de Sitter metric:

$$ds^2 = -dt^2 + \lambda(t)^2 \, d\vec{x}^2 \qquad d\vec{x}^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$$\lambda(t) = C e^{Ht}.$$

Define τ such that $d\tau = \lambda(t)^{-1} dt.$

$$\Rightarrow \quad \tau = -C^{-1} H^{-1} e^{-Ht}. \qquad -\infty < t < \infty$$

$$\Rightarrow \quad -\infty < \tau < 0.$$

$$ds^2 = H^{-2} \tau^{-2} \left(-d\tau^2 + d\vec{x}^2\right).$$

$$\tilde{\phi} = \phi - \bar{\phi}$$

$\hookrightarrow$ average value.

$$L = \frac{1}{2} \int d^3x \; H^{-2} \tau^{-2} \left\{ (\partial_\tau \tilde{\phi})^2 - (\vec{\nabla} \tilde{\phi})^2 \right\}$$

$$\tilde{\pi}(\vec{x}, \tau) = H^{-2} \tau^{-2} \partial_\tau \tilde{\phi}(\vec{x}, \tau).$$

$$\left[\tilde\phi(\vec x,\tau), \tilde\pi(\vec y,\tau)\right] = i\,\delta^{(3)}(\vec x-\vec y)$$

$$\Rightarrow \left[\tilde\phi(\vec x,\tau), \partial_\tau\tilde\phi(\vec y,\tau)\right] = i H^2\tau^2\,\delta^{(3)}(\vec x-\vec y)$$

$$\tau = -C^{-1}H^{-1}e^{-Ht}. \qquad d\tau = -H\tau\, dt$$

$$\partial_t\tilde\phi = -H\tau\,\partial_\tau\tilde\phi$$

$$\left[\tilde\phi(\vec x,t), \partial_t\tilde\phi(\vec y,t)\right] = -iH^3\tau^3\,\delta^{(3)}(\vec x-\vec y)$$

$$= -i\,\delta^{(3)}(H^{-1}\tau^{-1}(\vec x-\vec y)) \rightarrow 0 \text{ as } \tau\to\infty$$

$\left[\tilde\phi(\vec x,t), \partial_t\tilde\phi(\vec y,t)\right] \rightarrow 0$ as $t\to\infty$. $\Rightarrow \tilde\phi$ behaves as classical field.

$$\langle \tilde{\phi}(\vec{x},t)\ \tilde{\phi}(\vec{y},t)\rangle \xrightarrow[\text{Large } t]{} \phi_a(\vec{x},t)\ \phi_a(\vec{y},t)$$

$$\hat{\phi}(\vec{k},\tau) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\, \tilde{\phi}(\vec{x},\tau)$$

$$\Downarrow$$

Initial condition given by quantum result

Recall explicit result in momentum space:

$$\langle \hat{\phi}(\vec{k},\tau)\ \hat{\phi}(\vec{k}',\tau)\rangle = \frac{H^2\tau^2}{2\omega}\left(1+\frac{1}{\omega^2\tau^2}\right)(2\pi)^3\,\delta^{(3)}(\vec{k}+\vec{k}')$$

$$\xrightarrow[\omega\tau \ll 1]{} \frac{H^2}{2\omega^3}(2\pi)^3\,\delta^{(3)}(\vec{k}+\vec{k}')$$

$\omega^{-1} \gg \tau^{-1} = \lambda^{-1} H^{-1}$

In Minkowski space: $\frac{1}{2\omega}$

Classical eq. for $\hat{\phi}$:

$$S = \frac{1}{2}\int d\tau \int \frac{d^3k}{(2\pi)^3}\; H^{-2}\tau^{-2}\Big[\partial_\tau\hat{\phi}(-\vec{k},\tau)\,\partial_\tau\hat{\phi}(\vec{k},\tau) - \underbrace{\vec{k}^2}_{=\omega^2}\,\hat{\phi}(-\vec{k},\tau)\,\hat{\phi}(\vec{k},\tau)\Big]$$

$\Downarrow$ eq. of motion:

$$\partial_\tau\left(H^{-2}\tau^{-2}\partial_\tau\hat{\phi}\right) + \omega^2\hat{\phi}\,H^{-2}\tau^{-2} = 0$$

$\times H^2\tau^2$

$$\Rightarrow \partial_\tau^2\hat{\phi} + \omega^2\hat{\phi} - \frac{2}{\tau}\partial_\tau\hat{\phi} = 0$$

$$\Rightarrow \partial_\tau\hat{\phi} = \frac{\tau}{2}\left(\partial_\tau^2\hat{\phi} + \omega^2\hat{\phi}\right) \longrightarrow 0 \text{ as } \tau\to 0$$

We now repeat the analysis for metric fluctuation.

Average metric: $d\bar{s}^2 = -dt^2 + \lambda(t)^2 \, d\vec{x}^2$

$\lambda(t) = c e^{Ht}$.

Fluctuating metric:

$$ds^2 = -(1+2\Phi)\, dt^2 + 2\lambda(t)\, B_i\, dx^i\, dt$$
$$+ \lambda(t)^2 (\delta_{ij} + h_{ij})\, dx^i\, dx^j$$

Φ, h_{ij}, B_i depend on $\vec{x}, t$.

The background metric $d\bar{s}^2$ is rotationally invariant $x^i \to R_{ij} x^j \quad RR^T = I$

$\Rightarrow$ convenient to orginize the perturbations in representations of $SO(3)$.

$h_{ij} = \frac{3\times 4}{2} = 6$

$\psi, E : 1+1$

$F_i : 2, \; \hat{h}^T_{ij} = 6-3-1 = 2$

$$B_i = \partial_i B - S_i , \qquad \partial_i S_i = 0$$

Momentum space: $\hat{B}_i = ik_i \hat{B} - \hat{S}_i , \quad k_i \hat{S}_i = 0$

Similarly:

$$h_{ij} = -2\psi\delta_{ij} + 2\partial_i\partial_j E + (\partial_i F_j + \partial_j F_i) + h^T_{ij}$$

$\partial_i F_i = 0 \Rightarrow k_i \hat{F}_i = 0, \quad \partial_i h^T_{ij} = 0, \sum_{i=1}^{3} h^T_{ii} = 0 \Rightarrow k_i \hat{h}_{ij} = 0$

$$\sum_i \hat{h}_{ii} = 0.$$

Substitute into the Einstein action and expand up to quadratic order in perturbation.

$$\frac{1}{16\pi G}\int \sqrt{-\det g}\, R\, d^4x$$

Linear terms will vanish.

Quadratic terms are non-trivial.

In momentum space:

$$S = \int dt \int \frac{d^3k}{(2\pi)^3} \left[\cdots\cdots \right]$$

cannot contain $\times$ terms for different $SO(3)$ representations since indices cannot contract.

$\hat{\Phi}$ $\quad \hat{S}_i k_i = 0$ $\quad \hat{h}^T_{ij}$ $\quad \hat{S}_k$ $\quad \times$

We'll first analyze the action for $\hat{h}^T_{ij}$.

$\hat{h}^T_{ij}$ has 2 degrees of freedom.

Choose a basis of polarizations $e^{(s)}_{ij}, s=1,2$

$$k_i e^{(s)}_{ij}(\vec{k}) = 0, \quad \sum_{i=1}^{3} e^{(s)}_{ii} = 0 \quad \text{for } s = 1, 2.$$

Orthonormality condition:

$$\sum_{i,j} e^{(s)}_{ij} e^{(s')}_{ij} = \delta_{ss'}$$

degrees of freedom.

$$\hat{h}^T_{ij}(\vec{k},\tau) = \sum_{s=1}^{2} e^{(s)}_{ij}(\vec{k}) \, \hat{h}_s(\vec{k},\tau)$$

Ex. Check that the Einstein action for tensor perturbation, reduces to ($8\pi G = 1$):

$$S_{tensor} = \frac{1}{8} \int d\tau \int \frac{d^3k}{(2\pi)^3} \, H^{-2} \tau^{-2} \times \sum_{s=1}^{2} \left[\partial_\tau \hat{h}_s(-\vec{k},\tau) \, \partial_\tau \hat{h}_s(\vec{k},\tau) - \omega^2 \hat{h}_s(-\vec{k},\tau) \hat{h}_s(\vec{k},\tau) \right]$$

$$\omega^2 = \vec{k}^2$$

+ cubic & higher order terms.

$\frac{1}{2}\hat{h}_s$ behaves as a scalar $\hat{\phi}$.

$$\frac{1}{4} \langle \hat{h}_s(\vec{k},\tau) \, \hat{h}_{s'}(\vec{k}',\tau) \rangle = \delta_{ss'} (2\pi)^3 \, \delta^{(3)}(\vec{k}+\vec{k}') \, \frac{H^2 \tau^2}{2\omega} \left(1 + \frac{1}{\omega^2 \tau^2}\right)$$

Note:- We have used Burch-Davies vacuum for $\hat{h}_s$.

$\tau \to 0 \quad \langle \hat{h}_s(\vec{k},\tau)\, \hat{h}_{s'}(\vec{k}',\tau) \rangle$

$$= (2\pi)^3\, \delta^{(3)}(\vec{k}+\vec{k}')\, \delta_{ss'}\, \frac{4H^2}{2\omega^3}$$

→ tensor perturbation.

→ produces particular type of fluctuations in CMB polarization. (Discussed later)

→ not observed yet.

For scalar and vector fluctuations we need to take into account the effect of gauge transformation.

$$d\bar{s}^2 = -dt^2 + a(t)^2 d\vec{x}^2.$$

Change coordinates:

$$t \to t - \xi^0(\vec{x}, t), \quad x^i \to x^i - \xi^i(\vec{x}, t)$$

$$\xi^i = \partial_i \xi + \zeta_i, \quad \partial_i \zeta_i \equiv 0$$

$d\bar{s}^2 \to d\bar{s}^2 + (\cdots) \Rightarrow$ Produces fake changes.

$\Rightarrow$ Some of the deformations encoded in $\Phi, \Psi, E_i, F_i, S_{ij}, B, \ldots$ etc. are just gauge transformation (change of coordinates)

This does not affect h^T_{ij} since the gauge trs. parameters are scalars, or vectors.

Result: There are no dynamical scalar and vector modes in the metric.

Example: Vector perturbation:

$t \to t, \quad x^i \to x^i - \xi^i(\vec{x}, t), \quad \partial_i \xi^i = 0$

(Lower & upper i indices are the same)

$$d\bar{s}^2 \to -dt^2 + \lambda(t)^2 \delta_{ij}\left(dx^i - \partial_k \xi^i \, dx^k - \partial_t \xi^i dt\right)\left(dx^j - \partial_l \xi^j \, dx^l - \partial_t \xi^j dt\right)$$

$$\overset{Ex.}{=} d\bar{s}^2 - 2\lambda(t)^2 \partial_t \xi^i \, dt \, dx^i - \lambda(t)^2 (\partial_i \xi_j + \partial_j \xi_i) dx^i dx^j + O(\xi^2)$$

Compare with:

$$ds^2 = d\bar{s}^2 - 2\lambda(t) S_i \, dx^i dt + \lambda(t)^2 (\partial_i F_j + \partial_j F_i) \, dx^i dx^j$$

$$\Rightarrow S_i \to S_i + \lambda \partial_t \xi^i, \quad F_i \to F_i - \xi_i$$

$S_i \to S_i + \lambda \partial_t \zeta_i , \quad F_i \to F_i - \zeta_i$

$\Rightarrow \left(\frac{S_i}{\lambda} + \partial_t F_i \right)$ is gauge invariant.

Ex. Check that Einstein's action for vector perturbation depends only on $\left(\frac{S_i}{\lambda} + \partial_t F_i \right)$

Ex. Linearized Einstein's eq.

$\Rightarrow \vec{\nabla}^2 \left(\frac{S_i}{\lambda} + \partial_t F_i \right) = 0 \Rightarrow \vec{k}^2 \left(\frac{\hat{S}_i}{\lambda} + \partial_t \hat{F}_i \right) = 0$

$= M_i$

For $\vec{k} \neq 0$, $\frac{\hat{S}_i}{\lambda} + \partial_t \hat{F}_i = 0$ | $\int \hat{M}_i \vec{\nabla}^2 \hat{M}_i$