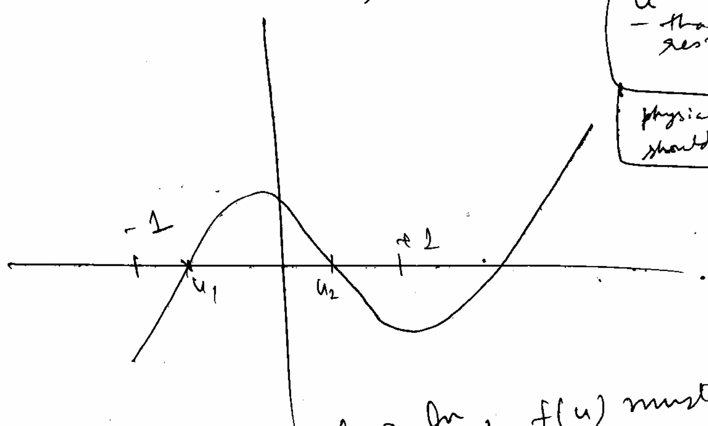
Imagine this, but it is impossible to achieve for any initial condn.
— so at least one root betn. -1 & 1
— if one root lies betn. -1 & 1 the 2 roots must lie betn. -1 + 1

one of the zeroes is betn. 1 or $+\infty$

[3 roots — either all real — or 1 real & 2 complex]

$\dot{u}^2$ is +ve — that further restricts u

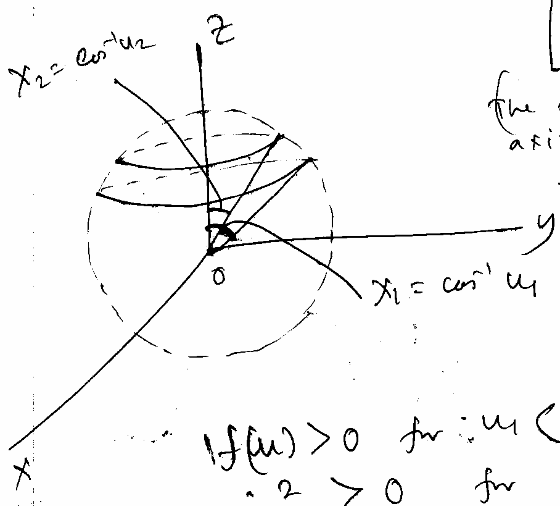
physical range of u should be betn. -1 & +1



i) For physical initial condn., $f(u)$ must have two zeroes at $-1 \leq u_1 \leq u_2 \leq 1$.

38) (2) For given E, p_ϕ & p_ψ, u must lie betn u_1 & u_2.

[In some special cases, we can reach -1 or $+1$ — then $p_\phi = p_\psi$ $u = \mp p_\psi$]



(the tip of the symmetry axis must lie in this band)

$f(u) > 0$ for $u_1 < u < u_2$

$\dot{u}^2 > 0$ for $u_1 < u < u_2$

$|\dot{u}| > 0$ for $u_1 < u < u_2$

$\dot{u} = 0$ at $u = u_1$ & u_2

but it doesn't vanish in betn.

it can't change its sign, (so $\dot{u}$ is continuous fn of time)

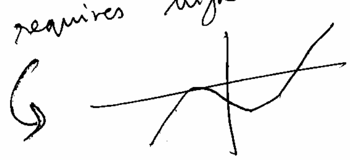
$\dot{u}$ increases or decreases monotonically as long as $u \neq u_1$ or u_2.

At $u = u_1$ or u_2, $\dot{u}$ changes its sign.

(As far as χ-angle is concerned, it oscillates betn χ_1 & χ_2)

[It can stay put at χ_1 & χ_2 — but that requires higher order zero of $f(u)$]

(Then we'll get a circle instead of a band)



$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\chi}{I_1 \sin^2\chi} = \frac{p_\phi - p_\psi u}{I_1 (1-u^2)}$$

(Denominator is always +ve)

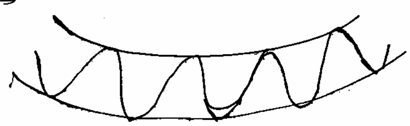
$\therefore \dot{\phi} = 0$ if $u = p_\phi / p_\psi$

Case 1 $\quad \frac{p_\phi}{p_\psi} > u_1, u_2$ or $\frac{p_\phi}{p_\psi} < u_1, u_2$.

In this case, $\dot{\phi}$ never vanishes $\Rightarrow \phi$ increases or decreases monotonically

Nutation during precession

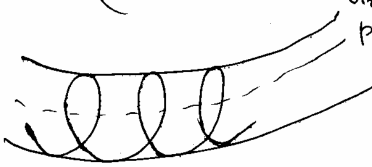
~~precession~~



(tip of the unit vector will have an oscillation like this)

Case 2 $\quad u_1 < \frac{p_\phi}{p_\psi} < u_2$

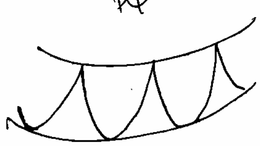
($\dot{\phi}$ changes sign $\because \dot{\phi} = 0$ for $u = p_\phi / p_\psi$)



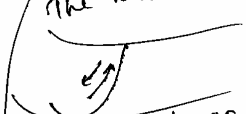
Case 3 $\quad \frac{p_\phi}{p_\psi} = u_2$

$\dot{\phi}$ is monotonic for $u_1 \le u \le u_2$

$\dot{\phi} = 0$ at $u = u_2$



The motion can't be :-



It can't go back
$\dot{\phi}$ doesn't change sign

Precession & Nutation

nutates back & forth betw. the 2 bands

40) Ex: Show that $\frac{p_\phi}{p_\psi} = u_1$ is not physically possible.

Ans — (I) For $p_\phi/p_\psi = u_1$, at $u = u_1$, $\dot{u} = 0$ & $\dot{\phi} = 0$
Also $\dot{\chi} = 0$ in this case.
Also, at $u = u_2$

$$E = \frac{p_\psi^2}{2I_3} + Mg\rho^{(0)} u_1$$

We also have in that case
$p_\psi = I_3 \dot{\psi}$
$p_\phi = I_3 u_1 \dot{\psi}$

If initially $\dot{\psi} \neq 0$ & > 0 we have $E \geq 0$, as it should be.

(II) For $p_\phi/p_\psi = u_1$, at $u = u_1$ & for $\dot{\psi} = 0$,

$E = Mg\rho^{(0)} u_1 < 0 \rightarrow$ physically impossible

$\because E > 0$ for all $\dot{\psi}$, this is physically not possible.

In E, ~~[illegible]~~ all terms $\uparrow$ as you move away from u_1. So energy conservation is violated.

19/10/05 Hamiltonian: $H(\vec{q}, \vec{p})$ $\quad \vec{q} = (q^1, \dots, q^N)$ 41
$\vec{p} = (p_1, \dots, p_N)$

Lagrangian:

$$L(\vec{q}, \vec{u}) = \sum_{i=1}^{N} u^i p_i - H(\vec{q}, \vec{p})$$

$$u^i = \frac{\partial H}{\partial p_i}$$

Divide the variables into two parts:–

$\tilde{q}^i, \tilde{p}_i \qquad i = 1, 2, \dots, M$

$\hat{q}^\alpha, \hat{p}_\alpha \qquad \alpha = M+1, \dots, N$

The Routhian is described as a function

$$R(\tilde{q}^i, \tilde{u}^i, \hat{q}^\alpha, \hat{p}_\alpha)$$
$$= \sum_{i=1}^{M} \tilde{u}^i \tilde{p}_i - H$$
$$\tilde{u}^i = \frac{\partial H}{\partial \tilde{p}_i}$$

If we take $M = N$, we get the usual Lagrangian. If we take $M = 0$, R is identical to H apart from a '$-$' sign

$$R(\vec{\tilde{q}}, \vec{\tilde{u}}, \vec{\hat{q}}, \vec{\hat{p}})$$
$$= \underbrace{\sum_{i=1}^{M} \tilde{u}^i \tilde{p}_i + \sum_{i=M+1}^{N} \hat{u}^i \hat{p}_i - H}_{L} - \sum_{i=M+1}^{N} \hat{u}^i \hat{p}_i$$

$$= L - \sum_{i=M+1}^{N} \hat{u}^i \hat{p}_i$$

So Routhian is a more general concept

Routhian is a Legendre transform of a subset of the momentum variables

(How do the eqns of motion look in the Routhian system?)

$$\cancel{\delta R = \sum_{i=1}^{M} \delta \tilde{u}^i \tilde{p}_i + \sum_{i=1}^{M} \tilde{u}^i \delta \tilde{p}_i}$$
$$\cancel{\sum_{i=1}^{M}}$$

42

$$\delta R = \sum_{i=1}^{M} \delta\tilde{u}^i\,\tilde{p}_i + \cancel{\sum_{i=1}^{M} \tilde{u}^i\,\delta\tilde{p}_i}$$

$$- \sum_{i=1}^{M}\left(\frac{\partial H}{\partial \tilde{q}^i}\,\delta\tilde{q}^i + \cancel{\frac{\partial H}{\partial \tilde{p}_i}\,\delta\tilde{p}_i}\right)$$

$$\tilde{u}^i$$

$$- \sum_{\alpha=M+1}^{N}\left(\frac{\partial H}{\partial \hat{q}^\alpha}\,\delta\hat{q}^\alpha + \frac{\partial H}{\partial \hat{p}_\alpha}\,\delta\hat{p}_\alpha\right)$$

Also, $$\delta R = \sum_{i=1}^{M}\left(\frac{\partial R}{\partial \tilde{q}^i}\,\delta\tilde{q}^i + \frac{\partial R}{\partial \tilde{u}^i}\,\delta\tilde{u}^i\right) + \sum_{\alpha=M+1}^{N}\left(\frac{\partial R}{\partial \hat{q}^\alpha}\,\delta\hat{q}^\alpha + \frac{\partial R}{\partial \hat{p}_\alpha}\,\delta\hat{p}_\alpha\right)$$

$$\boxed{\begin{aligned} \frac{\partial R}{\partial \tilde{q}^i} &= -\frac{\partial H}{\partial \tilde{q}^i} \\ \frac{\partial R}{\partial \tilde{u}^i} &= \tilde{p}_i \\ \frac{\partial R}{\partial \hat{q}^\alpha} &= -\frac{\partial H}{\partial \hat{q}^\alpha} \\ \frac{\partial R}{\partial \hat{p}_\alpha} &= -\frac{\partial H}{\partial \hat{p}_\alpha} \end{aligned}}$$

Eqns of motion

$$\frac{d\tilde{q}^i}{dt} = \frac{\partial H}{\partial \tilde{p}_i} \xrightarrow[\text{the form}]{\text{takes}} \boxed{\frac{d\tilde{q}^i}{dt} = \tilde{u}^i}$$

$$\frac{d\tilde{p}_i}{dt} = -\frac{\partial H}{\partial \tilde{q}^i} \Rightarrow \boxed{\frac{d}{dt}\frac{\partial R}{\partial \tilde{u}^i} = +\frac{\partial R}{\partial \tilde{q}^i}}$$

~~either~~ (Euler-Lag. eqns of the tilda variables, had Routhian been the Lagrangian)

$$\frac{d\hat{q}^\alpha}{dt} = \frac{\partial H}{\partial \hat{p}_\alpha} \Rightarrow \boxed{\frac{d\hat{q}^\alpha}{dt} = -\frac{\partial R}{\partial \hat{p}_\alpha}}$$

$$\frac{d\hat{p}_\alpha}{dt} = -\frac{\partial H}{\partial \hat{q}^\alpha} \Rightarrow \boxed{\frac{d\hat{p}_\alpha}{dt} = \frac{\partial R}{\partial \hat{q}^\alpha}}$$

(R is like −H for the hatted variables)

Why do we consider a mixed formalism like this ?

→ Allows us to utilise the advantages of both the formalism

[Lag. dyn. is easier to visualise bcos we are working in a space ½ the dim. of that for H

So, for solving the diff. eqns, Lag. formalism is more useful – no. of diff. eqns is less, though they are 2nd order.

Ham. ~~formalism~~ is more useful in finding ~~the eqns of motion~~ the conserved charges]

Application

Suppose we have a Hamiltonian system with 2N dimensional phase space and K conserved charges $I_1, \ldots, I_K$ such that

$$\{ I_\alpha, I_\beta \} = 0$$

If K = N then the system is integrable.

What if K < N? (then the system isn't manifestly integrable)

Recall the procedure for solving the system

Look for a canonical trsfn $(\vec{q}, \vec{p}) \to (\vec{\tilde{q}}, \vec{\tilde{p}})$

$$\tilde{q}^\alpha = \frac{\partial F_3(\vec{q}, \vec{\tilde{p}})}{\partial \tilde{p}_\alpha}, \quad p_\alpha = \frac{\partial F_3(\vec{q}, \vec{\tilde{p}})}{\partial q^\alpha}$$

such that $\tilde{p}_\alpha = \phi_\alpha\left(I_1^{(\vec{q},\vec{p})}, \ldots, I_N(\vec{q},\vec{p})\right)$ if K = N

Arbitrary fns (up to us to choose)

⇒ $\tilde{p}_\alpha$'s are conserved

⇒ $\frac{\partial H}{\partial \tilde{q}^\alpha} = 0$ ⇒ H → fn of $\vec{\tilde{p}}$ only

44) If $K<N$, look for a canonical transf.
$(\vec{q}, \vec{p}) \longrightarrow (\vec{\tilde{q}}, \vec{\tilde{p}})$ such that
$\tilde{p}^1, \ldots, \tilde{p}^K$ are fns of $I_1, \ldots, I_K$.

$\xrightarrow[\text{means}]{\text{this}}$ $\dfrac{\partial H}{\partial \tilde{q}^\alpha} = 0$ for $\alpha = 1, 2, \ldots, K$

Hence $\Rightarrow$ H is independent of $\underbrace{\tilde{q}^1, \ldots, \tilde{q}^K}$

these are called cyclic coordinates
[those coord. which don't appear explicitly in H
– their conjugate momenta can appear in H.
If there are conserved charges, there are corr. cyclic coordinates]

[When H is indep. of these coord., L will also be indep. of these coord., as is clear from the construction of L]

Consider a general Hamiltonian system of 2N-dimensional phase space:
$H(q^1, \ldots, q^M, p^1, \ldots, p^N)$

$\xrightarrow[\text{means}]{\text{this}}$ $q^{M+1}, \ldots, q^N$ are cyclic.

$$R(q^1, \ldots, q^M, u^1, \ldots, u^M, p^{M+1}, \ldots, p^N) = \sum_{i=1}^{M} u^i p_i - H$$

$$u^i = \frac{\partial H}{\partial p_i} \quad i = 1, \ldots, M$$

We treat $q^1, \ldots, q^M$ in the Lag. framework & $q^{M+1}, \ldots, q^N$ in the Ham. framework

We are treating the non-cyclic coordinates in Lag. formalism & the cyclic ones in the Ham. formalism.

Eqns of motion :-

A system of M coordinates

$$\begin{cases} \dfrac{dq^i}{dt} = u^i & i = 1, 2, \ldots, M \\[2mm] \dfrac{d}{dt}\dfrac{\partial R}{\partial u^i} = \dfrac{\partial R}{\partial q^i} & i = 1, 2, \ldots, M \end{cases}$$

$$\frac{dq^i}{dt} = \frac{\partial R}{\partial p_i} \qquad i = M+1, \ldots, N$$

$$\frac{dp_i}{dt} = \frac{\partial R}{\partial q^i} = 0 \qquad i = M+1, \ldots, N$$

$$p_i = p_i^{(0)} \rightarrow \text{constants} \Rightarrow R\left(q^1, \ldots, q^M, u^1, \ldots, u^M, p^{(0)}_{M+1}, \ldots, p^{(0)}_N\right)$$

($p^{(0)}_{M+1}, \ldots, p^{(0)}_N$ are just const. parameters & R ~~can~~ behaves just like a Lag. of M coordinates)

(So Cyclic coord. can be used to reduce the problem to a lower dimensional system)

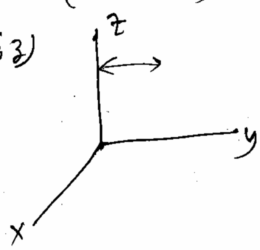
(Effectively we have reduced the no. of degrees of freedom using the cyclic coordinates)

[In the problem of axially sym. body in a grav. fields, we effectively used the Routhian procedure]

Example A particle in 3-dimension under an axially symmetric potential.

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V\left(\sqrt{x^2+y^2}, z\right)$$

Cylindrical polar coordinates (ρ, ϕ, z)



$$x = \rho \cos\phi$$
$$y = \rho \sin\phi$$
$$z$$

$$L = \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2\right) - V(\rho, z)$$

[Here $\dot{\rho} = u_\rho$, $\dot{\phi} = u_\phi$, $\dot{z} = u_z$]

ϕ is the cyclic coord. here.

In simple systems, a cyclic coord. can be found by inspection. In general, we find a conserved charge & then make a coord. cyclic ......

46) $p_\phi = \dfrac{\partial L}{\partial u_\phi} = m\rho^2 u_\phi$ $\Rightarrow u_\phi = \dfrac{p_\phi}{m\rho^2}$ (for the ϕ variable, we are going towards the Ham. framework)

$$R(\rho, z; u^\rho, u^z, p_\phi) = L - p_\phi u^\phi.$$

$$= \frac{1}{2} m\left(u_\rho^2 + u_z^2\right) - V(\rho, z) - \frac{p_\phi^2}{2m\rho^2}$$

$\Rightarrow$ As if we have two coordinates ρ, z with potential $V(\rho, z) + \dfrac{p_\phi^2}{2m\rho^2}$

(we can think it now as a Lag. system)

[Suppose, we had directly substituted $u_\phi = p_\phi/m\rho^2$ & written $L = \frac{1}{2} m(u_\rho^2 + u_z^2) - V(\rho, z) + p_\phi^2/2m\rho^2$ we would have got the wrong potential & a wrong answer.

The reason is that derivative at fixed ϕ doesn't mean taking derivative at fixed p_ϕ. so fixed p_ϕ doesn't help. L is a fn. of $\vec{q}$ & $\dot{\vec{q}}$, not $\vec{q}$ & $\vec{p}$]

It also follows from the variational principle. p_ϕ is conserved along the classical trajectory, but not so in a variation of this actual path

$$L = \frac{1}{2}m(u_\rho^2 + u_\phi^2) - V(\rho, z) + \frac{m}{2}\rho^2 u_\phi^2$$

can we do this change!

$$\frac{p_\phi^2}{2m^2\rho^2}$$

when we take $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial u_\rho}\right) - \dfrac{\partial L}{\partial \rho} = 0$, we will get a wrong result if we treat p_ϕ as fixed.

Another way of seeing it is that in the variational principle, we treat u_ϕ as $\dfrac{d\phi}{dt}$. But p_ϕ = constant will violate this & we'll get wrong result for an arbitrary variation of path

Time dependent Hamiltonian

$$\left.\begin{aligned} dq^\alpha/dt &= \frac{\partial H(\vec{q}, \vec{p}, t)}{\partial p_\alpha} \\ \frac{dp_\alpha}{dt} &= -\frac{\partial H(\vec{q}, \vec{p}, t)}{\partial q^\alpha} \end{aligned}\right\} \text{non-autonomous}$$

$\alpha = 1, \ldots, N$

For a general non-autonomous system

$$\frac{dx^i}{dt} = F^i(x, t) \qquad i = 1, 2, \ldots, n$$

~~Define~~ Introduce new coordinate x^{n+1} and new equations:

$$\left.\begin{aligned} \frac{dx^i}{dt} &= F^i(\vec{x}, x^{n+1}) \qquad i = 1, 2, \ldots, n \\ \frac{dx^{n+1}}{dt} &= 1 \end{aligned}\right\} \text{Autonomous system with } (n+1) \text{ variable}$$

(This essentially maps the problem to an autonomous system with one more variable)

(However, this isn't a Ham. system as we have $2n+1$ variables & we'll lose the adv. we had for ~~the~~ a Ham. system)

Introduce new coordinate q^{N+1} and new momentum p_{N+1}

New Hamiltonian

$$H_{New} = H(\vec{q}, \vec{p}, q^{N+1}) + p_{N+1}$$

(a time-indep. but ~~mom.~~ with ~~~~ $N+1$ coord. & $N+1$ mom.)

$$\frac{dq^{N+1}}{dt} = \frac{\partial H_{New}}{\partial p_{N+1}} = 1 \Rightarrow q^{N+1} = t + \text{constant}$$

$$\frac{dp_{N+1}}{dt} = -\frac{\partial H}{\partial q^{N+1}}$$

$$\frac{dq^i}{dt} = \frac{\partial H_{New}}{\partial p_i} = \frac{\partial H(\vec{q}, \vec{p}, q^{N+1})}{\partial p_i},$$

PTO

48)

$$\frac{dp_i}{dt} = - \frac{\partial H_{New}}{\partial q^i} = - \frac{\partial H(\vec{q}, \vec{p}, q^{N+1})}{\partial q^i} \cdot$$

[Step 1 $\rightarrow$ solve $\frac{dq^{N+1}}{dt} = \frac{\partial H_{New}}{\partial p_{N+1}} = 1 \ni q^{N+1} = t + \text{const.}$

Step 2 $\rightarrow$ solve $\frac{dq^i}{dt} = \frac{\partial H_{New}}{\partial p_i}$

& $\frac{dp_i}{dt} = - \frac{\partial H_{New}}{\partial q^i}$

& find that these are same as our original $\frac{dq^\alpha}{dt} = \frac{\partial H(\vec{q}, \vec{p}, t)}{\partial p_\alpha}$

$\frac{dp_\alpha}{dt} = - \frac{\partial H(\text{- -})}{\partial q^\alpha}$]

(The additional eqn. $\frac{dp_{N+1}}{dt} = \frac{\partial H}{\partial q^{N+1}}$, which don't affect at all the evolution of q^i & p_i — it gives some soln. which doesn't affect our original problem)

[If H is time-indep., t can be treated as a cyclic variable & we can easily integrate it out]

Ayan $\swarrow$ (The original Ham. can be considered as the Routhian of H_{New})

Small oscillations

Consider a generic Newtonian system with constraints.

$$L = \frac{1}{2}\sum m_\alpha (\dot{x}^\alpha)^2 - V(\vec{x})$$

Constraints : $x^\alpha = F^\alpha(\underbrace{q^1, \ldots, q^N}_{\text{independent variables}})$

$$\boxed{L = \frac{1}{2}\sum_{\alpha,\beta=1}^{N} f_{\alpha\beta}(\vec{q})\, \dot{q}^\alpha \dot{q}^\beta - V(\vec{q})}$$

→ Generic form of the Lag. of a constrained Newtonian system

Suppose at $\vec{q} = \vec{q}_0$,

$$\frac{\partial V}{\partial q^i} = 0 \quad \text{for } i = 1, 2, \ldots, N$$

In this case, $\vec{q} = \vec{q}_0$ is a soln of eqns of motion.

(Checking that it is so)

$$\vec{u} = \frac{d\vec{q}}{dt} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u^\alpha}\right) + \underbrace{\frac{\partial V}{\partial q^\alpha}}_{=0} = 0$$

$$\downarrow$$

$$\sum_\beta f_{\alpha\beta} u^\beta = 0 \quad (\text{for } \vec{u} = 0)$$

Goal:

Consider solutions of the form

$$\vec{q}(t) = \vec{q}_0 + \hat{\vec{q}}(t)$$

& study the eqns of motion for small $\hat{\vec{q}}(t)$.

20/10/05

Small oscillations

$$L = \frac{1}{2} f_{\alpha\beta}(\vec{q})\, \dot{q}^\alpha \dot{q}^\beta - V(\vec{q})$$

$f_{\alpha\beta}$ is a positive definite

[Ex. $\longrightarrow$ Prove the above.]

[The kinetic term is +ve definite. You are just constraining - so kinetic term can't become -ve]

$f_{\alpha\beta} = \sum \frac{\partial x_i}{\partial q^\alpha} \frac{\partial x_i}{\partial q^\beta} = \frac{\partial \vec{x}}{\partial q^\alpha} \cdot \frac{\partial \vec{x}}{\partial q^\beta}$

Let $M_{i\alpha} = \frac{\partial x_i}{\partial q^\alpha}$

$\therefore f_{\alpha\beta} = M_{i\alpha} M_{i\beta} = M^T_{\alpha i} M_{i\beta}$

$\Rightarrow f = \underbrace{M^T M}_{\text{This is +ve definite}}$

$\vec{q}^{(0)}$ is an extremium of $V(\vec{q})$.

$$V'(\vec{q}^{(0)}) = 0$$

$\Rightarrow \vec{q} = \vec{q}^{(0)}$, $\vec{u} = 0$ is a soln. of the eqns of motion.

Define : $\tilde{\vec{q}} = \vec{q} - \vec{q}^{(0)}$, $\tilde{\vec{u}} = \vec{u}$

study the motion for small $\tilde{\vec{q}}$. (a const. won't affect the eqns of motion)

$$L = \frac{1}{2} \sum_{\alpha,\beta} f_{\alpha\beta}(\vec{q}^{(0)})\, \tilde{u}^\alpha \tilde{u}^\beta - V(\vec{q}^{(0)})$$

$$- \frac{1}{2} \sum_{\alpha,\beta} \frac{\partial^2 V}{\partial q^\alpha \partial q^\beta}\bigg|_{\vec{q} = \vec{q}^{(0)}} \tilde{q}^\alpha \tilde{q}^\beta + 3^{rd} \text{ order terms}$$

[We can't diagonalise $\frac{1}{2} \sum f_{\alpha\beta}(\vec{q}^{(0)})\, \tilde{u}^\alpha \tilde{u}^\beta$ & $\frac{1}{2} \sum \frac{\partial^2 V}{\partial q^\alpha \partial q^\beta}\Big|_{\vec{q}^{(0)}} \tilde{q}^\alpha \tilde{q}^\beta$

simultaneously by an orthogonal transfn.

So, we have to use some trick]

Define $m_{\alpha\beta} = f_{\alpha\beta}(\vec{q}^{(0)})$ [$m_{\alpha\beta} \to$ a +ve definite matrix]

$$m = U^T m_{(d)} U$$

for some $SO(N)$ matrix U and ~~~~

some $m_{(d)} = \begin{pmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & \ddots & \\ 0 & & & m_N \end{pmatrix}$

$m_i > 0$ for $i = 1, 2, \ldots, N$ [$\because m_{\alpha\beta}$ is +ve definite]

$$L = \frac{1}{2} \cdot \tilde{u}^T \cdot U^T m_{(d)} U u - V(\vec{q}^{(0)}) - \frac{1}{2} \tilde{q}^T V^{(2)} \tilde{q}$$

Define : $\hat{q} = U\tilde{q}$, $\hat{q}^\alpha = U_{\alpha\beta} \tilde{q}^\beta$

($\because$ U is a constant matrix, if we want to preserve the eqns of motion, we must have $\rightarrow$)

$$\hat{u} = \frac{d\hat{q}}{dt} \Rightarrow \hat{u} = U\tilde{u}$$

$$\Rightarrow L = \frac{1}{2} \hat{u}^T m_{(d)} \hat{u} - V(\vec{q}^{(0)}) - \frac{1}{2} \hat{q}^T \underbrace{(U^T)^{-1} V^{(2)} U^{-1}}_{K} \hat{q}$$

(The kinetic term is now diagonal) (The potential term is still not diagonal)

$$L = \frac{1}{2} \sum_\alpha m_\alpha (\hat{u}^\alpha)^2 - V(\vec{q}^{(0)}) - \frac{1}{2} \sum_{\alpha,\beta} K_{\alpha\beta} \hat{q}^\alpha \hat{q}^\beta$$

K is a symmetric matrix. ($\because V^{(2)}$ is symmetric & we can check that $K^T = K$)

Define : $\bar{q}^\alpha = \sqrt{m_\alpha} \hat{q}^\alpha$

$$\bar{u}^\alpha = \frac{d\bar{q}^\alpha}{dt} = \sqrt{m_\alpha} \hat{u}^\alpha$$

$$L = \frac{1}{2} \sum_\alpha (\bar{u}^\alpha)^2 - V(\vec{q}^{(0)}) - \frac{1}{2} \sum_{\alpha,\beta} \underbrace{\frac{1}{\sqrt{m_\alpha m_\beta}} K_{\alpha\beta}}_{\bar{K}_{\alpha\beta}} \bar{q}^\alpha \bar{q}^\beta$$

$$L = \frac{1}{2} \bar{U}^T \bar{u} - \frac{1}{2} \bar{q}^T \bar{K} \bar{q} - V(\vec{q}^{(0)})$$

$\bar{K} = W^T \bar{K}_{(d)} W$

[$\bar{K}$ is a real symmetric matrix & $\frac{1}{2} \bar{q}^T \bar{K} \bar{q}$ can be diagonalized by an orthogonal transfn]

W is an SO(N) matrix.

$$K_{(d)} = \begin{pmatrix} K_1 & & & 0 \\ & K_2 & & \\ & & \ddots & \\ 0 & & & K_N \end{pmatrix}$$

52)

$$\therefore L = \frac{1}{2} \bar{u}^T u - \frac{1}{2} \bar{q}^T W^T \bar{K}_{(d)} W \bar{q} - V(\vec{q}^{(0)})$$

Define: $\check{q} = W\bar{q}$, $\check{u} = W\bar{u}$

'V' = check

$$L = \frac{1}{2} \check{u}^T \underbrace{(W^{-1})^T W^{-1}}_{\text{Identity matrix}} \check{u} - \frac{1}{2} \check{q}^T \bar{K}_{(d)} \check{q} - V(\vec{q}^{(0)})$$

[$\because$ W is an orthogonal matrix]

$$\therefore L = \frac{1}{2} \sum_{\alpha=1}^{N} (\check{u}^{\alpha})^2 - \frac{1}{2} \sum_{\alpha=1}^{N} K_\alpha (\check{q}^{\alpha})^2 - V(\vec{q}^{(0)})$$

$\xrightarrow[\text{gives us}]{\text{this}}$ N decoupled 1-dimensional systems (which we can solve explicitly)

(Nature of the soln will depend on the sign of K_α)

If $\vec{q}^{(0)}$ is a minimum of $V(\vec{q})$, then $V^{(2)}$ is positive definite.

Ex. $\Rightarrow$ Show that this in turn implies K and also $\bar{K}$ are positive definite & hence $K_\alpha > 0$ for $\alpha = 1, 2, \ldots, N$

$\xrightarrow[\text{implies}]{\text{this}}$ We have N different harmonic oscillators with period $\dfrac{2\pi}{\sqrt{K_\alpha}}$ $\alpha = 1, 2, \ldots, N$

General solution

$$\check{q}_\alpha = a_\alpha \cos\left(\frac{t}{\sqrt{K_\alpha}} + b_\alpha\right) \longrightarrow \text{normal modes (of the system)}$$

a_α, b_α are constants (which provide the initial condns)

If a_α is small for each α, then $\hat{q}^\alpha$ is small at all time

$\xrightarrow[\text{hence}]{\&}$ the approximation is valid.

$\xrightarrow[\text{known as}]{\text{this is}}$ small oscillations.

If some $K_\alpha < 0$, $K_\alpha = -h_\alpha$ (where $h_\alpha > 0$).

Then $\check{q}^\alpha = a_\alpha e^{t/\sqrt{h_\alpha}} + b_\alpha e^{-t/\sqrt{h_\alpha}}$

$\xrightarrow[\text{shows}]{\text{this}}$ For generic initial condition, $\check{q}^\alpha$ will grow for large t.

$\xrightarrow[\text{hence}]{\&}$ the approximation is not valid.

[So this approximation technique is used for the minimum of the system]

[application of this method is limited becos we have to study near the minimum — however this situation is encountered in many systems]

Perturbation Theory

$$H = \underset{\text{integrable}}{H_0} + \lambda H$$

$\lambda \rightarrow$ small parameter

[study the orbits of the system for this modified Hamiltonian]

[trying to develop the classical version of quantum mech. perturbation theory]

Action-Angle variables

Suppose we have an integrable system with N conserved charges $I_1(\vec{q}, \vec{p}), \ldots, I_N(\vec{q}, \vec{p})$

$$\{I_\alpha, I_\beta\} = 0$$

The motion of a trajectory lie inside an an N-dimensional subspace of the $2N$ dimensional phase space.

$$I_1 = C_1,\ I_2 = C_2,\ \ldots,\ I_N = C_N$$

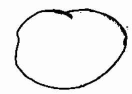
$C_1, \ldots, C_N$ constants
(this we know even without solving the system)

Suppose the motion of a given trajectory is bounded.
→ does not go to ∞
[in other ~~wo~~ words, the subspace ~~do~~ is bounded & doesn't go to infinity]

[e.g. → $x^2 - y^2 = s^2$ is a hyperbola & this subspace isn't bounded — it goes to infinity]

This bounded subspace is a closed compact space [we can't have a trajectory which ends somewhere)

[It has the topology ~~—~~ of a circle for $N=1$
By circle we mean sth which is closed

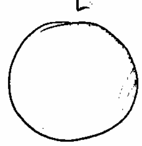

]

$N=1$

→ circle

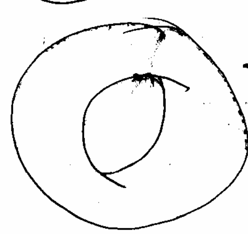
$N=2$ [2-dim subspace in a $4N$-dim phase space)

(1)



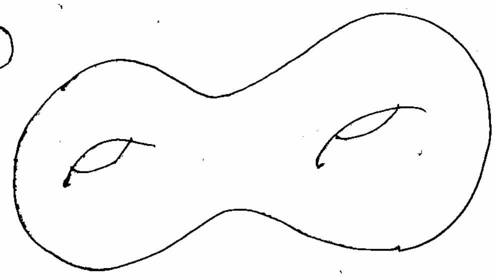
→ S^2 (sphere)

(2)



→ T^2 (2-dim. torus — a doughnut kind of thing)

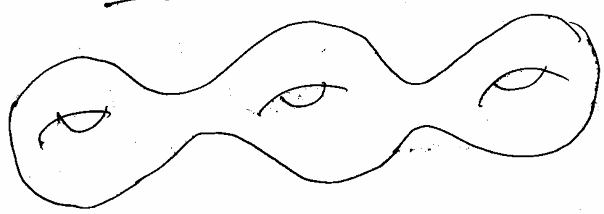
③



→ genus 2 surface

[Imagine 2 tori connected by a handle)

④ (In general, we can have a genus g surface)
— we have an infinite no. of possibilities



→ genus 3 surface etc.

[Only ② is allowed for a 4-dim. Ham. system, given the conds. $I_1, \ldots I_N$ & $\{I_\alpha, I_\beta\} = 0$]

[Neighbouring pts. must lie on the same subspace when we do have a canonical transfn. generated by a I_α. Each of the I_α's generates a vector field which moves you on the N-dim subspace]

[On each pt. of the 2-D torus, we have 2 independent vector fields & those 2 vector fields must commute. This can be used to prove that the only topological surface that satisfies this is the <u>torus</u>.

Torus is a flat surface (all the others have a curvature)]

Arnold
Proof is given.

Torus can be made into a flat space

56) A surface with no boundary & finite volume
→ compact space

For a general N the only possibility is T^N.
N-dimensional torus.

T^2: (θ^1, θ^2) $\quad 0 \le \theta^1 < 2\pi$, $\quad 0 \le \theta^2 < 2\pi$

(θ^1, θ^2) is the same as $(\theta^1 + 2\pi, \theta^2)$
and as $(\theta^1, \theta^2 + 2\pi)$.

[Clearly, it is topologically diff. from a sphere. We can't have 2 coord. for a sphere satisfying these prop. It has a more complicated dependence on θ & ϕ]

T^N: $(\theta^1, \dots, \theta^N)$ $\quad 0 \le \theta^\alpha < 2\pi$

$(\theta^1, \theta^2, \dots, \theta^N) \longleftrightarrow (\theta^1 + 2\pi, \theta^2, \dots, \theta^N)$
$\longleftrightarrow (\theta^1, \theta^2 + 2\pi, \theta^3, \dots, \theta^N)$
$\longleftrightarrow$ - - - - -

Product space :-
M_1: coordinate $\vec{x}$
M_2: coordinate $\vec{y}$
Product space is $(\vec{x}, \vec{y})$

$E = p^2/2 - q^2/2$
This surface is unbounded from below

For given $(c_1, \dots, c_N)$, we have a bounded torus.
Neighbouring $(c_1, \dots, c_N) \to$ a nearby torus.

Closed orbits will become open orbits if we give sufficient energy. But here we will have a closed orbit going to a ~~nearby~~ nearby torus

Consider the coordinate transformation

$$(\vec{q}, \vec{p}) \to (\vec{\tilde{q}}, \vec{\tilde{p}})$$ ~~such~~

such that $\tilde{p}^\alpha$ are conserved.

It is possible ~~[illegible]~~ to find a canonical transfn. such that

$$\tilde{p}_\alpha = h_\alpha(\vec{I})$$ ↳ arbitrary fns provided they are independent

$$p_\alpha = g_\alpha(\vec{q}, \vec{\tilde{p}}).$$

Find $F_3(\vec{q}, \vec{\tilde{p}})$ such that

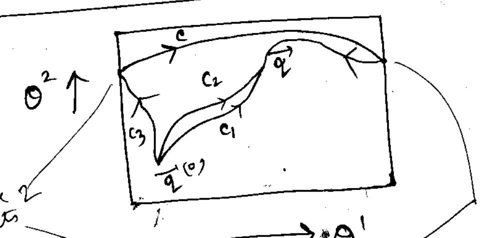
$$\frac{\partial F_3(\vec{q}, \vec{\tilde{p}})}{\partial q^\alpha} = p = g_\alpha(\vec{q}, \vec{\tilde{p}})$$

$$F_3(\vec{q}, \vec{\tilde{p}}) = \int_{\vec{q}^{(0)}}^{\vec{q}} g_\alpha(\vec{q}\,', \vec{\tilde{p}})\, dq'^\alpha$$

→ independent of the path due to the relation

$$\frac{\partial g_\alpha(\vec{q}, \vec{\tilde{p}})}{\partial q^\beta} = \frac{\partial g_\beta(\vec{q}, \vec{\tilde{p}})}{\partial q^\alpha}$$ ← follows from $\{I_\alpha, I_\beta\} = 0$

Consider a torus



(This has been done at fixed $\tilde{p}$'s & so that we don't go out of the surface)

These 2 points are the same for a torus

$$\int_{c_1} g_\alpha(\vec{q}\,', \vec{\tilde{p}})\, dq'^\alpha = \int_{c_2} g_\alpha(\vec{q}\,', \vec{\tilde{p}})\, dq'^\alpha \neq \int_{c_3} g_\alpha(\vec{q}\,', \vec{\tilde{p}})\, dq'^\alpha$$

58)

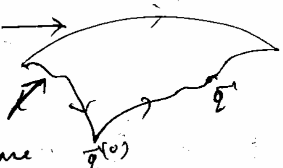
$$\int_{C_1} g_\alpha(\vec{q}\,', \vec{\tilde{p}})\, dq'^\alpha - \int_{C_3} g_\alpha(\vec{q}\,', \vec{\tilde{p}})\, dq'^\alpha$$

$$= \int_C g_\alpha(\vec{q}\,', \vec{\tilde{p}})\, dq'^\alpha$$

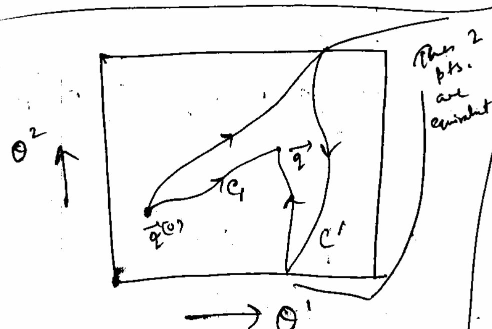
$$= \Psi_1(\vec{\tilde{p}})$$

Any continuous deformation of the contour won't affect the integral

so



One contour can be continuously deformed into another



$$\int_{C'} g_\alpha(\vec{q}\,', \vec{\tilde{p}})\, dq'^\alpha = \Psi_2(\vec{\tilde{p}})$$

[going about that contour will be just n times Ψ_1 or n times Ψ_2

so, once we know Ψ_1 & Ψ_2, we can find the number for the generic case of going n times]

Now recall $\tilde{p}_\alpha = h_\alpha(\vec{I})$

$I_\alpha = e_\alpha(\vec{\tilde{p}})$

choose $h_\alpha(\vec{I})$ such that

$$\Psi_1(\vec{\tilde{p}}) = \tilde{p}_1, \quad \Psi_2(\vec{\tilde{p}}) = \tilde{p}_2$$

For T^N

Define the cycle C_α which goes from
$(\theta_1, \theta_2, \ldots, \theta_\alpha, \ldots, \theta_N)$

$\downarrow$ to

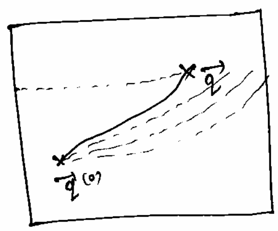
$(\theta_1, \theta_2, \ldots, \theta_\alpha + 2\pi, \ldots, \theta_N)$

[Any non-contractable cycle can be taken a linear combn. of C_α's]

$$\int_{C_\beta} g_\alpha(\vec{q}\,', \vec{p})\, dq'^\alpha = \tilde{p}_\beta$$

(Choice of $h_\alpha(\vec{I})$)

$$F_3(\vec{q}, \vec{\tilde{p}}) = \int_{\vec{q}^{(0)}}^{\vec{q}} dq'^\alpha\, g_\alpha(\vec{q}\,', \vec{\tilde{p}})$$



[Don't allow F_3 to jump. Shift $\vec{q}$ so that F_3 changes continuously. Ultimately $\vec{q}$ goes along a cycle & comes back to its original position]

In other words:–

$\frac{\partial F_3}{\partial q^\alpha} = p_\alpha$

$p_\alpha = g_\alpha(\vec{q}, \vec{\tilde{p}})$

$= K_\alpha(\vec{q}, \vec{I})$

$\frac{\partial F_3}{\partial q^\alpha} = K_\alpha(\vec{q}, \vec{I})$

$F_3 = \int dq'^\alpha K_\alpha(\vec{q}\,', \vec{I})$

Now, the $f_1(\vec{I})$ & $f_2(\vec{I})$ we get are declared as $\tilde{p}_1$ & $\tilde{p}_2$ at the end

Change in $F_3(\vec{q}, \vec{\tilde{p}})$ along the cycle C_α is $\tilde{p}_\alpha$.

Recall the canonical transfn. laws:

$$\tilde{q}^\alpha = \frac{\partial F_3}{\partial \tilde{p}_\alpha}, \quad p_\alpha = \frac{\partial F_3}{\partial q^\alpha}$$

Change in $\tilde{q}^\alpha$ as we move along C_β:

$$\Delta \tilde{q}^\alpha = \frac{\partial}{\partial \tilde{p}_\alpha} \Delta F_3 = \frac{\partial \tilde{p}_\beta}{\partial \tilde{p}_\alpha} = \delta_{\alpha\beta}$$

So, $\tilde{q}^\alpha = \frac{\theta_\alpha}{2\pi}$

 Under a translation $\theta_\beta \to \theta_\beta + 2\pi$,

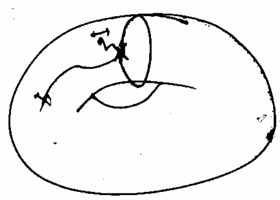
$\tilde{q}^\beta$ changes by 1.

Other $\tilde{q}^\alpha$'s do not change.

$\xrightarrow[\text{these are called}]{\text{for this reason}}$ angle variables.

$\tilde{p}_\alpha$'s are known as action angle variables.

[Essentially $\tilde{p}_\alpha$ measures the action along the path, bcos the change in $\tilde{q}_\alpha$ is trivial (since it is 1 — So p_α is called "action" angle variable)



21/10/05

Action angle variables

Given $H(\vec{q},\vec{p})$ & a set of conserved charges $I_1, \cdots, I_N$, we can find new coordinates phase space $(\vec{\tilde{q}}, \vec{\tilde{p}})$ such that ① $H(\vec{q},\vec{p}) = \tilde{H}(\vec{\tilde{p}})$

② $\tilde{q}^\alpha$ is a periodic variable with period 1.

Define $J_\alpha = \tilde{p}_\alpha / 2\pi$, $\theta^\alpha = 2\pi \, \tilde{q}^\alpha$

$\theta^\alpha \to$ periodic variable with period 2π

$$H(\vec{q}, \vec{p}) = H_0(\vec{J})$$

$$\frac{dJ_\alpha}{dt} = -\frac{\partial H_0(\vec{J})}{\partial \theta^\alpha} = 0 \Rightarrow J_\alpha = J_\alpha^{(0)} \text{ (constant)}$$

$$\frac{d\theta^\alpha}{dt} = \frac{\partial H_0}{\partial J_\alpha} = \omega_\alpha(\vec{J})$$

$$\Rightarrow \theta^\alpha = \theta^\alpha_{(0)} + \omega_\alpha(\vec{J})t \ldots$$

Q.) Is this motion periodic?

Suppose it is periodic with period T.

(we should come back to original pt., if it is periodic, after time T)

($J = J_\alpha^{(0)}$ of course doesn't change at all)

change of θ^α in a period T:

$$\Delta\theta^\alpha = \omega_\alpha(\vec{J})\, T = 2\pi n^\alpha$$

$n^\alpha \to$ integers

[θ doesn't have to return to original value. original value $+2\pi n^\alpha$ indicates that u return to same space in phase space]

(T has to be the same for all the coordinates)

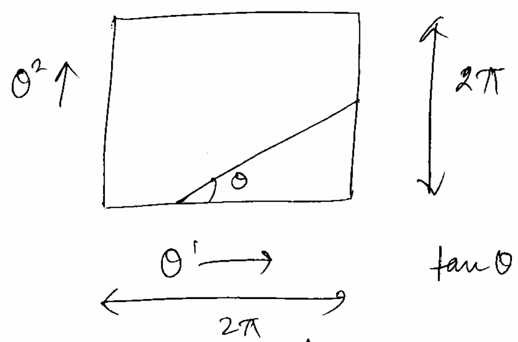
$$\Rightarrow \frac{\omega_\alpha(\vec{J})}{\omega_\beta(\vec{J})} = \frac{n^\alpha}{n^\beta} = \text{rational no.}$$

[Conversely, we can show that if $\frac{\omega_\alpha}{\omega_\beta}$ = rational no., then it is possible to find T]

Ex. If $\frac{\omega_\alpha}{\omega_\beta}$ = rational for every pair (α,β), then the motion is periodic. Find T. (Note, $\frac{\omega_\alpha}{\omega_\beta}$ are not indep.)

62) (Set of rational nos. is negligible compared to set of irrational nos.
We can take a volume around each rational no. & exclude it. We can make this volume arbitrarily small.
So, in the generic case, we will pick up an irrational no. on the real line.)

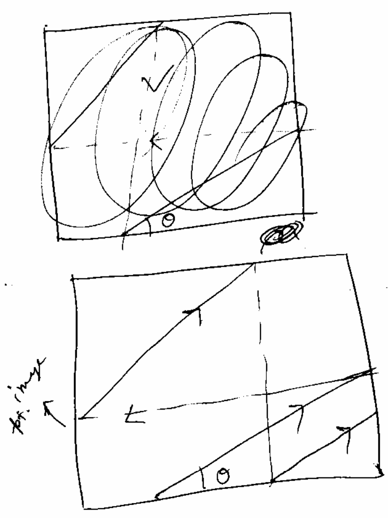
Two dimensional example



$\tan\theta = \omega_2/\omega_1$

say ...
$\tan\theta = \dfrac{\omega_2 t}{\omega_1 t}$
$= \theta^2/\theta^1$

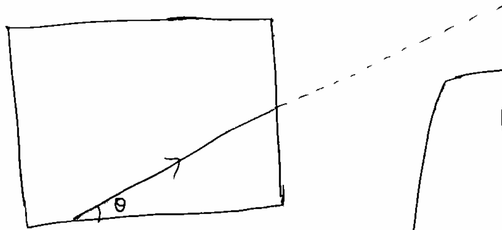
Suppose $\tan\theta = p/q$



1 way
Once you reach one end, you can start from its image pt. on the other end

infinite image pts. for irrational ratio

Consider the point $(\theta_1, \theta_2) = 2\pi(q, p) + (\theta_1^{(0)}, 0)$



(θ_1, θ_2) & $(\theta_1^{(0)}, 0)$ lie on the same st. line

After θ_1 has flipped $2\pi q$ times & θ_2 " " $2\pi p$ times, you return to the original pt.

There is a rational no. arbitrarily close to an irrational no. The approxn. to an irrational no. becomes better & better if we take the num. & den. larger & larger

(In the irrational case, you fill the whole torus in infinite time)

(In 1-D we get a trivial situation – motion is periodic)

Perturbation theory

$$H(\vec{q}, \vec{p}) = \underset{\text{integrable}}{H_0(\vec{q}, \vec{p})} + \lambda \underset{\text{small}}{H_1(\vec{q}, \vec{p})}$$

Suppose $(\vec{J}, \theta)$ are action angle variables for H_0.

In these variables, we have a new Hamiltonian $H(\vec{\theta}, \vec{J}) = \vec{H}_0(\vec{J}) + \lambda V(\vec{\theta}, \vec{J})$

Since θ^α ~~is a periodic~~ are angle variables, $V(\vec{\theta}, \vec{J})$ is periodic under $\theta^\alpha \to \theta^\alpha + 2\pi$.

The transfn. is still canonical becos a transfn. being " doesn't depend on the Hamiltonian.

64)

$$V(\vec{\theta}, \vec{J}) = \sum_{n_1, n_2, \ldots n_N} V^{(\vec{J})}_{(n_1, n_2, \ldots, n_N)} e^{i(n_1\theta^1 + \ldots + n_N\theta^N)}$$

($\because$ V is a periodic variable, it must admit a Fourier expansion of this type)

$n_1, n_2, \ldots \rightarrow$ all integers, +ve & −ve

$$= V_0(\vec{J}) + \sum_{(n_1, n_2, \ldots, n_N)}{}' V_{\vec{n}}(\vec{J}) e^{i\vec{n}\cdot\theta}$$

$\sum{}' \xrightarrow{\text{means}} \sum$ with $(n_1, n_2, \ldots, n_N) = (0, 0, \ldots, 0)$ removed

$$H(\vec{\theta}, \vec{J}) = \overrightarrow{H_0}(\vec{J}) + \lambda V_0(\vec{J}) + \lambda \sum_{\vec{n}}{}' V_{\vec{n}}(\vec{J}) e^{i\vec{n}\cdot\vec{\theta}}$$

H is real

$\Rightarrow V_{-\vec{n}} = (V_{\vec{n}})^*$

$V_0(\vec{J})$ is real

$\overline{H_0}(\vec{J})$ is real

$\lambda \rightarrow$ a real parameter

(Our aim is to show this system is integrable – i.e., there are N conserved charges)

~~Goal~~: Find a canonical transfn. such that the new Hamiltonian is a fn. of the momentum variables only.

Try to construct new action angle variables $\tilde{\vec{J}}, \tilde{\vec{\theta}}$ such that

$$H(\vec{\theta}, \vec{J}) = \tilde{H}(\vec{\tilde{J}})$$

We'll assume that the system is bounded

If we fail, then the system is non-integrable.

Suppose $F_3(\vec{\tilde{J}}, \vec{\theta})$ generates this canonical trs.

$$J_\alpha = \frac{\partial F_3(\vec{\tilde{J}}, \vec{\theta})}{\partial \theta^\alpha}, \quad \tilde{\theta}^\alpha = \frac{\partial F_3(\vec{\tilde{J}}, \vec{\theta})}{\partial \tilde{J}_\alpha}$$

$$F_3(\vec{\tilde{J}}, \vec{\theta}) = \tilde{J}_\alpha \theta^\alpha + \lambda f_3(\vec{\tilde{J}}, \vec{\theta}) + O(\lambda^2)$$

(In order that this canonical transfn. is a meaningful one, f_3 better be periodic in $\vec{\theta}$)

periodic in $\vec{\theta}$ (& hence can be Fourier expanded)

Choose:–

$$f_3(\vec{J}, \vec{\theta}) = g_0(\vec{J}) + \sum_{\vec{n}} g_{\vec{n}}(\vec{J}) e^{i\vec{n}\cdot\vec{\theta}}$$

$$J_\alpha = \frac{\partial F_3}{\partial \theta^\alpha} = \tilde{J}_\alpha + \lambda \frac{\partial f_3}{\partial \theta^\alpha}$$

$$= \tilde{J}_\alpha + \lambda \sum_{\vec{n}}{}' (i n_\alpha) e^{i\vec{n}\cdot\vec{\theta}} g_{\vec{n}}(\vec{J})$$

$$\tilde{\theta}^\alpha = \theta^\alpha + O(\lambda)$$

$$H(\vec{\theta}, \vec{J}) = \bar{H}(\vec{J}) + \lambda V_0(\vec{J}) + \lambda \sum_{\vec{n}}{}' V_{\vec{n}} e^{i\vec{n}\cdot\vec{\theta}}$$

$$= \bar{H}(\vec{\tilde{J}}) + \sum_{\alpha=1}^{N} \frac{\partial \bar{H}(\vec{\tilde{J}})}{\partial \tilde{J}_\alpha} \lambda \sum_{\vec{n}}{}' i n_\alpha e^{i\vec{n}\cdot\vec{\theta}} g_{\vec{n}}(\vec{J})$$

$$+ \lambda V_0(\vec{\tilde{J}}) + \lambda \sum_{\vec{n}}{}' V_{\vec{n}}(\vec{J}) e^{i\vec{n}\cdot\vec{\theta}}$$

$$+ O(\lambda^2)$$

can replace by $\vec{\tilde{J}}$

Choose $g_{\vec{n}}(\vec{\tilde{J}})$ such that

$$\sum_{\alpha=1}^{N} \frac{\partial \bar{H}(\vec{\tilde{J}})}{\partial \tilde{J}_\alpha} (i n_\alpha) g_{\vec{n}} = -V_{\vec{n}}(\vec{\tilde{J}})$$

66)

$$\frac{\partial \bar{H}(\vec{\tilde{J}})}{\partial \tilde{J}_\alpha} = \bar{\omega}^\alpha(\vec{\tilde{J}})$$

↳ angular frequencies of the unperturbed Hamiltonian

$$\therefore \quad \boxed{g_{\vec{n}}(\vec{J}) = i \frac{V_{\vec{n}}(\vec{J})}{\sum_{\alpha=1}^{n} \bar{\omega}^\alpha(\vec{J})\, n_\alpha}}$$

(It looks like in perturbation theory to first order λ, any perturbation is integrable — But there is a problem — the denominator may vanish)

If $\dfrac{\bar{\omega}^\alpha(\vec{J})}{\bar{\omega}^\beta(\vec{J})}$ = rational for any pair (α, β), we can have some $\vec{n}$ such that $\sum_{\alpha=1}^{N} \bar{\omega}^\alpha(\vec{J})\, n_\alpha = 0$

Even if one pair is rational, we can't do this

Even if $\bar{\omega}^\alpha / \bar{\omega}^\beta$ is irrational by choosing appropriate $\vec{n}$'s, we may be able to make

$$\sum_\alpha n_\alpha \bar{\omega}^\alpha(\vec{J})$$

as small as we like

⇒ corresponding $g_{\vec{n}}$ will be large

(any irrational pt. has a rational pt. very close to it)

The real no. line is dense in irrational nos

check

Does this mean ~~that~~ the perturbation theory fail everywhere?

Every irrational no. has a rational approximation

$1, \sqrt{2}$

$n_1(1) + n_2\sqrt{2} \rightarrow$ (want to make it as small as possible)

need $\left(\frac{n_1}{n_2} + \sqrt{2}\right)$ small

e.g. for approximating π to leading order, we take $22/7$. By ~~not~~ taking a larger numerator & a " denominator, we can get a better approximation

Making ~~this small~~

$\sum_\alpha \bar{\omega}^\alpha(\vec{J})\, n_\alpha$ small

requires large n_α's

(generically). [If the ratios are rational, even with small n_α's we can't achieve this]

Try to make $V_{\vec{n}}$ fall sufficiently fast for large $\vec{n}$ so that the fall of $V_{\vec{n}}$ is faster than the fall of $\sum_\alpha n_\alpha\, \bar{\omega}^\alpha(\vec{J})$

Take rational tori & use perturbation — won't work

[You can compare to a set of N harmonic oscillators] —

$\omega_1, \omega_2, \omega_3 - - -$

generic energy level $\sum n_\alpha \omega_\alpha$ $\quad \sum m_\alpha \omega_\alpha$

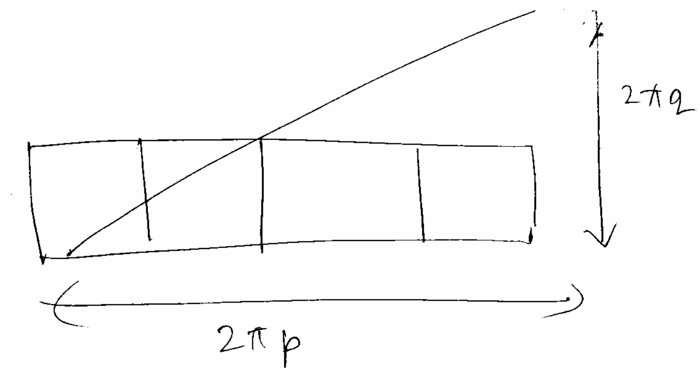
$m_\alpha \geq 0,\ n_\alpha \geq 0$

To have a degenerate energy level $\rightarrow \sum_\alpha n_\alpha - \sum_\alpha m_\alpha \omega_\alpha = 0$

$= \sum (m_\alpha - n_\alpha)\omega_\alpha = 0$

To have deg. energy --- use deg. perturbation theory

Even if ~~some~~ some system isn't integrable (intrinsically ") in all phase space, it can be integrable in some parts of phase space — that is what we are doing



25/10/05 Hamiltonian $H(\vec{q},\vec{p}) = H_0 + H_1$

$H_0 \to$ integrable system $\to \bar{H}_0(\vec{J})$, $\vec{J}$: action variable

$H_1 \to$ small perturbation

$$H = \bar{H}_0(\vec{J}) + \lambda V(\vec{J},\vec{\theta})$$

$$\lambda V_0(\vec{J}) + \lambda \sum_{\vec{n}}{}' V_{\vec{n}}(\vec{J}) e^{i\vec{n}\cdot\vec{\theta}}$$

where $\sum_{\vec{n}}{}' \ni$ exclude the $\vec{n}=0$ case

Look for new action - angle variables $(\vec{\tilde{J}}, \vec{\tilde{\theta}})$ such that

$$H = \tilde{H}(\vec{\tilde{J}})$$

Suppose $F_3(\vec{\tilde{J}},\vec{\theta})$ generates the canonical transformation from $(\vec{J},\vec{\theta})$ to $(\vec{\tilde{J}},\vec{\tilde{\theta}})$

$$J_i = \frac{\partial F_3(\vec{\tilde{J}},\vec{\theta})}{\partial \theta^i}, \quad \tilde{\theta}^i = \frac{\partial F_3(\vec{\tilde{J}},\vec{\theta})}{\partial \tilde{J}_i}$$

Take $F_3(\vec{\tilde{J}},\vec{\theta}) = \sum_{i=1}^{N} \tilde{J}_i \theta^i + \lambda \sum_{\vec{n}}{}' g_{\vec{n}}(\vec{\tilde{J}}) e^{i\vec{n}\cdot\vec{\theta}}$

Final result :

$$g_{\vec n}(\vec J) = i\,\frac{V_{\vec n}(\vec J)}{\sum_{\alpha=1}^{N} n^\alpha \omega_\alpha(\vec J)}$$

$$\omega_\alpha(\vec J) = \frac{\partial H_0}{\partial J_\alpha}(\vec J)$$

If $\frac{\omega_\alpha(\vec J)}{\omega_\beta(\vec J)}$ = rational for any pair (α, β),

then $g_{\vec n}(\vec J) \longrightarrow \infty$ for $\vec n$ satisfying

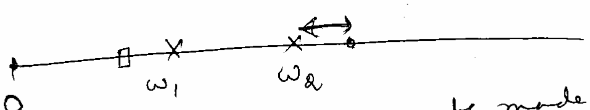
$$\sum_\alpha \omega_\alpha(\vec J)\, n_\alpha = 0 .$$

A ~~given~~ torus is given by a given $\vec J$ — changing $\vec J$ changes the torus. The ratio $\frac{\omega_\alpha}{\omega_\beta}$ also depends on $\vec J$ & of course on the given Hamiltonian

Rational points are dense in the space of real nos. (given any real no., u can find a rational no. arbit. close to it).

(So the pts. where perturbation fails is dense in the space of real nos.)

(That's why a generic pert. doesn't give an integrable system)



$(n_1\omega_1 + n_2\omega_2) \longrightarrow$ can be made arbit. small

chose m such that $m\omega_1$ is just larger than ω_2

& $(m-1)\omega_1$ would have smaller than ω_2

$n\omega_1$ for some n $(n\omega_1 - \omega_2)$ should be less than ω_1

70) (We can bring this as close to origin as we like
Had ω_1/ω_2 been rational, we could bring ~~that exa~~ $n_1\omega_1 + n_2\omega_2$ exactly to zero)

(For the special case $V_{\vec{n}} = 0$ whenever $\omega_\alpha/\omega_\beta$ is rational, that system is solvable for the that particular perturbation.
The very fact that the perturbation isn't valid for all $\vec{J}$'s means that the system isn't integrable. But, for the cases when the theory applies, we try to apply the tricks of an integrable system)

Condition on $V_{\vec{n}}$ for a sensible perturbation theory

Consider the $N=2$ case

$$g_{n_1,n_2}(\vec{J}) = i\,\frac{V_{n_1,n_2}(\vec{J})}{n_1\omega_1(\vec{J}) + n_2\omega_2(\vec{J})}$$

$$= i\,\frac{V_{n_1,n_2}(\vec{J})}{n_1\omega_2(\vec{J})\left(\frac{\omega_1(\vec{J})}{\omega_2(\vec{J})} + n_2/n_1\right)}$$

$f_{n_1,n_2}(\vec{J})$ $\qquad$ $\frac{\omega_1(\vec{J})}{\omega_2(\vec{J})} = x$

Conserved charges x, E

$$g_{n_1,n_2}(\vec{J}) = \frac{f_{n_1,n_2}(x, E)}{x + n_2/n_1}$$

$$g(\vec{\tilde{J}}, \vec{\theta}) = i \sum{}' \frac{f_{n_1,n_2}(x, E)}{x + n_2/n_1} e^{i(n_1\theta_1 + n_2\theta_2)}$$

$$F_3 = (\tilde{J}_1 \theta^1 + \tilde{J}_2 \theta^2) + \lambda g(\vec{\tilde{J}}, \vec{\theta})$$

Q) Under what condn is $g(\vec{\tilde{J}}, \vec{\theta})$ finite ?

Consider the region $|x| \leq 1$

x cannot be rational.

Let us consider those values of x for which $|x + n_2/n_1| > \frac{\eta}{(|n_1| + |n_2|)^\alpha}$ for every integer n_1, n_2 with $n_2 < n_1$.

η = A fixed but small no.

α = A ~~positive~~ integer

i.e. x does not lie in the range $\left(\frac{1}{2} - \frac{\eta}{3^\alpha}, \frac{1}{2} + \eta/3^\alpha\right)$

The length excluded becomes smaller & smaller as n_1, n_2 becomes larger & larger

(It is not guaranteed that we are left with a volume on the real line, which isn't excluded by this procedure)

$$\left|g(\vec{\tilde{J}}, \vec{\theta})\right| \leq \sum_{n_1,n_2}{}' \frac{|f_{n_1,n_2}(x, E)|}{|x + n_2/n_1|}$$

$$< \sum_{n_1,n_2}{}' \frac{|f_{n_1,n_2}(x, E)|}{\eta} (|n_1| + |n_2|)^\alpha$$

mod. of a no. is larger than the no. itself - so let's take the modulus

72) Suppose $\left| f_{n_1 n_2}(\lambda, E) \right| < \dfrac{K}{(|n_1| + |n_2|)^{\beta}}$

K : some finite no.
β = some ~~positive~~ integer

$$\Rightarrow \left| g(\vec{J}, \vec{\theta}) \right| < \frac{K}{\eta} \sum_{n_1, n_2}{}' \frac{1}{(|n_1| + |n_2|)^{\beta - \alpha}}$$

(The problem comes for large n_1, n_2.
So, we consider an integral instead of a sum :—)

$$\int_1^{\infty}\int_1^{\infty} \frac{dx\, dy}{(x+y)^{\beta-\alpha}}$$

$\rightarrow$ convergent for $(\beta - \alpha) > 2$

Ex. Prove it

$$\int_1^{\infty}\int_1^{\infty} \frac{dx\, dy}{(x+y)^{\beta-\alpha}}$$
$$= \frac{1}{\beta-\alpha-1}\int_1^{\infty} \frac{1}{(x+y)^{\beta-\alpha-1}}\Bigg|_{x=1}^{x=\infty} dy$$
$$= \frac{1}{\beta-\alpha-1}\int_1^{\infty} \frac{1}{(1+y)^{\beta-\alpha-1}}\, dy$$
$$= \frac{1}{(\beta-\alpha-1)(\beta-\alpha-2)} \frac{1}{(1+y)^{\beta-\alpha-2}}\Bigg|_1^{\infty}$$
$$= \;''\; \left(\frac{1}{2^{\beta-\alpha-2}} - 0\right)$$
if $(\beta - \alpha) > 2$

Conclusion :— If $(\beta - \alpha) > 2$, then the sum over n_1, n_2 converges.

[A condn. on β means a condn. of f & hence, a condn. on V $\rightarrow$ hence the perturbation ~~parts~~ itself]

Q) How small can we choose α?
We need to choose α such that ~~we have~~ some values of x are still allowed.

($\beta > 2 + \alpha$
So if α can be chosen as small as we like, β can be chosen as close to 2 as we like)

Total ~~volume~~ length of the excluded region betw. 0 and 1 :

$$< \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{n_1} \frac{2\eta}{(|n_1|+|n_2|)^{\alpha}}$$

(less than because the ~~cut off~~ excluded regions can overlap)

$$= \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{n_1} \frac{\eta}{(|n_1|+|n_2|)^{\alpha}} + \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{n_2} \frac{\eta}{(|n_1|+|n_2|)^{\alpha}}$$

bcos you can interchange n_1 & n_2

$$= \eta \times \text{finite no. if } \alpha > 2$$

$$< 1 \text{ for sufficiently small } \eta \ldots$$

For α greater than 2, the total length of excluded region < 1.

Actual volume is still smaller bcos we haven't taken into account the fact that $2/4 = 1/2$, etc.

$\Rightarrow$ there is a finite length region in x space for which perturbation theory is sensible provided

$$\beta > \alpha + 2.$$

$$\Rightarrow \beta > 4.$$

Higher order results (K A M)

Kolmogorov, Arnold, Moser

Under certain conditions, the perturbation theory is sensible to all orders

$\longrightarrow$ Need a reorganisation of the perturbation theory

74)

Now $\omega_i(\vec{J}) = \dfrac{\partial \bar{H}_0(\vec{J})}{\partial J_i} \to \dfrac{\partial}{\partial J_i}\left(\bar{H}_0(\vec{J}) + \lambda_0 \bar{V}_0(\vec{J})\right)$

Unperturbed motion

① Resonant torus: $\sum n_i \omega_i(\vec{J}) = 0$ for some n_i

completely destroyed by perturbation

② Non-resonant torus: $\sum n_i \omega_i(\vec{J}) \neq 0$ for any choice of n_i

Stable under perturbation

Unstable under perturbation

27/10/05

The question

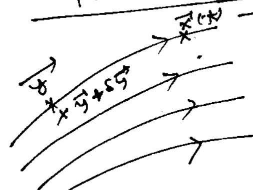
Phase space

$\vec{x}(t) \to$ obtd by solving the eqns of motion

$\vec{x}(t) = \vec{f}(\vec{y}, t)$

$\dfrac{\partial f^i(\vec{y}, t)}{\partial y^j} = M_{ij}(\vec{y}, t)$

Take $\vec{x}(0) = \vec{y}$



How fast do the neighbouring trajectories diverge as $t \to \infty$?

If the trajectories diverge exponentially as $t \to \infty$, then the motion is chaotic, otherwise the motion is regular.

Integrable systems have regular motion.

Use action angle variables

$J_i = J_i^{(0)}, \quad \theta^i = \omega_i(\vec{J}^{(0)})t + \theta^i_{(0)}$

$$J_i = J_i^{(0)} + \delta J_i^{(0)}$$

$$\theta^i = \omega_i\left(\vec{J}^{(0)} + \delta\vec{J}^{(0)}\right)t + \theta^i_{(0)} + \delta\theta^i_{(0)}$$

$$\delta J_i = \delta J_i^{(0)}$$

$$\delta\theta^i = \left(\omega_i\left(\vec{J}^{(0)} + \delta\vec{J}^{(0)}\right) - \omega_i\left(\vec{J}^{(0)}\right)\right)t + \delta\theta^i_{(0)}$$

θ^i dependence grows linearly in t — so motion is a regular motion

exp. div. is larger than power law divergence. If we change the initial condn. only slightly, the trajectories vary drastically 75

Chaos doesn't mean that the trajectories spread over the entire phase space. The ratio of final separation to the ratio of initial sep. may be infinity, though the final sep. isn't infinity

Formal analysis

Stability matrix

$\vec{x}$: coordinate in the phase space $(\vec{q}, \vec{p})$.

$\vec{x}(t)$: a trajectory.

$M_{ij}(\vec{y}, t)$ is defined as:

$\vec{y} \longrightarrow$ phase space coordinates

$$M_{ij}(\vec{x}(0), t) = \frac{\partial x^i(t)}{\partial x^j(0)}$$

The choice of coord. doesn't affect the regular or chaotic behaviour of a system as long as the coord. transfn. is non-singular — bcos the derivatives will have an upper bound as long as the motion is bounded

Time evolution of M

$$\frac{dx^i}{dt} = \Omega^{ij}\frac{\partial H}{\partial x^j}$$

$$\Omega = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$

$$\frac{d}{dt}\left(x^i(t) + \delta x^i(t)\right) = \Omega^{ij}\frac{\partial H(\vec{y})}{\partial y^j}\bigg|_{\vec{y} = \vec{x}(t) + \delta\vec{x}(t)}$$

We are choosing $\vec{q}, \vec{p}$ coord. system

$$= \Omega^{ij}\frac{\partial H(\vec{x})}{\partial x^j} + \Omega^{ij}\frac{\partial^2 H}{\partial x^j \partial x^k}\delta x^k(t) + O\left(\delta\vec{x}^2\right)$$

$$\frac{\partial^2 H}{\partial x^j \partial x^k} = H^{(2)}_{jk}(\vec{x})$$

(2) stands for 2nd order deriv. of H

$$\frac{d}{dt}\left(x^i(t) + M_{ij}(\vec{x}(0), t)\,\delta x^j(0)\right) + O(\delta\vec{x}^2)$$

$$= \frac{dx^i(t)}{dt} + \frac{\partial M_{ij}(\vec{x}(0), t)}{\partial t}\delta x^j(0) + O(\delta x^2)$$

76) Zeroth order term cancels on both sides.
Comparing the 1st order term, we get,

$$\frac{\partial M_{il}(\vec{x}(0),t)}{\partial t}\,\delta x^{l}(0)$$
$$= \Omega^{ij} H^{(2)}_{jk}\;\delta x^{k}(t)$$
$$= \Omega^{ij} H^{(2)}_{jk}\; M_{kl}(\vec{x}(0),t)\,\delta x^{l}(0)$$

This eqn. is valid for any choice of $\delta x^{l}(0)$ so must be valid for all $\delta \vec{x}(0)$

True for every $\delta\vec{x}(0)$

$$\Rightarrow \boxed{\frac{\partial M_{il}(\vec{x}(0),t)}{\partial t} = \Omega^{ij} H^{(2)}_{jk}\; M_{kl}(\vec{x}(0),t)}$$

(tells us how the Matrix M evolves with time)

$$\delta x^{i}(t) = M_{ij}(\vec{x}(0),t)\,\delta x^{j}(0) \Rightarrow \delta\vec{x}(t) = M\,\delta\vec{x}(0)$$

$$\Rightarrow |\delta\vec{x}(t)|^2 = \delta x^{i}(t)\,\delta x^{i}(t)$$

$$= \delta\vec{x}^{T}(0)\,(M^{T}M)\,\delta\vec{x}(0)$$
$$= \langle \delta\vec{x}(0)|\alpha\rangle\langle\alpha|M^{T}M|\beta\rangle\langle\beta|\delta\vec{x}(0)\rangle$$
$$= \langle\delta\vec{x}|\alpha\rangle \langle\alpha| M^TM|\beta\rangle$$

$$M^{T}M\,|\alpha\rangle = d^{2}_{(\alpha)}\,|\alpha\rangle$$

$|\phi\rangle$ → 2N dim. vector

Consider $\langle\alpha|M^{T}M|\alpha\rangle$
This is inner prdt. of $M|\alpha\rangle$ with itself & hence is > 0

+ve semidefinite means ≥ 0

+ve definite means > 0

Suppose $M^{T}M|\alpha\rangle = \lambda_\alpha|\alpha\rangle$

Then, $\lambda_\alpha\langle\alpha|\alpha\rangle \Rightarrow \lambda_\alpha =$ ratio of +ve nos. ≥ 0

$\therefore$ $M^{T}M$ is a positive semidefinite matrix $\Rightarrow$ eigenvalues of $M^{T}M$ are positive or zero

Denote them by $d^{2}_{(\alpha)}(t, \vec{x}(0))$

If we calculate the expectation value for any vector, it will be zero or +ve $\longrightarrow$ meaning of +ve semidefinite matrix

Anything we do on M, should depend on the initial choice of $\vec{x}(0)$

But in $d^2_{(\alpha)}(t, \vec{x}(0))$, we drop $\vec{x}(0)$ for convenience.

$$|\delta\vec{x}(t)|^2 = \langle\delta\vec{x}|\alpha\rangle\underbrace{\langle\alpha|M^TM|\beta\rangle}_{d^2_{(\beta)}(t)\,\underbrace{\langle\alpha|\beta\rangle}_{\delta_{\alpha\beta}}}\langle\beta|\delta\vec{x}(0)\rangle$$

Basis of eigenvectors of M^TM

$$= \sum_\beta \left(d_{(\beta)}(t)\right)^2 \langle\delta\vec{x}(0)|\beta\rangle\langle\beta|\delta\vec{x}(0)\rangle$$

time dependent

M^TM is symmetric
$|\alpha\rangle$, $|\beta\rangle$ are e.vectors of this sym. matrix
hence are orthonormal
Note: $(M^TM)^T = M^TM$

$\sum_i |\beta_i\rangle\langle\beta_i| = 1$
$\Rightarrow 1 - \sum_{i\neq i_0} |\beta_i\rangle\langle\beta_i| = |\beta_{i_0}\rangle\langle\beta_{i_0}|$
$\therefore \langle\delta\vec{x}(0)|\beta_{i_0}\rangle\langle\beta_{i_0}|\delta\vec{x}(0)\rangle = |\delta\vec{x}(0)|^2 - \sum_{i\neq i_0}\langle\delta\vec{x}(0)|\beta_i\rangle\langle\beta_i|\delta\vec{x}(0)\rangle$
$\Rightarrow 0 \le |\langle\beta_{i_0}|\delta\vec{x}(0)\rangle|^2 \le |\delta\vec{x}(0)|^2$

[$|\beta\rangle$ can't have exponential time-dependence because $|\beta\rangle$'s are normalised
So, the quantity $\langle\delta\vec{x}(0)|\beta\rangle\langle\beta|\delta\vec{x}(0)\rangle$ is bounded by $|\delta\vec{x}(0)|^2$]

$$|\langle\delta\vec{x}(0)|\beta\rangle| \le |\delta\vec{x}(0)|$$

$\Rightarrow$ Growth of $|\delta\vec{x}(t)|$ with t is controlled by the growth of $d^2_{(\beta)}(t)$ with t.

Liapunov function associated with the point $\vec{x}(0)$:

$$\lambda_\alpha(t) = \frac{1}{t}\ln d_{(\alpha)}(t)$$

$\alpha = 1, 2, \ldots, 2N$

There are $2N$ no. of Liapunov fns associated with a pt. in phase space $\because M$ is a $2N\times 2N$ matrix

78) Liapunov exponent: $\bar{\lambda}_\alpha = \lim_{t\to\infty} \lambda_\alpha(t)$

For large t, $\lambda_\alpha = \bar{\lambda}_\alpha$

$$d_{(\alpha)}(t) \sim \text{constant} \times e^{\bar{\lambda}_\alpha t}$$

If any $\bar{\lambda}_\alpha$ is positive, then the motion is chaotic.

Recall :– $M_{ij}(\vec{x}(0), t) = \dfrac{\partial x^i(t)}{\partial x^j(0)}$

Eqn. of motion:

$$\frac{dx^i(t)}{dt} = \{x^i(t), H\}$$

$$x^i(t+\delta t) = x^i(t) + \delta t\, \{x^i(t), H\} + O(\delta t^2)$$

Canonical transfn. of x^i generated by H

this implies $\Rightarrow x^i(t+\delta t)$ is related to $x^i(t)$ by a Canonical transfn.

Repeating this many times, we can see that $x^i(t)$ is related to $x^i(0)$ by a canonical transfn.

Any transfn. which preserves phase vol. isn't Canonical bcos vol. is sensitive only to the ~~matrix~~ det. of matrix Ω → only det. has to be 1

Successive canonical transfns. are canonical bcos all transfns. preserve the matrix Ω

Recall :– $y^i = f^i(\vec{x})$ is canonical iff

$$\frac{\partial y^i}{\partial x^k}\, \Omega^{kl}\, \frac{\partial y^j}{\partial x^l} = \Omega^{ij}$$

$$\Rightarrow \frac{\partial x^i(t)}{\partial x^k(0)}\, \Omega^{kl}\, \frac{\partial x^j(t)}{\partial x^l(0)} = \Omega^{ij}$$

$$\Rightarrow M_{ik}\, \Omega^{kl}\, M_{jl} = \Omega^{ij}$$

$\Rightarrow \cancel{M_{ik}\,\Omega^{lk}\,M_{jl} = \Omega^{ji}}$

Note: $\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \begin{pmatrix}-1&0\\0&-1\end{pmatrix} = -\mathbb{1}$

$\Omega^T = -\Omega$

Now, $-\Omega\,\Omega = \Omega^T\Omega = 1$

$\Rightarrow \Omega\Omega = -1$

$$\Rightarrow \boxed{M\Omega M^T = \Omega}$$

$$\Downarrow$$

$$M\Omega M^T\Omega M = \Omega\Omega M = -M$$

$$\Rightarrow M^{-1}M\Omega M^T\Omega M = -M^{-1}M$$

$$\Rightarrow -\Omega M^T\Omega M = -1$$

$$\Rightarrow M^T\Omega M = -\Omega^{-1}$$

$$\Rightarrow \boxed{M^T\Omega M = \Omega}$$

Suppose $v_{(\alpha)}$ is the eigenvector of M^TM with eigenvalue $d^2_{(\alpha)}(t)$.

$$M^TM\,v_{(\alpha)} = d_{(\alpha)}(t)^2\,v_{(\alpha)} \rightarrow \text{(2N-dim column vector)}$$

$$\Rightarrow (M^T)^{-1}v_{(\alpha)} = d_{(\alpha)}^{-2}\,M\,v_{(\alpha)}$$

Now,
$$M^TM\,\Omega\,v_{(\alpha)} = M^T\underbrace{M\Omega M^T}(M^T)^{-1}v_{(\alpha)}$$
$$= M^T\Omega\,(M^T)^{-1}v_{(\alpha)}$$
$$= M^T\Omega\,d_{(\alpha)}^{-2}\,M\,v_{(\alpha)}$$
$$= d_{(\alpha)}^{-2}\,M^T\Omega M\,v_{(\alpha)}$$
$$= d_{(\alpha)}^{-2}\,\Omega\,v_{(\alpha)}$$

$\Rightarrow \Omega v_{(\alpha)}$ is an eigenvector of M^TM with eigenvalue $d_{(\alpha)}^{-2}$.

\# If $d^2_{(\alpha)}$ is an eigenvalue of M^TM, so is $d_{(\alpha)}^{-2}$

$\Rightarrow$ If $\lambda_\alpha(t)$ is a Liapunov fn., then $-\lambda_\alpha(t)$ is also a Liapunov fn.

$\Rightarrow$ If $\bar{\lambda}_\alpha$ is a Liapunov exponent then $-\bar{\lambda}_\alpha$ is also a Liapunov exponent.

So, Liapunov exponents of a Ham. system come in equal & opposite pairs
80] always
— Not surprising because volume should remain the same.

e.g. → A square is changed to a rectangle.
If one dirn is expanding, in other dirn it should shrink

For an integrable system, all Liapunov exponents should vanish — Take action angle variables & prove it.

$$\overline{\lambda}_\alpha = \lim_{t\to\infty} \lambda_\alpha(t)$$

Under what condn does the limit exist?

$$\frac{\partial M}{\partial t} = \Omega H^{(2)} M$$

Sym. ∵ it is the 2nd deriv

$$\Rightarrow \frac{\partial M^T}{\partial t} = +M^T H^{(2)T} \Omega^T$$
$$= -M^T H^{(2)} \Omega$$

$$\therefore \frac{\partial}{\partial t}(M M^T) = \frac{\partial M}{\partial t} M^T + M \frac{\partial M^T}{\partial t}$$
$$= \Omega H^{(2)} M M^T - M M^T H^{(2)} \Omega$$

(We want to figure out how the e. values evolve with time)

$$\frac{\partial}{\partial t} \mathrm{Tr}\left((MM^T)^p\right)$$
$$= \mathrm{Tr}\left(\frac{\partial}{\partial t}(MM^T)(MM^T)^{p-1} + MM^T \frac{\partial}{\partial t}(MM^T)(MM^T)^{p-2} + \cdots + (MM^T)^{p-1}\frac{\partial}{\partial t}(MM^T)\right)$$
$$= p\,\mathrm{Tr}\left(\frac{\partial}{\partial t}(MM^T)(MM^T)^{p-1}\right)$$

$$\Rightarrow \frac{\partial}{\partial t} \mathrm{Tr}\left((MM^\dagger)^p\right) = p\,\mathrm{Tr}\left\{\Omega H^{(2)}(MM^\dagger)(MM^\dagger)^{p-1} - MM^\dagger H^{(2)}\Omega (MM^\dagger)^{p-1}\right\}$$

$$= p\,\mathrm{Tr}\left\{(\Omega H^{(2)} - H^{(2)}\Omega)(MM^\dagger)^p\right\}$$

Consider the basis of eigenvectors of $MM^\dagger$:

Then, $\frac{\partial}{\partial t}\left\{\sum_\alpha \left(d_\alpha^2(t)\right)^p\right\}$

$$= p\sum_\alpha \langle\alpha|\left(\Omega H^{(2)} - H^{(2)}\Omega\right)\underbrace{(MM^\dagger)^p}_{(d^2_\alpha(t))^p}|\alpha\rangle$$

$$= p\sum_\alpha \left(d_\alpha^2(t)\right)^p \langle\alpha|\left(\Omega H^{(2)} - H^{(2)}\Omega\right)|\alpha\rangle$$

suppose ~~$M^\dagger M|\alpha\rangle = d^2_{(\alpha)}|\alpha\rangle$~~

$M^\dagger M\, v_{(\alpha)} = d^2_{(\alpha)}\, v_{(\alpha)}$

E. vector of $M^\dagger M$: —

$M^\dagger M M^\dagger (M^\dagger)^{-1} v_{(\alpha)} = d^2_{(\alpha)} v_{(\alpha)}$

$\Rightarrow MM^\dagger (M^\dagger)^{-1} v_{(\alpha)} = d^2_{(\alpha)} (M^\dagger)^{-1} v_{(\alpha)}$

Infinite eqns. in finite no. variables — so either no soln. or if a soln. — then it is unique

p can take any value & so we have infinite no. eqns. in finite no. variables. However, all of them can't be indep. But we have enough eqns. to det. $\frac{\partial}{\partial t} d^2_\alpha(t)$

Now, $\frac{\partial}{\partial t}\left\{\sum_\alpha \left(d^2_\alpha(t)\right)^p\right\} = \sum_\alpha p\left(d^2_\alpha(t)\right)^{p-1}\frac{d}{dt}\left(d^2_\alpha(t)\right)$

$$\frac{\partial}{\partial t}\left(d^2_\alpha(t)\right) = d^2_\alpha(t)\,\langle\alpha|\left(\Omega H^{(2)} - H^{(2)}\Omega\right)|\alpha\rangle$$

→ doesn't depend on p & so if this is satisfied, the one above is satisfied for all p

Soln. → $\ln d^2_\alpha(t) = \int_0^t \langle\alpha(t')|\left(\Omega H^{(2)} - H^{(2)}\Omega\right)|\alpha(t')\rangle\, dt'$

→ (evaluating it along the trajectory)

(If $\langle \alpha(t') | (\Omega H^{(\alpha)} - H^{(\alpha)} \Omega) | \alpha(t') \rangle$ ~~has~~ is bounded ~~an upper bound~~, then it can grow at most linearly in t.)

If $(\Omega H^{(\alpha)} - H^{(\alpha)} \Omega)$ is bounded from above & ~~to~~ below, then $d_\alpha(t)$ grows at most linearly with ~~them,~~ time.

$\Rightarrow$ Liapunov exponents exist.

Unless the Ham. is singular, we expect $H^{(\alpha)}$ to be bounded for a ~~bounded~~ bounded motion

28/10/05

Liapunov function for integrable system in action-angle variables

①

$$\frac{\partial M}{\partial t} = \Omega H^{(2)} M$$

$$H^{(2)}_{ij} = \frac{\partial^2 H}{\partial x^i \partial x^j}$$

Variables: $\theta^1, \theta^2, \ldots, \theta^N, J_1, \ldots, J_N$

$$\frac{\partial^2 H}{\partial \theta^\alpha \partial \theta^\beta} = 0, \quad \frac{\partial^2 H}{\partial \theta^\alpha \partial J_\beta} = 0, \quad \frac{\partial^2 H}{\partial J_\alpha \partial J_\beta} = A_{\alpha\beta}$$

$$H^{(2)} = \begin{pmatrix} O_N & O_N \\ O_N & A \end{pmatrix}$$

O_N: $N \times N$ matrix with all entries zero

I_N: $N \times N$ identity matrix

(Now Ω will have a different structure bcos we have diff. ordering of θ^i & J^i)

$$\Omega = \begin{pmatrix} O_N & I_N \\ -I_N & O_N \end{pmatrix}$$

$$\Omega H^{(2)} = \begin{pmatrix} O_N & I_N \\ -I_N & O_N \end{pmatrix} \begin{pmatrix} O_N & O_N \\ O_N & A \end{pmatrix} = \begin{pmatrix} O_N & A \\ O_N & O_N \end{pmatrix}$$

$$(\Omega H^{(2)})^2 = \begin{pmatrix} O_N & A \\ O_N & O_N \end{pmatrix} \begin{pmatrix} O_N & A \\ O_N & O_N \end{pmatrix} = 0$$

$$\frac{\partial M}{\partial t} = \underbrace{\Omega H^{(2)}}_{\text{constants}} M$$

[$H^{(2)}$ is const. bcos along a trajectory, $H = H(J_1, \ldots, J_N)$ & J_i's are const.]

$$\Rightarrow M = e^{\Omega H^{(2)} t} \underbrace{M(t=0)}_{I_{2N}}$$

$$M_{ij} = \frac{\partial x^i(t)}{\partial x^j(0)} = \delta_{ij}$$

$M = M_0 e^{\Omega H^{(2)} t} \longrightarrow$ ✗ wrong

bcos then

$$\frac{\partial M}{\partial t} = M_0 \Omega H^{(2)} e^{\Omega H^{(2)} t}$$

$$= M_0 e^{\Omega H^{(2)} t} [\Omega H^{(2)}]$$

$M_0 H^{(2)}$ ~~$\Omega \ldots H^{(2)} t$~~

84) $M = e^{\Omega H^{(2)} t} = I_{2N} + \Omega H^{(2)} t + 0$ $[\because (\Omega H^{(2)})^2 = 0]$

(M grows linearly with t & so the Liapunov exponent will also grow linearly with t)

$$M = \begin{bmatrix} I_N & At \\ 0 & I_N \end{bmatrix}$$

$$MM^T = \begin{pmatrix} I_N & At \\ 0 & I_N \end{pmatrix} \begin{pmatrix} I_N & 0 \\ At & I_N \end{pmatrix}$$

[Used $A^T = A$ → follows from the defn. of A]

$$= \begin{pmatrix} I_N + A^2 t^2 & At \\ At & I_N \end{pmatrix}$$

($\because$ A is a real sym. matrix, it can be diagonalised by an orthogonal matrix)

$V^T V = I_N = V V^T$

$A = V A_d V^T$,

$$A_d = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N \end{pmatrix}$$

$$\begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} MM^T \begin{pmatrix} V^T & 0_N \\ 0_N & V^T \end{pmatrix} = \begin{bmatrix} V(I_N + A^2 t^2)V^T & VAV^T t \\ VAV^T t & VV^T \end{bmatrix}$$

2N × 2N orthogonal matrix $\because V$ is an N×N ort. mat.

(Note:- $VA^2V^T = VAV^TVAV^T = A_d^2$) i.e., V also diagonalises A^2

$$= \begin{bmatrix} I_N + A_d^2 t^2 & A_d t \\ A_d t & I_N \end{bmatrix}$$

[Each block of the matrix is diagonal bcos A_d &

sider

$$T = \begin{bmatrix} a_1 & & & b_1 & & \\ & a_2 \cdots & a_N & & b_2 \cdots & b_N \\ & & & d_1 & d_2 \cdots & d_N \\ c_1 & c_2 \cdots & c_N & & & \end{bmatrix}$$

→ How to find its e.values?

Let us look at
$$\begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix}$$
→ conjugate by the orthogonal matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Make a transf. such that $C_2 \leftrightarrow C_3$ & $R_2 \leftrightarrow R_3$, which doesn't change its e.values.

Then
$$\begin{bmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & d_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & c_2 & d_2 \end{bmatrix}$$

Now we can diagonalise each block — basically we have to diagonalise 2×2 matrices instead of a 4×4 matrix

X is transformed into

$$\begin{bmatrix} a_1 & b_1 & & & & & \\ c_1 & d_1 & & & & & \\ & & a_2 & b_2 & & & \\ & & c_2 & d_2 & & & \\ & & & & a_3 & b_3 & \\ & & & & c_3 & d_3 & \\ & & & & & & \ddots \end{bmatrix}$$

86) Let us now look at
$$\begin{bmatrix} I_N + A_d^2 t^2 & A_d t \\ A_d t & I_N \end{bmatrix}$$
which has the same structure as X.

Permutation of rows & columns

$$\begin{bmatrix} 1+\lambda_1^2 t^2 & \lambda_1 t & & & \\ \lambda_1 t & 1 & & & \\ & & 1+\lambda_2^2 t^2 & \lambda_2 t & \\ & & \lambda_2 t & 1 & \\ & & & & \ddots \end{bmatrix}$$

Eigenvalues from the α^{th} block:
$$\det \begin{bmatrix} 1+\lambda_\alpha^2 t^2 - \lambda & \lambda_\alpha t \\ \lambda_\alpha t & 1-\lambda \end{bmatrix} = 0$$
$$\Rightarrow \lambda^2 - (2+\lambda_\alpha^2 t^2)\lambda + 1 + \lambda_\alpha^2 t^2 - \lambda_\alpha^2 t^2 = 0$$
$$\Rightarrow \lambda^2 - (2+\lambda_\alpha^2 t^2)\lambda + 1 = 0$$
(It gives 2 e.values)

Prdt. of e.v. = 1 [consistent with our result that if d_α^2 is an e.value then d_α^{-2} is also an e.va

$$d_\alpha^{\pm 2} = \frac{1}{2}\left\{ 2+\lambda_\alpha^2 t^2 \pm \sqrt{(2+\lambda_\alpha^2 t^2)^2 - 4} \right\}$$
$$\lambda_\alpha^{(t)} = \frac{1}{t} \ln d_\alpha(t)$$
$$\bar{\lambda}_\alpha = \lim_{t\to\infty} \frac{1}{t} \underbrace{\ln d_\alpha(t)}_{\text{goes as } \ln t} = 0$$
~~[illegible]~~
([illegible] t depending

$\Rightarrow$ The Liapunov exponents are zero in the action-angle variables

For d_α^{-2}, $\sqrt{(2+\lambda_\alpha^2 t^2)^2 - 4}$

$$= (2+\lambda_\alpha^2 t^2)\left\{1 - \frac{4}{(2+\lambda_\alpha^2 t^2)^2}\right\}^{1/2}$$

$$\simeq (2+\lambda_\alpha^2 t^2)\left[1 - \frac{2}{(2+\lambda_\alpha^2 t^2)^2}\right]$$

[for large t]

[So, for large t,

$$d_\alpha^{-2} = \frac{1}{2}\left(2 - \frac{2}{2+\lambda_\alpha^2 t^2}\right) \to \text{so } d_\alpha^{-1} \text{ goes as } t^{-1}]$$

Coordinate dependence

$\vec{x}$: $(\vec{\theta}, \vec{J})$ coordinate system

$\vec{y}$: General coordinate system

$\dfrac{\partial y^i}{\partial x^j}$ & its inverse are finite matrices.

[$\because$ we are considering bounded motion, this is not an unnatural demand/assumption. If motion goes to infinity, then maybe we have to use a coord. system where the infinite pt. gets mapped to a finite pt. in the new coord. – singular transfn.]

<u>Recall</u>: $\delta x^i(t) = M_{ij}(t, \vec{x}(0))\, \delta x^j(0)$

$$\Rightarrow \delta y^i(t) = \left.\frac{\partial y^i}{\partial x^j}\right|_t \delta x^j(t)$$

$$= \left.\frac{\partial y^i}{\partial x^j}\right|_t M_{jk}(t, \vec{x}(0))\, \delta x^k(0)$$

$$\delta y^i(t) = \left.\frac{\partial y^i}{\partial x^j}\right|_{\vec{x}=\vec{x}(t)} M_{jk} \left.\frac{\partial x^k}{\partial y^l}\right|_{t=0} \delta y^l(0)$$

finite entries (for all time) acc. to our initial assumption

All we are saying is $\vec{x}(0) \xrightarrow[\text{mapped to}]{\text{gets}} \vec{y}(0)$

$\vec{x}(0) + \delta\vec{x}(0) \longrightarrow \vec{y}(0) + \delta\vec{y}(0)$

88)

e.g. of a singular transfn. →

$$y = \frac{1}{1+x^2}$$

$\therefore$ as $x \to \infty$, $y \to 1$

$1 + x^2 = \frac{1}{y} \Rightarrow x = \sqrt{\frac{1}{y} - 1}$

This is a singular map at $y = 1$

[this is a branch pt.; $\frac{dx}{dy}$ is div. at $y = 1$]

This is reflecting the fact that large change in x must be accompanied by small change in y.]

Now, $\delta y^i(t) = \tilde{M}_{il}(t, \vec{y}(0))\, \delta y^l(0)$,

where $\tilde{M}_{il}(t, \vec{y}(0)) = \left.\frac{\partial y^i}{\partial x^j}\right|_{\vec{x}' = \vec{x}(t)} M_{jk}(t, \vec{x}(0)) \frac{\partial x}{\partial y}$

this can grow, but can grow linearly in time $\therefore M = I + \Omega H^{(2)} t$

$\therefore$ $\tilde{M}$ has all entries bounded by $(-ct, ct)$ for some t.

Take ~~matrix~~ det

$$\det \begin{bmatrix} at - \lambda & bt \\ ct & dt - \lambda \end{bmatrix} = 0$$

$\Rightarrow \lambda^2 - (a+d)\lambda t + ad\,t^2 - bc\,t^2 = 0$

$\Rightarrow \lambda$ must be bounded by kt for some constant t

---

$\Rightarrow d_\alpha$ must be bounded by $k t$ for some const. t $(\because d_\alpha^2$ is bounded by $k_1 t^2)$

we must have $|d_\alpha| < Kt$ & $|d_\alpha| > \frac{1}{Kt}$

bcos in a Ham. system. the e.values come in pairs $\to d_\alpha$ &

General result for Hamiltonian system

① Liapunov functions depend on the choice of coordinate system but Liapunov exponents are independent of this choice. Ex.

$\bar{\lambda}_\alpha = \lim_{t\to\infty} \lambda_\alpha(t)$

② If the system has k conserved charges which have vanishing Poisson bracket with each other then at least $2k$ of the Liapunov exponents vanish.

It is sufficient to look at the d_α's larger than 1 & anyway the d_α's are bounded by Kt. We note that $\ln d_\alpha(t)$ can blow up either for large $d_\alpha(t)$ or $d_\alpha(t) < 1$. ~~If~~ $d_\alpha(t) < 1$ case then follows bcos d_α & $\frac{1}{d_\alpha}$ are both e.values

Reference :- H. D. Meyer, Journal of Chemical Physics. Volume 84, p 3147 (1986)

Consider a Hamiltonian system.
Conserved energy $= E$

$$\Gamma(E) = \int d^N p \, d^N q \; \delta\left(E - H(\vec{p}, \vec{q})\right)$$

$\longrightarrow$ Phase space volume of a subspace corresponding to energy E.

If we know other consvd. charges, we put another delta fn. & that'll restrict the phase space further

we f chaos (known as) $\to$ KS entropy
Kolmogorov Sinai

$$\mathcal{I}) = \int d^N p \, d^N q \;\; \delta\left(E - H(\vec{p}, \vec{q})\right) \sum_{\alpha=1}^{N} \overline{\lambda_{\alpha t}}(\vec{q}, \vec{p}) \geq 0$$

Liapunov exponents

90)

3/11/05 Numerical test of the integrability of the system

Example A Hamiltonian system with 4 dimensional phase space after elimination of the cyclic variables.

$H(q^1, q^2, p_1, p_2)$

Q.) Is there a second conserved charge $F(\vec{q}, \vec{p})$?

If there is a second conserved charge, then a phase curve with a given initial condition will lie inside a two dimensional subspace

$$\left.\begin{array}{l} H(\vec{q}, \vec{p}) = E \\ F(\vec{q}, \vec{p}) = C \end{array}\right\}$$

2 condns on 4 variables — so 2-D surface.

torus if the motion is bounded

Otherwise it will lie inside a 3-dimensional subspace $H(\vec{q}, \vec{p}) = E$.

Consider a trajectory with a given initial condition and follow it.

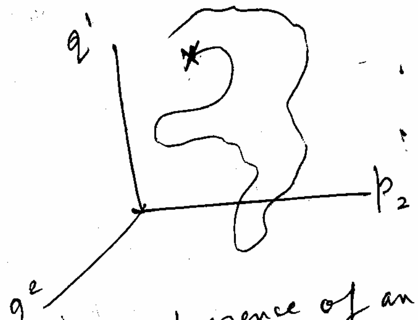
First eliminate one variable (say p^1) using the eqn $H(\vec{q}, \vec{p}) = E$.

this can always be done ∵ H expression is known anyway

We'll choose a specific branch.

e.g. $p^2 = \ldots$ will have 2 branches: $p = \pm\sqrt{\ldots}$

(q^1, q^2, p^2) are the coordinates of the 3-dim. subspace $H(\vec{q}, \vec{p}) = E$.



If in the presence of an additional conserved ~~charge~~ the intersection of the curve with the $q^1=0$ plane will lie along a 1-dim. curve. Otherwise it will be inside a 2-dim. region.

Accidental case → A 3-D ball trying to be put in a plane — that'll intersect the plane in a 2-D region

Algorithm

① Begin with a specific initial condn & follow the trajectory numerically integrating the eqn. of motion.

② Note down the q^2, p_1, p_2 coordinates whenever $q^1=0$.

If bounded motion — coord. will oscillate back & forth & $q^1=0$ will come many times — though motion itself may not be periodic

⇒ a set of values of (q^2, p_1, p_2)

All of these satisfy
$$H(q^1=0, q^2, p_1, p_2) = E.$$

Of these points, discard those which are not on the branch we are looking at.

bcos we have already chosen a specific branch

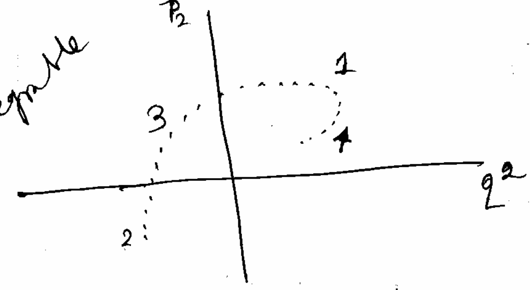
We can verify whether we are on the right branch bcos we know ~~q_1, p_1, p_2~~ q^1, q^2, p_1, p_2

H is a single-valued fn. of coord. & so no ambiguity in [illegible]ically integrating

This gives a set of values of (q^2, p_2).

③ Plot these in the (q^2, p_2) plane.

Integrable



If we don't choose a specific branch, we may get more than 1 curve

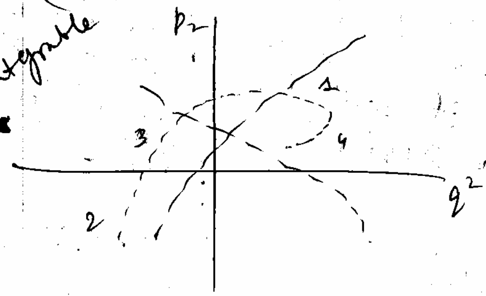
$H = p_1^2 + p_2^2 + p_3^2 + V(q^1, q^2, q^3)$

$p_1 = \pm\sqrt{E - V - p_2 - p_3^2}$

$dq/dt = p_1$

(If we don't pick branch) →

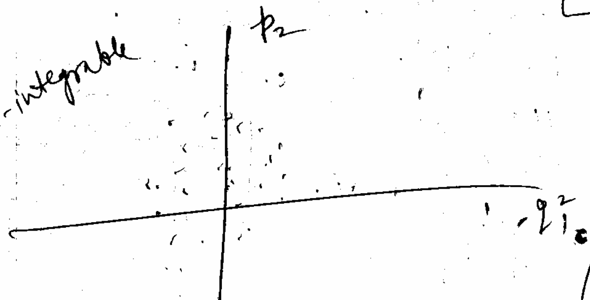
Integrable



For less data, the more curves you have, the more confused you get

No ordering in getting these pts. on these curves

Non-integrable



(Won't see any pattern at all)

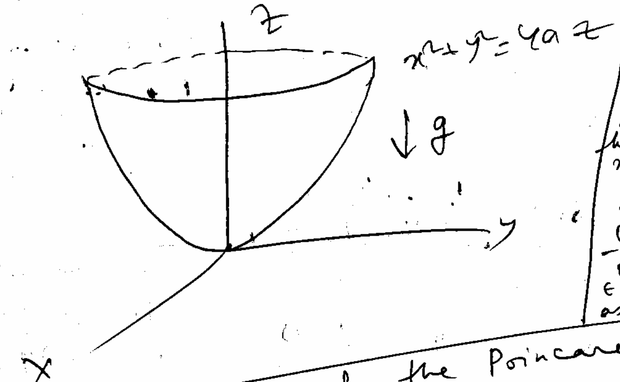
Getting a curve for some initial condns doesn't guarantee a global conserved charge — as we have seen in the past. theory

(The curve itself may be very complicated)

(So ~~don't~~ wait for sufficient time to see this pattern)

In local region the system may appear to have a cond. charge — system may be integrable in some regions.

H.W. Test the integrability of a 3-D particle moving under gravity with a parabolic reflecting wall.



For higher dim. system, we have to impose more conditions e.g. $q^1=q^2=0$ & pts. will be sparse (harder to come by) — then we demand q^1 & q^2 lie with some ϵ & try to reduce ϵ as much as possible.

(This map is called the Poincare map).

Poincare map :- Suppose we are given E, the coordinates (q^2, p_2) and the branch of p_1 (In this case) Then for $q^1=0$, the values of p_1, q^2, p_2 are fixed.

Thus we can follow the trajectory & determine the values of (q^2, p_2) when $q^1=0$ again & p_1 is on the right branch.

This gives a map from any point in the (q^2, p_2) plane to another point in the (q^2, p_2) plane.

$\Rightarrow$ Poincare map.

H.W. Henon-Heiles system

$$H = \frac{1}{2}\left(p_1^2+p_2^2\right) + \frac{1}{2}\left\{(q^1)^2+(q^2)^2+2(q^1)^2 q^2 - \frac{2}{3}(q^2)^3\right\}$$

For q^1, q^2 small, pert. theory can be applied

94) Study this using Poincare map.

Results

$E = \frac{1}{12}$: Most trajectories intersect (p_2, q^2) plane along a curve.

$E = \frac{1}{8}$: some trajectories intersect the (p_2, q^2) plane along a curve, others along a 2-D region.

$E = 1/6$: Most trajectories intersect the (p_2, q^2) plane in a 2-D region.

check this numerically.

One way to ensure q^1, q^2 small is keep the energy small.

$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q^1)^2 + (q^2)^2$

will then be a small amplitude oscillation

In this problem, in a sense all tori are resonant, ang. freq's same indep. of energy

We can write H as

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q^1)^2 + \frac{1}{2}(1-\epsilon)(q^2)^2 + \underbrace{\left[\epsilon (q^2)^2 + 2(q^1)^2 q^2 - 2/3\,(q^2)^3\right]}_{\text{Perturbation}}$$

Then the problem of resonant tori can be avoided — Problem: this depends very much on what you call ~~resona~~ ~~perturbed~~ perturbation

(Poincare map isn't continuous bcos the next pt. won't in general lie near the previous one — commonly known as discrete time dynamics)

Poincare map is an example of a

Poincare map

discrete time dynamics.

$$\vec{x}_{k+1} = \vec{F}(\vec{x}_k) \rightarrow \text{discrete time dynamics}$$

$\vec{x} = (x^1, x^2, \ldots, x^n)$ pt. on this manifold

(Rule to go from one pt. to another — continuum dynamics is the dyn. system we are studying)

This is a 2nd order non-autonomous dynamical system (95)

$\ddot{x} + \dot{x} + \sin x = \cos t$

$\rightarrow$ 2nd order non-autonomous or 3rd order autonomous system.

Study the Poincare map for this problem.

96)

10/11/05

$\vec{x}(t)$ $\frac{dx^i}{dt} = f^i(\vec{x})$

$x^i_{n+1} = f^i(\vec{x}_n)$

$\frac{x^i_{n+1} - x^i_n}{h} = f^i(\vec{x}_n)$

h: interval between two successive times

$x^i_{n+1} = x^i_n + h f^i(\vec{x}_n)$

Discrete time dynamics.
e.g. → what is the population of the next yr same time given the population this yr.

Simplest case:

$x_{n+1} = F(x_n)$

Fixed point: $x_{n+1} = x_n$

$\Rightarrow F(x_F) = x_F$

x_F is a fixed point $\Rightarrow F(x_F) = x_F$

Suppose $x_n = x_F + \underbrace{\epsilon}_{\text{small}}$

$x_{n+1} = F(x_F + \epsilon) = \underbrace{F(x_F)}_{x_F} + \epsilon F'(x_F) + O(\epsilon^2)$

$\Rightarrow x_{n+1} - x_F \simeq F'(x_F)\,\epsilon$

$= F'(x_F)\,(x_n - x_F)$

$|F'(x_F)| < 1 \xrightarrow{\text{gives}}$ Stable fixed point.

bcos difference is going to decrease in successive steps

$|F'(x_F)| > 1 \Rightarrow$ unstable fixed point.

Example ①

$F(x) = ax + b$ (we take the fn. F to be linear)

$x_F = a x_F + b \Rightarrow x_F = \frac{b}{1-a}$

Now, $F'(x_F) = a$

$\therefore$ for $|a| > 1$ unstable

for $|a| < 1$ stable.

Define $y = x - \frac{b}{1-a}$

$$\therefore\ y_{n+1} = x_{n+1} - \frac{b}{1-a} \qquad \text{(by defn)}$$
$$= ax_n + b - \frac{b}{1-a}$$
$$= a\left(y_n + \frac{b}{1-a}\right) + b - \frac{b}{1-a}$$
$$= ay_n + \underbrace{\frac{ab}{1-a} + b - \frac{b}{1-a}}_{0}$$

$\Rightarrow\ y_{n+1} = a y_n$

(If $|a| > 1$, after successive steps it goes to infinity
If $|a| < 1$, " " steps, it goes to zero)

$\boxed{\text{So zero is a stable fixed pt. for } |a| < 1}$

$\boxed{\text{You can do it also graphically}}$

Next example

$$F(x) = -(ax^2 + bx + C)$$
$$F(x) - x = -\{ax^2 + (b+1)x + C\}$$
$$= -a(x-\alpha)(x-\beta)$$

assuming that in $F(x)$ the coeff. of x^2 is $-$ve. If not we redefine our variable to get it

$\alpha, \beta \to$ 2 roots of this eqn

α, β : roots of $ax^2 + (b+1)x + c$.
$\to$ assumed to be real.

For complex roots it means that on the real line there won't be any roots — so we'll consider cases for real roots

$$x_{n+1} = ax_n^2 + bx_n + c$$

Define : $y = -x$

$$\therefore\ y_{n+1} = -x_{n+1} = -ax_n^2 - bx_n - c$$
$$= -ay_n^2 + by_n - c$$

Sign of a can be changed by coordinate redefinition.

$$x_{n+1} = F(x_n) = x_n - a(x_n - \alpha)(x_n - \beta)$$
$$F(x) = x - a(x-\alpha)(x-\beta)$$

'$-$' sign chosen bcos the final form will look simpler fixed points : $F(x_F) = x_F$

$$\Rightarrow x_F = \alpha \text{ or } \beta$$

(we change the coord. so that one f.p. is at 0 ~~& the other at 1~~)

Define : $y = x - \alpha$

$$\begin{aligned}
y_{n+1} &= x_{n+1} - \alpha \\
&= x_n - \alpha - a(x_n - \alpha)(x_n - \beta) \\
&= (x_n - \alpha)\left(1 - a(x_n - \beta)\right) \\
&= y_n\left(1 - a(y_n + \alpha - \beta)\right)
\end{aligned}$$

Define : $z = \dfrac{y}{\beta - \alpha - 1/a}$

$$\begin{aligned}
z_{n+1} &= \frac{y_{n+1}}{\beta - \alpha - 1/a} \\
&= \frac{1}{\beta - \alpha - 1/a}\, y_n\left(1 - a y_n - a(\alpha - \beta)\right) \\
&= \frac{y_n\left(1 - a(\alpha - \beta)\right)}{\beta - \alpha - 1/a} - \frac{a y_n^2}{\beta - \alpha - 1/a} \\
&= \underbrace{\left(1 + a(\alpha - \beta)\right)}_{\mu} z_n(1 - z_n)
\end{aligned}$$

New recursion relation $\rightarrow$ $z_{n+1} = \mu z_n(1 - z_n)$

(Let us assume μ to be +ve — if not we can redefine ...)

Take $\mu > 0$

If $0 \le z_n \le 1$,

then $0 \le z_n(1 - z_n) \le 1/4$

The scaling is chose so that if we start at $z_n = 1$, in the next step we reach 0 $\rightarrow$ check

We use a linear transfn. from y to z variables – reqd. 2 parameters, which we adjusted such that in the final thing we have only 1 parameter

Any general problem of the given type can be brought into the form $z_{n+1} = \mu z_n(1 - z_n)$

If $\mu \leq 4$ then

$$0 \leq \mu z_n(1-z_n) \leq 1$$

~~It rules out~~ (So if we start within 0 & 1, we remain within 0 & 1)

Rough

$y = x - x^2$

$y' = 1 - 2x$

$y' = 0$ for $x = 1/2$

$\therefore y_{max} = \frac{1}{2} - 1/4 = 1/4$

The motion is bounded if $0 \leq z \leq 1$.

So if we are ~~stayi~~ studying motion in this region, we are sure that the pt. won't run off to infinity

Fixed points

$$z_F = \mu z_F (1 - z_F) = F(z)$$

$$\Rightarrow z_F(\mu - 1 - \mu z_F) = 0$$

$$z_F = 0, \quad z_F = \frac{\mu - 1}{\mu} = 1 - 1/\mu$$

2 fixed points

Lots of interesting things happen when the motion is bounded

Stability :- $F'(z) = \mu - 2\mu z = \mu(1 - 2z)$

$F'(0) = \mu$ stable for $0 \leq \mu < 1$

$$F'(1 - \tfrac{1}{\mu}) = \mu(1 - 2 + 2/\mu) = 2 - \mu$$

stable for : $1 < \mu < 3$

We can show that for $0 \leq \mu < 1$, we approach 0 & for $1 < \mu < 3$ we approach $1 - 1/\mu$

Q) What happens if $3 < \mu < 4$?
(Motion can't converge to any of these f.p. & the f.p. is unstable for this)

Second generation map :

$$F^{(2)}(z) = F(F(z))$$

$$F(z) = \mu z(1 - z)$$

$$z_{n+2} = F(z_{n+1}) = F(F(z_n)) = F^{(2)}(z_n)$$

$F^2 \rightarrow$ fourth order polynomial in z

Fixed point eq. for $F^{(2)}$:

$z_F = F^{(2)}(z_F) \rightarrow$ fourth order eqn.

So this has 4 solutions, but not all of them necessarily real.

Suppose $F(z_F) = z_F$
$\rightarrow$ fixed point of F

$$F^{(2)}(z_F) = F\left(F(z_F)\right) = F(z_F) = z_F .$$

$\Rightarrow$ fixed points of F are also fixed points of $F^{(2)}$. (but $F^{(2)}$ may have more f.p.'s)

Suppose z_F is a fixed point of $F^{(2)}$ but not a fixed point of F.
Motion beginning at such a fixed point will oscillate between z_F and $F(z_F)$.
(This is called <u>period doubling</u>. You come back to the same pt. after 2 such intervals. For a f.p. of $F(z)$, we find that after ~~a~~ unit ~~of~~ ~~one~~ interval, we are at the same pt.)

<u>Stability of fixed points of second generation map:-</u>

$$F^{(2)'}(z_F) \qquad = F'\left(F(z)\right)\, F'(z)$$

$$F^{(2)'}(z) = \frac{d}{dz} F\left(F(z)\right)$$

$$F^{(2)'}(z_F) = F'\left(F(z_F)\right) F'(z_F)$$

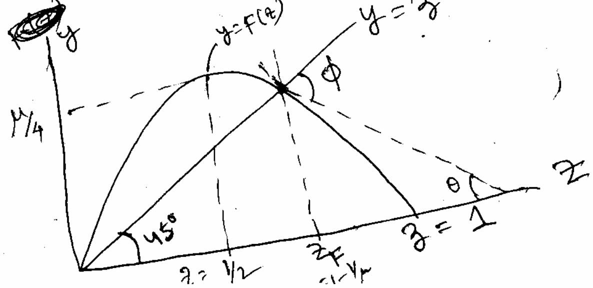
If z_F is a fixed point of F, then $F(z_F) = z_F$.

$$F^{(2)'}(z_F) = \left(F'(z_F)\right)^2$$

An unstable f.p. ~~will also be~~ ~~an~~ of F' is also an unstable f.p. of F^2 & vice versa

A stable f.p. — — —

Bcos if modulus of -- is less than 1, the square of mod. is also < 1.



(For larger μ, the curve gets steeper, though shape remains the same)

$F'(z_F) = 2-\mu$

$\mu < 3 \Rightarrow f'(z_F) > -1$ ($\theta < 45°$)

$\mu < 3 \Rightarrow \theta < 45°$

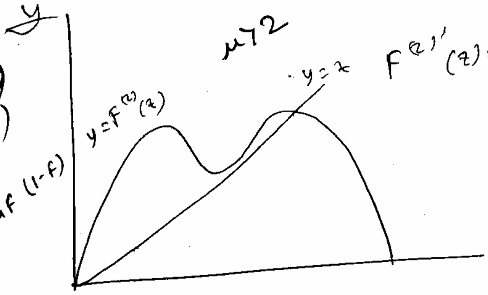
$\mu = 3 \Rightarrow \theta = 45°$

$\phi = 90°$

Extrema of $F^{(2)}$: for ~~zeros of $F^{(2)}$~~, zeros of $F^{(2)'}$

1) zero of $F'(z)$

2) $F(x) = 1/2$ $\therefore f^{(2)}(z) = \mu f(1-f)$



$F^{(2)'}(z) = f'(F(z))\,F'(z)$

$F(z) = 1/2$

$F^{(2)''}(z) = f''(z)\,f'(f(z)) + f''(f(z))\,(f'(z))^2$

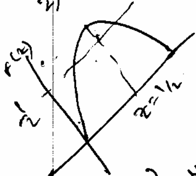
For $z = 1/2$, $f'(z) = 0$

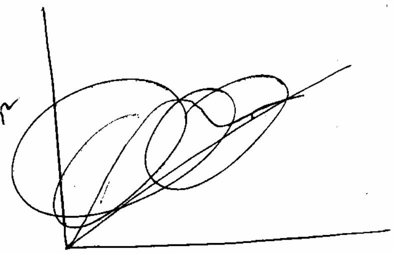
$\therefore F^{(2)''}(1/2) = f''(1/2)\,f'(f(1/2))$

$= f''(1/2)\,f'(\mu/4)$

$= f''(1/2)\,\mu(1-\mu/2)$

-ve, -ve for $\mu > 2$





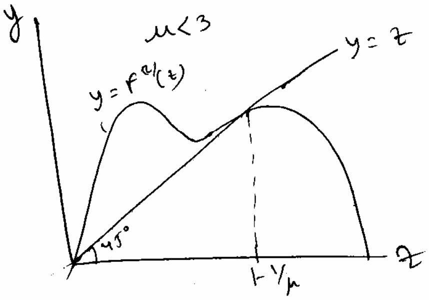
$F(z) = \mu^2 z(1-z)$

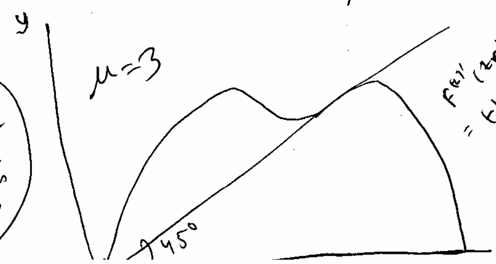
$f(1/2) = \frac{\mu}{4} > \frac{1}{2}$ if $\mu > 2$

$F^{(2)'}$ is a cubic polynomial bcos $F^{(2)}$ is a quartic polynomial

$1 - 1/\mu$

$F^{(2)'}(z_F) = (f'(z_F))^2 < 1$ for $\mu < 3$

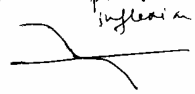




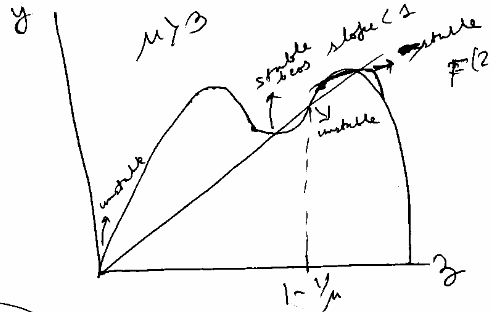
$F^{(2)'}(z_F) = (-1)^2 = 1$

For $\mu = 3$, $F^{(2)'}(z_F) = 1$ ⇒ slope is 45°

pt. of inflexion



→ a cubic curve (3 roots are coinciding)



$F^{(2)\prime}(z_F) = (F'(z_F))^2 > 1$

at $z_F = 1 - 1/\mu$

The roots lie betw. the 2 extrema of $F^{(2)}(z)$ → Ayon

We are trying to find the soln. to a quartic eqn.

The other 2 roots which came down from complex plane now start separating.

Exactly at $\mu = 3$, $F^{(2)}$ gets two additional fixed points.

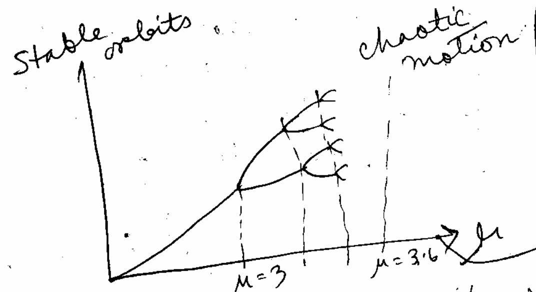
(For $\mu > 3$, there are no stable fixed pts., but we have a stable 2-cycle)

For $\mu > 3$, the original fixed points of F are unstable but $F^{(2)}$ now has two new stable fixed points.
⇒ Motion is attracted to a period two orbit.

Both fixed points have same $F^{(2)\prime}(z_F)$.

(Once the f.p.'s of $F^{(2)}$ get unstable, 2 new stable f.p.'s of $F^{(4)}$ will appear — clear from our local analysis — each unstable f.p. of $F^{(2)}$ will split into 2 new stable f.p.'s of $F^{(4)}$.)

Both fixed points have same $F^{(2)\prime}(z_F)$ —
If we increase μ further, at some point these new fixed points of $F^{(2)}$ become unstable ⇒ New fixed points of $F^{(4)}$ appear.
Motion is attracted to a period 4 orbit — period doubling.



The splitting takes place at the same value of μ

$\mu = 3.84$ period 3 orbit $F^{(3)}$

$F^{(4)\prime}(z_F) = F'(z_F)\, F'(F(z_F))$

$F'(F(F(z_F)))\, F'(F(F(F(z_F))))$

& then again starts branching

Motion is interspersed ~~with~~ with chaotic & period motion

In discrete time dynamics we have a complicated motion compared to continuum dynamics

2 other f.p.'s of $F^{(2)}$ which are not f.p.'s of f are:—

$F^{(2)}(z) = f(f(z))$

We must have for some $f(z) = \alpha$ (say) such that $f(\alpha) = \alpha$

Let $f(z) = \alpha$ or $z = z_1, z_2$ where $z_1 \neq z_2$

$\therefore f(f(z_i)) = f(z_j)$

$\because$ ~~$i \neq j$~~ We must have

$f(f(z_1)) = f(z_2)$ or $f(f(z_2)) = f(z_1)$

$\Rightarrow z_2 = f(z_1)$; $z_1 = f(z_2)$

Now, $F^{(2)\prime}(z) = \dfrac{df(f(z))}{d(f(z))}\, \dfrac{df(z)}{dz}$

$\therefore F^{(2)\prime}(z_1) = \left.\dfrac{df(f(z))}{d(f(z))}\right|_{z=z_1} \times \left.\dfrac{df(z)}{dz}\right|_{z=z_1}$

$= \left.\dfrac{df(f(z))}{d(f(z))}\right|_{f(z)=z_2} \times \left.\dfrac{df(u)}{du}\right|_{u=z_2}$

$\therefore F^{(2)\prime}(z_2) = \left.\dfrac{df(u)}{du}\right|_{u=z_1} \times \left.\dfrac{df(z)}{dz}\right|_{z=z_2}$

Same

Problem set 2 (1)

104)

We are to ensure that the restoring force must be smaller than that of a harmonic oscillator, & hence the period should be smaller.

$$T = \frac{4}{\sqrt{2}} \int_0^a \frac{1}{\sqrt{mgAa^{2k}}} \left[1 - (x/a)^{2k}\right]^{-1/2} dx$$

$f(x) = \frac{x^{2k}}{(2k)!} f^{2k}(0)$

$f(a) = \frac{a^{2k}}{(2k)!} f^{2k}(0)$

For small $a \to$ estimate T

Now $\left(1 - (x/a)^{2k}\right)^{-1/2} \longrightarrow$ Bounded betn 1 & 0

Approximation $\to T \sim a^{-2k+1}$,

$\therefore m = 2k - 1 \Rightarrow 2k = m + 1$

$\therefore f(x) \sim x^{2m+2}$

$$T = \frac{K_1}{a^k} \int_0^1 \frac{a\, dz}{\sqrt{1 - z^{2k}}}$$

$x/a = z$

$\therefore dx = a\, dz$

$$= \frac{K_1}{a^{k-1}} \int_0^1 \frac{dz}{\underbrace{(1-z)^{1/2}}_{\text{divergence will come from this term}} (1 + z + z^2 + \cdots + z^{2k-1})^{1/2}}$$

geometric series & convergent

$$\sim \frac{K_1}{a^{k-1}} \int_0^1 \frac{dz}{\sqrt{1-z}}$$

$1 - z = u$

$\therefore dz = -du$

$$= \frac{K_1}{a^{k-1}} \int_1^0 \frac{-du}{u^{1/2}}$$

$$= \frac{K_1}{a^{k-1}} \int_0^1 \frac{du}{u^{1/2}} = \frac{K_1}{2a^{k-1}} u^{1/2} = \frac{K_1}{2a^{k-1}}$$

$f = -\frac{\partial V}{\partial x}$

Flatter the potential, larger the time period.

for small x, x^3 is flatter than x^2 — etc.

Show that the momentum conjugate to a coord. (say p_η) changes its sign when you add an arbitrarily large pot. at $\eta = \text{const}$. & all other momenta remain the same.

$(\rho, \phi, z) \rightarrow \; z = \frac{1}{2}(\xi - \eta) + a$

$\rho = \sqrt{\xi\eta}$

$\phi = \phi$

17/11/05

Shamik

$2N$-dimensional Hamiltonian system having $\{I_i\}_{i=1}^{N}$ consvd. charges such that $\{I_i, I_j\} = 0$.

$$x'^a = x^a + \epsilon^{(i)} \Omega^{ab} \frac{\partial I^{(i)}}{\partial x^b} + O(\epsilon^2)$$

$i = 1, 2, \ldots N$

& $\epsilon^{(i)}$'s are indep.

$$\Omega = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \ldots \\ & & \end{pmatrix}$$

A transfn.

$$x^a = f^a(\vec{\epsilon}, \vec{\eta})$$

$\vec{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_N)$

$\vec{\eta} = (\eta_1, \eta_2, \ldots, \eta_N)$

At present, these $\vec{\epsilon}$ has nothing to do with the previous $\epsilon^{(i)}$'s.

$$\frac{\partial x^a}{\partial \epsilon^{(i)}} = \Omega^{ab} \frac{\partial I^{(i)}}{\partial x^b}$$

$2N \times N$ such eqns are there

For a fixed a, there are N such eqns

we are imposing this condn. that whatever f^a be, this has to be satisfied

$$\Rightarrow \frac{\partial I^{(j)}}{\partial x^a} \frac{\partial x^a}{\partial \epsilon^{(i)}} = \frac{\partial I^{(j)}}{\partial x^a} \Omega^{ab} \frac{\partial I^{(i)}}{\partial x^b}$$

$$= \{I^{(j)}, I^{(i)}\} = 0 \quad (1)$$

$$\Rightarrow \frac{\partial I^{(j)}}{\partial x^a} \frac{\partial x^a}{\partial \epsilon^{(i)}} = 0 \quad \text{(true for all } i \,\&\, j\text{)}$$

Consider $x \to x + \delta x$ & find the change in I_i:-

$$dI^{(i)} = \frac{\partial I^{(i)}}{\partial x^a} dx^a$$

$$= \frac{\partial I^{(j)}}{\partial x^a} \left(\frac{\partial x^a}{\partial \epsilon^{(i)}} d\epsilon^{(i)} + \frac{\partial x^a}{\partial \eta^{(i)}} d\eta^{(i)} \right)$$

$$\Rightarrow dI^{(i)} = \cancel{\left(\frac{\partial I^{(i)}}{\partial x^a}\frac{\partial x^a}{\partial \epsilon^{(i)}}\right)}^{\to 0} d\epsilon^{(i)} + \frac{\partial I^{(i)}}{\partial x^a}\frac{\partial x^a}{\partial \eta^{(i)}} d\eta^{(i)}$$

[using ①]

$$\Rightarrow dI^{(i)} = \left(\frac{\partial I^{(i)}}{\partial x^a}\frac{\partial x^a}{\partial \eta^{(i)}}\right) d\eta^{(i)}$$

(true for arbitrary η & ϵ)

$$\Rightarrow \frac{\partial I^{(i)}}{\partial \epsilon^{(i)}} = 0$$

$$\Rightarrow I^{(i)} = I^{(i)}(\eta^{(i)})$$

$$\Rightarrow dI^{(i)} = \left|\left|\frac{\partial I}{\partial \eta}\right|\right|_{ji} d\eta^{(i)}$$

↘ Jacobian

If $dI^{(i)} = 0$, then $d\eta^{(i)} = 0$.

ϵ parametrize the given manifold

So, η = constant or equivalently I_i = constant, you move on a given manifold — changing η's, you go to ~~another~~ another one.

For any fixed $I^{(i)}$, you can think the $\epsilon^{(i)}$ as the time coordinate

fixing $i = 1$

$$\frac{\partial x^a}{\partial \epsilon^{(i)}} = \Omega^{ab}\frac{\partial I^{(i)}}{\partial x^b}$$

$$\Rightarrow \frac{\partial x^a}{\partial \epsilon^{(i)}} = \Omega^{ab}\frac{\partial I^{(i)}}{\partial x^b}$$

~~$I^{(i)} \quad \epsilon \to \epsilon + \delta\epsilon$~~

~~$I^{(i)}$~~

108)

$$I^{(i)} \Rightarrow \epsilon + \delta\epsilon_1$$

Now, $\{I_i, I_j\} = 0$

x_0 $\quad$ in the ith position

$$\phi_\epsilon^{(i)}(\vec{0}) = (0, \ldots, \epsilon, \ldots 0)$$

$$\phi^{(j)}_{\epsilon_j}\, \phi^{(i)}_{\epsilon_i}(\vec{0}) = (0, \ldots, \underset{i\text{-th}}{\epsilon_i}, \ldots, \underset{j\text{-th}}{\epsilon_j}, \ldots 0)$$

$\phi_{\vec{\epsilon}}(x_0) = x_0$ Discrete subgroup

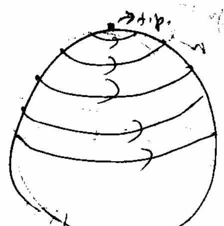
$R^N \to$ Abelian group under addition

$\{e_1, e_2, \ldots, e_k\}$ $k \leq N$

any discrete str. of R^N has this str. $\leftarrow (n_1 e_1 + n_2 e_2 + \cdots + n_k e_k)$ $n_i \in \mathbb{Z}$

Take R^N. ~~Suppose~~ Take any discrete subgroup of R^N.

Claim: U can choose k basis vectors with $k \leq N$ such that you can span that discrete subgroup



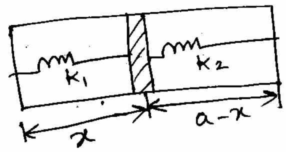
~~Can~~ must be able to find a vector field on the surface which is neither zero nor infinity.

For spiral – it'll spiral onto itself.
On a torus you can spiral without intersecting itself

First we'd like to argue that $k = N$, i.e., you must be able to choose N periodic dirns. That is clear otherwise ~~motion~~ will be unbounded in the $(N-k)$ dirns.

No space filling surfaces – Redn. from $2N$ to N-dim. space is genuine

Problem set 1



$$\omega = \begin{pmatrix} 0 & 1/T \\ -1/T & 0 \end{pmatrix}$$

$$\partial_i \omega_{jk} + \partial_j \omega_{ki} + \partial_k \omega_{ij} = 0$$

$$\partial_i \omega_{jk} + \partial_j \omega_{ki} + \partial_k \omega_{ij}$$

for $j = k$

$$0 + \partial_j \omega_{ki} + \partial_k \omega_{ij} = \partial_j \underbrace{\omega_{ji} + \partial_j \omega_{ij}}_{\text{antisym.}}$$

$$= 0$$

All 3 indices can't be diff.
so trivially satisfied for all values of i, j, k
$\because$ i, j, k can only ~~~~ take values 1 & 2.

$$x(a-x)\left[k_2(a-x) - k_1 x\right] + \left[n_1(a-x) - n_2 x\right] \frac{2}{3n_1 + 5n_2}\left[E - \frac{1}{2}k_1 x^2 - \frac{1}{2}k_2(a-x)^2\right] = 0$$

$$f(x) = -k_1 x + k_2(a-x) + \frac{n_1 R T}{x} - \frac{n_2 R T}{a-x} = 0$$

as $x \uparrow$, each term $\downarrow$

$f(x)$ monotone $\downarrow$ fn. of x & can't cut the real axis more than once

(when $T > 0 \to$ physical region)

110)

If we have $(ax+b) = 0$

& $(cx+d) = 0$

Then $k_1^2(ax+b) + k_2^2(cx+d) = 0$ will have a root betw. $x = -b/a$ & $x = -d/c$

So, if we first set $n_1 = n_2 = 0$ (no gas — only spring)

$\longrightarrow$ one root (r_1)

If we ~~fix~~ then set $k_1 = k_2 = 0$ (no spring — only gas)

$\rightarrow$ another root (r_2)

$\because$ the original eqn is a weighted sum of these 2 eqns & with the weights, the roots must lie betw. r_1 & r_2.

$g(x) = ax + b + cx + d$
$= (a+c)x + (b+d)$

$\therefore g(x) = 0$
gives $x = -\frac{b+d}{a+c} = -\frac{b}{a+c} + \frac{d}{a+c}$

$x_1 = -b/a$
$x_2 = +d/c$

$\therefore$ to show
$x_1 < x < x_2$

$\Omega^2 = f'/m$

$\because f$ is monotonically $\downarrow$

f' is $-ve$

& $\therefore$ we have elliptic f.p.'s.

112) 0.1 — 0

2 — 0

3.2 — ·513045

3.2 — 0.79945

0.5 — 0

1 — 0

1.2 — 1.66667

1.5 — 0.333333

1.9 — 0.493684

2.2 — 0.545455

2.5 — 0.6

2.9 — 0.655172

3 — 0.65612

3 — 0.676878

3.3 — 0.479421

3.3 — 0.823603

3.4 — 0.842157

3.4 — 0.451963

3.45 — 0.847468

3.45 — 0.445968

3.45 — 0.852428

3.45 — 0.433992

3.5 — 0.874997

3.5 — 0.38282

3.5 — 0.826941

3.5 — 0.500884

3.55 0.881684

" 0.370328

" 0.827805

" 0.506031

" 0.887371

" 0.3548

" 0.812656

" 0.540475

3.84 0.959447
0.149407
4 0.488004
~~0.488004~~

$f(x) = \mu(1-x)x$

$f(f(x)) = \mu f(x)(1-f(x))$
$= \mu\, x(1-x)(1-\mu x(1-x))$
$= \mu(x-x^2)(1-\mu x+\mu x^2)$

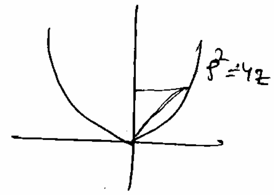
$f^{(2)}(z) = f(f(z)) = \mu\, f(z)(1-f(z))$
$= \mu^2 z(1-z)(1-f(z))$

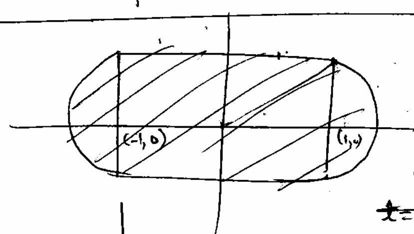
114) ~~ρ ...~~ $\rho = \rho' t + \rho$

$z = z' t - 5t^2 + z$

Take $\rho'(0) = 1$, $z'(0) = 1$, $\rho(0) = z(0) = 0$.

$(0,0) \xrightarrow{\ell} t = 0$





$x = x' t + x$

$y = y' t + y$

~~0.5, 1, 0, 0~~

~~$t = 1$, $x = 0.5$~~

~~$y = \dots$ $y = \dots$~~

$x' = -0.5$

$(0,0) \to t = 0$

1, 1, 0, 0

$t = 1$, $x = 1$

$y = 1$

$x' = 1$

$y' = -1$

Problems
Action Angle variables

Linear Harmonic Oscillator

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 = E \longrightarrow \text{conserved charge}$$

Then $\cancel{p(q) = p(q,\gamma)}$

$$p(q,E) = \sqrt{2mE - (m\omega q)^2}$$

to find the action angle variable :—

F_3 generating fn.

Now, to find the action angle variable :—

$$J = \oint p\, dq$$
$$= \oint \sqrt{2mE - (m\omega q)^2}\, dq$$
$$= \sqrt{2mE} \oint \sqrt{1 - \frac{m\omega^2 q^2}{2E}}\, dq$$
$$= 2\sqrt{2mE} \int_{-a}^{a} \sqrt{- -}\, dq$$
$$= 4\sqrt{2mE} \int_0^a \sqrt{- -}\, dq$$
$$= 4\sqrt{2mE} \int_0^{\sqrt{\frac{2E}{m\omega^2}}} \sqrt{1 - \frac{m\omega^2 q^2}{2E}}\, dq$$

$\frac{1}{2}m\omega^2 a^2 = E$
$\Rightarrow a = \sqrt{\frac{2E}{m\omega^2}}$

$q\sqrt{\frac{m\omega^2}{2E}} = \sin\theta$
$\therefore dq = \cos\theta \sqrt{\frac{2E}{m\omega^2}}\, d\theta$

$$= 4\sqrt{2mE}\sqrt{\frac{2E}{m\omega^2}} \int_0^{\pi/2} \cos^2\theta\, d\theta$$
$$= 4 \times \frac{2E}{\omega} \times \frac{1}{2} \int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$
$$= \frac{4E}{\omega}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = = \frac{2E\pi}{\omega}$$
$$= \frac{4E}{\omega}\,\frac{\pi}{2}$$

$$\therefore \boxed{J = E\frac{2\pi}{\omega}}$$

$H(J) = E = \frac{\omega J}{2\pi}$ $\longrightarrow$ finding $H(J)$

Construct $\frac{d\theta}{dt} = -\frac{\partial H}{\partial J} = -\frac{\omega}{2\pi} = \nu$

Cycloid Pendulum

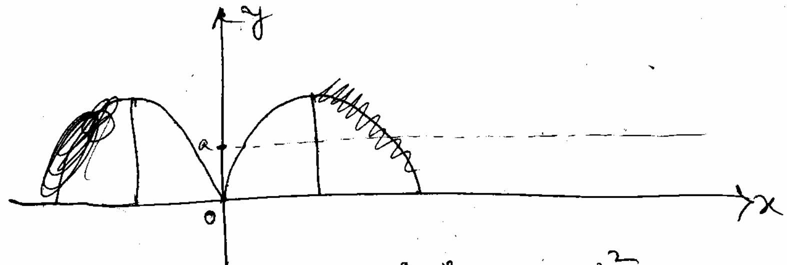
$$\left.\begin{aligned} x &= a(\theta + \sin\theta) \\ y &= a(1-\cos\theta) \end{aligned}\right\} \text{ Eqns for a cycloid}$$

$$-\pi \leq \theta \leq \pi$$

$a\sin\theta = x - a\theta$

$-a\cos\theta = y - a$

$\therefore (x-a\theta)^2 + (y-a)^2 = a^2$



$$ds^2 = (dx)^2 + (dy)^2 = (v\,dt)^2$$
$$= a^2\left[(1+\cos\theta)^2 + \sin^2\theta\right](d\theta)^2 = 2a^2(1+\cos\theta)(d\theta)^2$$

$\therefore$ the K.E. is $T = \frac{1}{2}mv^2 = \frac{m}{2}\,2a^2(1+\cos\theta)\left(\frac{d\theta}{dt}\right)^2$

$$\Rightarrow T = ma^2(1+\cos\theta)\dot{\theta}^2$$

Now, $V = mgy = mga(1-\cos\theta)$

$\therefore L = T - V = ma^2\left[(1+\cos\theta)\dot{\theta}^2 - g/a\,(1-\cos\theta)\right]$

$\therefore$ the canonical momentum is

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = 2ma^2(1+\cos\theta)\dot{\theta}$$

$$\text{or } \dot{\theta} = \frac{p_\theta}{2ma^2(1+\cos\theta)}$$

So the Hamiltonian can be written as

$$H(\theta, p_\theta) = T + V = \frac{\cancel{(ma^2)(1+\cos\theta)}p_\theta^2}{4(ma^2)^2(1+\cos\theta)^2} + V$$

$$= \frac{p_\theta^2}{4ma^2(1+\cos\theta)} + mga(1-\cos\theta)$$

We need p_θ to build $\oint p_\theta \, d\theta$.

H is conserved — in particular, for $\theta = \pm\pi$, $p_\theta = 0$

$\therefore \; H = E = 2mga$

$$\therefore \; 2mga = \frac{p_\theta^2}{4ma^2(1+\cos\theta)} + mga(1-\cos\theta)$$

$$\Rightarrow \; 8m^2 g a^3 (1+\cos\theta) - 4m^2 a^3 g (1-\cos^2\theta) = p_\theta^2$$

$$\Rightarrow \; p_\theta^2 = 4m^2a^3 g + 8m^2a^3 g\cos\theta + 4m^2a^3 g\cos^2\theta$$

$$= 4m^2a^3 g \,(\cos^2\theta + 1 + 2\cos\theta)$$

$$= 4m^2a^3 g \,(1+\cos\theta)^2 .$$

$$\therefore \; p_\theta = \pm\, 2ma\sqrt{ag}\,(1+\cos\theta)$$

$$\# \quad J = \oint p_\theta \, d\theta = 2\int_{-\pi}^{\pi} |p_\theta| \, d\theta$$

$$= 4ma\sqrt{ag}\int_{-\pi}^{\pi}(1+\cos\theta)\,d\theta$$

$$= 8ma\sqrt{ag}\,\left[\theta + \sin\theta\right]_0^{\pi} = 8ma\sqrt{ag}\,\pi .$$

$$\cancel{= 8m\sqrt{g}\sqrt{\frac{E^3}{8m^3g^3}}\,\pi}$$

$$= 4\pi\sqrt{\frac{a}{g}}\,H$$

$$\cancel{H = 2mga} \quad H = \frac{1}{4\pi}\sqrt{g/a}\;J = \frac{1}{2\pi}\sqrt{\frac{g}{4a}}\;J$$

$$\# \quad \nu = \frac{\partial H}{\partial J} = \frac{1}{2\pi}\sqrt{g/4a} \longrightarrow \text{freq. indep. of amplitude}$$

---

simple H.O. (linear)

$$q(t) = \sqrt{\frac{2E}{m\omega^2}}\,\sin(\omega t+\alpha) \; ; \; p(t) = \sqrt{2mE}\,\cos(\omega t + \alpha)$$

We have, $H(J) = \frac{\omega}{2\pi}\,J = E$ (already found)

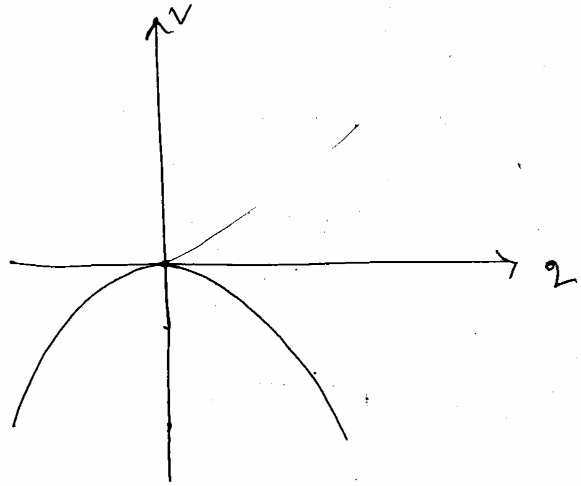
Also, $Q = \nu t + \beta \; ; \; 2\pi Q = \omega t + \alpha$

$$\therefore \quad q(t) = \sqrt{\frac{J}{\pi\omega}}\,\sin(2\pi Q) \quad \Big| \; p(t) = \sqrt{\frac{m\omega J}{\pi}}\;\cos(2\pi Q$$

118)

$V = -\frac{1}{2}q^2$

$H = \frac{p^2}{2} - \frac{1}{2}q^2$



Commutation relations for the Poisson brackets:–

$\{f, g\} = -\{g, f\}$

$\{f, c\} = 0$

$\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\}$

$\{f_1 f_2, g\} = f_1 \{f_2, g\} + f_2 \{f_1, g\}$

Jacobi's identity $\{f, \{g,h\}\} + \{g, \{h,f\}\} + \{h, \{f,g\}\} = 0$ — (1)

Poisson's theorem:–

Put $h = H$ in (1)

If $\{g, H\} = \{H, f\} = 0$,

then $\{H, \underbrace{\{f, g\}}_{\text{also a conserved charge.}}\} = 0$

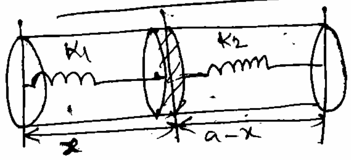
(119)

Problem set 1

$C_v = \frac{3R}{2}$, $C_p = \frac{5R}{2}$ } monatomic gases

$C_v = \frac{5R}{2}$, $C_p = \frac{7R}{2}$ } polyatomic gases

(1)



(a) $E = \frac{3}{2} N_1 k T + \frac{5}{2} N_2 k T + \frac{p^2}{2m} + \frac{1}{2}K_1 x^2 + \frac{1}{2} K_2 (a-x)^2$ —— (1)

one gas

$$\boxed{\begin{array}{l}\text{Entropy } S = \int C_v \frac{dT}{T} + nR\ln V + S_0 \\ \text{If } C_v \text{ is constant,} \\ S = C_v \ln T + nR \ln V + S_0'\end{array}}$$

from (1), $\left(\frac{3}{2}N_1 + \frac{5}{2}N_2\right) k \frac{\partial T}{\partial x} + K_1 x - K_2(a-x) = 0$

$\Rightarrow \frac{3N_1+5N_2}{2} k \frac{\partial T}{\partial x} = K_2(a-x) - K_1 x$

$\therefore \boxed{\frac{\partial T}{\partial x} = 2\times\frac{K_2(a-x)-K_1x}{k(3N_1+5N_2)}}$ —— (2)

Eqns of motion:

$\boxed{dx/dt = p/m}$ —— (3)

$dp/dt = -K_1x + K_2(a-x) + P_1A - P_2A$

$= -K_1 x + K_2(a-x) + \frac{N_1kT}{x} - \frac{N_2kT}{a-x}$

$= -(K_1+K_2)x + K_2 a + \ldots$

$\Rightarrow \boxed{dp/dt = K_2 a - (K_1+K_2)x + \left(\frac{N_1}{x} + \frac{N_2}{x-a}\right)kT}$ —— (4)

(b) Total entropy is

$S = \left(\frac{3}{2}N_1 k + \frac{5}{2}N_2 k\right)\ln T + N_1 k \ln x + N_2 k\ln(a-x) + \text{constant}$

$\therefore \frac{\partial S}{\partial p} = \frac{3N_1+5N_2}{2} k/T \, \frac{\partial T}{\partial p}$

Also, $\frac{3N_1+5N_2}{2} k \frac{\partial T}{\partial p} = -p/m$

$\therefore \frac{\partial S}{\partial p} = -\frac{p}{mT}$

(120)

$$\Rightarrow \boxed{\frac{dx}{dt} = -T \, \partial S/\partial p} \quad \text{— (a)}$$

Again, $\frac{\partial S}{\partial x} = \frac{N_1 k}{x} - \frac{N_2}{a-x} + \frac{3N_1 k + 5N_2}{2} \frac{k}{T} \frac{\partial T}{\partial x}$

$$= \frac{N_1 k}{x} - \frac{N_2}{a-x} + \frac{K_2(a-x) - K_1 x}{T}$$

$$\Rightarrow T \frac{\partial S}{\partial x} = dp/dt$$

$$\Rightarrow \boxed{dp/dt = T \, \partial S/\partial x} \quad \text{— (b)}$$

$$\therefore \frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix} \begin{pmatrix} \partial S/\partial x \\ \partial S/\partial p \end{pmatrix}$$

$$\Omega = \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix} \longrightarrow \text{antisymmetric}$$

$$\omega = \frac{1}{T^2}\begin{pmatrix} 0 & -(-T) \\ -T & 0 \end{pmatrix} = \frac{1}{T^2}\begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/T \\ -1/T & 0 \end{pmatrix}$$

Now $f = \frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j}$

for $i = j$, we have

$$f = 0 + \frac{\partial \omega_{ik}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^i}$$

$$= \frac{\partial \omega_{ik}}{\partial x^i} - \frac{\partial \omega_{ik}}{\partial x^i}$$

$$= 0$$

So all i, j, k must be different to get a non-zero result. But, for a 2-dim. Ham. system, this is not possible. Hence it is trivially satisfied.

Now, $\frac{dS}{dt} = \frac{\partial S}{\partial p}\frac{dp}{dt} + \frac{\partial S}{\partial x}\, dx/dt$

$= \frac{\partial S}{\partial p}\left(T\,\partial S/\partial x\right) + \frac{\partial S}{\partial x}\left(-T\,\partial S/\partial p\right)$

$= 0$

$\therefore$ S is conserved

$dx/dt = p/m = f_1(x, p)$

$\therefore \partial f_1/\partial x = 0$ & $\partial f_1/\partial p = 1/m$

$p = 0 \longrightarrow$ for fixed ~~point~~

$\cancel{\frac{d}{dt} x = }$ $dp/dt = K_2 a - (K_1 + K_2)x + \left(\frac{N_1}{x} + \frac{N_2}{x-a}\right)kT$

For f.p. $\rightarrow K_2 a - (K_1 + K_2)x + \left(\frac{N_1}{x} + \frac{N_2}{x-a}\right)kT = 0$
one real root $= \phi(x)$

$\because \phi(x)$ is a monotonically $\downarrow$ fn., $\therefore \phi'(x) < 0$

$\frac{d}{dt}(x - x_F) = \frac{dx}{dt} = f_1(x_F, p_F) + (x - x_F)\partial f_1/\partial x + (p - p_F)\frac{\partial f_1}{\partial p} + \ldots$

$= 0 + p/m$

$\Rightarrow \boxed{\frac{d}{dx}(x - x_F) = p/m}$

$\frac{d}{dt}(y - y_F) = dy/dt = \cancel{f_2(x_F, p_F)}^{0} + (x - x_F)\left.\frac{\partial f_2}{\partial x}\right|_{x_F, 0}$

$+ (p - p_F)\left.\frac{\partial f_2}{\partial p}\right|_{x_F, 0}$

$= (x - x_F)\left[-(K_1 + K_2) - \left\{\frac{N_1}{x^2} + \frac{N_2}{(x-a)^2}\right\}kT + \left(\frac{N_1}{x} + \frac{N_2}{x-a}\right)k\frac{\partial T}{\partial x}\right]$

$+ 0$

$= -c^2(x - x_F)$

$\therefore \frac{d}{dt}\begin{pmatrix} x - x_F \\ p - p_F \end{pmatrix} = \begin{pmatrix} 0 & 1/m \\ -c^2 & 0 \end{pmatrix}\begin{pmatrix} x - x_F \\ p - p_F \end{pmatrix}$

$\begin{vmatrix} -\lambda & 1/m \\ -c^2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + c^2/m = 0$

$\lambda \rightarrow$ imaginary.

122)

$$\frac{dS}{dt} = \frac{5N_2 kc(k_1+k_2)x}{m(3N_1+5N_2)}\left(\frac{3N_1}{T_1}+\frac{5N_2}{T_1+cp/m}\right)$$

$$-\frac{k_2 a p/m}{3N_1+5N_2}\left(\frac{3N_1}{T_1}+\frac{5N_2}{T_1+cp/m}\right)$$

$$-\left[\frac{N_1 kT_1}{x}+\frac{N_2 k(T_1+cp/m)}{x-a}\right]\left[\left(\frac{5N_2 kc/m+p/m}{3N_1+5N_2}\right)\left(\frac{3N_1}{T_1}+\frac{5N_2}{T_1+cp/m}\right)+\frac{5kN_2 c}{2(mT_1+cp)}\right]$$

At f.p., $p=0$

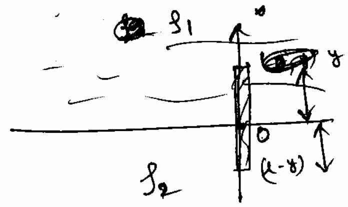
Then
$$\frac{dS_f}{dt} = \frac{5N_2 kc(k_1+k_2)x_f}{m(3N_1+5N_2)}\,\frac{3N_1+5N_2}{T_1}$$

$$-\left(\frac{N_1 kT_1}{x}+\frac{N_2 kT_1}{x-a}\right)\left[\frac{5N_2 kc/m}{3N_1+5N_2}\,\frac{3N_1+5N_2}{T_1}+\frac{5kN_2 c}{2mT_1}\right]$$

$$= \frac{5N_2 kc(k_1+k_2)x_f}{mT_1}-\left(\frac{N_1 k}{x}+\frac{N_2 k}{x-a}\right)\left(\frac{5N_2 kc}{m}+\frac{5N_2 kc}{2m}\right)$$

$$= \frac{5N_2 kc(k_1+k_2)x_f}{mT_1}-\frac{11N_2 k^2 c}{2m}\left(\frac{N_1}{x_f}+\frac{N_2}{x_f-a}\right)$$

(2)



$\rho_2 > \rho_1$

$$dy/dt = p/m = f_1$$

$$dp/dt = -\ell A\rho g + yA\rho_1 g + (\ell-y)A\rho_2 g - \alpha_1 \tfrac{p}{m} y - \alpha_2 \tfrac{p}{m}(\ell-y)$$

$$= -y(\rho_2-\rho_1)Ag + (\rho_2-\rho)\ell Ag - \alpha_1 y \tfrac{p}{m} - \alpha_2 \tfrac{p}{m}(\ell-y) = f_2$$

f.p. $\Rightarrow$ $p = 0$

& $-y_F(\rho_2-\rho_1)\cancel{Ag} + (\rho_2-\rho)\ell\cancel{Ag} = 0$

$$\Rightarrow \boxed{y_F = \frac{(\rho_2-\rho)\ell}{\rho_2-\rho_1}}$$

$$\boxed{\begin{array}{c}\rho > \rho_1 \\ \rho_2-\rho_1 > \rho_2-\rho \\ 0 < \frac{\rho_2-\rho}{\rho_2-\rho_1}\ell < \ell\end{array}}$$

$\partial f_1/\partial y = 0$, $\partial f_1/\partial p = 1/m$

$$\partial f_2/\partial y\big|_F = \left[-(\rho_2-\rho_1)Ag - \alpha_1 p/m + \alpha_2 p/m\right]_F$$

$$= -(\rho_2-\rho_1)Ag$$

$$\partial f_2/\partial p = -\alpha_1 y/m - \frac{\alpha_2}{m}(\ell-y)$$

$$= +\frac{(\alpha_2-\alpha_1)}{m}\frac{(\rho_2-\rho)\ell}{\rho_2-\rho_1} - \frac{\alpha_2 \ell}{m}$$

$$= +\frac{\cancel{\alpha_2\rho_2} - \alpha_2\rho - \alpha_1\rho_2 + \alpha_1\rho - \cancel{\alpha_2\rho_2} + \alpha_2\rho_1}{m(\rho_2-\rho_1)}\ell$$

$$\cancel{= + (\alpha_2 - \alpha_1)\rho_1 + \alpha_1\rho}$$

$$= \frac{\alpha_2(\rho_1-\rho) + \alpha_1(\rho-\rho_2)}{m(\rho_2-\rho_1)}\ell$$

$$= -\frac{\alpha_2(\rho-\rho_1)+\alpha_1(\rho_2-\rho)}{m(\rho_2-\rho_1)}\ell = c$$

$$M = \begin{pmatrix} 0 & 1/m \\ -\sigma^2 & c \end{pmatrix}$$

124

Roots $\Rightarrow \begin{vmatrix} -\lambda & 1/m \\ -a^2 & c-\lambda \end{vmatrix} = 0$

$\Rightarrow -c\lambda + \lambda^2 + a^2/m = 0$

$\Rightarrow \lambda^2 - c\lambda + a^2/m = 0$

$\Rightarrow \lambda = \dfrac{c \pm \sqrt{c^2 - 4a^2/m}}{2}$

$= \dfrac{c}{2} \pm \sqrt{c^2/4 - a^2/m}$

$< c/2$

$\therefore \sqrt{c^2/4 - a^2/m} < c/2$

Let $\lambda_1 > \lambda_2$.

$c^2/4 > a^2/m$

If $c > 0$: $\lambda_1 = c/2 + \sqrt{c^2/4 - a^2/m} > 0$

$\lambda_2 = c/2 - \sqrt{c^2/4 - a^2/m} > 0$

If $c < 0$: $\lambda_1 = -\dfrac{|c|}{2} + \sqrt{c^2/4 - a^2/m} < 0$

$\lambda_2 = -\dfrac{|c|}{2} - \sqrt{c^2/4 - a^2/m} < 0$

If $c = 0$: $\lambda_1 = i\sqrt{a^2/m}$

$\lambda_2 = -i\sqrt{a^2/m}$

$c^2/4 < a^2/m$: $\lambda = \dfrac{c}{2} \pm i\sqrt{a^2/m - c^2/4}$

$\lambda_1 = c/2 + i\sqrt{a^2/m - c^2/4}$

$\lambda_2 = c/2 - i\sqrt{a^2/m - c^2/4}$

For $\alpha_1 = \alpha_2 = 0$, $c = 0$

Then $M = \begin{bmatrix} 0 & 1/m \\ -a^2 & 0 \end{bmatrix}$.

$$\begin{vmatrix} -\lambda & 1/m \\ -a^2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + a^2/m = 0$$

$$\therefore \lambda = \pm i\sqrt{a^2/m}$$

$$dy/dt = p/m = f_1$$

$$dp/dt = -y(\rho_2-\rho_1)Ag + (\rho_2-\rho)\,lAg = f_2$$

~~Let us assume~~ $\cancel{\partial H/\partial y = p/m}$

$$\cancel{\partial H/\partial p = y(\rho_2-\rho_1)Ag - (\rho_2-\rho)lAg}$$

~~Then~~ $\cancel{H = yp/m + f(p)}$

$$\cancel{H = y p(\rho_2-\rho_1)Ag - (\rho_2-\rho_1)l p Ag}$$

Let us assume $\partial H/\partial p = p/m$.

$$\partial H/\partial y = y(\rho_2-\rho_1)Ag - (\rho_2-\rho)\,lAg$$

$$\therefore H = p^2/2m + f(y)$$

$$\therefore df/dy = y(\rho_2-\rho_1)Ag - (\rho_2-\rho)\,lAg$$

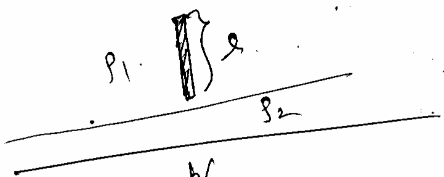
$$\Rightarrow f = \frac{y^2}{2}(\rho_2-\rho_1)Ag - y(\rho_2-\rho)\,lAg$$

$$\therefore \boxed{H = \frac{p^2}{2m} + \frac{y^2}{2}(\rho_2-\rho_1)Ag - y(\rho_2-\rho)\,lAg}$$

$$\partial H/\partial p = p/m$$

$$\partial H/\partial y = y(\rho_2-\rho_1)Ag - (\rho_2-\rho)\,lAg$$

Case I



$dy/dt = p/m$

$dp/dt = -\rho A l g + \rho_1 A l g - \alpha_1 l p/m$

$= (\rho_1 - \rho) A l g - \alpha_1 l p/m$

$\frac{\partial H}{\partial p} = p/m \Rightarrow H = \frac{p^2}{2m} + f(y)$

$df/dy = \alpha_1 l p/m - (\rho_1 - \rho) A l g$

$= \alpha_1 l p/m + (\rho - \rho_1) A l g$

$\therefore \boxed{H = p^2/2m + \alpha_1 l p y/m + (\rho - \rho_1) A l g y}$

$\partial H/\partial p = p/m$

$\partial H/\partial y = \alpha_1 l p/m + (\rho - \rho_1) A l g$

Eqn $\Rightarrow \frac{p^2}{2m} = H - \left[\alpha_1 l p/m + (\rho - \rho_1) A l g\right] y$

For $\alpha_1 = 0$

$H = p^2/2m + (\rho - \rho_1) A l g y$

$\Rightarrow p^2/2m = H - (\rho - \rho_1) A l g y$

$= -(\rho - \rho_1) A l g \left[y - \frac{E}{(\rho - \rho_1) A l g}\right]$

Case II

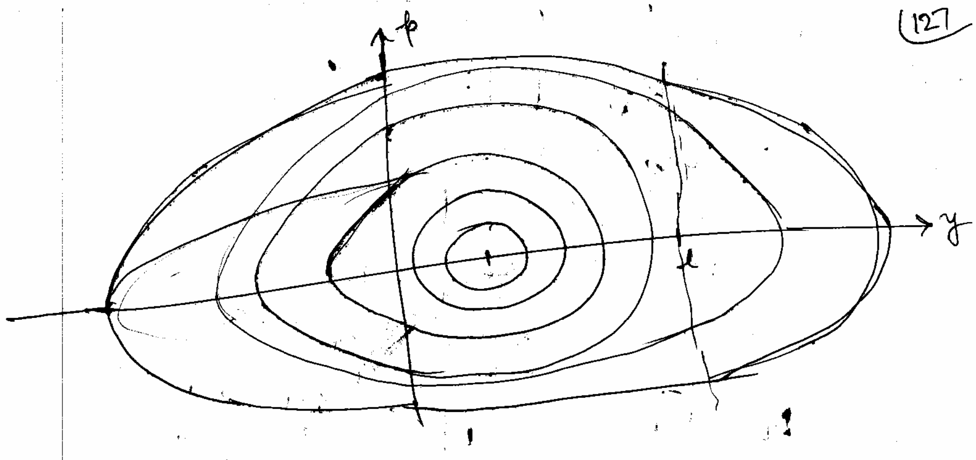
$dy/dt = p/m$ (for $\alpha_2 = 0$)

$dp/dt = (\rho_2 - \rho) A l g$

$H = p^2/2m - (\rho_2 - \rho) A l g y$

$p^2/2m = H + (\rho_2 - \rho) A l g y$

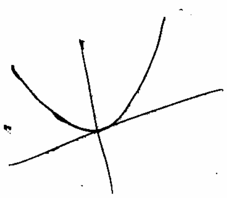
$= (\rho_2 - \rho) A l g \left[y + \frac{E}{(\rho_2 - \rho) A l g}\right]$



$$\frac{p^2}{2m} + \underbrace{\frac{y^2}{2}(\rho_2-\rho_1)Ag - yl(\rho_2-\rho)Ag}_{(\rho_2-\rho_1)\frac{Ag}{2}\left[y^2 - \frac{2yl\cancel{Ag}}{\cancel{Ag}}\frac{(\rho_2-\rho)}{\rho_2-\rho_1}\right]} = E$$

$$= (\rho_2-\rho_1)\frac{Ag}{2}\left[y^2 - 2y\,l(\rho_2-\rho) + l^2\left(\frac{\rho_2-\rho}{\rho_2-\rho_1}\right)^2\right] - \cancel{(\rho_2-\rho_1)}\frac{Ag}{2}\,l^2\frac{(\rho_2-\rho_0)^2}{\rho_2-\rho_1}$$

$$\therefore \frac{p^2}{2m} + \left[y - \frac{(\rho_2-\rho)l}{\rho_2-\rho_1}\right]^2\frac{(\rho_2-\rho_1)Ag}{2} = E + \frac{(\rho_2-\rho_0)^2 Ag\,l^2}{2(\rho_2-\rho_1)}$$



$y = x^2$

(3)
128

$$\frac{d^2\theta}{dt^2} + a\,\frac{d\theta}{dt} + b\sin\theta = 0$$

$$\boxed{d\theta/dt = p} = f_1$$

$\therefore\ dp/dt + ap + b\sin\theta = 0$

$$\Rightarrow \boxed{dp/dt = -ap - b\sin\theta} = f_2$$

$\partial f_1/\partial\theta = 0,\quad \partial f_1/\partial p = 1$

$\partial f_2/\partial\theta = -b\cos\theta,\quad \partial f_2/\partial p = -a$

$$M = \begin{bmatrix} 0 & 1 \\ -b\cos\theta & -a \end{bmatrix}$$

f.p. $\rightarrow p = 0$
& $-b\sin\theta = 0$
$\Rightarrow \theta = \pm n\pi, 0$
$\therefore \cos\theta = (-1)^n$

Let $M_1 = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$ $\quad M_2 = \begin{bmatrix} 0 & 1 \\ b & -a \end{bmatrix}$

$\cos\theta = 1$

$$\begin{vmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 + a\lambda + b = 0$

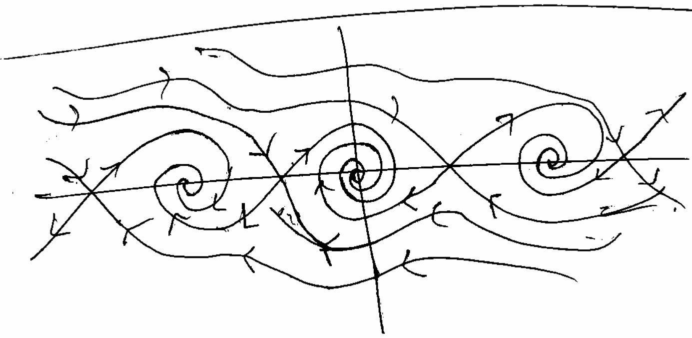
$\lambda = -\frac{a}{2} \pm \sqrt{a^2/4 - b}$

$\cos\theta = -1$

$$\begin{vmatrix} -\lambda & 1 \\ b & -a-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 + a\lambda - b = 0$

$\lambda = -a/2 \pm \sqrt{a^2/4 + b}$



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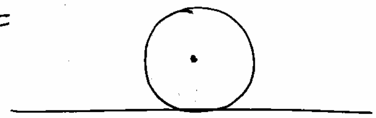
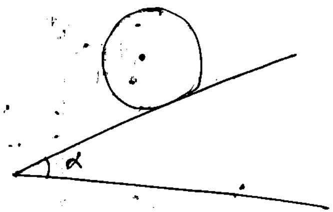
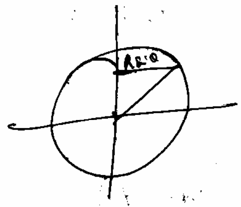






$$\iint R^2\sin^2\theta \, \frac{M}{4\pi \cancel{R^2}} \cancel{R^2} \sin\theta \, d\theta \, d\phi$$

$$= \frac{MR^2}{4\cancel{\pi}} \, 2\cancel{\pi} \int (1-\cos^2\theta)\sin\theta \, d\theta$$

$$= \frac{MR^2}{2} \int (\cos^2\theta - 1)(-\sin\theta)\, d\theta$$

$$= \frac{MR^2}{2} \left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^\pi$$

$$= \frac{MR^2}{2}\left(-\frac{1}{3} + 1 - \frac{1}{3} + 1\right)$$

$$= \frac{MR^2}{\cancel{2}} \times \frac{2}{3} \cancel{\times 2}$$

$$= \frac{2}{3} MR^2$$

$$\iiint r^2 \sin^2\theta \, \frac{M}{\frac{4}{3}\pi R^3} \cancel{r^2} \sin\theta \, d\theta \, d\phi \, dr$$

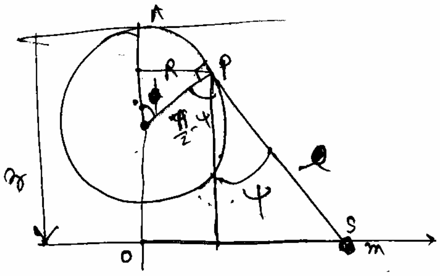
$$\cancel{= \frac{3M}{4\pi} \, 2\pi \, \frac{R^3}{3} \int \sin^2\theta \, (1-\cos^2\theta)\sin\theta \, d\theta}$$

$$\cancel{= \frac{M}{2} R^3}$$

$$= \frac{3M}{4\pi R^3} \, \frac{R^5}{5} \times 2\pi \int (1-\cos^2\theta)\sin\theta \, d\theta$$

$$= \frac{3MR^2}{2\times 5} \times 2 \times \frac{2}{3} \qquad = \frac{2}{5} MR^2 .$$

130)



$\dot{\pi} R < 2l$

$\phi + \psi = \pi/2$

$AP = \phi R$

$PS = l - \phi R$

$z = R - R\cos\phi + PS\cos\psi$

$= R - R\cos\phi + (l - \phi R)\cos(\pi/2 - \phi)$

$$\boxed{z = R(1-\cos\phi) + (l-\phi R)\sin\phi}$$

$OS = R\sin\phi + PS\sin\psi$

$= R\sin\phi + (l-\phi R)\sin(\pi/2 - \phi)$

$$\Rightarrow \boxed{x = R\sin\phi + (l - \phi R)\cos\phi}$$

$\cancel{2T/m = \dot{x}^2 + \dot{z}^2 = (R\cos\phi\dot\phi - \dot\phi R\sin\phi - \phi\cos\phi R\dot\phi)^2 + (R\cos\phi\dot\phi - \dot\phi\cos\phi + \sin\phi\dot\phi\phi)^2}$

$\cancel{\dot z = +R\sin\phi\dot\phi \ldots + (l-\phi R)\cos\phi\dot\phi - R\sin\phi\dot\phi}$

$\cancel{= (l-\phi R)\cos\phi\dot\phi}$

$\cancel{\dot x = R\cos\phi\dot\phi - (l-\phi)\sin\phi\dot\phi - \dot\phi\cos\phi}$

$\cancel{= R\cos\phi = (R-1)\cos\phi = l}$

$\dot x = R\cos\phi\dot\phi - (l-\phi R)\sin\phi\dot\phi - \dot\phi R\cos\phi$

$$\Rightarrow \boxed{\dot x = -(l-\phi R)\sin\phi\dot\phi}$$

$\dot z = +R\sin\phi\dot\phi + (l-\phi R)\cos\phi\dot\phi - \dot\phi R\sin\phi = (l-\phi R)\cos\phi\dot\phi$

$\therefore T = \frac{m}{2}(l-\phi R)^2\dot\phi^2$ | $V = -mgz$

$$\frac{\partial T}{\partial \dot\phi} = m(l-\phi R)^2\dot\phi = p$$

$$\therefore \dot\phi = \frac{p}{m(l-\phi R)^2}$$

$$\therefore H = \frac{p^2}{2m(l-\phi R)^2} - mg\left[R(1-\cos\phi) + (l-R\phi)\sin\phi\right]$$

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m(l-\phi R)^2} = f_1$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial \phi} = -\left[\frac{p^2(2)R}{2m(l-\phi R)^3} - mg\left[R\sin\phi - R\sin\phi + (l-R\phi)\cos\phi\right]\right]$$

$$= \frac{-p^2 R}{m(l-\phi R)^3} + mg(l-R\phi)\cos\phi = f_2$$

F.P.

$p = 0$

$(l-R\phi)\cos\phi = 0$

$\phi = l/R$ or $\phi = (2n+1)\pi/2$

$$\frac{d}{dt}(\phi-\phi_F) = (p-0)\frac{1}{m(l-\phi_F R)^2} = \frac{p}{m(l-\phi_F R)^2}$$

$$\frac{d}{dt}(p-p_F) =$$

$$\left.\frac{\partial f_2}{\partial p}\right|_F = \frac{-2pR}{m(l-\phi_F)^3} = 0$$

$$\left.\frac{\partial f_2}{\partial \phi}\right|_F = mg(l-R\phi_F)\sin\phi_F - mgR\cos\phi_F$$

$$l - \frac{3\pi}{2}R = \frac{2l-3\pi R}{2} = \left(\frac{2l-\pi R}{2}\right) - \pi R$$

$$M = \begin{bmatrix} 0 & \frac{1}{m(l-\phi_F R)^2} \\ mg(l-R\phi_F)\sin\phi_F - mgR\cos\phi_F & 0 \end{bmatrix}$$

if $\cos\phi_F = 0$, $M = \begin{bmatrix} 0 & \frac{1}{m(l-R\pi/2)^2} \\ mg(l-R\pi/2) & 0 \end{bmatrix}$

$$\lambda^2 = \frac{g}{l-\pi R/2}$$

$$\lambda = \pm i\sqrt{\frac{2g}{2l-\pi R}}$$

132)

$$\frac{d\phi}{dt} = \frac{p}{m(R\phi - l)^2} = f_1$$

$$\begin{cases} \partial f_1/\partial \phi \big|_F = 0 \\ \partial f_1/\partial p \big|_F = \dfrac{1}{m(R\phi_F - l)^2} \end{cases}$$

$$dp/dt = \frac{p^2 R}{m\,\cancel{(R\phi)}\,(R\phi - l)^2} - mg\,(l - \phi R)\cos\phi$$

Stable f.p. $\longrightarrow \left(\pi/2, 0\right)$

$$\frac{d}{dt}(\phi - \phi_F) = \frac{(p - p_F)}{m(l - \pi R/2)^2}$$

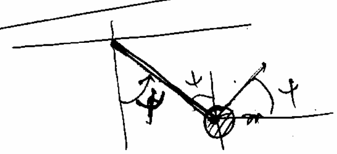
$$\frac{d}{dt}(p - p_F) = -(\phi - \phi_F)\,mg\,(l - \pi R/2)$$

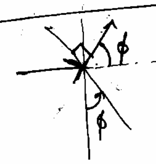
$$\therefore \cancel{d\phi} = -(\phi - \phi_F)\frac{mg\,\cancel{(l - \pi R/2)}}{m(l - \pi R/2)^{\cancel{2}}}$$

$$\therefore \frac{d^2}{dt^2}(\phi - \phi_F) = -\frac{g}{(l - \pi R/2)}(\phi - \phi_F)$$

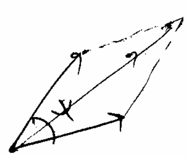
$$\therefore \omega = \sqrt{\frac{g}{l - R\pi/2}}$$

18





$$T = \frac{m}{2}\left(l^2\dot\psi^2 + \dot r^2 + 2l\dot\psi\,\dot r\cos\psi\right)$$



$$p = \frac{\partial T}{\partial \dot\psi} = \frac{m}{2}\left(2l^2\dot\psi + 2l\dot r\cos\psi\right)$$

$$v_x = \dot r + l\dot\psi\cos\psi$$

$$v_y = l\dot\psi\sin\psi$$

$$v_x^2 + v_y^2 = \dot r^2 + l^2\dot\psi^2 + 2l\dot r\dot\psi\cos\psi$$

$$L = T + mg\,l\cos\psi$$

$$\frac{\partial L}{\partial \dot r} = m\dot r + \cancel{2}\,l\,\frac{m}{2}\,\dot\psi\cos\psi \qquad \frac{\partial L}{\partial r} = 0$$

$$\therefore\ \cancel{m}\ddot r + \cancel{m}\,l\,\ddot\psi\cos\psi - \cancel{m}\,l\,\dot\psi^2\sin\psi = 0$$

$$\Rightarrow\ \frac{d}{dt}\left(m\dot r + ml\,\dot\psi\cos\psi\right) = 0$$

$$\therefore\ \cancel{m}\dot r + l\,\dot\psi\cos\psi = C$$

$$\Rightarrow\ \dot r = C - l\,\dot\psi\cos\psi$$

$$\therefore\ T = \frac{m}{2}\left[l^2\dot\psi^2 + \left(C - l\dot\psi\cos\psi\right)^2 + 2l\dot\psi\left(C - l\dot\psi\cos\psi\right)\cos\psi\right]$$

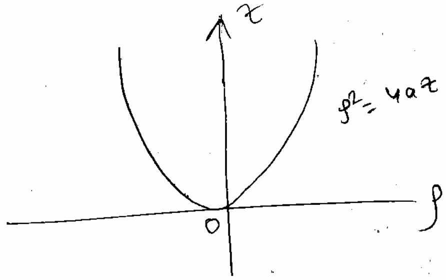
$$p = \frac{\partial T}{\partial \dot\psi} = m\,l^2\dot\psi + m\left(C - l\dot\psi\cos\psi\right)\left(-l\cos\psi\right) + \frac{m}{2}\cancel{2}\,l\left(C - l\dot\psi\cos\psi\right)\cos\psi - \cancel{2}\,l^2\dot\psi\cos^2\psi\,\frac{m}{2}$$

$$\Rightarrow\ \frac{p}{ml} = l\dot\psi - \cos\psi\,\cancel{\left(C - l\dot\psi\cos\psi\right)} + \cancel{\left(C - l\dot\psi\cos\psi\right)}\cos\psi - l\dot\psi\cos^2\psi$$

$$\Rightarrow\ l\left(\dot\psi - \dot\psi\cos^2\psi\right) = \frac{p}{ml}$$

$$\Rightarrow\ \dot\psi = \frac{p}{ml^2\sin^2\psi}$$

$$\therefore\ T = \frac{m}{2}\left[\frac{p^2}{m^2l^2\sin^4\psi} + \left(C - \frac{p\,l\cos\psi}{ml^2\sin^2\psi}\right)^2 + \frac{2lp}{ml^2\sin^2\psi}\right.$$



$T = \frac{m}{2}(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2)$

$V = mgz$

$p_\phi = \frac{\partial T}{\partial \dot\phi} = m\rho^2\dot\phi, \quad p_\rho = m\dot\rho, \quad p_z = m\dot z$

$L = \frac{m}{2}(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2) - mgz$

$\phi \to$ cyclic coordinate $\therefore p_\phi$ is conserved.

Note :- Hamilton - Jacobi :-

(1) Spherical polar

If $U = a(r) + \frac{b(\theta)}{r^2} + \frac{c(\phi)}{r^2 \sin^2\theta}$

(2) Parabolic

If $U = \frac{a(\xi) + b(\eta)}{\xi + \eta}$

$= \frac{a(r+z) + b(r-z)}{2r}$

(3) Elliptic

If $U = \frac{a(\xi) + b(\eta)}{\xi^2 - \eta^2}$

$= \frac{\sigma^2}{r_1 r_2}\left\{ a\frac{(r_2+r_1)}{2\sigma} + b\frac{(r_2 - r_1)}{2\sigma} \right\}$

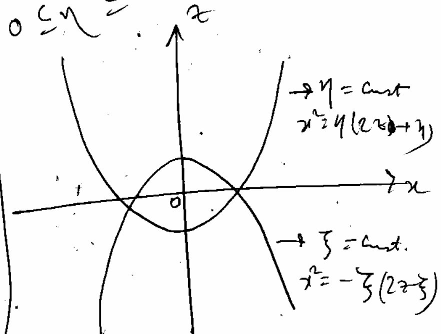
Parabolic coordinates :-

$z = \frac{1}{2}(\xi - \eta)$

$\rho = \sqrt{\xi\eta}$

$0 \le \xi \le \infty$

$0 \le \eta \le \infty$



$r = \sqrt{z^2 + \rho^2} = \sqrt{\frac{1}{4}(\xi^2 + \eta^2 - 2\xi\eta + 4\xi\eta)} = \frac{1}{2}(\xi + \eta)$

$\xi = r + z$

$\eta = r - z$

In our problem, we take

$z = \frac{1}{2}(\xi - \eta) + a$ $\quad \boxed{2(z-a) = \xi - \eta \Rightarrow \xi = \eta + 2(z-a)}$

$\rho = \sqrt{\xi\eta}$

$\phi = \phi$

Then $\rho^2 = 4az$ is given by

$\xi\eta = \frac{4a}{2}(\xi - \eta) + 4a^2 = 2a(\xi-\eta) + 4a^2$

or, $\xi\eta = 2a\xi - \eta + 4a^2$

For ~~$\xi = 2a$~~, $\eta = 2a$, ~~$2a\xi = 2a\xi - \eta + 4a^2$~~

~~we have~~ ~~$2a\xi$~~

$\rho^2 = \xi\eta = (\eta + 2(z-a))\eta = \eta^2 + 2(z-a)\eta$

$\eta =$ const gives paraboloids.

For $\rho^2 = 4az$, we have

$\eta^2 + 2(z-a)\eta = 4az$

$\Rightarrow \eta^2 - 2a\eta + 2z\eta = 4az$

$\eta = 2a$ gives ~~$4a^2 - 4a^2$~~ LHS

LHS $4a^2 - 4a^2 + 4az = 4az$.

$\therefore$ we ~~choose~~ $\boxed{\eta = 2a}$ gives our paraboloid.

Region of motion $\Rightarrow \eta \leq 2a$.

Now, $\dot{z} = \frac{1}{2}(\dot{\xi} - \dot{\eta})$, $\dot{\rho} = \frac{\xi\dot{\eta} + \dot{\xi}\eta}{2\sqrt{\xi\eta}}$

$\therefore T = \frac{m}{2}\left[\frac{\xi^2\dot{\eta}^2 + \dot{\xi}^2\eta^2 + 2\xi\dot{\xi}\eta\dot{\eta}}{4\xi\eta} + \xi\eta\dot{\phi}^2 + \frac{\dot{\xi}^2 + \dot{\eta}^2 - 2\dot{\xi}\dot{\eta}}{4}\right]$

~~$= \frac{m}{2}$~~ $+ \frac{2\dot{\xi} + 2\dot{\eta}}{4}$

~~$p_{\xi} = \frac{\partial T}{\partial \dot{\xi}} = \frac{m}{2}\left[\frac{2\dot{\xi}\eta^2 + 2\xi\eta\dot{\eta}}{2\xi\eta} + \dots\right] = \frac{m}{2}\left[\frac{\dot{\xi}\eta^2 + \xi\eta\dot{\eta}}{\xi\eta} + \frac{\dot{\xi} - \dot{\eta}}{2}\right]$~~

$\Rightarrow$ 136) $\cancel{p_\xi = \frac{m}{4\xi\eta}\left[2\dot\xi\eta^2 + 2\xi\eta\dot\eta + \xi\eta\dot\xi - \xi\eta\dot\eta\right]}$

$$T = \frac{m}{2}\left[\frac{\xi^2\dot\eta^2 + \dot\xi^2\eta^2 + \cancel{2\xi\dot\xi\dot\eta\eta} + \dot\xi^2\xi\eta + \dot\eta^2\xi\eta - \cancel{2\dot\xi\dot\eta\xi\eta}}{4\xi\eta} + \xi\eta\dot\phi^2\right]$$

$$= \frac{m}{2}\left[\frac{\xi^2\dot\eta^2 + \dot\xi^2\eta^2 + \dot\xi^2\xi\eta + \dot\eta^2\xi\eta}{4\xi\eta} + \xi\eta\dot\phi^2\right]$$

$$p_\xi = \frac{\partial T}{\partial\dot\xi} = \frac{m}{\cancel{2}}\frac{\cancel{2}\dot\xi(\eta^2+\xi\eta)}{4\xi\eta}$$

$$= \frac{m\dot\xi}{4\xi}(\eta+\xi) = \frac{m(\xi+\eta)\dot\xi}{4\xi}$$

$$p_\eta = \frac{\partial T}{\partial\dot\eta} = \frac{m}{\cancel{2}}\frac{\cancel{2}\dot\eta}{4\xi\eta}(\xi^2+\xi\eta) = \frac{m(\xi+\eta)\dot\eta}{4\eta}$$

$$p_\phi = \frac{\partial T}{\partial\dot\phi} = \xi\eta\dot\phi m$$

$$\therefore\ T = \frac{m}{2}\left[\dot\xi^2\frac{(\eta^2+\xi\eta)}{4\xi\eta} + \dot\eta^2\frac{(\xi^2+\xi\eta)}{4\xi\eta} + \cancel{\xi}\cancel{\eta}\frac{p_\phi^2}{m^2\xi^{\cancel{2}}\eta^{\cancel{2}}}\right]$$

$$\Rightarrow T = \frac{\cancel{m}}{\cancel{2}}\frac{\overset{2}{\cancel{16}}\xi^{\cancel{2}}p_\xi^2}{m^{\cancel{2}}(\xi+\eta)^{\cancel{2}}}\frac{\cancel{(\eta+\xi)}}{\cancel{4\xi}} + \frac{m}{2}\frac{16\eta^2p_\eta^2}{m^{\cancel{2}}(\xi+\eta)^{\cancel{2}}}\frac{\cancel{(\eta+\xi)}}{4\eta} + \frac{p_\phi^2}{2m\xi\eta}$$

$$= \frac{2(\xi p_\xi^2 + \eta p_\eta^2)}{m(\xi+\eta)} + \frac{p_\phi^2}{2m\xi\eta}$$

$$V = mgz = \frac{mg}{2}(\xi - \eta) + \text{constant}$$

$$H = C_0 = \frac{2(\xi p_\xi^2 + \eta p_\eta^2)}{m(\xi+\eta)} + \frac{p_\phi^2}{2m\xi\eta} + \frac{mg}{2}(\xi - \eta)$$

$q: p_\phi$ conserved

$$C_0(\xi+\eta) = \frac{2\xi}{m}p_\xi^2 + \frac{2\eta}{m}p_\eta^2 + \frac{p_\phi^2}{2m}\left(\frac{\xi+\eta}{\xi\eta}\right) + \frac{mg}{2}(\xi^2 - \eta^2)$$

$$\ni \frac{2\xi p_\xi^2}{m} - C_0\xi + \frac{p_\phi^2}{2m\xi} + mg\xi^2/2$$

$$= C_0\eta - \frac{2\eta p_\eta^2}{m} - \frac{p_\phi^2}{2m\eta} + mg\eta^2/2$$

$$F_3 = F_3(q, \mathcal{P}) \ni p = \frac{\partial F_3}{\partial q}, \quad Q = \frac{\partial F_3}{\partial \mathcal{P}}$$

$$F_3 = F_3^{0}(\xi) + \tilde{F}_3(\eta)$$

$$\frac{2\xi}{m}\left(\frac{dF_3^0}{d\xi}\right)^2 - C_0\xi + \frac{p_\phi^2}{2m\xi} + mg\xi^2/2 = C_q \rightarrow \text{conserved charge}$$

$$C_0\eta - \frac{2\eta}{m}\left(\frac{d\tilde{F}_3}{d\eta}\right)^2 - \frac{p_\phi^2}{2m\eta} + mg\eta^2/2 = C_q$$

Elliptic coordinates

$$\rho = \sigma\sqrt{(\xi^2-1)(1-\eta^2)} \qquad 1 \le \xi \le \infty$$

$$z = \sigma\xi\eta \qquad -1 \le \eta \le 1$$

$$\phi = \phi$$

Surfaces of constant ξ are the ellipsoids

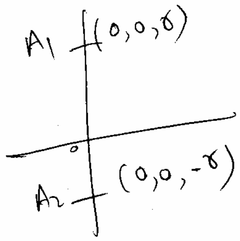
$$\frac{z^2}{\sigma^2\xi^2} + \frac{\rho^2}{\sigma^2(\xi^2-1)} = 1$$

$A_1, A_2 \rightarrow$ foci

Surfaces of constant η are hyperboloids

$$\frac{z^2}{\sigma^2\eta^2} - \frac{\rho^2}{\sigma^2(1-\eta^2)} = 1 \qquad A_1, A_2 \rightarrow \text{foci}$$

138) $OA_1 = r_1 = \sqrt{(z-\sigma)^2+\rho^2}$



$OA_2 = r_2 = \sqrt{(z+\sigma)^2+\rho^2}$

$\cancel{r_1 = \sigma(\xi-\eta)}$

$\therefore r_1 = \sqrt{\sigma^2(\xi\eta-1)^2+\sigma^2(\xi^2-1)(1-\eta^2)}$

$= \sigma\sqrt{\cancel{\xi^2\eta^2} - 2\xi\eta + \xi^2 + \eta^2 \cancel{-1} \cancel{-\xi^2\eta^2} \cancel{+1}}$

$\boxed{r_1 = \sigma(\xi-\eta)}$

$r_2 = \sqrt{\sigma^2(\xi\eta+1)^2+\sigma^2(\xi^2-1)(1-\eta^2)}$

$= \sigma\sqrt{\cancel{\xi^2\eta^2}\cancel{+1} + 2\xi\eta + \xi^2 + \eta^2 \cancel{-1} \cancel{-\xi^2\eta^2}}$

$\boxed{r_2 = \sigma(\xi+\eta)}$

$\dot\rho = \frac{\sigma\,\cancel{2}\xi\dot\xi(1-\eta^2)}{\cancel{2}\sqrt{(\xi^2-1)(1-\eta^2)}} - \frac{\sigma\eta\dot\eta(\xi^2-1)}{\sqrt{(\xi^2-1)(1-\eta^2)}}$

$= \frac{\sigma\left[\xi\dot\xi(1-\eta^2) - \eta\dot\eta(\xi^2-1)\right]}{\sqrt{(\xi^2-1)(1-\eta^2)}}$

$\dot z = \sigma(\xi\dot\eta + \dot\xi\eta)$

$\therefore \frac{\dot\rho^2+\dot z^2}{\sigma^2} = \frac{\xi^2\dot\xi^2(1-\eta^2)^2 + \eta^2\dot\eta^2(\xi^2-1)^2 - 2\xi\dot\xi\eta\dot\eta(1-\eta^2)(\xi^2-1)}{(\xi^2-1)(1-\eta^2)} + (\xi\dot\eta+\dot\xi\eta)^2$

$= \xi^2\dot\xi^2\frac{(1-\eta^2)}{\xi^2-1} + \eta^2\dot\eta^2\frac{(\xi^2-1)}{1-\eta^2} - \cancel{2\xi\dot\xi\eta\dot\eta}$
$+ \xi^2\dot\eta^2 + \dot\xi^2\eta^2 + \cancel{2\xi\dot\xi\eta\dot\eta}$

$\therefore T = \frac{m}{2}\sigma^2\left[\xi\dot\xi^2\frac{(1-\eta^2)}{\xi^2-1} + \eta^2\dot\eta^2\frac{(\xi^2-1)}{1-\eta^2} + \xi^2\dot\eta^2 + \dot\xi^2\eta^2 + (\xi^2-1)(1-\eta^2)\dot\phi^2\right]$

$$p_\xi = \frac{\partial T}{\partial \dot\xi} = m\sigma^2\left[\xi^2\dot\xi\,\frac{(1-\eta^2)}{\xi^2-1} + \dot\xi\,\eta^2\right]$$

$$= \frac{m\sigma^2}{\xi^2-1}\dot\xi\left[\xi^2 - \cancel{\xi^2\eta^2} + \cancel{\xi^2\eta^2} - \eta^2\right]$$

$$= \frac{m\sigma^2}{(\xi^2-1)}(\xi^2-\eta^2)\dot\xi$$

$$p_\eta = \frac{\partial T}{\partial \dot\eta} = m\sigma^2\left[\eta^2\dot\eta\,\frac{(\xi^2-1)}{1-\eta^2} + \xi^2\dot\eta\right]$$

$$= \frac{m\sigma^2\dot\eta}{1-\eta^2}\left[\cancel{\eta^2\xi^2} - \eta^2 + \xi^2 - \cancel{\xi^2\eta^2}\right]$$

$$= \frac{m\sigma^2}{1-\eta^2}\dot\eta\,(\xi^2-\eta^2)$$

$$\therefore T = \frac{\cancel{(m\sigma^2)}}{2}\left[\xi^2\frac{(1-\eta^2)}{\xi^2-1} + \eta^2\right]\frac{p_\xi^2}{(m\sigma^2)^{\cancel{2}}}\frac{(\xi^2-1)^2}{(\xi^2-\eta^2)^2}$$

$$= \frac{\xi^2 - \cancel{\xi^2\eta^2} + \cancel{\xi^2\eta^2} - \eta^2}{2m\sigma^2(\xi^2-\eta^2)^2}\,p_\xi^2\,(\xi^2-1)$$

$$= \frac{p_\xi^2}{2m\sigma^2(\xi^2-\eta^2)}(\xi^2-1)$$

Also $\dfrac{p_\eta^2}{2m\sigma^2(\xi^2-\eta^2)}(1-\eta^2)$

$$\therefore T = \frac{1}{2m\sigma^2(\xi^2-\eta^2)}\left[(\xi^2-1)p_\xi^2 + (1-\eta^2)p_\eta^2\right] + \cancel{\frac{(m\sigma^2)(\xi^2-1)(1-\eta^2)}{2}\frac{p_\phi^2}{(m\sigma^2)^2(\xi^2-1)^2(1-\eta^2)^2}}$$

$$\frac{\partial T}{\partial\dot\phi} = m\sigma^2(\xi^2-1)(1-\eta^2)\dot\phi$$

$$= --- + \frac{(\xi^2-1)+(1-\eta^2)}{2m\sigma^2(\ )(\ )}p_\phi^2$$

$$= --- + \frac{1}{2m\sigma^2(\xi^2-\eta^2)}\left[\frac{1}{1-\eta^2} + \frac{1}{\xi^2-1}\right]p_\phi^2$$

Problem set 2

(4) $T = \frac{1}{2}\,\mathrm{Tr}\left(\frac{dS}{dt} K_A \frac{dS^T}{dt}\right) + \frac{1}{2} M_B \left(\frac{d\vec{R}}{dt}\right)^2$
$+ M_B \frac{d\vec{R}}{dt}\cdot\left(\frac{dS_1}{dt}\vec{\rho}_B^{\,cm}\right) + \frac{1}{2}\,\mathrm{Tr}\left(\frac{dS_1}{dt} K_B \frac{dS_1^T}{dt}\right)$

$$\frac{\partial T}{\partial \dot\alpha^i} = \frac{1}{2}\,\mathrm{Tr}\left(\frac{\partial S}{\partial \alpha^i} K_A \frac{\partial S^T}{\partial t}\right) + \frac{1}{2}\,\mathrm{Tr}\left(\frac{dS}{dt} K_A \frac{\partial S^T}{\partial \alpha^i}\right) + M_B\left(\frac{d\vec{R}}{dt}\right)^T\left(\frac{\partial S}{\partial \alpha^i} H\right) + M_B\left(\frac{\partial S}{\partial \alpha^i}H\right)^T\cdot\left(\frac{dS_1}{dt}\vec{\rho}_B^{\,cm}\right)$$

$$\frac{\partial T}{\partial \dot\rho^i} = M_B \frac{d\vec{R}}{dt}\cdot\left(\frac{\partial S_1}{\partial \rho^i}\vec{\rho}_B^{\,cm}\right) + \mathrm{Tr}\left(\frac{\partial S_1}{\partial \rho^i} K_B \frac{dS_1^T}{dt}\right)$$

$$g = \mathrm{Tr}\left(\Phi_1 K_A \frac{dS^T}{dt}\right) + M_B\left(\frac{dR}{dt}\right)^T\left(\Phi_1 H\right) + M_B\left(\Phi_1 H\right)^T\left(\frac{dS_1}{dt}\vec{\rho}_B^{\,c}\right)$$
$$+ M_B\left(\frac{dR}{dt}\right)^T\left(\Phi_2\,\vec{\rho}_B^{\,c}\right) + \mathrm{Tr}\left(\Phi_2 K_B \frac{dS_1^T}{dt}\right)$$

$S = R_3(\phi)\,R_1(\chi)\,R_3(\psi)$

$S_1 = R_3(\phi_1)\,R_2(\chi_1)\,R_3(\psi_1)$

$\phi - \phi_1 = \tilde\phi \rightsquigarrow$ cyclic coordinate $\qquad \dot\phi = \dot{\tilde\phi} + \dot\phi_1$

$\therefore\ \phi = \tilde\phi + \phi_1$

$\therefore\ S = R_3(\tilde\phi + \phi_1)\,R_1(\chi)\,R_3(\psi)$

$S_1 = R_3(\phi_1)\,R_2(\chi_1)\,R_3(\psi_1)$

Now, $R = \frac{1}{2}(M_B h^2 + I_1)\left[\dot\chi^2 + \sin^2\chi\,(\dot{\tilde\phi} + \dot\phi_1)^2\right]$
$+ M_B h \rho_2\ \sin\tilde\phi\ \cos\chi\ \sin\chi_1\ \dot\chi\ \dot\phi_1$
$+ M_B h \rho_2\ \cos\tilde\phi\ \sin\chi\ \sin\chi_1\ \dot\phi_1\ (\dot{\tilde\phi} + \dot\phi_1)$
$+ M_B h \rho_2\ [\cos\tilde\phi\ \cos\chi\,\chi_1 + \sin\chi\ \chi_1]\ \dot\chi\ \dot\chi_1$
$- M_B h \rho_2\ \sin\tilde\phi\ \dot\chi_1\ \dot\phi_1$
$- M_B h \rho_2\ \sin\tilde\phi\ \dot\chi_1\ \dot\phi_1$

Terms containing ϕ & ϕ_1 $\quad \cos\tilde{\phi}\, c\chi\, c\chi_1\, \dot{\chi}\dot{\chi}_1 + \sin\tilde{\phi}\cos\chi\sin\chi_1\, \dot{\chi}\,\dot{\phi}_1$ $\quad$ (141)

$$+ \cos\tilde{\phi}\, \sin\chi\, \sin\chi_1 \left(\dot{\tilde{\phi}} + \dot{\phi}_1\right)\dot{\chi}$$

$$- \sin\tilde{\phi}\, \sin\chi\, c\chi_1 \left(\dot{\tilde{\phi}} + \dot{\phi}_1\right)\dot{\chi}_1$$

$$K.E. = \frac{1}{2}\, \mathrm{Tr}\left(\frac{dS}{dt}\, K\, dS^T/dt\right)$$

$$S = R_3(\phi)\, R_1(\chi)\, R_3(\psi)$$

$$S^{-1} = S^T = R_3^{-1}(\psi)\, R_1^{-1}(\chi)\, R_3^{-1}(\phi)$$

$$= R_3(-\psi)\, R_1(-\chi)\, R_3(-\phi)$$

$U(\theta) = R(\theta)$
$U(-\theta) = R^{-1}(\theta) = R(-\theta)$

$$= \begin{bmatrix} \cos\phi\, c\psi - c\chi\, s\psi\, s\phi & -c\phi\, s\psi - c\chi\, c\psi\, s\phi & +s\phi\, s\chi \\ s\phi\, c\psi + c\chi\, s\psi\, c\phi & -s\phi\, s\psi + c\chi\, c\psi\, c\phi & c\phi\, s\chi \\ +s\chi\, s\psi & -s\chi\, c\psi & c\chi \end{bmatrix}$$

Marion $\quad S^{-1} = \begin{bmatrix} c\psi\, c\phi - c\theta\, s\phi\, s\psi & c\psi\, s\phi + c\theta\, c\phi\, s\psi & s\psi\, s\theta \\ -s\psi\, c\phi - c\theta\, s\phi\, c\psi & -s\psi\, s\phi + c\theta\, c\phi\, c\psi & c\psi\, s\theta \\ \sin\theta\, s\phi & -s\theta\, c\phi & c\theta \end{bmatrix}$

Siv $\quad$ Marion
$-\psi \equiv +\psi$
$-\chi \equiv +\theta$
$-\phi \equiv +\phi$

Marion $\qquad\qquad\qquad\qquad$ Siv

$$\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi}\, s\theta\, s\psi + \dot{\theta}\, c\psi = -\dot{\phi}_1 - \dot{\chi}_1 - \dot{\psi}_1 = -\dot{\phi}\, s\chi\, s\psi - \dot{\chi}\cos\psi$$

$$\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi}\, s\theta\cos\psi - \dot{\theta}\, s\psi = +\dot{\phi}\, s\chi\cos\psi - \dot{\chi}\, s\psi$$

$$\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi}\, c\theta + \dot{\psi} = -\dot{\phi}\, c\chi - \dot{\psi}$$

$$= \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2$$

$(I_1 = I_2)$

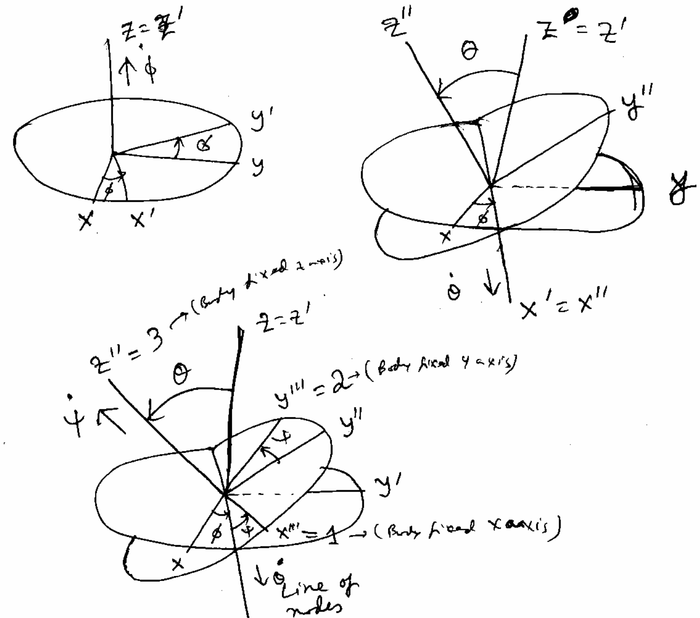
$$\therefore K.E. = \frac{1}{2} I_1 \left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2} I_3 \omega_3^2$$

$$= \frac{I_1}{2}\left[\dot{\phi}^2 s^2\chi\, s^2\psi + \dot{\chi}^2 c^2\psi + \cancel{2\dot{\phi}\dot{\chi}\, s\chi\, s\psi\, c\psi} + \dot{\phi}^2 s^2\chi\, c^2\psi + \dot{\chi}^2 s^2\psi - \cancel{2\dot{\phi}\dot{\chi}\, s\chi\, s\psi\, c\psi}\right] + \frac{1}{2} I_3 \left(\dot{\phi}^2 c^2\chi + \dot{\psi}^2 - 2\dot{\phi}\dot{\psi}\, c\chi\right)$$

$$= \frac{I_1}{2}\left(\dot{\chi}^2 + \dot{\phi}^2 s^2\chi\right) + \frac{I_3}{2}\left(\dot{\phi}\, c\chi + \dot{\psi}\right)^2$$

142) Marion

$\lambda = \lambda_\psi \lambda_\theta \lambda_\phi$



We may associate the time derivatives of these rotation ∠'s with the comp. of the angular vel. vector $\vec{\omega}$.

Thus
$$\left.\begin{aligned} \omega_\phi &= \dot\phi \\ \omega_\theta &= \dot\theta \\ \omega_\psi &= \dot\psi \end{aligned}\right\}$$

We express $\vec{\omega}$ in the body fixed axes.

The ang. vel. $\dot\phi, \dot\theta, \dot\psi$ are directed along the following axes:—

$\dot\phi \rightarrow$ along the z (fixed) axis

$\dot\theta \rightarrow$ " " line of nodes

$\dot\psi \rightarrow$ " " 3 ~~(fixed) axis~~ (body) axis

The components of these ang. vel. along the body coord. axes are:-

$$\left.\begin{aligned}\dot\phi_1 &= \dot\phi \sin\theta \sin\psi \\ \dot\phi_2 &= \dot\phi \sin\theta \cos\psi \\ \dot\phi_3 &= \dot\phi \cos\theta\end{aligned}\right\}$$

$$\left.\begin{aligned}\dot\theta_1 &= \dot\theta \cos\psi \\ \dot\theta_2 &= -\dot\theta \sin\psi \\ \dot\theta_3 &= 0\end{aligned}\right\}$$

$$\left.\begin{aligned}\dot\psi_1 &= 0 \\ \dot\psi_2 &= 0 \\ \dot\psi_3 &= \dot\psi\end{aligned}\right\}$$

Collecting the indiv. components of $\bar\omega$, we have, finally,

$$\omega_1 = \dot\phi_1 + \dot\theta_1 + \dot\psi_1 = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$

$$\omega_2 = \dot\phi_2 + \dot\theta_2 + \dot\psi_2 = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$$

$$\omega_3 = \dot\phi_3 + \dot\theta_3 + \dot\psi_3 = \dot\phi \cos\theta + \dot\psi$$