Scalar part of the metric fluctuation

$$ds^2 = -(1+2\Phi)\,dt^2 + 2\lambda(t)\,\partial_i B\, dx^i\, dt$$
$$+ \lambda(t)^2\left(\delta_{ij} - 2\Psi\,\delta_{ij} + 2\partial_i\partial_j E\right) dx^i dx^j$$

$$t \to t - \xi^0(\vec{x},t), \qquad x^i \to x^i - \partial_i \xi(\vec{x},t)$$

$$d\bar{s}^2 \to -\left(dt - \partial_t\xi^0\, dt - \partial_i\xi^0\, dx^i\right)^2 +$$
$$\left(\lambda^2 + 2\lambda\dot{\lambda}(-\xi^0)\right)\left(dx^i - \partial_k\partial_i\xi\, dx^k - \partial_t\partial_i\xi\, dt\right)\left(dx^i - \partial_l\partial_i\xi\, dx^l - \partial_t\partial_i\xi\, dt\right)$$
$$= d\bar{s}^2 + 2\partial_t\cancel{\xi^0}\, dt^2 + 2\,dt\,\cancel{dx^i}\,\partial_i\xi^0 - 2\lambda(t)^2\,\cancel{\partial_k}\partial_i\xi\, dx^i dx^k$$
$$- 2\lambda(t)^2\,\cancel{\partial_t}\partial_i\xi\, dt\, dx^i - 2\lambda\dot{\lambda}\,\xi^0\, dx^i dx^i$$

$$\delta\Phi = -\partial_t\xi^0, \quad \delta B = \lambda^{-1}\xi^0 - \lambda\,\partial_t\xi, \quad \delta E = -\xi, \quad \delta\Psi = \frac{\dot{\lambda}}{\lambda}\xi^0$$

$\delta\Phi = -\partial_t \xi^0$, $\delta B = \bar{\lambda}^{-1}\xi^0 - \bar{\lambda}\partial_t \xi$, $\delta\Psi = H\xi^0$, $\delta E = -\xi$

$\dot{\bar{\lambda}}/\bar{\lambda}$

One possible gauge choice:

$E = 0$, $B \equiv 0$ $\qquad \bar{\lambda} = c\, e^{2Ht}$

Gauge invariant combinations:

$$\Phi_B = \Phi - \frac{d}{dt}\left(\bar{\lambda}^2 \dot{E} - \bar{\lambda} B\right) \qquad \dot{E} = \frac{d}{dt}E$$

$$\Psi_B = \Psi + \bar{\lambda}^2 H\left(\dot{E} - \frac{B}{\bar{\lambda}}\right)$$

$$\dot{E} - \frac{B}{\bar{\lambda}} \to \dot{E} - \frac{B}{\bar{\lambda}} - \partial_t \xi - \left(\frac{\xi_0}{\bar{\lambda}^2} - \partial_t \xi\right) = \dot{E} - \frac{B}{\bar{\lambda}} - \frac{\xi_0}{\bar{\lambda}^2}$$

Study Linearized Einstein's eq:

Ex. In momentum space:

$$k_i \left\{ \frac{d}{dt} \hat{\Psi}_B + \mathcal{H} \hat{\Phi}_B \right\} = 0$$

$$3H \left(\frac{d\hat{\Psi}_B}{dt} + H \hat{\Phi}_B \right) + \frac{\vec{k}^2}{\lambda^2} \hat{\Psi}_B = 0$$

$\Rightarrow$ For $\vec{k} \neq 0$, $\frac{d}{dt} \hat{\Psi}_B + H \hat{\Phi}_B = 0$

$\Rightarrow \quad \hat{\Psi}_B = 0 \longrightarrow \hat{\Phi}_B = 0.$

No dynamics in the scalar mode.

Now suppose that we also have a scalar field ϕ with constant $V(\phi)$.

Ex. In the action there will be no mixing between modes of $\tilde{\phi} = \phi - \bar{\phi}$ and the metric to quadratic order.

$$\langle \hat{\phi}(\vec{k},\tau)\, \hat{\phi}(\vec{k}',\tau) \rangle = (2\pi)^3\, \delta^{(3)}(\vec{k}+\vec{k}')\, \frac{H^2}{2\omega^3}.$$

Now consider the case of slow roll inflation.

$\bar{\phi}(t)$ $\hookrightarrow$ fn. of t.

$\dot{\bar{\phi}} \neq 0$

$$S_\phi = \frac{1}{2}\int d^4x \sqrt{-\det g}\,\left(-g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right)$$

$$\phi = \bar{\phi} + \tilde{\phi}$$

$$-2\underbrace{g^{0\nu}}_{b}\,\underbrace{\partial_t\bar{\phi}}_{}\,\partial_\nu\tilde{\phi}$$

Linear in fluctuation

fixed by the backgnd.

$\Rightarrow \tilde{\phi}$ begins to mix with Ψ, Φ, E, B.

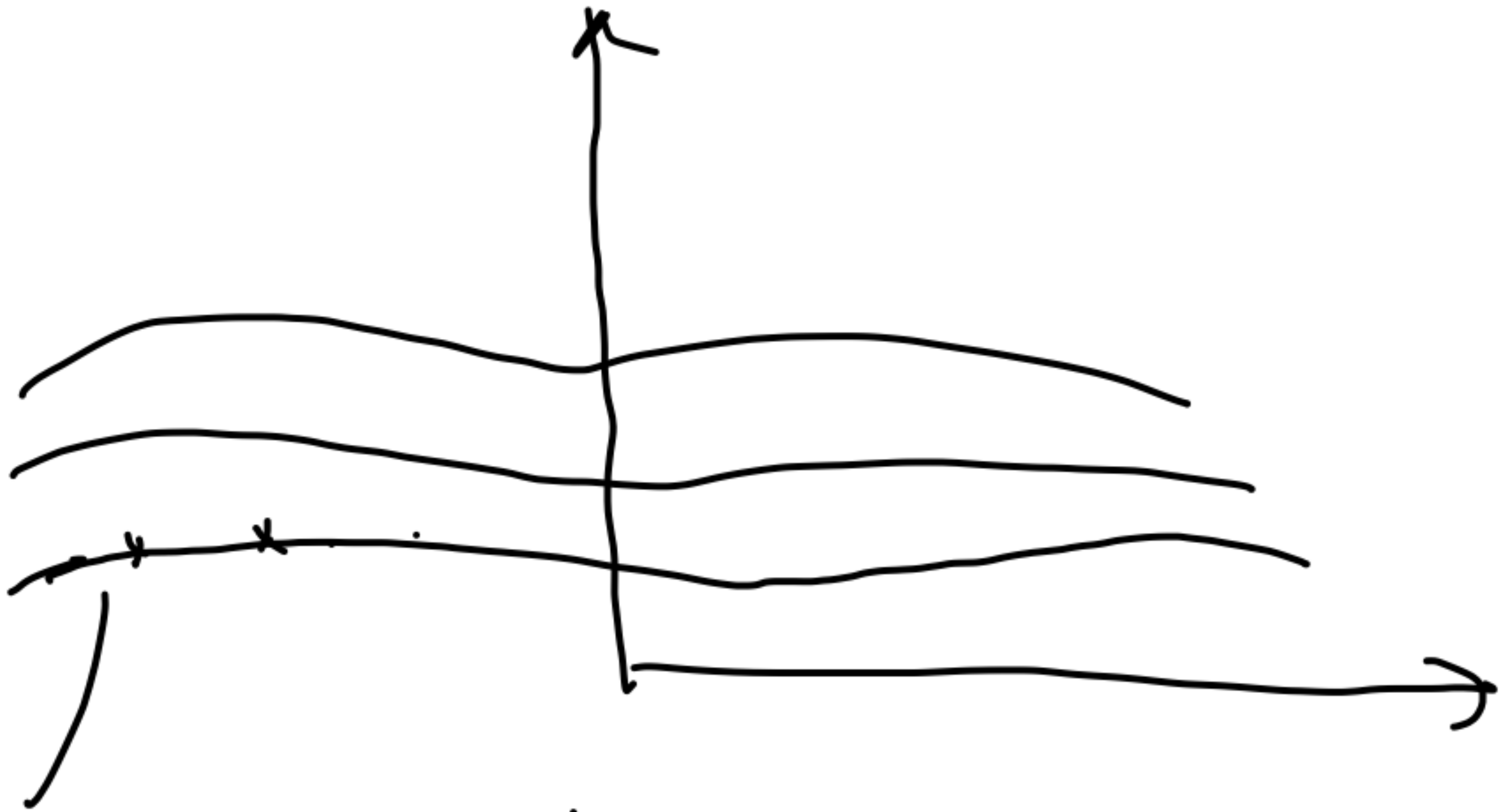
Gauge trs. law of $\tilde{\phi}$:

$$\bar{\phi}(t) \longrightarrow \bar{\phi}(t-\xi^0) = \bar{\phi}(t) - \xi^0 \partial_t \bar{\phi}.$$

$$\phi = \bar{\phi} + \tilde{\phi} \qquad \delta\tilde{\phi} = -\xi^0 \partial_t \bar{\phi}$$

By choosing ξ^0 appropriately, we can set $\tilde{\phi}$ to 0.

$\rightarrow$ scalar fluctuations can be set to 0.



In the absence of fluctuation we choose t s.t.

$$ds^2 = -dt^2 + \lambda(t)^2 d\vec{x}^2$$

ϕ = constant on a constant time slice.

$$\delta\tilde{\phi} = -\partial_t \bar{\phi}\, \xi^0 , \quad \delta\Psi = +H \xi^0$$

$$\zeta = -\left(\Psi + \frac{H}{\dot{\phi}} \tilde{\phi}\right) \text{ is gauge invariant.}$$

$$\mathcal{R} = -\zeta = \left(\Psi + \frac{H}{\dot{\phi}} \tilde{\phi}\right).$$

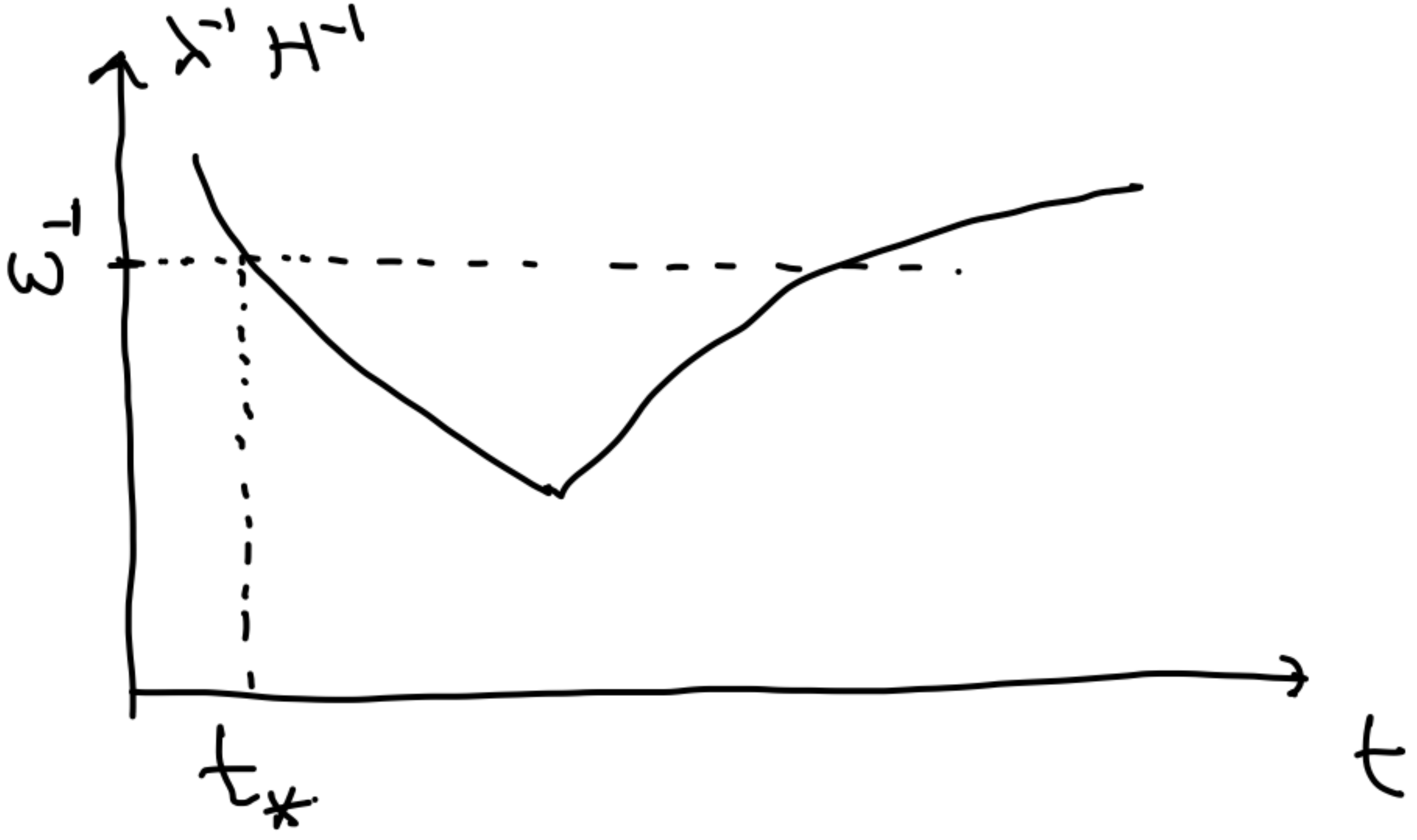
Ex. In the combined action:

If we use $E=0$, $B=0$ gauge, then Einstein's eq. still set $\Psi=0$, $\Phi=0$

The dynamics of $\tilde{\phi}$ remains the same as before.

$$\langle \hat{\phi}(\vec{k},\tau)\, \hat{\phi}(\vec{k}',\tau) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}+\vec{k}') \frac{H^2}{2\omega^3}$$

$$\Rightarrow \boxed{\langle \hat{R}(\vec{k},\tau)\, \hat{R}(\vec{k}',\tau) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}+\vec{k}') \frac{H^2}{2\omega^3} \frac{H^2}{\dot{\phi}^2}}$$



$$\left(1 + \frac{1}{\omega^2\tau^2}\right)$$

$\downarrow$

dominates when

$\omega\tau \ll 1$

$\omega^{-1} \gg \tau = \lambda^{-1}H^{-1}$

$\dot{\bar{\phi}}$ and H change little | H $\quad c\,e^{Ht}$

during time H^{-1}.

We need to calculate $\dot{\bar{\phi}}$ and H at t_*.

Some definitions

$$\langle \hat{R}(\vec{k},\tau)\,\hat{R}(\vec{k}',\tau)\rangle = (2\pi)^3\,\delta^{(3)}(\vec{k}+\vec{k}')\,\frac{H^4}{2\dot{\phi}^2\,\omega^3}$$

P_R : scalar power spectrum

$$\langle \hat{h}_s(\vec{k},\tau)\,\hat{h}_{s'}(\vec{k}',\tau)\rangle = (2\pi)^3\,\delta^{(3)}(\vec{k}+\vec{k}')\,\frac{2H^2}{\omega^3}\,\delta_{ss'}$$

$$\sum_{s=1}^{2}\langle \hat{h}_s(\vec{k},\tau)\,\hat{h}_s(\vec{k}',\tau)\rangle = (2\pi)^3\,\delta^{(3)}(\vec{k}+\vec{k}')\,\frac{4H^2}{\omega^3}$$

P_T : tensor power spectrum

Some more definitions

$$\Delta_s^2 = \frac{\omega^3}{2\pi^2} \rho_R = \frac{H^4}{4\pi^2 \dot{\Phi}^2}$$

$$\Delta_T^2 = \frac{\omega^3}{2\pi^2} \rho_T = \frac{2H^2}{\pi^2}$$

ω-independent Δ_s, Δ_T is known as scale invariant spectrum.

$\rightarrow$ Harrison-Zeldovich

Δ_s, Δ_T are approximately ω independent.

t_* depends on ω via $\omega^{-1} = \lambda(t_*)^{-1} H(t_*)^{-1}$

$H, \dot{\Phi}$ depends on t_* mildly.

Deviation from scale invariance (mild dependence on ω) $\to$ spectral index.

to be defined later.