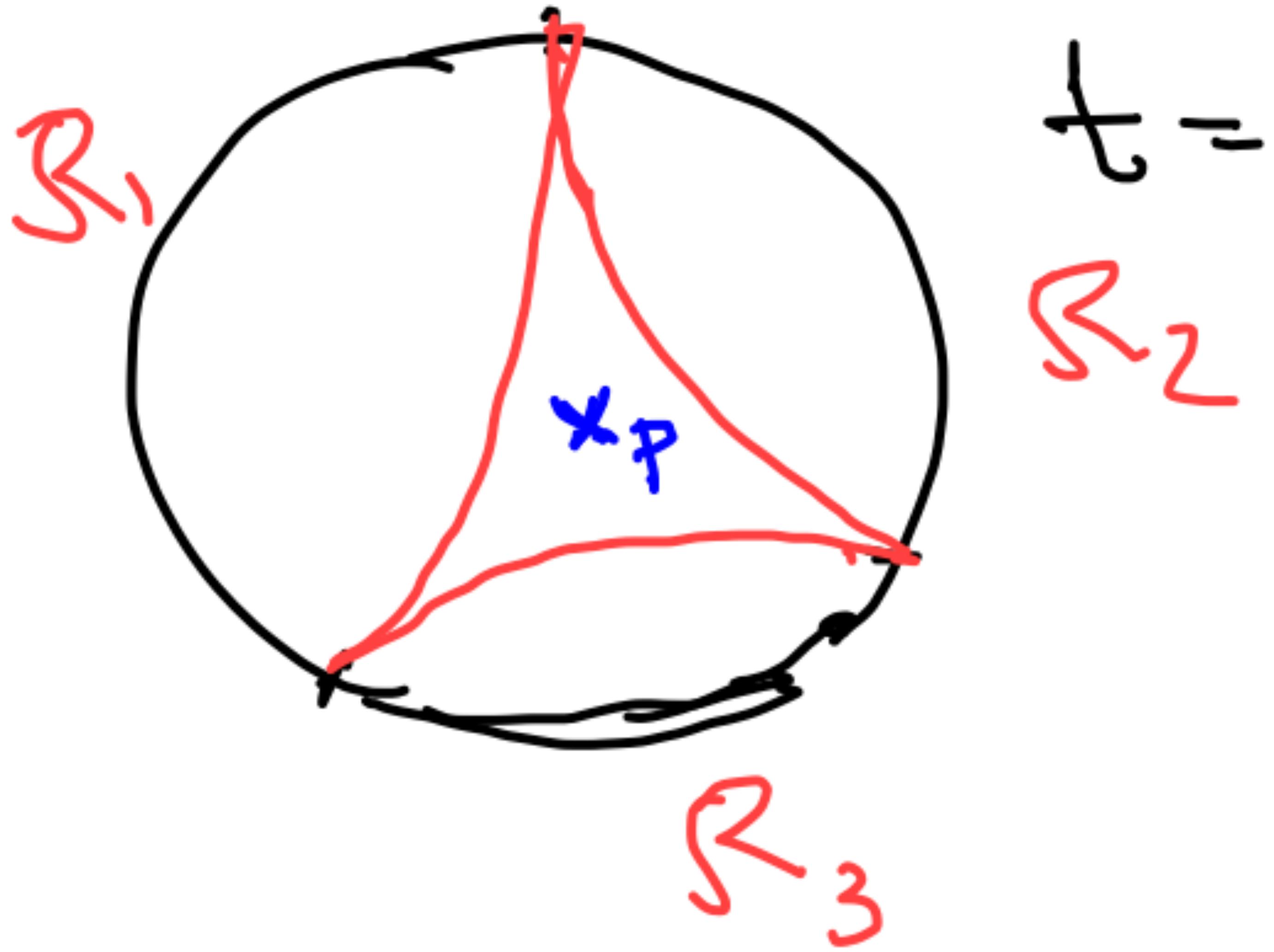
The situation in AdS/CFT



$t=0$ slice

None of R_1, R_2 or R_3 contain information about p but $R_1 \cup R_2$, $R_2 \cup R_3$ or $R_3 \cup R_1$ do.

Q. How is the information encoded?

We'll model the system using a quantum error correction code.

Encode a state $|\psi\rangle \in \mathcal{H}$ in a bigger Hilbert space such that even if we loose some part of the information on the bigger Hilbert space we can recover $|\psi\rangle$.

Note: We cannot copy $|\psi\rangle$ (no-cloning thm.)

$$|\psi\rangle \to |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \quad \times \text{ not possible.}$$

We'll work with qudits → a d-state system (d ≠ dimension ~~of~~ space).

Hilbert space in which the original state $|\psi\rangle$ lives is taken to be a product of k qudits. ⇒ d^k dimensional space.

⇒ logical qudits (bulk of AdS).

Hilbert space in which we encode the state $|\psi\rangle$ as $|\tilde{\psi}\rangle$ is taken to be a product of n qudits $(n > k)$ CFT

→ physical qudits (boundary)

$|\psi\rangle \to |\tilde{\psi}\rangle$ a linear map.

Code subspace: subspace of states of the physical qudits spanned by the $|\tilde{\psi}\rangle$'s. (d^k dimensional subspace of d^n dimensional space).

We'll illustrate this using the qutrit code. $|0\rangle, |1\rangle, |2\rangle \to$ 3 states of a qutrit.

Encode one logical qutrit into 3 physical qutrits.

$$|\psi\rangle = \sum_{i=0}^{2} c_i |i\rangle \longrightarrow |\tilde{\psi}\rangle = \sum_{i=0}^{2} c_i |\tilde{i}\rangle$$

$|\tilde{0}\rangle, |\tilde{1}\rangle, |\tilde{2}\rangle$ span the code subspace.

Hilbert space of physical qutrits is $3^3 = 27$ dimensional.

<u>Prescription:</u>

$$|\tilde{0}\rangle = \frac{1}{\sqrt{3}}\left(|000\rangle + |111\rangle + |222\rangle\right)$$

$$|\tilde{1}\rangle = \frac{1}{\sqrt{3}}\left(|012\rangle + |120\rangle + |201\rangle\right)$$

$$|\tilde{2}\rangle = \frac{1}{\sqrt{3}}\left(|021\rangle + |102\rangle + |210\rangle\right)$$

Invariant under cyclic perm of the phys. qutrits.

$$|\tilde{i}\rangle = U_{12}\left(|i\rangle_1 \otimes |\chi\rangle_{23}\right)$$

$$|\chi\rangle_{23} = \frac{1}{\sqrt{3}}\left(|00\rangle_{23} + |11\rangle_{23} + |22\rangle_{23}\right)$$

U_{12} : unitary operator acting on the 12 Hilbert space:

$U: \quad |00\rangle \to |00\rangle$

$|11\rangle \to |20\rangle \to |21\rangle \to |02\rangle \to |22\rangle \to |10\rangle$

$\to |12\rangle \to |01\rangle \to |11\rangle$

$$|\tilde{0}\rangle = U_{12}\left(|0\rangle_1 \otimes \frac{1}{\sqrt{3}}\left(|00\rangle_{23} + |11\rangle_{23} + |22\rangle_{23}\right)\right)$$

$$|\tilde{i}\rangle = U_{12}\left(|i\rangle_1 \otimes |\chi\rangle_{23}\right)$$

$$|\tilde{i}\rangle = U_{23}\left(|i\rangle_2 \otimes |\chi\rangle_{31}\right)$$

$$|\tilde{i}\rangle = U_{31}\left(|i\rangle_3 \otimes |\chi\rangle_{12}\right)$$

follow from cyclicity.

$$U_{23} \frac{1}{\sqrt{3}} \sum_{k=0}^{2} |k\rangle_1 |i\rangle_2 |k\rangle_3$$

Recovery

$$U_{12}^{\dagger}|\tilde{i}\rangle = |i\rangle_1 \otimes |\chi_{23}\rangle$$

Apply $U_{12}^{\dagger}$ and read out the state in the first qudit.

$$U_{12}^{\dagger}|\tilde{\Psi}\rangle = \sum_i c_i |i\rangle_1 \otimes |\chi\rangle_{23}$$

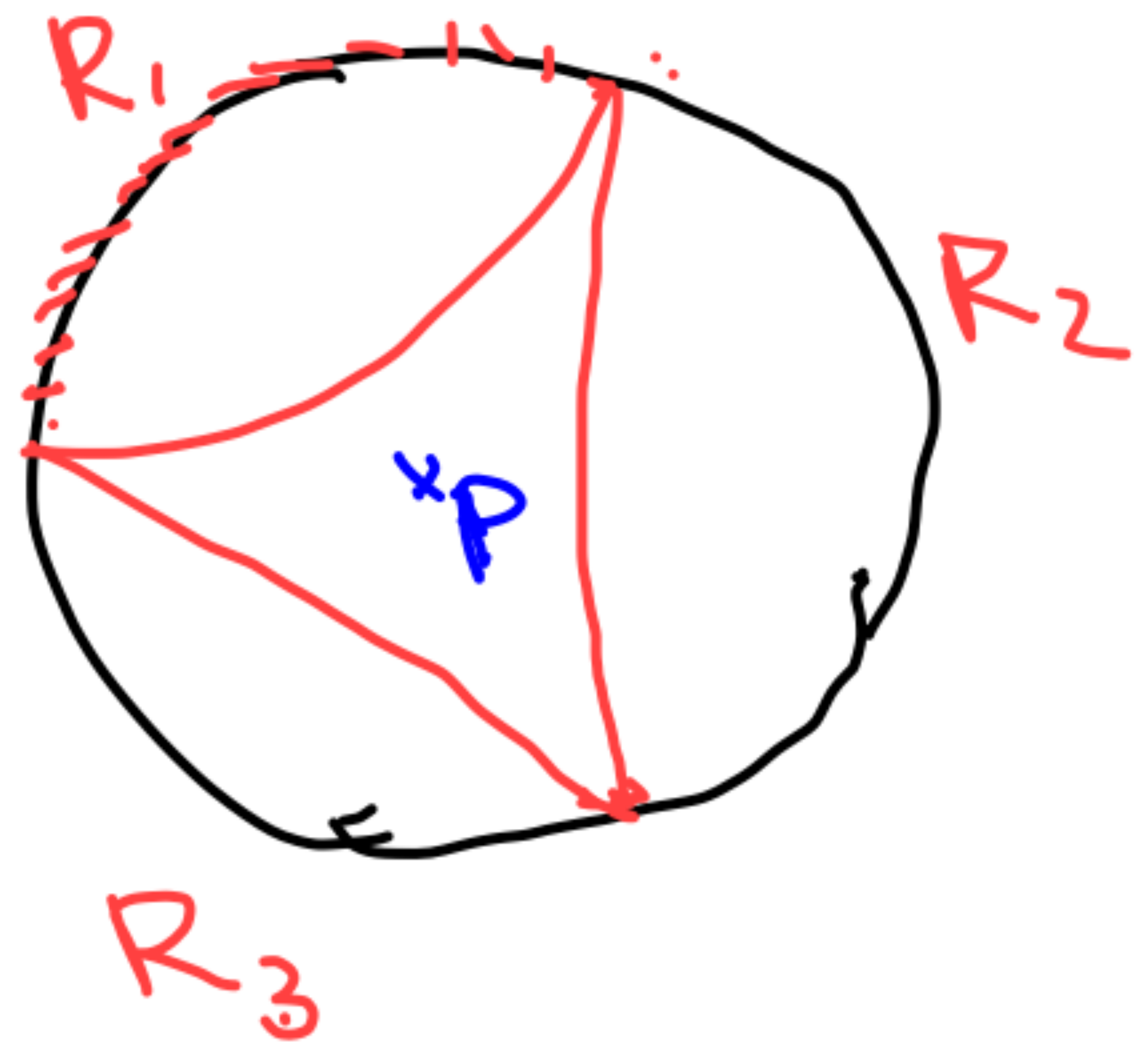
$$\not\downarrow$$

$$\sum_i c_i |\tilde{i}\rangle$$

$$U_{23}^{\dagger}|\tilde{\Psi}\rangle = \sum_i c_i |i\rangle_2 \otimes |\chi\rangle_{31}$$

$$U_{31}^{\dagger}|\tilde{\Psi}\rangle = \sum_i c_i |i\rangle_3 \otimes |\chi\rangle_{12}$$

$\Rightarrow$ If we lose any one of the 3 qudits we can still recover the information.



Logical operator: $G|i\rangle = \sum_{j=0}^{2} G_{ji}|j\rangle$ ← numbers.

Represent G on physical qutrits.

→ Want to construct $\tilde{G}$ such that

$$\tilde{G}|\tilde{i}\rangle = \sum_{j=0}^{2} G_{ji}|\tilde{j}\rangle$$

Can choose

$$\tilde{O}_{12} = U_{12} \, O_1 \, U_{12}^\dagger$$

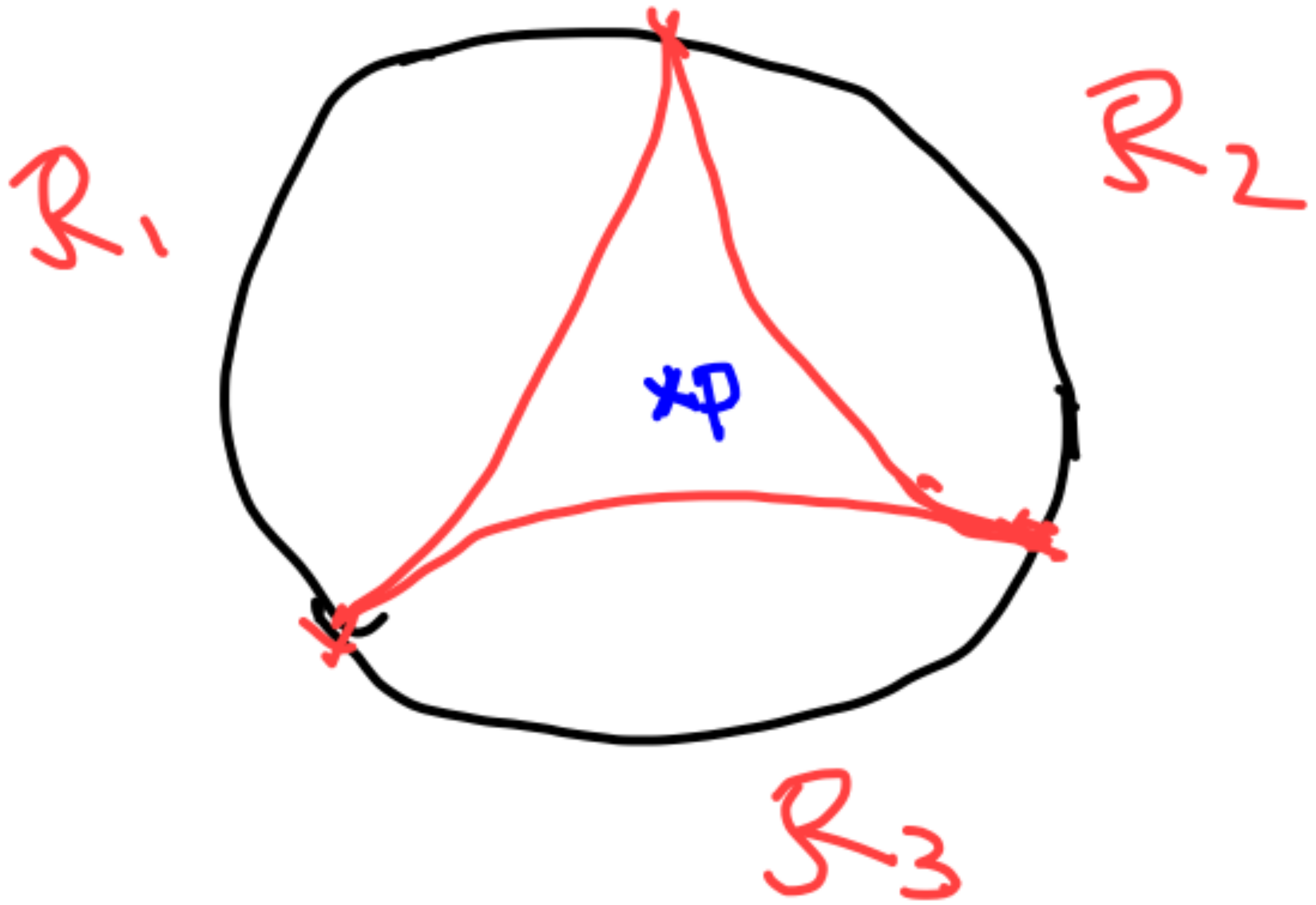
$$\tilde{O}_{12} |\tilde{i}\rangle = U_{12} \, O_1 \, U_{12}^\dagger \; U_{12} \left(|i\rangle_1 \otimes |\chi\rangle_{23} \right)$$

$$= U_{12} \sum_j O_{ji} |j\rangle_1 \otimes |\chi\rangle_{23}$$

$$= \sum_j O_{ji} \; |\tilde{j}\rangle$$

$$\tilde{O}_{23} = U_{23} \, O_2 \, U_{23}^\dagger , \quad \tilde{O}_{31} = U_{31} \, O_3 \, U_{31}^\dagger$$

- Different operators but they act on the code subspace in the same way.



Take a logical operator O.

X_3 : some operator on R_3 (3rd qutrit)

In what sense is $[X_3, \tilde{O}] = 0$?

$\langle \tilde{\psi} | [X_3, \tilde{O}] | \tilde{\phi} \rangle = 0$ if $|\tilde{\psi}\rangle$ and $|\tilde{\phi}\rangle$ are in the code subspace.

$$\langle \tilde{\Psi} | [X_3, \tilde{G}] | \tilde{\Phi} \rangle$$

$$= \langle \Psi | X_3 \tilde{G} | \tilde{\Phi} \rangle - \langle \tilde{\Psi} | \tilde{G} X_3 | \tilde{\Phi} \rangle.$$

$$\Downarrow$$

$$\langle \tilde{\Phi} | X_3^{\dagger} \tilde{G}^{\dagger} | \tilde{\Psi} \rangle^{*}$$

$$= \langle \Psi | X_3 \tilde{G}_{12} | \tilde{\Phi} \rangle - \langle \tilde{\Phi} | X_3^{\dagger} \tilde{G}_{12}^{\dagger} | \tilde{\Psi} \rangle^{*}$$

$$= \langle \tilde{\Psi} | X_3 \tilde{G}_{12} | \tilde{\Phi} \rangle - \langle \tilde{\Psi} | \tilde{G}_{12} X_3 | \tilde{\Phi} \rangle.$$

$$= \langle \tilde{\Psi} | [X_3, \tilde{G}_{12}] | \tilde{\Phi} \rangle = 0$$

Define:

$$T_{i_1 i_2 i_3 j} = \sqrt{3}\; {}_1\langle i_1 | \otimes {}_2\langle i_2 | \otimes {}_3\langle i_3 | \tilde{j} \rangle$$

basis in code subspace.

$T_{i_1 i_2 i_3 j}$ is a "perfect tensor."

$T_{i_1 \cdots i_{2n}}$ is called a perfect tensor if any balanced bipartition of indices give a unitary matrix.

$M_{i_1 i_2, i_3 j} = T_{i_1 i_2 i_3 j}$ should be unitary

Proof.

$$|\chi\rangle_{23} = \sum_k \frac{1}{\sqrt{3}} |k\rangle_2 \otimes |k\rangle_3$$

$$T_{i_1 i_2 i_3, j} = \sum_{k=0}^{2} {}_1\langle i_1| \otimes {}_2\langle i_2| \otimes {}_3\langle i_3| \underbrace{U_{12} |j\rangle_1 \otimes |k\rangle_2 \otimes |k\rangle_3}_{|\tilde{j}\rangle}$$

$${}_3\langle i_3|k\rangle_3 = \delta_{i_3 k}$$

$$= {}_1\langle i_1| \otimes {}_2\langle i_2| \; U_{12} \, |j\rangle_1 \otimes |i_3\rangle_2$$

$$= (U_{12})_{i_1 i_2, j i_3} \rightarrow \text{Unitary matrix.}$$

→ Similar analysis holds for other partitions.