$$\langle \hat{\mathcal{R}}(\vec{k},\tau)\, \hat{\mathcal{R}}(\vec{k}',\tau) \rangle = (2\pi)^3\, \delta^{(3)}(\vec{k}+\vec{k}')\, \frac{H^4}{2\dot{\phi}^2 \omega^3}$$

$$\rho_R$$

$$\sum_s \langle h_s(\vec{k},\tau)\, h_s(\vec{k}',\tau) \rangle = (2\pi)^3\, \delta^{(3)}(\vec{k}+\vec{k}')\, .\, \frac{4H^2}{\omega^3}$$

$$\rho_T$$

$$\Delta_S^2 = \frac{\omega^3}{2\pi^2}\rho_R = \frac{H^4}{4\pi^2\dot{\phi}^2}, \quad \Delta_T^2 = \frac{\omega^3}{2\pi^2}\rho_T = \frac{2H^2}{\pi^2}$$

$$r = \frac{\Delta_T^2}{\Delta_S^2} = \frac{8\dot{\phi}^2}{H^2} \to \text{tensor to scalar ratio.}$$

observation: $r < .2$, $\Delta_S \sim 10^{-5} \Rightarrow \Delta_T^2 \lesssim 2\times 10^{-11}$

$$\Rightarrow \frac{2H^2}{\pi^2} \lesssim 2\times 10^{-11}, \quad H^2 = \frac{V}{3} \Rightarrow V \lesssim \frac{3\pi^2}{2} \times 2\times 10^{-11} \sim 3\times 10^{-10}$$

If M_I is the mass scale of inflation

then $V \sim M_I^4$

$$M_I \lesssim \left(3 \times 10^{-10}\right)^{1/4} \frac{M_{Pl}}{\sqrt{8\pi}} \qquad M_{Pl} = 1.22 \times 10^{19}\ \text{GeV}.$$

$$\lesssim 10^{16}\ \text{GeV}.$$

"Theoretical upper bound" on r

$$r = \frac{8\dot{\phi}^2}{H^2} \Rightarrow \dot{\phi} = \sqrt{\frac{r}{8}}\, H$$

Assume slow roll $\Rightarrow \dot{\phi}$ and H changes slowly compared to $H \to$ more or less constant during the last 60 e-folding.

$$\dot{\phi} = \sqrt{\frac{r}{8}}\, H$$

$$\int_{t_i}^{t_f} \dot{\phi}\, dt = \sqrt{\frac{r}{8}} \int_{t_i}^{t_f} H\, dt$$

$$\lambda = c\, e^{Ht}$$

$$\frac{\lambda(t_i)}{\lambda(t_f)} = e^{H(t_f - t_i)}$$

$$\Rightarrow H(t_f - t_i) \simeq \ln \frac{\lambda(t_f)}{\lambda(t_i)}$$

$$\phi(t_f) - \phi(t_i) = \sqrt{\frac{r}{8}}\, N$$

$N \rightarrow$ # of efolding.

60.

$$N > 60 \Rightarrow \Delta\phi > \sqrt{\frac{r}{8}}\, N$$

It is natural to require that $\Delta\phi \lesssim 1$.

For generic scalar field quantum gravity corrections to correct the potential by term $\sim M_{Pl}^{-n}$ for $n > 0$.

$$V(\phi) = V_{infl}(\phi) + \sum \frac{\phi^{n+4}}{M_{Pl}^{n}}$$

When $\phi > M_{Pl}$, then the quantum gravity corrections would become Large destroying flatness of $V(\phi)$. This is the reason for demanding $\Delta\phi < M_{Pl}$.

$\Rightarrow \sqrt{\frac{r}{8}}\, N < 1$ since $\Delta\phi \simeq \sqrt{\frac{r}{8}}\, N$

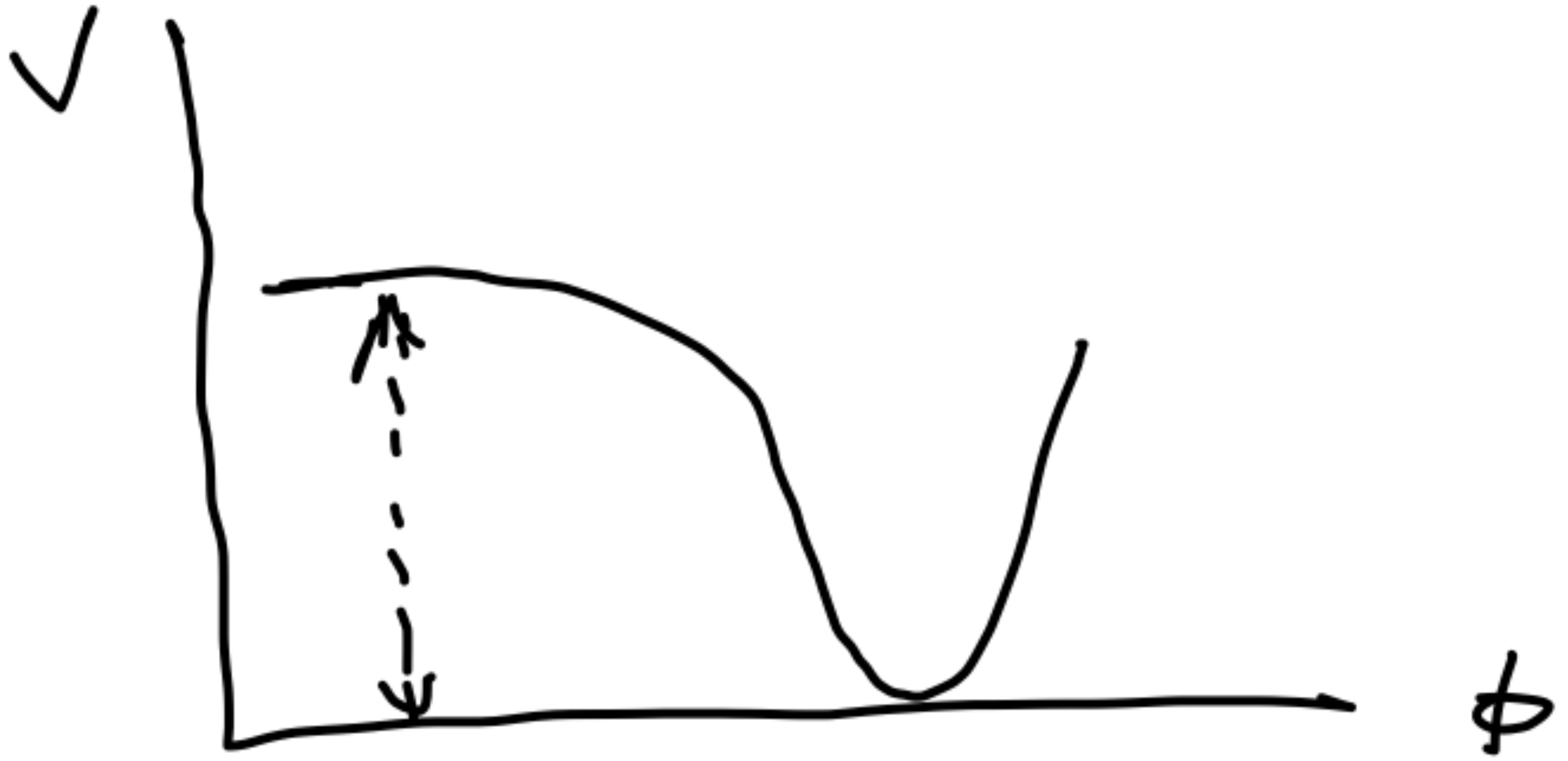
and $\Delta\phi < 1$

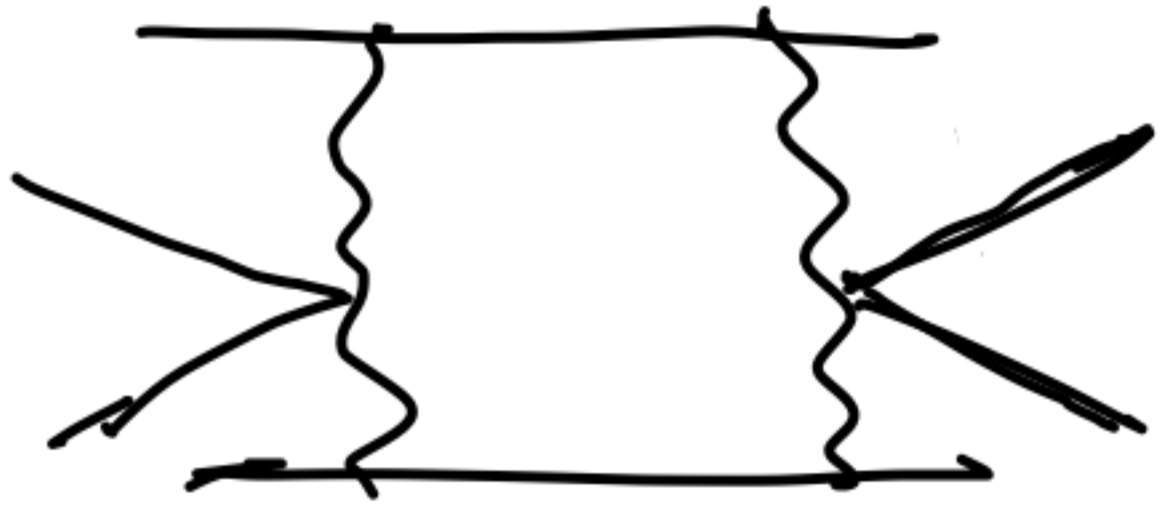
$N > 60$

$\Rightarrow r < \frac{8}{3600} \sim \frac{1}{500} \to$ make it impossible to observe r in the near future.

Lyth bound $\to$ natural but can be over come:

$\Rightarrow$ If do observe tensor modes, it will put strong constraint on the kind of potential we can use for inflation.





$$\phi^6 \frac{1}{M_{Pl}^2} + \phi^8 \frac{1}{M_{Pl}^4} + \cdots$$

$$(m_0 + \cdots) \qquad \lambda \phi^4$$

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \alpha \frac{\phi^6}{M_{Pl}^2}$$

can provide inflation over some range (ϕ_i, ϕ_f) for small enough m.

Spectral index

$$\Delta_S^2 = \frac{H^4}{4\pi^2 \dot{\Phi}^2}, \quad \Delta_T = \frac{2H^2}{\pi^2} \quad \text{at } t_*$$

Scalar spectral index

$$n_S = 1 + \frac{d \ln \Delta_S^2}{d(\ln \omega)}, \quad n_T = \frac{d \ln \Delta_T^2}{d \ln \omega}$$

convention

$$\Rightarrow \Delta_S^2 \sim \omega^{n_S - 1} \quad " \quad \Delta_T^2 \sim \omega^{n_T}$$

if n_S, n_T are ω independent.

Recall the equation that determines t_*.

$$\omega^{-1} = \lambda^{-1} H^{-1}, \quad \omega = \lambda H.$$

$$d\omega = \dot{\lambda} H\, dt + \lambda \dot{H}\, dt = \lambda H^2 dt \left(1 + \frac{\dot{H}}{H^2}\right)$$

$$\dot{\lambda} \Downarrow \lambda H \qquad \simeq \lambda H^2 dt \qquad \frac{\dot{H}}{H^2} \text{ small}$$

$$\Rightarrow \omega \frac{d}{d\omega} = \lambda H \cdot \frac{1}{\lambda H^2} \frac{d}{dt} = H^{-1} \frac{d}{dt}$$

$$\omega \frac{d}{d\omega} \ln \Delta_s^2 = H^{-1} \frac{d}{dt} \ln \left(\frac{H^4}{(\partial_t \bar{\phi})^2} \right) = H^{-1} \left(\frac{4\dot{H}}{H} - 2\frac{\ddot{\bar{\phi}}}{\dot{\bar{\phi}}} \right)$$

$$= \frac{4\dot{H}}{H^2} - 2\frac{\ddot{\bar{\phi}}}{\dot{\bar{\phi}}} = (-4\epsilon + 2\eta) \text{ at } t_*.$$

$$n_s = 1 - 4\epsilon_* + 2\eta_*$$

$$\omega \frac{d}{d\omega} \ln \Delta_T^2 = H^{-1} \frac{d}{dt} \ln H^2$$

$$= H^{-1} \frac{2\dot{H}}{H} = -2\epsilon \quad \Rightarrow \quad n_T = -2\epsilon_*$$

$\Rightarrow$ n_s, n_T gives us direct information about $V(\phi)$ at the time of horizon exit.

n_T has not been observed.

n_s is a little less than 1 $\rightarrow$ observed.

Physical interpretation of $\mathcal{R}$:

$$\mathcal{R} = \Psi + \frac{H}{\partial_t \bar{\phi}} \tilde{\phi}$$

Gauge trs: $\tilde{\phi} \to \tilde{\phi} - \xi^0 \partial_t \bar{\phi}$, $\Psi \to \Psi + H \xi^0$

$$ds^2 = -(1+2\Phi)\,dt^2 + 2\lambda\,\partial_i B\,dx^i\,dt \;? + \lambda(t)^2\,dx^i\,dx^j\,(\delta_{ij} - 2\Psi\,\delta_{ij} + 2\partial_i\partial_j E)$$

In $B = E = 0$ gauge, dynamics set $\Phi = 0$, $\Psi = 0$

$\Rightarrow$ $\mathcal{R}$ gets contribution only from $\tilde{\phi}$.

ϕ is ahead in some regim $\rightarrow$ inflation ends early $\rightarrow$ T starts falling early

behind in some other region $\rightarrow$ " " late. $\therefore$ T starts falling late.

In some region T will be smaller
In other regions T " " larger.

can be eventually translated to fluctuations in CMB temperature.

We could also choose a gauge in which $\tilde{\phi} = 0$.

$\mathcal{R} = \Psi \Rightarrow$ Fluctuation in $\mathcal{R}$ comes from Ψ only.

$$ds^2 = -dt^2 + a(t)^2 (1-2\Psi) d\vec{x}^2.$$

In this gauge ϕ is uniform.
$\Rightarrow$ Inflation ends everywhere at the same time.
$\Rightarrow$ No temperature fluctuation of the kind we discussed, but the metric fluctuates.

spatial metric: $a(t)^2 \, \underbrace{d\vec{x}^2 (1-2\Psi)}$

Ricci scalar for the spatial metric

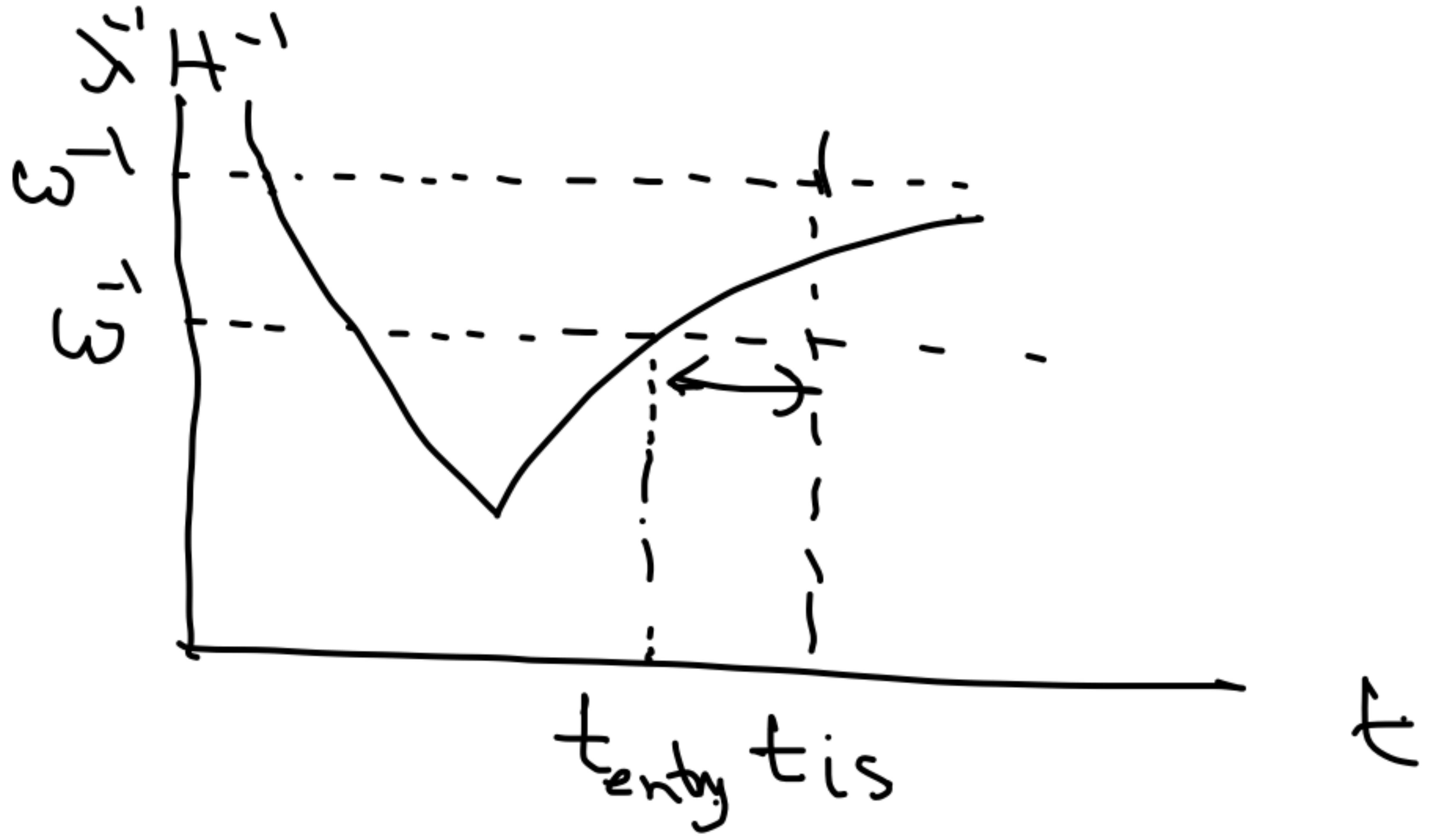
$\propto -\vec{\nabla}^2 \Psi \sim \vec{k}^2 \Psi .$

This fluctuation of the spatial metric affects what we observe as CMB temperature.

First task: Study how $\mathcal{R}$ evolves from t_* to t_{LS}.

Then study its effect on CMB temperature.

should be simpler for ω which remain outside horizon till t_{LS}.



Step 1: classical evolution.

Solve Einstein's equation including fluctuating $T_{\mu\nu} \to \bar{T}_{\mu\nu} + \tilde{T}_{\mu\nu} \to$ scalar, vector, tensor.