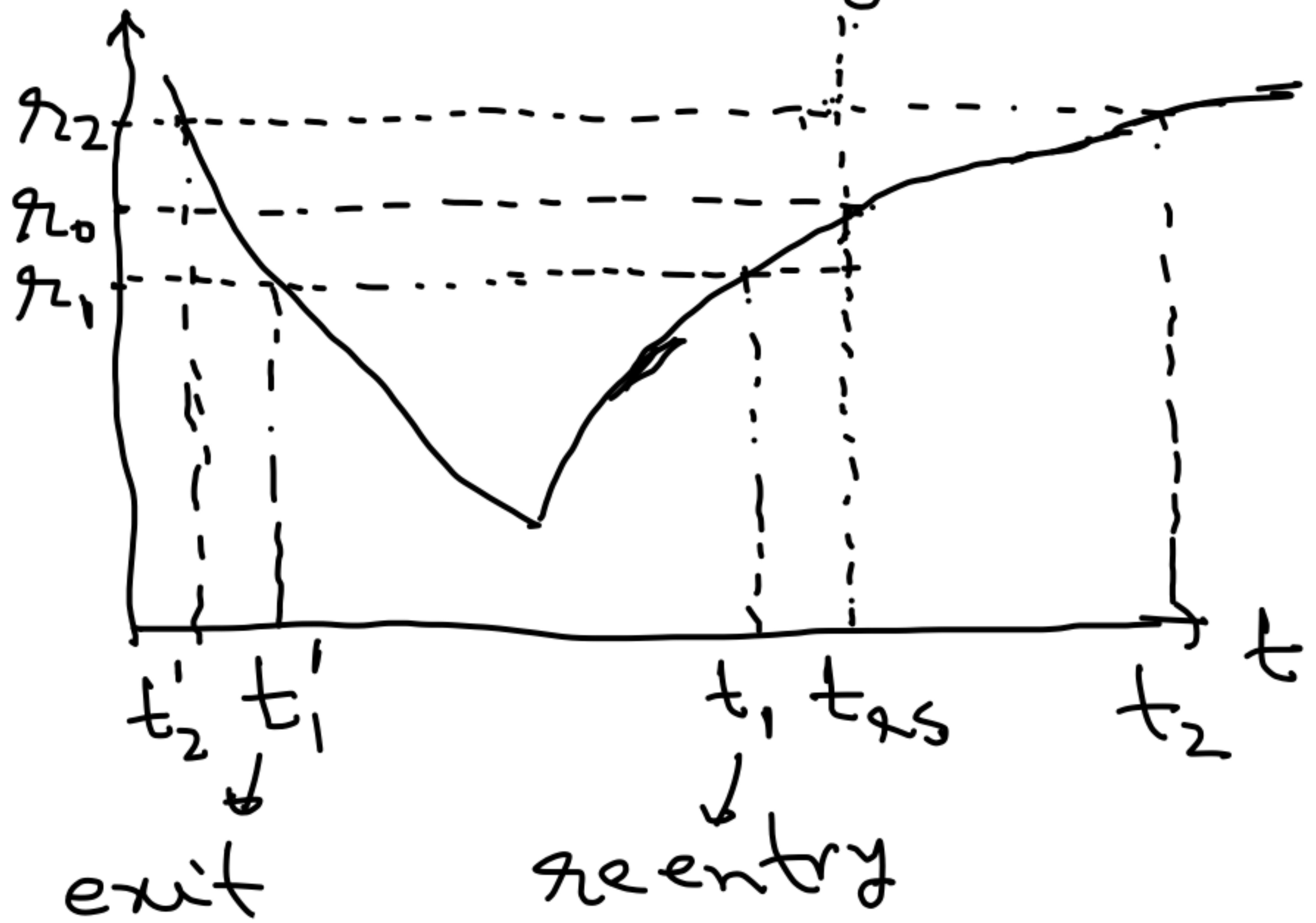
$\lambda^{-1}H^{-1} \to$ comoving distance scale that light can travel in time H^{-1}

$\downarrow$

microscopic physics can affect fluctuations over this scale.



Q. What produces these fluctuations during inflation?

A. Quantum fluctuations of fields at the time of inflation.

Units: So far we used $\hbar = c = 1$.

Mostly displayed G explicitly.

From now on we'll set $G = \frac{1}{8\pi}$.

→ Reduced Planck units.

Reduced Planck mass $= \frac{1}{\sqrt{8\pi}}$ Planck mass.

" " length $= \sqrt{8\pi}$ " length

" " time $= \sqrt{8\pi}$ " time.

Defns. Suppose $X_i(\vec{x},t)$ for $i = 1, 2, \ldots$ are some local operators

$\rightsquigarrow \phi(\vec{x},t),$ metric, $\ldots$, $T_{\mu\nu}(\vec{x},t), \ldots$

$$\overline{X}_i(t) = \langle X_i(\vec{x},t) \rangle \rightarrow \text{quantum average}$$

$$\tilde{X}_i(\vec{x},t) = X_i(\vec{x},t) - \overline{X}_i(\vec{x},t)$$

Quantity of interest $\langle \tilde{X}_i(\vec{x},t)\, \tilde{X}_j(\vec{y},t) \rangle = f_{ij}(\vec{x}-\vec{y},t)$

quantum average

In actual experiment we average over space:

Fourier transform:

$$\hat{X}_i(\vec{k},t) = \int d^3x \; e^{-i\vec{k}\cdot\vec{x}} \; \tilde{X}_i(\vec{x},t)$$

$$\tilde{X}_i(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \; e^{i\vec{k}\cdot\vec{x}} \; \hat{X}_i(\vec{k},t).$$

$$\langle \hat{X}_i(\vec{k},t) \; \hat{X}_j(\vec{k}',t) \rangle$$

$$= \int d^3x \; d^3y \; e^{-i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{y}} \; f_{ij}(\vec{x}-\vec{y},t)$$

Define $\vec{x}-\vec{y} = \vec{z}$

$$= \int d^3y \; d^3z \; e^{-i\vec{y}\cdot(\vec{k}+\vec{k}') - i\vec{k}\cdot\vec{z}} \; f_{ij}(\vec{z},t)$$

$$= (2\pi)^3 \, \delta^{(3)}(\vec{k}+\vec{k}') \; \hat{f}_{ij}(\vec{k},t)$$

$$ds^2 = -dt^2 + \lambda(t)^2 \underbrace{(dr^2 + r^2 d\Omega^2)}$$

Comoving distance $\quad d\vec{x}^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

$\to$ distance measured in $\vec{x}$.

$\vec{k}$: conjugate variable to $\vec{x}$

Small $\vec{k}$ $\sim$ large distance.

Large $\vec{k}$ $\sim$ small distance.

Previous argument $\to$ fluctuations at scale $|\vec{k}|^{-1}$ should not be affected when $|\vec{k}|^{-1} \gg \lambda^{-1} H^{-1}$.

Massless scalar in de Sitter, $V(\phi) = V_0$.

$$\left(\frac{\dot{\lambda}}{\lambda}\right)^2 = \frac{8\pi G}{3} V_0 = \frac{V_0}{3}, \quad H = \frac{\dot{\lambda}}{\lambda} = \sqrt{\frac{V_0}{3}}$$

Solution $\lambda(t) = c\, e^{Ht}$.

$$ds^2 = -dt^2 + \lambda(t)^2 d\vec{x}^2$$

Define $\tau = f(t)$ such that $\frac{dt}{\lambda(t)} = d\tau$.

$$ds^2 = \lambda(t)^2 \left(-d\tau^2 + d\vec{x}^2\right)$$

τ : conformal time.

$$d\tau = c^{-1}\, dt\, e^{-Ht}$$

$$\Rightarrow \tau = -c^{-1} H^{-1} e^{-Ht}$$

$\tau \to -\infty$ as $t \to -\infty$.

$\to 0$ as $t \to \infty$.

$$-\infty < \tau < 0.$$

$$ds^2 = \frac{1}{H^2\tau^2}\left(-d\tau^2 + d\vec{x}^2\right)$$

Since $V(\phi) = V_0$, $\langle \phi(\vec{x}, t) \rangle = \bar{\phi}$ can take any value.

$$\tilde{\phi} = \phi - \bar{\phi}$$

$$S = -\frac{1}{2}\int d^4x \sqrt{-\det g}\; g^{\mu\nu}\,\partial_\mu\tilde{\phi}\,\partial_\nu\tilde{\phi}$$

$$= \frac{1}{2}\int d\tau\, d^3x\, \frac{1}{H^4\tau^4} H^2\tau^2\left[(\partial_\tau\tilde{\phi})^2 - (\vec{\nabla}\tilde{\phi})^2\right]$$

$$L = \frac{1}{2}\int d^3x\, H^{-2}\tau^{-2}\left[(\partial_\tau\tilde{\phi})^2 - (\vec{\nabla}\tilde{\phi})^2\right]$$

$$L = \frac{1}{2}\int d^3x \; H^{-2}\tau^{-2}\left[(\partial_\tau \tilde{\phi})^2 - (\vec{\nabla}\tilde{\phi})^2 \right]$$

$\Rightarrow$ $H^2 \tau^2$ is playing the role of $\hbar$.

As $\tau \to 0$, the theory should become classical...

$$\tau = \lambda^{-1} H^{-1}$$

$$\tilde{\pi}(\vec{x},\tau) = \frac{\delta L}{\delta(\partial_\tau \tilde{\phi}(\vec{x},\tau))} = H^{-2}\tau^{-2}\,\partial_\tau \tilde{\phi}(\vec{x},\tau)$$

$$[\tilde{\phi}(\vec{x},\tau), \tilde{\pi}(\vec{y},\tau)] = i\,\delta^{(3)}(\vec{x}-\vec{y}).$$

$$[\tilde{\phi}(\vec{x},\tau), \partial_\tau \tilde{\phi}(\vec{y},\tau)] = i H^2 \tau^2 \delta^{(3)}(\vec{x}-\vec{y}) \to 0 \text{ as } \tau \to 0.$$

$$L = \frac{1}{2}\int d^3x \, \frac{1}{H^2\tau^2}\left[(\partial_\tau \tilde{\Phi})^2 - (\vec{\nabla}\tilde{\Phi})^2 \right]$$

Define $\tilde{\chi}(\vec{x},\tau) = (H\tau)^{-1}\tilde{\Phi}(\vec{x},\tau)$

$$L = \frac{1}{2}\int d^3x \left[\left(\partial_\tau \tilde{\chi} + \frac{1}{\tau}\tilde{\chi}\right)^2 - (\vec{\nabla}\tilde{\chi})^2 \right]$$

$$\partial_\tau \tilde{\Phi} = \partial_\tau (H\tau\tilde{\chi}) = H\tau\left(\partial_\tau \tilde{\chi} + \frac{1}{\tau}\tilde{\chi}\right)$$

H is τ dependent.

Approaches the usual free field Hamiltonian as $\tau \to -\infty$.

In momentum space

$$L = \frac{1}{2}\int \frac{d^3 k}{(2\pi)^3}\left[\left(\partial_\tau \hat{x}(-\vec{k},\tau) + \frac{1}{\tau}\hat{x}(-\vec{k},\tau)\right)\left(\partial_\tau \hat{x}(\vec{k},\tau) + \frac{1}{\tau}\hat{x}(\vec{k},\tau)\right) - \underbrace{\vec{k}^2}_{\omega_{\vec{k}}^2 \to \omega^2}\hat{x}(-\vec{k},\tau)\,\hat{x}(\vec{k},\tau)\right]$$

1. Find the analog of the plane wave soln. (including time dependence)
2. Explane $\hat{x}(\vec{k},\tau)$ in this basis.
3. Classical eq → time independent coefficients.
4. Make them into quantum operators and find commutation relns..

Eq. of motion for $\hat{x}(\vec{k},\tau)$

Ex. $\left(-\partial_\tau+\frac{1}{\tau}\right)\left\{\left(\partial_\tau+\frac{1}{\tau}\right)\hat{x}(\vec{k},\tau)\right\}-\omega^2\hat{x}(\vec{k},\tau)=0$

Ex. Check that the solns are:

$$v(\vec{k},\tau)=\frac{1}{\sqrt{2\omega}}(2\pi)^{3/2}e^{-i\omega\tau}\left(1+\frac{1}{i\omega\tau}\right)$$

$v(\vec{k},\tau)^*$ is the other soln.

Note: $v(-\vec{k},\tau)=v(\vec{k},\tau)$.

Ex. $v(\vec{k},\tau)\,\partial_\tau v(\vec{k},\tau)^*-\left(\partial_\tau v(\vec{k},\tau)\right)v(\vec{k},\tau)^*$
$=(2\pi)^3 i$

$$\hat{\chi}(\vec{k},\tau) = a(\vec{k})\, v(\vec{k},\tau) + a(-\vec{k})^{*}\, v(\vec{k},\tau)^{*}$$

$$\tilde{\chi}(\vec{x},\tau)^{*} = \tilde{\chi}(\vec{x},\tau) \Rightarrow \hat{\chi}(\vec{k},\tau)^{*} = \hat{\chi}(-\vec{k},\tau).$$

Ex. check that first line satisfies.

Quantum theory:

$$\hat{\chi}(\vec{k},\tau) = a(\vec{k})\, v(\vec{k},\tau) + a(-\vec{k})^{\dagger}\, v(\vec{k},\tau)^{*}$$

$$\partial_\tau \hat{\chi}(\vec{k},\tau) = a(\vec{k})\, \partial_\tau v(\vec{k},\tau) + a(-\vec{k})^{\dagger}\, \partial_\tau v(\vec{k},\tau)^{*}$$

Now use the commutation relns. involving $\tilde{\Phi}$, $\tilde{\pi} = H^{-2}\tau^{-2}\partial_\tau \tilde{\Phi}$

Ex. check that this gives:

$$[a(\vec{k},\tau), a(\vec{k}',\tau)] = 0, \quad [a(\vec{k},\tau)^\dagger, a(\vec{k},\tau)^\dagger] = 0$$

$$[a(\vec{k},\tau), a(\vec{k}',\tau)^\dagger] = \delta^{(3)}(\vec{k}-\vec{k}')$$

→ Like free field.

Choose a vacuum defined to be the lowest energy state of $\mathcal{H}$.

Here H is time dependent.

→ defin. of $|0\rangle$ time dependent.

We define the vacuum as the lowest energy state of H as $\tau \to -\infty$ (far past).

In this limit $H = \int d^3k\, |\vec{k}|\, a(\vec{k})^\dagger a(\vec{k})$

$$H|0\rangle = 0 \quad \Rightarrow \quad a(\vec{k})|0\rangle = 0 \text{ for every } \vec{k}.$$

$\downarrow$

Bunch-Davies vacuum

Other choices are possible α-vacuum.

$$\tilde{a}(\vec{k}) = p\, a(\vec{k}) + q\, a(-\vec{k})^\dagger, \quad \tilde{a}(\vec{k})^\dagger = q\, a(-\vec{k}) + p\, a(\vec{k})^\dagger$$

→ One parameter family of choices.

$$\tilde{a}(\vec{k},\alpha)\;,\;\tilde{a}(\vec{k},\alpha)^{\dagger}.$$

$|\alpha\rangle$ is defined as $\tilde{a}(\vec{k},\alpha)|\alpha\rangle = 0$ for all $\vec{k}$.

α-vacuum.

Ex. Since $\hat{\phi}$ is known in terms of $\hat{\chi}$ which is known in terms of $a(\vec{k}), a(-\vec{k})^{\dagger}$

we can calculate $\langle \hat{\phi}(\vec{k},\tau)\,\hat{\phi}(\vec{k}',\tau)\rangle$.

$$\langle 0|\hat{\phi}(\vec{k},\tau)\,\hat{\phi}(\vec{k}',\tau)|0\rangle$$

$$= (2\pi)^3\,\delta^{(3)}(\vec{k}+\vec{k}')\,\frac{1}{2\omega}H^2\tau^2\left(1+\frac{1}{\omega^2\tau^2}\right)$$

$$\langle \hat\phi(\vec k,\tau)\,\hat\phi(\vec k',\tau)\rangle = (2\pi)^3\,\delta^{(3)}(\vec k+\vec k')\,\frac{1}{2\omega}H^2\tau^2\left(1+\frac{1}{\omega^2\tau^2}\right)$$

$$\xrightarrow{\omega^2\tau^2 \ll 1} (2\pi)^3\,\delta^{(3)}(\vec k+\vec k')\,H^2/(2\omega^3)$$

classical evolution.

$$\langle \hat\phi(\vec k,\tau)\,\hat\phi(\vec k',\tau)\rangle$$

$$\hat\phi_a(\vec k,\tau)\,\hat\phi_a(\vec k',\tau)$$

$$\omega^{-2} \gg \tau^2 = \lambda^{-2}H^{-2}$$

$$|k|^{-1} \gg \lambda^{-1}H^{-1}$$

In flat space $\frac{1}{2\omega}$

We have enhancement by (H^2/ω^2) $\to$ specific to de Sitter.