Scalar perturbation

6 gauge invariant variables:

$\Phi_B, \Psi_B, S, \mathcal{R}, \tilde{\Sigma}, p_e$

Perfect fluid ansatz: $\tilde{\Sigma} = 0$

Equation of state $\Rightarrow p_e = 0$.

Einstein's eq. $K_{\mu\nu} = 0$, $K_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 8\pi G T_{\mu\nu}$

Four scalars K_{00}, $K_{i0} \to 1$ scalar, $K_{ij} \to 2$ scalars

$\Rightarrow$ Four equations.

Work in momentum space, drop the $\hat{}$ on top of Fourier transformed variables

① $3H(\dot{\Psi}_B + H\Phi_B) + \frac{\vec{k}^2}{\lambda^2}\Psi_B = 4\pi G\,\frac{\partial_t\bar{\rho}}{H}(S+\Psi_B)$

② $\dot{\Psi}_B + H\Phi_B = 4\pi G\,\frac{\bar{\rho}+\bar{p}}{H}(\mathcal{R}-\Psi_B)$

③ $\Psi_B - \Phi_B = 8\pi G\lambda^2\tilde{\Sigma} = 0$

④ $\dot{S} = -H\frac{p_e}{\bar{\rho}+\bar{p}} - \Pi = -\Pi$

$$\Pi/H = -\frac{k^2}{3\lambda^2H^2}\left\{S - \Psi_B\left(1 - \frac{2\bar{\rho}}{9(\bar{\rho}+\bar{p})}\frac{\vec{k}^2}{\lambda^2H^2}\right)\right\}$$

superhorizon perturbation: $\vec{k}^2 \ll \lambda^2H^2 \Rightarrow \frac{\Pi}{H}$ small.

④ $\Rightarrow \dot{S} \simeq 0$ ③ $\Rightarrow \Psi_B = \Phi_B$, ① $- 3H\times$ ② $\Rightarrow \partial_t\bar{\rho}\cdot(S+\Psi_B) = 3H(\bar{\rho}+\bar{p})(\mathcal{R}-\Psi_B)$

$\partial_t(\bar{\rho}\lambda^3) = -3\bar{p}\lambda^2\dot{\lambda} \Rightarrow \partial_t\bar{\rho} = -3(\bar{\rho}+\bar{p})H \Rightarrow S+\Psi_B = -\mathcal{R}+\Psi_B$

$\Rightarrow S = -\mathcal{R} \Rightarrow \dot{\mathcal{R}} = 0$

$S = -\mathcal{R}, \quad \dot{S} = 0 \Rightarrow \dot{\mathcal{R}} = 0, \quad \Psi_B = \Phi_B.$

$$② \quad \dot{\Phi}_B + H\Phi_B = 4\pi G \frac{\bar{\rho} + \bar{p}}{H}(\mathcal{R} - \Phi_B)$$

Consider the period when we are inside a given era (only one component of $T_{\mu\nu}$ dominate)

$\bar{p} = \omega\bar{\rho}$, $\quad \omega = 0$ for matter, $\frac{1}{3}$ for radiation, -1 for cos. const.

$$\dot{\Phi}_B + H\Phi_B + 4\pi G H \frac{(\omega+1)\bar{\rho}}{H^2}\Phi_B = 4\pi G H \frac{\bar{\rho}(1+\omega)}{\frac{8\pi G}{3}\bar{\rho}}\mathcal{R}$$

$$H^2 = \frac{8\pi G}{3}\bar{\rho}$$

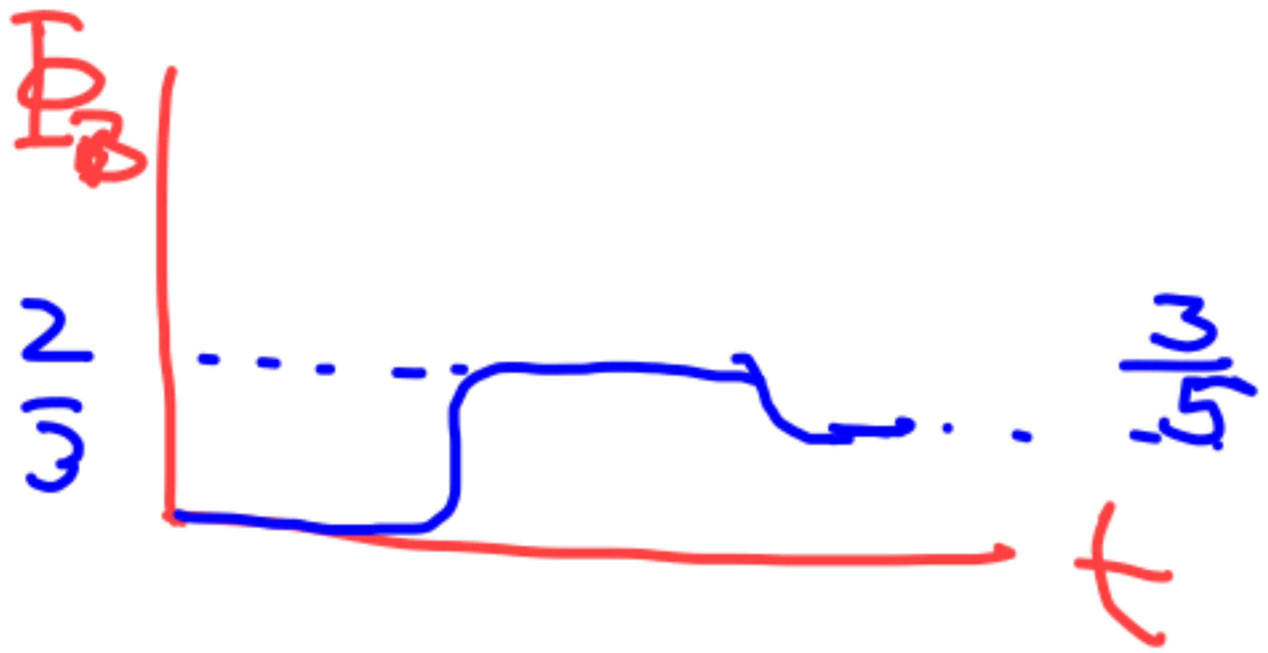
$$\Rightarrow \dot{\Phi}_B + H\Phi_B\left(1 + \frac{3(\omega+1)}{2}\right) = \frac{3(\omega+1)}{2}H\mathcal{R}$$

$$\dot{\Phi}_B + \frac{3\omega+5}{2}H\Phi_B = \frac{3(\omega+1)}{2}H\mathcal{R}$$

$$H^{-1}\dot{\Phi}_B + \frac{3w+5}{2}\Phi_B = \frac{3(1+w)}{2}\mathcal{R}$$

In matter/radiation dominated era

$$H \sim \frac{K}{t+C}, \quad K, C \text{ constants.}$$



$$\frac{(t+C)}{K}\dot{\Phi}_B + \frac{3w+5}{2}\Phi_B = \frac{3(1+w)}{2}\mathcal{R}$$

$$\Rightarrow (t+C)\dot{\Phi}_B + A\Phi_B = B\mathcal{R}, \quad A = \frac{3w+5}{2}K, \; B = \frac{3(1+w)}{2}K$$

$$\times (t+C)^{A-1} \Rightarrow \frac{d}{dt}\left((t+C)^A\Phi_B\right) = B\mathcal{R}(t+C)^{A-1}$$

$$(t+C)^A\Phi_B = \frac{B}{A}\mathcal{R}(t+C)^A + D \Rightarrow \Phi_B = \frac{B}{A}\mathcal{R} + \frac{D}{(t+C)^A} \xrightarrow[t\to\infty]{} \frac{3(1+w)}{3w+5}\mathcal{R}$$

$$\Phi_B = \frac{3(1+w)}{5+3w} \mathcal{R} \underset{\substack{t=t_{ls} \\ w=0}}{=} \frac{3}{5} \mathcal{R}$$

At $t = t_{ls}$, $\Phi_B = \Psi_B = \frac{3}{5} \mathcal{R}_I$, $\mathcal{R} = \mathcal{R}_I$, $\zeta = -R_I$

$\langle \times \quad \times \rangle \big|_{t_{ls}}$ determined from $\langle \mathcal{R}(\vec{k},t)\, \mathcal{R}(\vec{k}',t) \rangle \big|_{t=t_*}$

Calculated for a given model of slow roll inflation.

In $E = B = 0$ gauge

$\Phi_B = \Phi$

$ds^2 = -dt^2(1+2\Phi) + \cdots$

Vector perturbation:

4 vectors : 2 from metric, 2 from $T_{\mu\nu}$

$$ds^2 = -dt^2 - 2\lambda(t)\, dx^i\, dt\, S_i + dx^i dx^j(\delta_{ij} + \partial_i F_j + \partial_j F_i)$$

$$\partial_i S_i = 0, \quad \partial_i F_i = 0$$

$$T_i{}^0 = Q_i, \quad T_i{}^j = \partial_i \Sigma_j + \partial_j \Sigma_i, \quad \partial_i Q_i = 0, \quad \partial_i \Sigma_i = 0.$$

One gauge parameter : ξ^i

Gauge invariant quantities, $Q_i, \Sigma_i, \dot{F}_i + \frac{S_i}{\lambda}$

Perfect fluid ansatz: $\Sigma_i = 0$; Einstein's eq. has two vector components K_{i0}, K_{ij}

Ex. The Einstein's eqs give:

$$\dot{\Theta}_i + 3H\Theta_i = \vec{k}^2 \Sigma_i = 0$$

$$\vec{k}^2 (\dot{F}_i + S_i/\lambda) = 16\pi G\, \Theta_i$$

$$\Theta_i(t) = \Theta_i(t_*) \exp\left[-3\int_{t_*}^{t} dt'\, H(t')\right]$$

$\Rightarrow$ $\Theta_i(t) \to 0$ as $\cancel{t} \to \infty$

$\Rightarrow$ $\dot{F}_i + S_i/\lambda \to 0$

$\Bigg\}$ No vector perturbation.

Tensor perturbation

Gauge invariant variables: h^T_{ij}, Σ^T_{ij}

Perfect fluid ansatz $\Rightarrow$ $\Sigma^T_{ij} = 0$

$\Rightarrow$ Einstein's eq. is source free.

Choose a basis: $e^{(s)}_{ij}(\vec{k})$ $\quad s = 1, 2$

$$k_i e^{(s)}_{ij} = 0, \quad e^{(s)}_{ii} = 0, \quad h^T_{ij} = \sum_{s=1}^{2} h^{(s)}(\vec{k}, t)\, e^{(s)}_{ij}(\vec{k})$$

Ex- Einstein's eq $K^T_{ij} = 0$

$$\Rightarrow \partial_t^2 h^{(s)} + 3H \partial_t h^{(s)} + \frac{\vec{k}^2}{\lambda^2} h^{(s)} = 0$$

For $\vec{k}^2/\lambda^2 H^2 \ll 1$

$$\partial_t^2 h^{(s)} + 3H \partial_t h^{(s)} = 0$$

$$\Rightarrow \partial_t h^{(s)} = \partial_t h^{(s)}\Big|_{t_*} = 0$$

$$\times \exp\left[-3\int_{t_*}^{t} dt'\, H(t')\right]$$

$$\Rightarrow \partial_t h^{(s)} = 0 \Rightarrow h^{(s)}(t) = h^{(s)}(t_*)$$

$$h^{(s)}(t_{LS}) = h^{(s)}(t_*)$$

$$\langle h^{(s)}(\vec{k}, t_{LS})\, h^{(s')}(\vec{k}', t_{LS})\rangle = \langle h^{(s)}(\vec{k}, t_*)\, h^{(s')}(\vec{k}', t_*)\rangle$$

Conclusion: All fluctuations at the time of last scattering are determined from $\langle \mathcal{R}(\vec{k}, t_*)\, \mathcal{R}(\vec{k}', t_*)\rangle$ and

$\langle h^{(s)}(\vec{k}, t_*)\, h^{(s)}(\vec{k}', t_*)\rangle$.

$\rightarrow$ calculated during inflation.

Relate to observations:

Suppose $X(\vec{k},t)$ is some observable whose fluctuations we can observe.

Ex. $X = \delta T$

In general we expect:

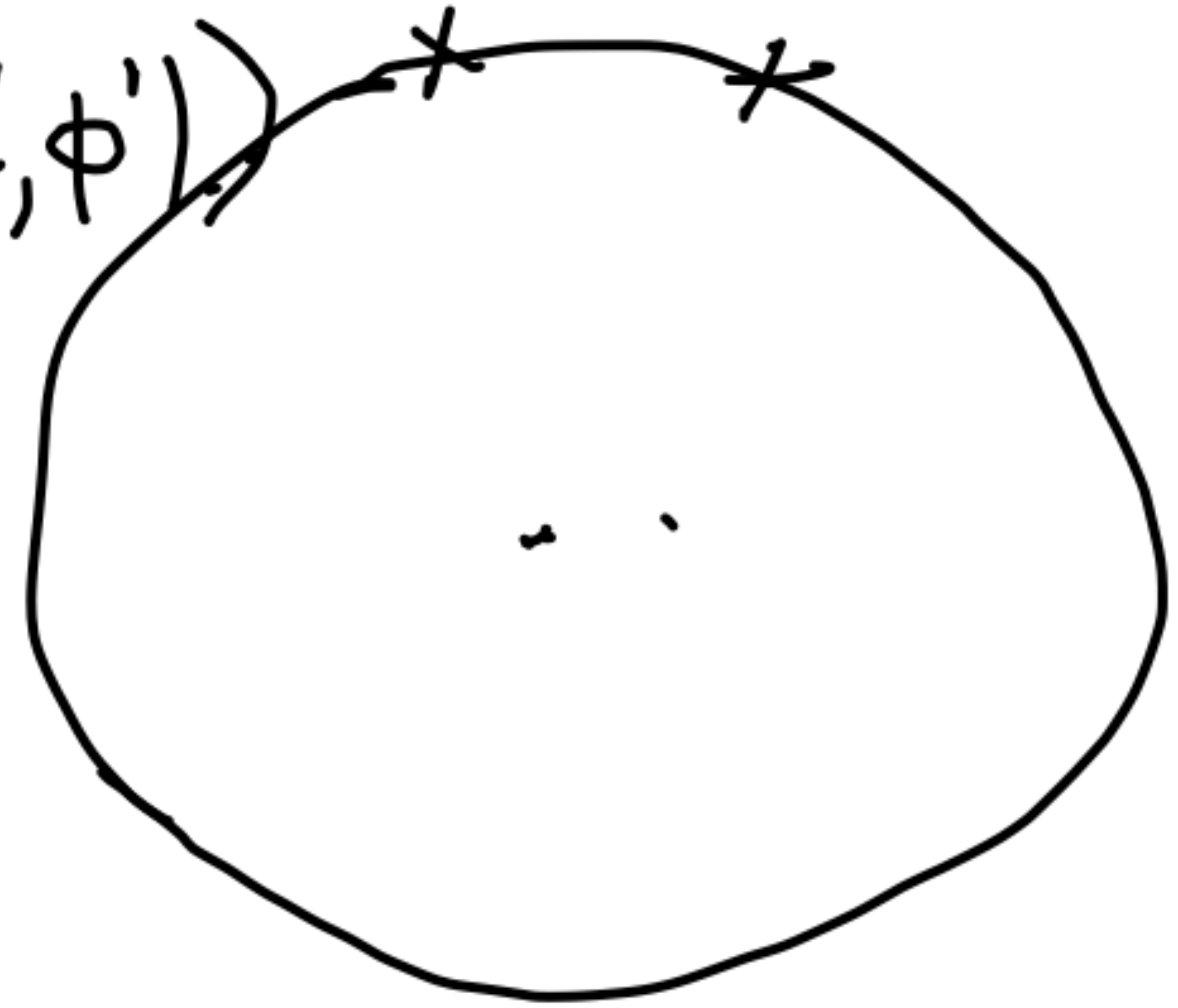
$$X(\vec{k},t)_{ls} = f(\vec{k})\, \mathcal{R}(\vec{k},t_*) \quad \text{for scalar fluctuation.}$$

$$= g(k)\, h^{(s)}(\vec{k},t_*) \quad \text{for tensor fluctuations}$$

should be simple for superhorizon scale. | can be complicated for subhorizon scales

What we observe in CMB is angular correlation, not spatial correlation.

$$\langle X(\theta,\phi)\, X(\theta',\phi')\rangle$$



Need to make an effort to convert $|\vec{k}|$ dependence of $\langle \delta_I(\vec{k},t)\, \delta_I(\vec{k}',t)\rangle$ to angular correlations.

$$\langle \delta T(\theta,\phi)\, \delta T(\theta',\phi')\rangle \to f(\lambda)$$