Clarification on
Axion-dilaton eq:

$$S \propto \int \sqrt{\det \tilde{g}}\, \tilde{g}^{\mu\nu} \frac{\partial_\mu \tau\, \partial_\nu \bar{\tau}}{\tau - \bar{\tau}}$$

Vary $\bar{\tau}$:

$$\delta S = \int \sqrt{\det \tilde{g}} \left\{ \tilde{g}^{\mu\nu} \left[\frac{\partial_\mu \tau\, \partial_\mu (\delta \bar{\tau})}{(\tau - \bar{\tau})^2} + 2 \frac{\partial_\mu \tau\, \partial_\mu \bar{\tau}}{(\tau - \bar{\tau})^3} \delta \bar{\tau} \right] \right.$$

$$= \int \sqrt{\det \tilde{g}} \left[- \tilde{\nabla}^\mu \left(\frac{\partial_\mu \tau}{(\tau - \bar{\tau})^2} \right) + 2 \tilde{g}^{\mu\nu} \frac{\partial_\mu \tau\, \partial_\nu \bar{\tau}}{(\tau - \bar{\tau})^3} \right]$$

$$= \int \sqrt{\det \tilde{g}} \left[- \frac{1}{(\tau - \bar{\tau})^2} \tilde{\nabla}^2 \tau + 2 \tilde{g}^{\mu\nu} \frac{\partial_\mu \tau\, \partial_\nu (\tau - \bar{\tau})}{(\tau - \bar{\tau})^3} + 2 \tilde{g}^{\mu\nu} \frac{\partial_\mu \tau\, \partial_\nu \bar{\tau}}{(\tau - \bar{\tau})^3} \right]$$

$$= \int \sqrt{\det \tilde{g}} \left[- \frac{1}{\tau - \bar{\tau}} \tilde{\nabla}^2 \tau + 2 \frac{\tilde{g}^{\mu\nu} \partial_\mu \tau\, \partial_\nu \tau}{(\tau - \bar{\tau})^2} \right]$$

$$\Rightarrow \tilde{\nabla}^2 \tau - \frac{2}{\tau - \bar{\tau}} \tilde{g}^{\mu\nu} \partial_\mu \tau\, \partial_\nu \tau = 0$$

Flat metric, holomorphic τ:

$$\tilde{\nabla}^2 \tau = 0. \qquad \tilde{g}^{\mu\nu} \partial_\mu \tau\, \partial_\nu \tau \propto (\partial_z \tau\, \partial_{\bar{z}} \tau) = 0$$

$\Rightarrow$ Holomorphic τ satisfies eq. of motion

$$\cancel{T_{mn} \propto \left(\partial_m \tau\, \partial_n \bar{\tau} - \frac{1}{2} \tilde{g}_{mn}\, \partial_p \tau\, \partial^p \bar{\tau} \right)}$$

~~$\rightarrow$ vanishes for holomorphic τ.~~

~~$\rightarrow$ metric remains flat in the bulk.~~

Some clarification about the coordinate system:

Metric: Flat in torus coordinate ω.

$\rightarrow$ identified under $\omega \to -\omega$.

~~The~~ $ds^2 = |d\omega|^2$

~~The~~ Near $\omega = 0$, single valued coordinate

is $z = \omega^2$. $\omega = z^{1/2}$

$z \to e^{2\pi i} z \Rightarrow \omega \to -\omega$

identified.

$$ds^2 = |d\omega|^2 = \frac{1}{4} \frac{dz\, d\bar{z}}{z^{1/2}\, \bar{z}^{1/2}}$$

not relevant.

On tetrahedron:

$$ds^2 = \prod_{i=1}^{4} \frac{dz\, d\bar{z}}{(z-\omega_i)^{1/2} (\bar{z}-\bar{\omega}_i)^{1/2}}$$

$\omega_1, \ldots \omega_4$: Locations of O7-planes.

Note: as $z \to \infty$ $ds^2 = \frac{dz\, d\bar{z}}{z^2\, \bar{z}^2} = dy\, d\bar{y}$, $y \equiv \frac{1}{z}$

Metric is regular at ∞.

In z-coordinate system: $\tau = \frac{1}{2\pi i}\left(\sum_{i=1}^{16} \ln(z - z_i) - 4 \sum_{i=1}^{4} \ln(z - \omega_i)\right)$

F - theory:

Consider some Calabi-Yau manifold M ~~M_n~~ of $2n$ real dimensions which is "elliptically fibered".

→ A fiber bundle with base a complex manifold B of real dim. $2n-2$ and the fiber a torus T^2 with modular parameter τ

→ τ varies holomorphically with the base coordinate $\vec{z}$.

→ T^2 can degenerate at codimension 2 subspaces where $\tau \to i\infty$ or one of its $SL(2,\mathbb{Z})$ images.

(Total space can still be non-singular)

Consider M-theory on M.

Locally M-theory on $T^2 \equiv$ IIB on S^1

T^2 size $\to 0$ $\Rightarrow$ S^1 size $\to \infty$.

Thus M-theory on $\mathcal{M}$

$\equiv$ IIB on a manifold $\mathcal{N}$ obtained by fibering S^1 over B.

$\tau \Rightarrow$ axion-dilaton moduli of IIB

$\rightarrow$ varies as we move on the base.

Now consider $\mathcal{M}$ and adjust its Kahler modulus such that T^2 size $\to 0$. at every point in B.

$\Rightarrow$ In IIB, S^1 size $\to \infty$ at every point in B.

$\Rightarrow$ effectively IIB on B with τ varying over B.

(2n-2) dimensional.

$\Rightarrow$ F-theory compactification.

$\rightarrow$ A mechanism by which we use 2n-dim. elliptically fibered CY to generate a compactification of IIB on a (2n-2) dim manifold.

Example: $CY_1 = T^2 \Rightarrow$ Base is a point.

$CY_2 = K3$

Not all K3 are elliptically fibered but a subset is.

Base: $S^2 = CP^1$

Coordinate z.

τ varies over z

$\tau(z)$ is not arbitrary but chosen so that the total space is K3.

How to do this?

Weierstrass form of T^2:

A torus can be described by the eq:

$$y^2 = x^3 + \underbrace{f x + g}_{\text{constants}}$$

x, y: complex variables $\in S^2$

1 eq. in 2 variables $\Rightarrow$ 1 complex dim.

$$y^2 = (x-x_1)(x-x_2)(x-x_3).$$

Away from x_1, x_2, x_3, x is a good local coordinate system.

Near $x = x_i$, we have two points for each x. (x, y) & $(x, -y)$

$\Rightarrow$ A sphere with 4 square root branch points at x_1, x_2, x_3, ∞.

$\Rightarrow$ Riemann surface with two branches. $\Rightarrow$ a torus.

Q. How is τ related to f and g?

$$j(\tau) = \frac{4 \cdot (24f)^3}{27g^2 + 4f^3}$$

$$\Downarrow$$

$$\frac{\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8}{\eta(\tau)^{24}} \to q^{-1} \text{ as } q = e^{2\pi i \tau} \to 0$$

$\to$ pole.

$$j(i) = (24)^3. \quad j(e^{i\pi/3}) = 0.$$

Now consider ~~a~~ elliptically fibered K3.

$\tau(z)$ varies as a fn of z

$\Rightarrow f(z), g(z)$ vary as fn of z.

$y^2 = x^3 + f(z)x + g(z)$ One eq. in 3 variables $\Rightarrow$ 2 complex dim manifold.

Under what condition does this describe a K3 (CY_2)?

~~Ans~~ Ans: $f(z)$ is a polynomial of degree 8 and $g(z)$ is a polynomial of degree 12.

Proof: Use homogeneous coordinates on $\mathbb{CP}^1$ $z = \frac{z^1}{z^2}$ $(z^1, z^2) \equiv \lambda(z^1, z^2)$

$$z_2^{12} y^2 = x^3 \cancel{z}z_2^{12} + z_2^4 x \underbrace{f(z_1, z_2)}_{\Downarrow \text{ Hom. of deg. 8}} + \underbrace{g(z_1, z_2)}_{\Downarrow \text{ Hom. of deg. 12}}$$

$\tilde{y} = z_2^6 y, \quad \tilde{x} = z_2^4 y$

$$\tilde{y}^2 = \tilde{x}^3 + \tilde{x} f(z_1, z_2) + g(z_1, z_2)$$

Scaling $(z_1, z_2, \tilde{x}, \tilde{y}) = (\lambda z_1, \lambda z_2, \lambda^4 \tilde{x}, \lambda^6 \tilde{y})$

$\Rightarrow$ An eq. in weighted projective space

A degree d eq. in weighted projective space of weight $(\omega_1, \ldots \omega_n)$:

c_1 = coefficient of J in

$$\frac{\prod_{i=1}^{n}(1+\omega_i J)}{(1+dJ)} \Rightarrow \sum_{i=1}^{n} \omega_i - d$$

Here $\sum_{i=1}^{4} \omega_i = 1+1+4+6 = 12$

$d = 12$

$\Rightarrow c_1 = 0.$

$$j(\tau(z)) = \frac{(24)^3 \, 4f^3(z)^3}{4f(z)^3 + 27g(z)^2}$$

$$\Delta(z) = 4f(z)^3 + 27g(z)^2$$

Zeroes of Δ $\Rightarrow$ poles of j $\Rightarrow$ fiber degenerates.

How many such zeroes?

Δ: deg. 24 $\Rightarrow$ 24 zeroes.

We shall see that degenerate fiber

$\Rightarrow$ 7 brane locations on B

(not necessarily D7)

Now consider a special case:

$$f(z) = a \prod_{i=1}^{4} (z-\omega_i)^2 \qquad \Delta = (4a^3+27b^2) \prod_{i=1}^{4} (z-z_i)^6$$

$$g(z) = b \prod_{i=1}^{4} (z-\omega_i)^3.$$

$$j(\tau(z)) = \frac{4 \cdot (24a)^3}{4a^3+27b^2} \Rightarrow z \text{ independent.}$$

$\Rightarrow \tau(z)$ is constant on S^2.

Monodromy: What happens as z goes around a ω_i?

$$y^2 = x^3 + 4 f(z) x + g(z)$$

$\to$ Surface eq.

$$z = z_i + \epsilon e^{i\phi}$$

$$f(z) = \epsilon^2 k_1 e^{2i\phi} \qquad g(z) = k_2 \epsilon^3 e^{3i\phi}$$

To satisfy the eq:

$$x = k_3 \epsilon e^{i\phi} \qquad y = k_4 \epsilon e^{3i\frac{\phi}{2}}$$

As $\phi \to \phi + 2\pi$, $z \to z$, $x \to x$, $y \to -y$

What is the $y \to -y$ on T^2?

Reversing the sign of both coordinates

$\Rightarrow SL(2,\mathbb{Z})$ trs. $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow (-1)^{F_L} \cdot \Omega_2$.

Metric on the base:

$$ds^2 \propto \frac{dz\, d\bar{z} \quad \tau_2(z) \, |\eta(\tau(z))|^2 \, |\eta(\bar{\tau}(\bar{z}))|^2}{\prod_{\alpha=1}^{24} (z - z_\alpha)^{1/12} (\bar{z} - \bar{z}_\alpha)^{1/12}}$$

$\to$ In general.

In this special case i, the zeroes of Δ are grouped in 6 at one place.

$$ds^2 \propto \frac{dz \, d\bar{z}}{\prod_{i=1}^{4} (z - \omega_i)^{1/2} (\bar{z} - \bar{\omega}_i)^{1/2}} \times \tau_2 \, |\eta(\tau)|^2 \, |\overline{\eta(\tau)}|^2$$

$\Rightarrow$ Tetrahedron.

Thus at this point in the moduli space F-theory on elliptically fibered K3 $\equiv$ IIB on $T^2/(-1)^{F_L} \cdot \Omega \cdot \mathcal{I}_2$

What happens when we deform it?