$\rho_m,\ \rho_\Lambda,\ \rho_r,\qquad \Omega_i = \frac{\rho_{i0}}{\rho_c}\qquad i = m, r, \Lambda$

$\rho_c = \frac{3H_0^2}{8\pi G}$ → critical density.

$$\left(\frac{\dot\lambda}{\lambda}\right)^2 + \frac{k}{a_0^2\lambda^2} = \frac{8\pi G}{3}\left(\frac{\rho_{m0}}{\lambda^3} + \frac{\rho_{r0}}{\lambda^4} + \rho_{\Lambda 0}\right) = H_0^2\left(\frac{\Omega_m}{\lambda^3} + \frac{\Omega_r}{\lambda^4} + \Omega_\Lambda\right)$$

$\Omega_r = 5\times 10^{-5}$, $\Omega_m = .3 \to .05 + .25$

ordinary matter   dark matter.

Determination of Ω_Λ:

$$\frac{d}{dt}\left(\lambda^2 \times \text{Friedmann eq}\right) \Rightarrow 2\ddot\lambda\cancel{\dot\lambda} = H_0^2\left(-\frac{\Omega_m}{\lambda^2} - \frac{2\Omega_r}{\lambda^3} + 2\Omega_\Lambda\lambda\right)\cancel{\dot\lambda}$$

Set $\lambda = 1$ $(t = t_0)$ $\Rightarrow \ddot\lambda = -q_0 H_0^2$

$$\Rightarrow -2q_0\cancel{H_0^2} = \left(-\Omega_m - 2\Omega_r + 2\Omega_\Lambda\right)\cancel{H_0^2}$$

Result: $\Omega_\Lambda \simeq .7$

$$\Omega_r + \Omega_m + \Omega_\Lambda \simeq 1$$

$$\left(\frac{\dot{\lambda}}{\lambda}\right)^2 + \cancel{\frac{k}{a_0^2 \lambda^2}} = H_0^2 \left(\frac{\Omega_m}{\lambda^3} + \frac{\Omega_r}{\lambda^4} + \Omega_\Lambda \right) \qquad a_0 = a(t_0)$$

$\frac{k}{a_0^2}$ is indistinguishable from 0.

In the past, $\frac{\Omega_m}{\lambda^3}$ dominates over $\frac{k}{a_0^2}$.

In the future: Ω_Λ dominates over $\frac{k}{a_0^2}$

$$H_0 \simeq 70\ \mathrm{Km/sec/Mpc}$$

$$= 3.086\times 10^{13}\ \mathrm{Km} \times 10^{6}$$

$$= \frac{70}{3.086\times 10^{13}\times 10^{6}\ \mathrm{sec}} = \frac{70\times 60\times 60\times 24\times 365}{3\times 10^{19}\ \mathrm{years}}$$

$$\left.\frac{\dot{\lambda}}{\lambda}\right|_{t=t_0} \sim \frac{\cancel{3}\times 10^{5}\times 10^{4}}{\cancel{3}\times 10^{19}\ \mathrm{years}} \sim \frac{1}{10^{10}\ \mathrm{years}}$$

$$\rho_c = \frac{3H_0^2}{8\pi G} = \cancel{3}\times\left(\frac{70}{3\times 10^{19}\ \mathrm{sec}}\right)^2 \times \frac{1}{8\cancel{\pi}}$$

$G = 1$

$$= \frac{1}{10}\times\left(\frac{70}{3\times 10^{19}}\times 5.4\times 10^{-44}\right)^2 \times \frac{m_{pl}}{l_{pl}^3}$$

$m_{pl} = 1.22\times 10^{19}\ \mathrm{Gev}/c^2$

$l_{pl} = 1.62\times 10^{-33}\ \mathrm{cm}$

$$\sim 10^{6}\ \mathrm{Gev/cm^3} \sim 1\ \mathrm{Gev/m^3}$$

$\rho_m = \rho_c \frac{\Omega_m}{\lambda^3}$, $\rho_r = \rho_c \frac{\Omega_r}{\lambda^4}$, $\rho_\Lambda = \rho_c \Omega_\Lambda$

Today $\lambda \sim 1$, $\Omega_m \sim .3$, $\Omega_\Lambda \sim .7$, $\Omega_r \sim 5 \times 10^{-5}$

For $\lambda >> 1$, ρ_Λ dominates.

For $\lambda << 1$, $\rho_m >> \rho_\Lambda$

$\rho_m >> \rho_r$ if $\frac{\Omega_m}{\lambda^3} >> \frac{\Omega_r}{\lambda^4} \Rightarrow \lambda >> \frac{\Omega_r}{\Omega_m} \sim \frac{5 \times 10^{-5}}{.3} \sim 10^{-4}$

λ_{eq} defined as $\frac{\Omega_m}{\lambda_{eq}^3} = \frac{\Omega_r}{\lambda_{eq}^4}$.

$\lambda << \lambda_{eq}$, ρ_r dominates.

The estimate $\lambda_{eq} \sim 10^{-4}$ is a rough estimate.

Reason: We also have neutrino radiation.

not observed but theoretically predicted.

In the far past when radiation dominated

$$\rho_r = \Omega_r \rho_c \frac{1}{\lambda^4}$$

$$\rho_r \simeq \frac{C}{\lambda^4}$$

C will in general differ from $\Omega_r \rho_c$

For now we shall use

$$\left(\frac{\dot{\lambda}}{\lambda}\right)^2 = H_0^2\left(\frac{\Omega_m}{\lambda^3} + \frac{\Omega_r}{\lambda^4} + \Omega_\Lambda\right)$$

assuming $\Omega_m, \Omega_r, \Omega_\Lambda$ constants all through time & proceed.

① Age of the universe:

t_0 in the convention that $\lambda = 0$ at $t=0$

$$\int_0^{t_0} dt = \int_0^1 \frac{d\lambda}{\dot{\lambda}} = H_0^{-1}\int_0^1 d\lambda \left(\Omega_m \lambda^{-1} + \Omega_r \lambda^{-2} + \Omega_\Lambda \lambda^2\right)^{-1/2}$$

$$t_0 = H_0^{-1}\int_0^1 d\lambda\, \big(\underbrace{\Omega_m \lambda^{-1} + \Omega_r \lambda^{-2} + \Omega_\Lambda \lambda^2}_{\text{As } \lambda \to 0 \quad \approx \lambda}\big)^{-1/2}$$

$$= H_0^{-1}\left[\underbrace{\int_0^{\lambda_{eq}} d\lambda\, \Omega_r^{-1/2} \lambda}_{2} + \int_{\lambda_{eq}}^1 \left(\Omega_m \lambda^{-1} + \Omega_\Lambda \lambda^2\right)^{-1/2} d\lambda\right]$$

$$\Omega_r^{-1/2}\, \frac{\lambda_{eq}^2}{2} \qquad \lambda_{eq} = \frac{\Omega_r}{\Omega_m}$$

$$\sim \frac{\Omega_r^{3/2}}{2\,\Omega_m^2} \Rightarrow \text{small} \ll 1$$

$$\simeq H_0^{-1} \int_{\lambda_{eq}}^1 \left(\Omega_m \lambda^{-1} + \Omega_\Lambda \lambda^2\right)^{-1/2} d\lambda$$

$$\simeq .96 \times H_0^{-1}$$

$$\simeq 13.8 \times 10^9 \text{ years}$$

Horizon at time t

= maximum distance a signal can travel since the big bang ($\lambda = 0$, $t = 0$).

Start at $t = 0$, $r = 0$, $\theta = 0$, $\phi = 0$.

Suppose at t it reaches $r = r_1$, $\theta = 0$, $\phi = 0$.

$$-dt^2 + a(t)^2\, dr^2 = 0 \qquad (k/a_0^2 = 0)$$

$$\Rightarrow \int_0^{r_1} dr = \int_0^t \frac{dt'}{a(t')} = \frac{1}{a_0}\int_0^t \frac{dt'}{\lambda(t')}$$

$$d_H(t) = a(t)\int_0^{r_1} dr = \lambda(t)\int_0^t \frac{dt'}{\lambda(t')}$$

$$d_H(t) = \lambda(t) \int_0^t \frac{dt}{\lambda(t)}$$

Comoving horizon: Horizon, measured in todays metric.

$$r_H(t) = a_0 \int_0^{r_1} dr = a_0 \int_0^t \frac{dt'}{a(t')} = \int_0^t \frac{dt'}{\lambda(t')}$$

change variable from t' to $\lambda' = \lambda(t')$

$$d\lambda' = \dot{\lambda}(t')\, dt' = dt'\, H_0 \left(\frac{\Omega_m}{\lambda'} + \frac{\Omega_r}{\lambda'^2} + \Omega_\Lambda \lambda'^2 \right)^{1/2}$$

$$\Rightarrow r_H = \int_0^\lambda \frac{d\lambda'}{H_0 \lambda'} \left(\frac{\Omega_m}{\lambda'} + \frac{\Omega_r}{\lambda'^2} + \Omega_\Lambda \lambda'^2 \right)^{-1/2}$$

$$r_H = H_0^{-1}\int_0^{\lambda} \frac{d\lambda'}{\lambda'}\left(\frac{\Omega_m}{\lambda'} + \frac{\Omega_r}{\lambda'^2} + \Omega_\Lambda \lambda'^2\right)^{-1/2}$$

Today $\lambda = 1$.

$$r_H(t_0) = H_0^{-1}\int_0^{1} \frac{d\lambda'}{\lambda'}\left(\frac{\overset{=.3}{\Omega_m}}{\lambda'} + \cancel{\frac{\Omega_r}{\lambda'^2}} + \overset{=.7}{\Omega_\Lambda} \lambda'^2\right)^{-1/2}$$

$$\simeq 3H_0^{-1} \simeq 45\times 10^9 \text{ light years.}$$

How far can we send signal if we do it today?

$$H_0^{-1}\int_1^{\infty} \frac{d\lambda'}{\lambda'}\left(\frac{\Omega_m}{\lambda'} + \cancel{\frac{\Omega_r}{\lambda'^2}} + \Omega_\Lambda \lambda'^2\right)^{-1/2} \simeq H_0^{-1}$$

$2H_0^{-1}$