Euclidean geometry:

Consider two infinitesimally close points in a 3-dimensional space:

$$(x^1, x^2, x^3) \quad \& \quad (x^1+dx^1, x^2+dx^2, x^3+dx^3)$$

Distance between the two points ds:

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

Generalization to N dimensional space:

Distance between $(x^1, x^2, \cdots x^N)$ & $(x^1+dx^1, \cdots x^N+dx^N)$

$$ds^2 = (dx^1)^2 + \cdots (dx^N)^2 = \sum_{i=1}^{N} (dx^i)^2$$

Riemannian Geometry

$$ds^2 = \sum_{i,j} g_{ij}(\vec{x})\, dx^i dx^j$$

$g_{ij}(\vec{x})$: A function of $\vec{x}$ for each pair (ij)
(symmetric in i & j)

Euclidean Geometry: Special case of Riemannian geometry:

$$g_{ij}(\vec{x}) = \delta_{ij}$$

Different functions $g_{ij}(x)$

$\Rightarrow$ Different Riemannian manifold

But this is not always true.

Effect of change of coordinates:

Instead of using $(x^1, \cdots x^N)$ as coordinates, choose a different set of coordinates

$$\left.\begin{aligned} x'^1 &= f^1(\vec{x}) \\ x'^2 &= f^2(\vec{x}) \\ &\vdots \\ x'^N &= f^N(\vec{x}) \end{aligned}\right\} \underset{\substack{\text{Inverse}\\\text{relation}}}{\Longrightarrow} \begin{aligned} x^1 &= \phi^1(\vec{x}') \\ x^2 &= \phi^2(\vec{x}') \\ &\vdots \\ x^N &= \phi^N(\vec{x}') \end{aligned}$$

$$ds^2 = \sum_{i,j} g_{ij}(\vec{x})\, dx^i\, dx^j$$

$$= \sum_{i,j} g_{ij}(\vec{x}) \sum_k \frac{\partial x^i}{\partial x'^k} dx'^k \sum_l \frac{\partial x^j}{\partial x'^l} dx'^l$$

$$= \sum_{k,l} \underbrace{\left(\sum_{i,j} g_{ij}(\vec{x}) \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l} \right)}_{\substack{\equiv \\ g'_{kl}(\vec{x}')}} dx'^k\, dx'^l$$

$$ds^2 = \sum_{k,l} g'_{kl}(\vec{x}')\, dx'^k\, dx'^l$$

The same geometry corresponds to different metrices in different coordinate systems.

Example: 3-dimensional Euclidean metric in spherical polar coordinates:

$$ds^2 = \sum_{i=1}^{3} (dx^i)^2$$

Polar coordinate: (r, θ, ϕ).

$$x^3 = r\cos\theta$$
$$x^1 = r\sin\theta\cos\phi$$
$$x^2 = r\sin\theta\sin\phi$$

$$dx^1 = dr\sin\theta\cos\phi + d\theta\, r\cos\theta\cos\phi - d\phi\, r\sin\theta\sin\phi$$

$$dx^2 = dr\sin\theta\sin\phi + d\theta\, r\cos\theta\sin\phi + d\phi\, r\sin\theta\cos\phi$$

$$dx^3 = dr\cos\theta - d\theta\, r\sin\theta$$

$\Rightarrow$
$$ds^2 = dr^2 + r^2\sin^2\theta\, d\phi^2 + r^2 d\theta^2$$

$$\Rightarrow \quad g_{rr} = 1, \quad g_{\phi\phi} = r^2\sin^2\theta, \quad g_{\theta\theta} = r^2$$

This is the 3-dimensional Euclidean metric in spherical polar coordinates.

Example of a non-Euclidean space:

Surface of a two dimensional sphere:

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = a^2$$

Need to choose two independent coordinates on the surface:

① (θ, ϕ) as independent coordinates.

$r = a \Rightarrow$ Equation of the sphere.

$$ds^2 = a^2 \sin^2\theta \, d\phi^2 + a^2 \, d\theta^2$$

$$\Rightarrow \quad g_{\theta\theta} = a^2, \quad g_{\phi\phi} = a^2 \sin^2\theta$$

② x^1, x^2 as independent coordinates.

$$x^3 = \sqrt{a^2 - (x^1)^2 - (x^2)^2}$$

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$$= (dx^1)^2 + (dx^2)^2 + \left(\frac{-2x^1 dx^1 - 2x^2 dx^2}{2\sqrt{a^2 - (x^1)^2 - (x^2)^2}} \right)^2$$

$$= (dx^1)^2 + (dx^2)^2 + \frac{(x^1)^2 (dx^1)^2 + (x^2)^2 (dx^2)^2 + 2x^1 x^2 dx^1 dx^2}{a^2 - (x^1)^2 - (x^2)^2}$$

$$= \frac{1}{a^2 - (x^1)^2 - (x^2)^2} \Big\{ (a^2 - (x^2)^2)(dx^1)^2 + (a^2 - (x^1)^2)(dx^2)^2 + 2x^1 x^2 \, dx^1 dx^2 \Big\}$$

$$g_{11} = \frac{a^2-(x^2)^2}{a^2-(x^1)^2-(x^2)^2}, \qquad g_{22} = \frac{a^2-(x^1)^2}{a^2-(x^1)^2-(x^2)^2}$$

$$g_{12} = \frac{x^1 x^2}{a^2-(x^1)^2-(x^2)^2}$$

Coordinate transformation between (x^1, x^2) & (θ, ϕ):

$$x^1 = a \sin\theta \cos\phi$$

$$x^2 = a \sin\theta \sin\phi$$

Ex. Check that under this coordinate trs. the metric transforms correctly according to the given rules.

Claim: There is no coordinate transformation which can make this metric Euclidean ie which can make

$$g_{ij} = \delta_{ij}.$$

How can we prove this?

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① Try to construct appropriate combinations of the metric and its derivatives which does not transform under coordinate trs.

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② Show that this combination is zero for the Euclidean metric and non-zero for the metric on the sphere

This will prove the desired result.

Metric transformation law:

$$g'_{ij}(x') = \sum_{k,l} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} g_{kl}(\vec{x})$$

// Shorthand notation

$$\frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} g_{kl}(\vec{x})$$

Any index which is repeated twice is automatically summed over from 1 to N.

Consider any q index combination $A_{i_1 \cdots i_q}$ of metric and its derivatives such that

$$A'_{i_1 \cdots i_q}(\vec{x}') = \frac{\partial x^{j_1}}{\partial x'^{i_1}} \frac{\partial x^{j_2}}{\partial x'^{i_2}} \cdots \frac{\partial x^{j_q}}{\partial x'^{i_q}} A_{j_1 \cdots j_q}(\vec{x})$$

$\Rightarrow$ $A_{i_1 \cdots i_q}$ is called a rank q covariant tensor.

Metric is a symmetric rank 2 covariant tensor.

Note: Indices of a covariant tensor are written at the bottom.

With this convention repeated indices in this eqn. always comes in pairs with one on top and one at the bottom.

Also $1 \leftrightarrow 1$ correspondence between the bottom indices on both sides.

Consider a q-index object $B^{i_1 \cdots i_q}(\vec{x})$ such that

$$B'^{i_1 \cdots i_q}(\vec{x}') = \frac{\partial x'^{i_1}}{\partial x^{j_1}} \frac{\partial x'^{i_2}}{\partial x^{j_2}} \cdots \frac{\partial x'^{i_q}}{\partial x^{j_q}} B^{j_1 \cdots j_q}(\vec{x})$$

B is called a contravariant rank q **tensor**.

Example : Define:

$$g^{ij} = ij \text{ component of } g^{-1}$$

Matrix inverse

$$g^{ij}(\vec{x})\, g_{jk}(\vec{x}) = \delta^i{}_k$$

↳ Kronecker δ function

Ex. Show that g^{ij} transforms as a contravariant rank 2 tensor.

Mixed tensor:

$$C^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(\vec{x})$$

---

subject to the transformation law:

$$C'^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(\vec{x}') = C^{k_1 \cdots k_p}{}_{l_1 \cdots l_q}(\vec{x})$$

$$\cdot \frac{\partial x'^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial x'^{i_p}}{\partial x^{k_p}} \frac{\partial x^{l_1}}{\partial x'^{j_1}} \cdots \frac{\partial x^{l_q}}{\partial x'^{j_q}}$$

$\rightarrow$ rank (p, q) tensor.

Rule for ~~the~~ transformation law of product of tensors:

$$A^{i_1 \cdots i_{p_1}}{}_{j_1 \cdots j_{q_1}} \, B^{k_1 \cdots k_{p_2}}{}_{l_1 \cdots l_{p_2}}$$

$$\equiv C^{i_1 \cdots i_{p_1} k_1 \cdots k_{p_2}}{}_{j_1 \cdots j_{q_1} l_1 \cdots l_{p_2}}$$

Ex. C is a rank $(p_1 + p_2, q_1 + q_2)$ tensor.

Define:

$$D^{i_2 \cdots i_{p}}{}_{j_2 \cdots j_q} = A^{i_1 i_2 \cdots i_{p}}{}_{i_1 j_2 \cdots j_q}$$

Ex. If A is a rank (p, q) tensor then D is a rank $(p-1, q-1)$ tensor.

This is called contracting the indices i_1 and j_1.

General rules

① Contract ~~every~~ any ~~only~~ upper index with only a lower index in an arbitrary product of tensors.

② If the result has p upper index and q lower index then it is a rank (p,q) tensor.

This makes it easy to construct new tensors out of products of tensors.

Example:

$$A_{ijkl} = g_{ij}\, g_{kl}$$

$\rightarrow$ A rank $(0,4)$ tensor.

$B^{ij}{}_{kl} = g^{ij}\, g_{kl}$: A rank $(2,2)$ tensor.

$C^{i}{}_{l} = B^{ij}{}_{jl}$: A rank $(1,1)$ tensor.

In this case $C^{i}{}_{l} = \delta^{i}{}_{l}$

$\hookrightarrow$ Kronecker δ

Conclusion: The Kronecker δ is a rank $(1,1)$ tensor.

Note: The Kronecker δ is a given matrix and does not depend on the coordinate system.

$\Rightarrow$ Called an invariant tensor

$$\delta^{i}{}_{j} = \delta^{k}{}_{l}\, \frac{\partial x^{l}}{\partial x'^{j}}\, \frac{\partial x'^{i}}{\partial x^{k}}$$

~~Weinberg, Gravitation and Cosmology~~

~~Cosmology: [illegible] cannot~~

Strategy: Form a rank (0,0) tensor $\phi(x)$ out of $g_{\mu\nu}(x)$ and its derivatives.

$$\phi'(x') = \phi(x)$$

$\rightarrow$ does not transform under coordinate transformation

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If ϕ is different for two different metrics, then it is not possible to relate them via coordinate transformation.

How do we find such a ϕ?

Contract indices of various tensors formed out of g_{ij}.

e.g. $\quad g^{ij} g_{ij}$

$$\parallel$$

$$g^{ij} g_{ji} = \delta^i_i = N.$$

$\rightarrow$ independent of the metric.

$\rightarrow$ cannot be used to test if two metrics describe different geometric spaces.

A general result. All ~~[illegible]~~ scalars formed out of the metric (without derivatives) are of this type.

Resolution: Construct tensors out of derivatives of the metric and then contract indices.

Define $\partial_i = \dfrac{\partial}{\partial x^i}$, $\partial'_i = \dfrac{\partial}{\partial x'^i}$

Transformation law of $\partial_i g_{jk}$.

$$\partial'_i g'_{jk}(x') = \frac{\partial x^l}{\partial x'^i} \frac{\partial}{\partial x^l}\left(g_{mn}(\vec{x}) \frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^k}\right)$$

$$= \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^k} \frac{\partial g_{mn}(\vec{x})}{\partial x^l}$$

$$+ \frac{\partial x^l}{\partial x'^i} g_{mn}(\vec{x}) \frac{\partial}{\partial x^l}\left(\frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^k}\right)$$

//

$$g_{mn}(\vec{x}) \frac{\partial}{\partial x'^i}\left(\frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^k}\right)$$

Extra term.

$\partial_i g_{jk}$ is not a tensor.

Similarly $\partial_i g^{jk}$ is also not a tensor.

In fact one can show that it is not possible to construct a ~~~~ tensor out of a single derivative of g_{ij}

Need to try terms with two derivatives of g_{ij}.

Define:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}).$$

Ex. Show that

$$\Gamma'^l_{mn}(\vec{x}') = \frac{\partial x'^l}{\partial x^i} \frac{\partial x^j}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \Gamma^i_{jk}(\vec{x})$$

$$+ \frac{\partial x'^l}{\partial x^k} \frac{\partial^2 x^k}{\partial x'^m \partial x'^n}$$

⇓

Extra term.

Define:

$$R^i_{jkl} = \partial_l \Gamma^i_{jk} - \partial_k \Gamma^i_{jl} + \Gamma^m_{jk} \Gamma^i_{lm} - \Gamma^m_{jl} \Gamma^i_{km}$$

Ex. Show that R^i_{jkl} transforms as a rank (1, 3) tensor.

Given this we can now construct many different scalars:

$$R^i_{jkl} R^{i'}_{j'k'l'} g_{ii'} g^{jj'} g^{kk'} g^{ll'}$$

$$R^i_{jil} g^{jl}$$

$$R^i_{jil} R^k_{mkn} g^{jm} g^{ln}$$

etc.

Comparing these quantities we can try to see if ~~the~~ two ~~manifolds~~ spaces are geometrically different.

For Euclidean geometry

$$R^{i}{}_{jkl} = 0$$

since $g_{ij} = \delta_{ij}$ is a constant.

$\Rightarrow$ All these invariants vanish.

Thus if these invariants are non-zero for some ~~manifold~~ space then that space is geometrically distinct from Euclidean space

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Ex. Show that for 2-dimensional sphere

$$R^{i}{}_{jkl} \, R^{i'}{}_{j'k'l'} \, g_{ii'} \, g^{jj'} \, g^{kk'} \, g^{ll'}$$

$$\propto a^{-4} \quad \& \text{ non zero.}$$

$\Rightarrow$ ① This is not Euclidean ~~space~~ geometry.

---

② Different values of a correspond to different geometric spaces.

Proof of the statement:

It is not possible to construct any non-trivial tensor out of a single derivative of g_{ij}.

Assume the contrary.

There is a tensor $T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(\vec{x})$ constructed from $\partial_k g_{ij}$.

$\vec{x}_0$: An arbitrary point.

We shall show that it is possible to choose a coordinate system $\vec{x}'$ such that

$$\partial'_k g'_{ij}(\vec{x}_0') = 0.$$

$$\Rightarrow \quad T'^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(\vec{x}_0') = 0.$$

$\Rightarrow \quad T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(\vec{x}_0) = 0$ if T is a tensor.

But $\vec{x}_0$ is an arbitrary point in the space.

$\Rightarrow$ T vanishes at all points.

$\Rightarrow$ T is trivial.

Thus it remains to show that there is a coordinate system in which

$$\partial'_k \, g'_{ij} (\vec{x}'_0) = 0.$$

Choose: $\vec{x}'^k$ such that

$$x^i = x_0^i + x'^i - x_0'^i + \tfrac{1}{2} A^i_{jk} (x'^j - x_0^j)(x'^k - x_0^k)$$

At $\vec{x}' = \vec{x}'_0$, $\vec{x} = \vec{x}_0$

$$\frac{\partial x^i}{\partial x'^m} = \delta^i_m + A^i_{mk} (x'^k - x_0'^k)$$

$$g'_{mn} (\vec{x}') = \frac{\partial x^i}{\partial x'^m} \frac{\partial x^j}{\partial x'^n} g_{ij} (\vec{x})$$

$$= g_{mn} (\vec{x}) + A^i_{mk} (x'^k - x_0'^k) \, g_{in} (\vec{x})$$

$$+ A^j_{nk} (x'^k - x_0'^k) \, g_{mj} (\vec{x})$$

$$+ A^i_{mk} (x'^k - x_0'^k) \, A^j_{nl} (x'^l - x_0'^l) \, g_{ij} (\vec{x})$$

$$\partial'_s \, g'_{mn} (\vec{x}'_0) = \partial'_s \, g_{mn} (\vec{x}) \Big|_{\vec{x} = \vec{x}_0}$$

$$+ A^i_{ms} \, g_{in} (\vec{x}_0) + A^j_{ns} \, g_{mj} (\vec{x}_0)$$

$$= \frac{\partial x^r}{\partial x'^s}\Big|_{\vec{x} = \vec{x}_0} \partial_r \, g_{mn} (\vec{x}_0) + A^i_{ms} \, g_{in} (\vec{x}_0) + A^j_{ns} \, g_{mj} (\vec{x}_0$$

$$\frac{\partial x^r}{\partial x'^s} = \delta^r_s + A^r_{sm}\,(x'^m - x_0^m)$$

$$= \delta^r_s \quad \text{at } \vec{x}' = \vec{x}_0'.$$

$$\Rightarrow \quad \partial'_s\, g'_{mn}(\vec{x}_0') = \partial_s g_{mn}(\vec{x}_0) + A^i_{ms}\, g_{in}(\vec{x}_0)$$

$$+ A^j_{ns}\, g_{mj}(\vec{x}_0).$$

Choose: $A^i_{ms} = -\cancel{\tfrac{1}{2}(\partial_s g_{mp}(\vec{x}_0))\, g^{pi}(\vec{x}_0)}\ \Gamma^i_{ms}(\vec{x}_0)$

$$\Rightarrow \quad \partial'_s\, g'_{mn}(\vec{x}_0') = 0$$

$$\cancel{= \partial_s g_{mn}(\vec{x}_0) - \tfrac{1}{2}(\partial_s g_{mp}(\vec{x}_0))\, g^{pi}(\vec{x}_0)\, g_{in}(\vec{x}_0)}$$

$$\cancel{- \tfrac{1}{2}(\partial_s g_{np}(\vec{x}_0))\, g^{pj}(\vec{x}_0)\, g_{mj}(\vec{x}_0)}$$

$$\cancel{= \partial_s g_{mn}(\vec{x}_0) - \tfrac{1}{2}\partial_s g_{mn}(\vec{x}_0) - \tfrac{1}{2}\partial_s g_{nm}(\vec{x}_0)}$$

$$\cancel{= 0}$$

Thus in the $\vec{x}'$ coordinate system choosen this way

$$\partial'_s\, g'_{mn}(\vec{x}_0') = 0.$$

$$\Rightarrow \quad T'^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(\vec{x}_0') = 0.$$

$$\Rightarrow \quad T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(\vec{x}_0) = 0$$

Covariant derivative:

$\partial_k A^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$ is not a tensor.

Define:

$$
\begin{aligned}
& D_k A^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \\
&= \partial_k A^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} + \Gamma^{i_1}_{k l_1} A^{l_1 i_2 \cdots i_p}{}_{j_1 \cdots j_q} \\
&\quad + \Gamma^{i_2}_{k l_2} A^{i_1 l_2 i_3 \cdots i_p}{}_{j_1 \cdots j_q} + \cdots + \Gamma^{i_p}_{k l_p} A^{i_1 \cdots i_{p-1} l_p}{}_{j_1 \cdots j_q} \\
&\quad - \Gamma^{m_1}_{k j_1} A^{i_1 \cdots i_p}{}_{m_1 j_2 \cdots j_q} - \Gamma^{m_2}_{k j_2} A^{i_1 \cdots i_p}{}_{j_1 m_2 j_3 \cdots j_q} \\
&\quad - \cdots - \Gamma^{m_q}_{k j_q} A^{i_1 \cdots i_p}{}_{j_1 \cdots j_{q-1} m_q}
\end{aligned}
$$

Ex. show that $D_k A^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$ is a rank $(p, q+1)$ tensor.

$\Rightarrow$ $D_k g_{ij}$ is a rank 3 covariant tensor.

Ex. show that $D_k g_{ij} = 0$ identically.

Q. How ~~exact~~ close to the Euclidean metric can we bring a general metric?

Consider a specific point $\vec{x}_0$.

Can we make $g'_{ij}(\vec{x}_0') = \delta_{ij}$ by a coordinate transformation?

$$x^i = x_0^i + \underbrace{S_{ij}}_{\text{constant}}(x'^j - x_0'^j)$$

$$\frac{\partial x^i}{\partial x'^j} = S_{ij}$$

$$g'_{ij}(x') = \frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j} g_{mn}(\vec{x}_0)$$

$$= S_{mi} S_{nj} g_{mn}(\vec{x})$$

$$g'_{ij}(\vec{x}_0') = S_{mi}\, g_{mn}(\vec{x}_0)\, S_{nj}$$

$$= \left(S^T g(\vec{x}_0)\, S\right)_{ij}$$

$$g(\vec{x}_0) = U^T \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} U$$

$$U^T U = \mathbb{1}$$

① Suppose λ_i are all positive ⇒ Euclidean signature

Take $S = U^{-1}\begin{pmatrix} (\sqrt{\lambda_1})^{-1} & & \\ & \ddots & \\ & & (\sqrt{\lambda_N})^{-1} \end{pmatrix}$

$$\Rightarrow g'(\vec{x}_0') = \mathbb{1} \Rightarrow g'_{ij}(\vec{x}_0') = \delta_{ij}$$

① One of the λ_i (say λ_1) negative.
(Minkowski signature)

Take $$S = U^{-1}\begin{pmatrix} (\sqrt{-\lambda_1})^{-1} & & & \\ & (\sqrt{\lambda_2})^{-1} & & \\ & & \ddots & \\ & & & (\sqrt{\lambda_n})^{-1} \end{pmatrix}$$

$$\Rightarrow \quad g'(\vec{x}_0{}') = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \equiv \eta$$

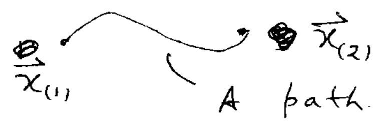
$\Rightarrow$ Minkowski metric.

Thus by a coordinate trs. we can bring the metric at a specific point to be of the form δ_{ij} (or η_{ij}) for Euclidean (or Minkowski) signature.

Geodesic : Generalization of a straight line to curved space.

Straight line in Euclidean spaces

→ Shortest path between two points.



$$x^i(u) \qquad 0 \le u \le 1$$

$$x^i(u=0) = x^i_{(1)}$$
$$x^i(u=1) = x^i_{(2)}$$

$$L = \int_0^1 ds = \int_0^1 \sqrt{g_{ij}(x)\,\frac{dx^i}{du}\frac{dx^j}{du}}\; du$$

Find $x^i(u)$ subject to the boundary condition that minimizes L.

$$\delta L = \int_0^1 du\,\frac{1}{2}\left(g_{mn}(x)\,\frac{dx^m}{du}\frac{dx^n}{du}\right)^{-1/2}$$

$$\left[\ \frac{\partial g_{ij}(x)}{\partial x^k}\,\delta x^k\,\frac{dx^i}{du}\,\frac{dx^j}{du}\right.$$

$$\left. +\ 2 g_{kj}(\vec{x})\,\frac{d}{du}(\delta x^k)\,\frac{dx^j}{du}\ \right]$$

$$\cancel{+\ g_{ij}(x)\,\frac{dx^i}{du}\,\frac{d}{du}(\delta x^j)}$$

G.21

$$= \int_0^1 du \left[\frac{1}{2} \left(g_{mn} \frac{dx^m}{du} \frac{dx^n}{du} \right)^{-1/2} \right.$$

$$\left\{ \frac{\partial g_{ij}}{\partial x^k} \delta x^k \frac{dx^i}{du} \frac{dx^j}{du} \right.$$

$$\left. - 2 \delta x^k \frac{d}{du} \left(g_{kj}(\vec{x}) \frac{dx^j}{du} \right) \right\}$$

$$- \delta x^k g_{kj}(\vec{x}) \frac{dx^j}{du}$$

$$\left. \frac{d}{du} \left(g_{mn}(\vec{x}) \frac{dx^m}{du} \frac{dx^n}{du} \right)^{-1/2} \right]$$

We choose the parameter u such that

$$g_{mn}(\vec{x}) \frac{dx^m}{du} \frac{dx^n}{du} = \text{constant}.$$

Equal interval in u $\equiv$ Equal distance.

$\Rightarrow$

$$\delta L = \int_0^1 du \; \frac{1}{2} \left(g_{mn} \frac{dx^m}{du} \frac{dx^n}{du} \right)^{-1/2}$$

$$\delta x^k g_{kj} \left(\frac{d^2 x^j}{du^2} + \Gamma^j_{i\ell} \frac{dx^i}{du} \frac{dx^\ell}{du} \right)$$

$\delta L = 0$

$$\Rightarrow \frac{d^2 x^j}{du^2} + \Gamma^j_{ik} \frac{dx^i}{du} \frac{dx^k}{du} = 0. \Rightarrow \text{Geodesic equation.}$$

Check that it implies $\frac{d}{du}\left(g_{mn} \frac{dx^m}{du} \frac{dx^n}{du}\right) = 0$

In the coordinate system $\Gamma^j_{ik} = 0$

$x^j = a^j u + b^j$ Eliminate u $\Rightarrow$ straight lines.

Ex. Check invariance under coord. trs.

Consider a geodesic $x^i(u)$.

u_0 : Some specific value of u.

$\vec{x}(u_0) \equiv \vec{x}_0$.

Choose a new coordinate system $\vec{x}'$ such that

$$\Gamma'^i_{jk}(\vec{x}'_0) = 0.$$

Geodesic equation around $\vec{x}' = \vec{x}'_0$:

$$\frac{d^2 x'^i}{du^2} + \underbrace{\Gamma'^i_{jk}(\vec{x}')}_{=\,O(\vec{x}' - \vec{x}'_0)} \frac{dx'^j}{du} \frac{dx'^k}{du} = 0$$

Solution:

$$x'^i = a^i(u - u_0) + x'^i_0 + c^i(u - u_0)^3 + \cdots$$

Note: No $b_i(u-u_0)^2$ term.

Will not satisfy the eqn. at $u = u_0$

$$u - u_0 = \frac{1}{a^1}(x'^1 - x'^1_0) + O\left((x'^1 - x'^1_0)^3\right)$$

$$\Rightarrow x'^i = \frac{a^i}{a^1}(x'^1 - x'^1_0) + O\left((x'^1 - x'^1_0)^3\right)$$

$\Rightarrow$ approximately a straight line.

Tangent space:

Consider a specific point $\vec{x}_0$ in the coordinate space.

Take a geodesic passing through it.

$$\frac{d^2x^i}{du^2} + \Gamma^i_{jk} \frac{dx^i}{du}\frac{dx^j}{du} = 0.$$

~~Take~~ Requires 2N boundary condition.

At $u=0$ $x^i = x^i_0 \Rightarrow N$ b.c.

Other b.c.'s:

$$\frac{dx^i}{du} = \underbrace{n^i}_{\text{Constants.}} \quad \text{at } u=0$$

$v = \underbrace{\alpha}_{\text{constant}} u$: Reparameterization.

$\Downarrow$ preserves the geodesic eqs.

$n^i \to \alpha^{-1} n^i$

$\Rightarrow$ Can be used to set the overall normalization of n^i.

$\vec{n}$: An N-dimensional vector.

This vector space is known as the tangent space.

Ex. check that the geodesic eqn. is invariant under coordinate trs.

Given any vector $\vec{n}$ at the tangent space at $\vec{x}_0$

$\Rightarrow$ A geodesic passing through $\vec{x}_0$.

A geodesic passing through $\vec{x}_0$

$\Rightarrow$ A vector in the tangent space at $\vec{x}_0$ up to an overall normalization.

Coordinate trs:

$$\vec{x} \to \vec{x}'$$

$$n'^i = \frac{dx'^i}{du} = \frac{\partial x'^i}{\partial x^k} \frac{dx^k}{du}$$

$$= \frac{\partial x'^i}{\partial x^k} n^k.$$

Note: We should distinguish this from a tensor of the kind described earlier.

" ~~A~~ A function of the metric defined at all $\vec{x}$.

In contrast $\vec{n}$ denotes the direction of a geodesic at a specific point $\vec{x}_0$.

$\vec{n}$ is not a function of $\vec{x}$, nor is it determined in terms of the metric.

Consider two points $\vec{x}_1$ & $\vec{x}_2$

$\vec{x}_1$ $\vec{x}_2$

$\vec{n}_1$: A vector in the tangent space at $\vec{x}_1$

$\vec{n}_2$: A vector in the tangent space at $\vec{x}_2$

Is there any sense in which we can ask if they are parallel?

Try: $\vec{n}_1 \parallel \vec{n}_2$ in the usual sense.

Ratios of components are equal.

But this definition is not invariant under a coordinate transformation.

$$n'^{i}_{1} = \left.\frac{\partial x'^{i}}{\partial x^{j}}\right|_{\vec{x}_1} n_1^j$$

$$n'^{i}_{2} = \left.\frac{\partial x'^{i}}{\partial x^{j}}\right|_{\vec{x}_2} n_2^j$$

Different matrices.

If $\vec{n}_1 \parallel \vec{n}_2$ it does not imply that $\vec{n}'_1 \parallel \vec{n}'_2$

In Euclidean space we are lucky because we had a preferred coordinate system in which g_{ij} is constant.

(G-26)

Strategy: Since it is difficult to compare two tangent vectors at far away points, let us try to compare them at nearby points.

$\vec{n}$ at $\vec{x}_0$, $\vec{m}$ at $\vec{x}_0+\delta\vec{x}_0$

Go to the special frame $\vec{x}'$ in which

$$\Gamma'^{i}_{jk}(\vec{x}'_0)=0$$

$$\vec{n}\to\vec{n}' \qquad \vec{m}\to\vec{m}'$$

related by parallel transport

We shall say that $\vec{n}$ and $\vec{m}$ are ~~vectors~~ if

$$\vec{n}' = \vec{m}' + O\left(\delta x_0\right)^2$$

Note: When we go back to a general coordinate system:

$$n^i = \left.\frac{\partial x^i}{\partial x'^j}\right|_{\vec{x}'_0} n'^j \quad , \quad m^i = \left.\frac{\partial x^i}{\partial x'^j}\right|_{\vec{x}'_0+\delta x'_0} m'^j$$

$$\Rightarrow \quad \vec{n} - \vec{m} = O\left(\delta\vec{x}_0\right)$$

Let us calculate this quantity:

$$m^i - n^i = \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\,\delta x'^k_0\, n'^j$$

$$\Gamma'^{i}_{jk}(\vec{x}'_0) = \frac{\partial x'^i}{\partial x^m}\frac{\partial x^n}{\partial x'^j}\frac{\partial x^l}{\partial x'^k}\Gamma^m_{ln} + \frac{\partial x'^i}{\partial x^m}\frac{\partial^2 x^m}{\partial x'^j\partial x'^k}$$

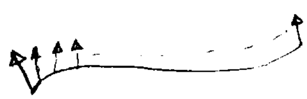
$$\Rightarrow \quad \frac{\partial^2 x^m}{\partial x'^j \partial x'^k} = - \frac{\partial x^n}{\partial x'^j} \frac{\partial x^l}{\partial x'^k} \Gamma^m_{ln}$$

$$\Rightarrow \quad m^i - n^i = - \frac{\partial x^n}{\partial x'^j} \frac{\partial x^l}{\partial x'^k} \Gamma^i_{ln} \, \delta x'^k_0 \, n'^j$$

$$= - \Gamma^i_{ln}(\vec{x}_0) \delta x_0^l \, n^k$$

Thus two vectors $\vec{m}$ and $\vec{n}$ are related by parallel transport if they satisfy the above equation.

Parallel transport along a curve:



$$\delta n^i + \Gamma^i_{lk}(\vec{x}_0) \delta x_0^l \, n^k = 0$$

$$\frac{dn^i}{du} + \Gamma^i_{lk}(\vec{x}) \frac{dx^l}{du} n^k = 0.$$

$\Rightarrow$ Equation for parallel transport along a curve.

Check: ① The eqn is invariant under a coordinate transformation

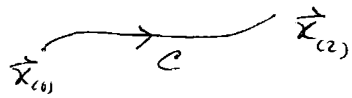
② $\frac{d}{du}\left(n^i n^j g_{ij}(x)\right) = 0$

$\Rightarrow$ Length of the vector does not change under parallel transport.

① Parallel trs. rules linear. Work in fixed coord. system.

If $\vec{n}(u)$ & $\vec{k}(u)$ satisfy ∥ trs. eqs.

then $\vec{n}_i(u) + \vec{k}_i(u)$ also does



$M(\vec{x}_{(1)}, \vec{x}_{(2)}; C)$: $N \times N$ matrix s.t.

n^i at $\vec{x}_{(1)}$ gets transported to

$M^i{}_j(\vec{x}_{(1)}, \vec{x}_{(2)}; C)\, n^j$ at $\vec{x}_{(2)}$.

Note:

$$M(\vec{x}_{(1)}, \vec{x}_{(2)}; C) = M^{-1}(\vec{x}_{(1)}, \vec{x}_{(2)}; -C)$$

in opposite direction.

Proof: Parallel transport is reversible.

Invariant under $u \to 1-u$.

If $\vec{n}$ gets transported to $\vec{m}$, then $\vec{m}$ gets transported back to $\vec{n}$.

$$\vec{m} = M\vec{n}$$
$$\vec{n} = M^{-1}\vec{m}$$

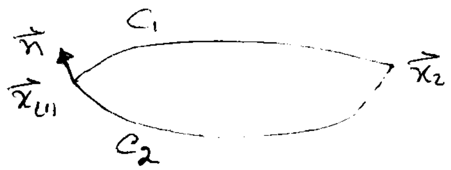
If C is closed $M(\vec{x}_{(1)}; \vec{x}_{(1)}; C)$ is known as holonomy matrix.

Ex. Check that

$$\partial_{\not{k}\,\ell}\left(g_{ij}(\vec{x})\, n^i n^j\right) = 0 \quad \text{if}$$

$$\frac{dn^i}{du} + \Gamma^i_{jk}\,\frac{dx^j}{du}\, n^k = 0$$

Q. 1 Take two curves from $\vec{x}_{(1)}$ to $\vec{x}_{(2)}$



Take a vector $\vec{n}$ at $\vec{x}_{(1)}$

Parallel transport it to $\vec{x}_2$ along C_1 & along C_2.

Result: $\vec{m}_1$ and $\vec{m}_2$ (say)

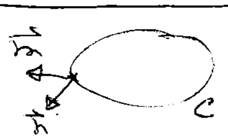
Q. Is $\vec{m}_1 = \vec{m}_2$?

Is $M(\vec{x}_{(1)}, \vec{x}_{(2)}; C_1) = M(\vec{x}_{(1)}, \vec{x}_{(2)}; C_2)$

Related Question:

Q.2. Take a closed curve C.

Parallel transport a vector $\vec{n}$ along C and return to the original point.



Result: $= \vec{m}$ (say)

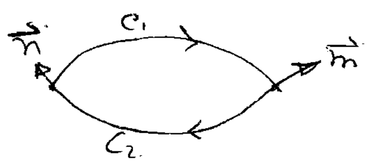
Q. Is $\vec{m} = \vec{n}$?

Is $M(\vec{x}_{(1)}, \vec{x}_{(1)}; C) = I$?

The two questions are equivalent.

Suppose the answer to 2 is yes for any $\vec{n}$ and any closed curve C.

Now ask Q. I.



$\vec{n} \to \vec{m}$ along C_1.
$\vec{m} \to \vec{n}$ along C_2.

But parallel transport is a reversible process.

If $\vec{m}$ gets parallel transported to $\vec{n}$ along reversed C_2, then ~~m~~ $\vec{n}$ gets parallel transported to $\vec{m}$ along C_2.

$$\frac{dn^i}{du} + \Gamma^i_{jk} \frac{dx^j}{du} n^k = 0$$

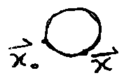
inv. under $u \to -u$.

$\Rightarrow$ $\vec{n}$ gets parallel transported to the same vector along C_1 & C_2.

$$M(\vec{x}_{(1)}, \vec{x}_{(1)}; C) = M(\vec{x}_{(1)}, \vec{x}_{(2)}; C_1) M(\vec{x}_{(2)}, x_{(1)}; C_2) = I$$

~~We~~ ~~Shall~~ Instead of trying to analyze $\parallel$ transport along a finite closed loop, we shall analyze it for an infinitesimal closed loop.

Parameter u = arc length.



$\vec{x}(u=0)=\vec{x}_0$, $\vec{x}(u=\epsilon)=\vec{x}_0$

$\epsilon =$ circumference ~~radius~~ of the loop.

$\frac{dx^i}{du}$ = finite

$$\frac{dn^i}{du} = -\Gamma^i_{jk}(\vec{x}(u))\, n^j(u)\, \frac{dx^k}{du}$$

$n^i = n^i_0$ at $u=0$.

We shall compute $n^i(u=\epsilon)-n^i(u=0)$ to second order in ϵ i.e. to the order of the area of the loop.

Use iterative scheme.

Set $n_i = n^i_0$, $\vec{x}(u)=\vec{x}_0$

$$\Rightarrow \frac{dn^i}{du} = -\Gamma^i_{jk}(\vec{x}_0)\, n^j_0 \frac{dx^k}{du}$$

$$\Rightarrow n^i = n^i_0 - \Gamma^i_{jk}(\vec{x}_0)\, n^j_0 (x^k - x^k_0) + O(\epsilon^2)$$

Substitute this and evaluate the r.h.s. to first order in ϵ

$$\frac{dn^i}{du} = -\{\Gamma^i_{jk}(\vec{x}_0) + \partial_l \Gamma^i_{jk}(\vec{x}_0)(x^l - x^l_0)\}$$
$$\{n^j_0 - \Gamma^j_{st}(\vec{x}_0)\, n^s_0 (x^t - x^t_0)\}\frac{dx^k}{du} + O(\epsilon^2)$$

$$= -\Gamma^i_{jk}\, n^j_0 \frac{dx^k}{du} - \partial_l \Gamma^i_{jk}(\vec{x}_0)(x^l - x^l_0)\frac{dx^k}{du} n^j_0$$
$$+ \Gamma^i_{jk}(\vec{x}_0)\, \Gamma^j_{st}(\vec{x}_0)\, n^s_0 (x^t - x^t_0)\frac{dx^k}{du} + O(\epsilon^2)$$

(G.31)

$$= -\Gamma^i_{jk}(\vec{x}_0) n_0^j \frac{dx^k}{du} + \left(\partial_t \Gamma^i_{sk}(\vec{x}_0) - \Gamma^i_{jk}(\vec{x}_0)\Gamma^j_{st}(\vec{x}_0)\right) n_0^s (x^t - x_0^t) \frac{dx^k}{du} + O(\epsilon^2)$$

$$n^i(u) = n_0^i - \Gamma^i_{jk} n_0^j (x^k(u) - x_0^k)$$

$$- \left\{\partial_t \Gamma^i_{sk}(\vec{x}_0) - \Gamma^i_{jk}(\vec{x}_0)\Gamma^j_{st}(\vec{x}_0)\right\} n_0^s \int_0^u (x^t(u') - x_0^t) \frac{dx^k(u')}{du'} du'$$

Antisymmetric in t, k

$x^k(u=\epsilon) - x_0^k = 0.$ $\quad n_i(u=0) = n_0^i$

$n^i(u=\epsilon) - n_i(u=0)$

$$= -\left\{\partial_t \Gamma^i_{sk}(\vec{x}_0) - \Gamma^i_{jk}(\vec{x}_0)\Gamma^j_{st}(\vec{x}_0)\right\} n_0^s$$

$$\times \underbrace{\int_0^\epsilon (x^t(u') - x_0^t) \frac{d}{du'}(x^k(u') - x_0^k)\, du'}_{\text{Antisymmetric in } k, t}$$

Do integration by parts.

Boundary terms vanish.

$n_0^i(u=\epsilon) - n^i(u=0)$

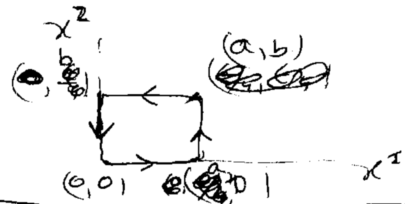
$$= -\frac{1}{2} R^i{}_{skt} n_0^s \int (x^t(u') - x_0^t) \frac{d}{du'}(x^k(u') - x_0^k)\, du'$$

$$\oint (x^t(u') - x_0^t)\, \frac{d}{du'}\, (x^k(u') - x_0^k)\, du'$$

$\neq 0$ in general.

---

related to the area of the ~~loop~~ loop.



$\vec{x}_0 = (0,0)$.

$$\oint x^t(u')\, \frac{d}{du'}\, x^k(u')\, du' \quad \text{for } t=1,\ k=2$$

$$= \underbrace{\int_{(0,0)}^{(a,0)} x^1\, dx^2}_{=0} + \underbrace{\int_{(a,0)}^{(a,b)} x^1\, dx^2}_{a\cdot b} + \underbrace{\int_{(a,b)}^{(0,b)} x^1\, dx^2}_{=0} + \underbrace{\int_{(0,b)}^{(0,0)} x^1\, dx^2}_{=0}$$

$\Rightarrow$ area of the loop.

Thus parallel transport along a small loop ~~vanishes~~ gives us back the original vector only if

$$R^i{}_{skt} = 0.$$

i.e. the Riemann tensor vanishes.

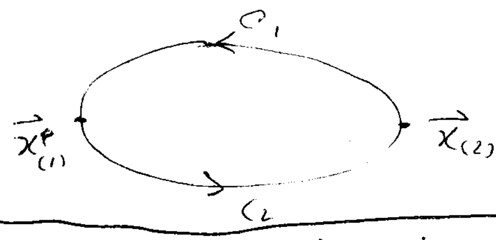
Otherwise we get non-trivial result.

~~Hence~~ If along every small loop there is no change under parallel transport, then even along big loops there is no change under $\parallel$ transport $\prod_{i=1}^{N} (I + c_i \epsilon^3) \to I$ if $N \to \infty$ with $\epsilon^2 N$ fixed.

Reduce the loop bit by bit. Need $N \sim \frac{1}{\epsilon^2}$ steps.

If $M(\vec{x}_{(1)}; \vec{x}_{(1)}; C) = I$, then for any other point $\vec{x}_{(2)} \in C$, $M(\vec{x}_{(2)}, \vec{x}_{(2)}; C) = I$.

Proof:



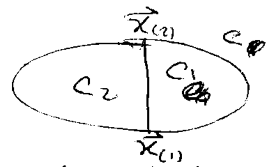
$$M(\vec{x}_{(2)}, \vec{x}_{(2)}; C) = M(\vec{x}_{(2)}, \vec{x}_{(1)}; C_1)\, M(\vec{x}_{(1)}, \vec{x}_{(2)}; C_2)$$

$$= M(\vec{x}_{(2)}, \vec{x}_{(1)}; C_1)\, M(\vec{x}_{(1)}, \vec{x}_{(2)}; C_2)$$

$$M(\vec{x}_{(2)}, \vec{x}_{(1)}; C_1)\, M^{-1}(x_{(2)}, x_{(1)}; C_1)$$

$$= M(\vec{x}_{(2)}, \vec{x}_{(1)}; C_1)\, M^{-1}(x_{(2)}, x_{(1)}; C_1) = I$$

(G.33)



If parallel transport along C_1 and C_2 does not affect $\vec{n}$, then neither ~~does~~ does, ∥ transport along C.

Now we can divide C_1 and C_2 each to half.

If the ∥ transport along C_3, C_4, C_5, C_6 does not change the vector then neither does ∥ transport along C.

Divide into N parts.

Area ~~of~~ of each $\sim 1/N$.

Circumference $\sim \frac{1}{\sqrt{N}}$

We have seen that if $R^i{}_{jkl} = 0$, then ∥ transport of a vector produces a change of order $\frac{1}{N^{3/2}}$ at most.

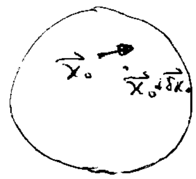
Total change under ∥ transport along $C \sim N \times \frac{1}{N^{3/2}} = \frac{1}{N^{1/2}}$.

N can be taken as large as we want.

Take $N \to \infty$.

$\Rightarrow$ Total change $= 0$.

Parallel transport on a sphere (Geometric meaning)



$\vec{n}(\vec{x}_0)$: A tangent vector at $\vec{x}_0$.

Can be regarded as a 3-d vector.

Take ~~a~~ vector $\parallel$ $\vec{n}$ (in 3-d sense at $\vec{x}_0+\delta\vec{x}_0$).

$\Rightarrow$ This vector is not a tangent vector along the sphere.

$\Rightarrow$ We need to take the projection of this along the tangent plane.

This is the meaning of $\parallel$ transport.

Does this definition agree with the intrinsic definition using the intrinsic metric?

It does.

y^μ $(\mu=1,2,3)$: Coordinates of Euclidean space.

x^i $(i=1,2)$: Intrinsic coordinates on the sphere.

$n^i(\vec{x}_0)$: An intrinsic tangent vector at $\vec{x}_0$.

(G-35)

Ex. Show that

① The vector $n^i(\vec{x}_0)$ corresponds to a 3-d vector

$$N^\mu(\vec{x}_0) = n^i(\vec{x}_0)\left(\frac{\partial y^\mu}{\partial x^i}\right)_{\vec{x}_0} \quad \text{in 3-dimension}$$

---

② ~~Let~~ Now consider the same 3-dimensional vector $N^\mu(\vec{x}_0)$ at $\vec{x}_0 + \delta\vec{x}_0$. Show that its projection in the tangent plane ~~corr~~ corresponds to a ~~two~~ three ~~dimens~~ dimensional vector:

$$N^\mu(\vec{x}_0 + \delta\vec{x}_0) = N^\nu(\vec{x}_0) \left.\frac{\partial y^\nu}{\partial x^i}\right|_{\vec{x}_0+\delta\vec{x}_0} g^{ij}(\vec{x}_0+\delta\vec{x}_0)\left(\frac{\partial y^\mu}{\partial x^j}\right)_{\vec{x}_0+\delta\vec{x}_0}$$

$g_{ij}(\vec{x})$: Induced metric on the sphere from 3-d Euclidean metric:

$$g_{ij}(\vec{x}) = \frac{\partial y^\mu}{\partial x^i}\frac{\partial y^\mu}{\partial x^j}$$

---

③ This can now be regarded as a two dimensional tangent vector at $\vec{x}_0 + \delta\vec{x}_0$:

$$n^j(\vec{x}_0 + \delta\vec{x}_0) = N^\nu(\vec{x}_0)\left(\frac{\partial y^\nu}{\partial x^i}\right)_{\vec{x}_0+\delta\vec{x}_0} g^{ij}(\vec{x}_0+\delta\vec{x}_0)$$

---

④ Show that:

$$n^i(\vec{x}_0 + \delta\vec{x}_0) - n^i(\vec{x}_0) = -\Gamma^i_{jk}(\vec{x}_0)\, n^j\, \delta x_0^k$$

→ The usual ‖ transport formula.

Properties of the Riemann Tensor:

① Symmetry properties:

Define:

$$R_{ijkl} = g_{is} R^{s}{}_{jkl}$$

Ex. Show that

(a) $R_{ijkl} = -R_{ijlk} = -R_{jikl} = R_{jilk}$.

(b) $R_{ijkl} = R_{klij}$

(c) $R_{ijkl} + R_{iklj} + R_{iljk} = 0$

② Definition:

(a) $R_{ij} = R^{k}{}_{ikj} = g^{kl} R_{likj}$

→ Ricci Tensor

(b) $R = g^{ij} R_{ij}$ → scalar curvature

Differential Identities: | Ex. $R_{ij} = R_{ji}$

Ex. show that

$$D_s R_{ijkl} + D_k R_{ijls} + D_l R_{ijsk} = 0.$$

→ Bianchi Identity.

Ex. Show that:

$$D_i \left(A^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \; B^{k_1 \cdots k_{p'}}{}_{\ell_1 \cdots \ell_{q'}} \right)$$

$$= \left(D_i \; A^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \right) B^{k_1 \cdots k_{p'}}{}_{\ell_1 \cdots \ell_{q'}}$$

$$+ A^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \; D_i \; B^{k_1 \cdots k_{p'}}{}_{\ell_1 \cdots \ell_{q'}}$$

Corrolary:

$$D_s R_{ij} = D_s \, g^{k\ell} R_{ki\ell j}$$

$$= \underbrace{(D_s g^{k\ell})}_{=0} R_{ki\ell j} + g^{k\ell} D_s R_{ki\ell j}$$

Take the Bianchi identity and contract with g^{ik}.

$$g^{ik} \left[D_s R_{ijk\ell} + D_k R_{ij\ell s} + D_\ell R_{ijsk} \right] = 0$$

$$\Rightarrow \quad D_s R_{j\ell} + g^{ik} D_k R_{ij\ell s} - D_\ell R_{js} = 0$$

Multiply by g^{js}

$$g^{js} D_s R_{j\ell} + g^{ik} D_k R_{i\ell} - D_\ell R = 0.$$

$$\Rightarrow \quad g^{ik} D_k R_{i\ell} - \tfrac{1}{2} D_\ell R = 0.$$

$$D_k \, g^{ki} \left[R_{i\ell} - \tfrac{1}{2} g_{i\ell} R \right] = 0$$

Einstein tensor

Using these techniques we can now consider Riemannian generalization of Minkowski space.

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu \qquad \eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$x^0 = ct$, x^1, x^2, x^3 · space coordinates

$\Rightarrow \quad ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu = -d\tau^2$ Proper time

$\Downarrow$

Signature $(-1, 3)$.

Do these spaces have any physical significance?

Answer: They describe the presence of gravitational field.

In the absence of gravity a particle travels in a straight line.

In the presence of gravity a particle travels along a geodesic.

$ds^2 < 0$ along a time-like geodesic

$$\frac{d^2 x^\mu}{du^2} + \Gamma^\mu_{\nu e} \frac{dx^\mu}{du} \frac{dx^\nu}{du} = 0$$

Take $du^2 = -ds^2$

$\Rightarrow g_{\mu\nu} \frac{dx^\mu}{du}\frac{dx^\nu}{du} = -1 = \text{const.}$

To show that this describes the effect of gravity, we need to show that at least in some limit this describes the motion of a particle under Newtonian gravitational potential.

Which limit?

① Non relativistic : $\left|\frac{dx^0}{du}\right| >> \left|\frac{dx^i}{du}\right|$

② Static Potential $g_{\mu\nu}$ is time independent.

③ Weak gravitational field

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + \underbrace{h_{\mu\nu}}_{\text{small}}$$

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{00}\left(\frac{dx^0}{du}\right)^2 = 0$$

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\nu}(\underbrace{\partial_0 g_{\nu 0}}_{0} + \underbrace{\partial_0 g_{\nu 0}}_{0} - \partial_\nu g_{00})$$

$$= -\frac{1}{2} g^{\mu\nu}\partial_\nu g_{00}$$

$$g = \eta + h = \eta(1 + \eta^{-1}h)$$

$$g^{-1} = (1+\eta^{-1}h)^{-1}\eta^{-1}$$

$$= (1-\eta^{-1}h)\eta^{-1} = \eta^{-1} - \eta^{-1}h\eta^{-1}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\rho}h_{\rho\sigma}\eta^{\sigma\nu}$$

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00}$$

$$\Gamma^0_{00} = 0, \quad \Gamma^i_{00} = -\frac{1}{2}\eta^{ij}\partial_j h_{00} = -\frac{1}{2}\partial_i h_{00}$$

$$\frac{d^2x^0}{du^2} = 0 \Rightarrow x^0 = au$$

$$\frac{d^2x^i}{du^2} - \frac{1}{2}\partial_i h_{00}\, a^2 = 0 \Rightarrow \frac{1}{c^2}\frac{d^2x^i}{dt^2} - \frac{1}{2}\partial_i h_{00} = 0$$

$$\frac{d^2x^i}{dt^2} = +\frac{1}{2}\,\partial_i\left(c^2 h_{00}\right) = -\partial_i V$$

$V = -\frac{1}{2} c^2 h_{00} \rightarrow$ Newtonian gravitational potential

---

Note: Finite $V \Rightarrow$ small h_{00} since c is large in ordinary units (cm, sec etc.)

$\Rightarrow$ Eqn. of motion of a particle under a Newtonian gravitational potential.

Note: Geodesic is a property of the space and not of the particle moving on it.

$\Rightarrow$ All particles (irrespective of their mass, charge etc.) follow the same trajectory.

This is also a property of Newtonian gravity.

$(x^0_{(0)}, \cdots x^3_0)$: A space-time point.

We can choose a coordinate system $\vec{x}'$ such that in this coordinate system:

$$\Gamma'^{\mu}_{\nu\rho}(\vec{x}'_{(0)}) = 0.$$

$\Rightarrow$ Geodesic equation:

$$\frac{d^2 x'^{\mu}}{du^2} + \Gamma'^{\mu}_{\nu\rho} \frac{dx'^{\rho}}{du} \frac{dx'^{\nu}}{du} = 0$$

$\Rightarrow$ At $\vec{x}'_{(0)}$ $\quad \dfrac{d^2 x'^{\mu}}{du^2} = 0.$

$\Rightarrow$ Equation of motion of a particle in the absence of gravity.

Thus :

For a particle travelling under the action of gravity, and a given space-time point $\vec{x}_{(0)}$ on its trajectory, it is always possible to find a coordinate frame in which the equations of motion of the particle at $\vec{x}_0$ takes the form of the equations of motion of a free particle in the absence of gravity.

For $u = \tau$

$\Rightarrow$ Principle of equivalence. | 'Not automatic consequ... of g.c.i. e.g. $\frac{d^2x^{\mu}}{du^2} + \Gamma^{\mu}_{\nu\rho}\frac{dx'^{\nu}}{du}\frac{d...}{d...} + R^{\mu}_{\nu}\frac{dx...}{d...}$

This particular coordinate system is known as locally inertial frame...

① $D_i g_{kl} = 0$.

② Parallel transport of a tangent vector
$\Rightarrow$ a tangent vector.

③ Equality of the two defns of ∥ transport
for spaces embedded in Euclidean space

④ Symmetries of R_{ijkl}:

$$R_{ijkl} = -R_{jikl} = R_{klij}$$

⑤ $R_{ijkl} + R_{iklj} + R_{iljk} = 0$.

⑤ Bianchi Identities

$$D_s R_{ijkl} + D_k R_{ijls} + D_l R_{ijsk} = 0.$$

Requirements of an eqn. of motion.
1. G.C.I.
2. Free particle eqn. for $g_{\mu\nu}=\eta_{\mu\nu}$
3. Principle of equivalence

Massless particles

σ can no longer be taken to be proper time.

$$d\tau^2 = -g_{\mu\nu}\, dx^\mu\, dx^\nu = 0$$

Use Principle of equivalence to write down the eqs of motion in x' coordinate system.

Then go back to the original coordinates.

$$\frac{d^2 x'^\alpha}{du^2} = 0 \qquad \text{At } \vec{x}' = \vec{x}'_{(0)}$$

$du^2 = -\eta_{\alpha\beta}\, dx'^\alpha\, dx'^\beta$ for massive particle

$$\eta_{\alpha\beta} \frac{dx'^\alpha}{du} \frac{dx'^\beta}{du} \begin{matrix} = 0 \rightarrow \text{massless} \\ < 0 \rightarrow \text{massive} \end{matrix}$$

$u = x'^0$ for massless particle

$$\frac{d}{du}\left(\frac{\partial x'^\alpha}{\partial x^\beta} \frac{dx^\beta}{du} \right) = 0$$

$$\Rightarrow \frac{\partial x'^\alpha}{\partial x^\beta} \frac{d^2 x^\beta}{du^2} + \frac{\partial^2 x'^\alpha}{\partial x^\beta \partial x^\gamma} \frac{dx^\gamma}{du} \frac{dx^\beta}{du} = 0$$

$$\Rightarrow \frac{d^2 x^\sigma}{du^2} + \frac{\partial x^\sigma}{\partial x'^\alpha} \frac{\partial^2 x'^\alpha}{\partial x^\beta \partial x^\gamma} \frac{dx^\gamma}{du} \frac{dx^\beta}{du} = 0 \qquad \text{At } \vec{x}_{(0)}$$

$$\Gamma^\alpha_{\beta\gamma} = \underbrace{\Gamma'^\mu_{\nu\sigma}}_{0 \text{ at } \vec{x}' = \vec{x}'_{(0)}} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\beta} \frac{\partial x'^\sigma}{\partial x^\gamma} + \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\gamma}$$

$$\Rightarrow \Gamma^\alpha_{\beta\gamma} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\gamma} \qquad \text{at } \vec{x}_{(0)}$$

$\Rightarrow$ Eqs of motion

$$\frac{d^2 x^\sigma}{du^2} + \Gamma^\sigma_{\beta\gamma} \frac{dx^\beta}{du} \frac{dx^\gamma}{du} = 0$$

$$ds^2 = \eta_{\alpha\beta}\, dx'^\alpha\, dx'^\beta = g_{\alpha\beta}(\vec{x})\, dx^\alpha\, dx^\beta$$

< 0 for massive
$= 0$ for massless

$$\eta_{\alpha\beta} \frac{dx'^{\alpha}}{du} \frac{dx'^{\beta}}{du} = 0$$

$$\Downarrow$$

$$\underbrace{\eta_{\alpha\beta} \frac{\partial x'^{\beta}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}}}_{g_{\mu\nu}(\vec{x}) \text{ at } \vec{x} = \vec{x}(u)} \frac{dx^{\mu}}{du} \frac{dx^{\nu}}{du} = 0$$

$$\Rightarrow \quad g_{\mu\nu}(\vec{x}) \frac{dx^{\mu}}{du} \frac{dx^{\nu}}{du} = 0.$$

$\Rightarrow$ Condition on the trajectory of a massless particle.

An application of the eqs. of motion of a free particle

→ Gravitational red-shift.

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$d\tau^2 = -\, g_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$\Rightarrow \quad g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1.$$

Consider a particle at rest : $\Rightarrow \frac{dx^i}{d\tau} = 0.$

$$g_{00} \left(\frac{dx^0}{d\tau}\right)^2 = -1$$

$$\frac{dx^0}{d\tau} = (-\, g_{00})^{-1/2}$$

Take a clock

$\Delta\tau$ = Proper time taken by the second hand to move one step.

= 1 second.

(In the locally inertial frame the second hand takes one second to move one step).

Time elapsed during this process:

$$\Delta x^0 = (-g_{00})^{-1/2} \times 1 \text{ second}.$$

If $(-g_{00})^{1/2} \leq 1$, the clock moves slower.

if $(-g_{00})^{1/2} > 1$ the clock moves faster

But everything else also moves slower. So we shall not feel it.

In order to see this we have to compare clocks at two different points with different values of g_{00}.

$$\Delta x^0 = \left(-g_{00}(\vec{x}_1)\right)^{-1/2} \times 1 \text{ second} \qquad \Delta x^0 = \left(-g_{00}(\vec{x}_2)\right)^{-1/2} \times 1 \text{ second}$$

Thus the ratio of the ~~rate at which the~~ time taken by the second hand to move 1 second:

$$\frac{\Delta t^{(1)}}{\Delta t^{(2)}} = \left(\frac{g_{00}(\vec{x}_1)}{g_{00}(\vec{x}_2)}\right)^{-1/2}$$

Larger $g_{00} \Rightarrow$ lower $\Delta t^{(1)}$.

In the non-relativistic limit:

$$g_{00} = -1 - 2V/c^2$$

$$\frac{\Delta t^{(1)}}{\Delta t^{(2)}} = \left\{\frac{1 + 2V(\vec{x}_1)/c^2}{1 + 2V(\vec{x}_2)/c^2}\right\}^{-1/2}$$

$$= 1 + \frac{V(\vec{x}_2) - V(\vec{x}_1)}{c^2}, \ldots$$

$\nu = 1/\Delta t$

$$\frac{\nu_2}{\nu_1} = 1 + \frac{\Delta V}{c^2} \qquad \Delta V = V(\vec{x}_2) - V(\vec{x}_1)$$

$\nu_2 = \nu + \Delta\nu \qquad \nu_1 = \nu$

$$\frac{\Delta\nu}{\nu} = \frac{\Delta V}{c^2}$$

Lower potential $\Rightarrow$ lower frequency.

Gravitational potential near a point mass M:

$$V = -\frac{GM}{r^2}$$

$$\frac{\Delta\nu}{\nu} = -\frac{GM}{c^2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

$\rightarrow$ Can be experimentally verified.

$\Delta\nu \rightarrow$ Frequency of a spectral line, or of an accurate clock.

Note: V near a point mass $< V$ at ∞.

$\Rightarrow \nu$ near a point mass $< \nu$ at ∞.

$\Rightarrow$ Gravitational red shift.

Universality of Principle of equivalence.

→ Can be applied to any system, not just a free particle.

Example I. Consider a charged particle moving under the simultaneous influence of electromagnetic and gravitational field..

Q. What will be its equations of motion?

A: ① In the locally inertial frame the eq. of motion will be that of a relativistic particle moving under the influence of the electromagnetic field.

② Convert this equation back to the general coordinate system.

• Eq. of motion of a relativistic charged particle in background electromagnetic field:

$$m \frac{d^2 x^\mu}{d\tau^2} = f^\mu$$

↳ relativistic force.

$$f^\mu = e\, \eta^{\mu\nu} F_{\nu\rho} \frac{dx^\rho}{d\tau}$$

[set $c = 1$]

$F_{\nu\rho}$: Electromagnetic field strength tensor:

$$F_{\nu\rho} = \partial_\nu A_\rho - \partial_\rho A_\nu$$

(A_0, A_1, A_2, A_3) Vector potential.

$$E_i \equiv -F_{0i} = -\partial_0 A_i + \partial_i A_0$$

$$B_i \equiv \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk} F_{jk}$$

$$\epsilon_{123} = 1$$

~~$\epsilon_{ijk} = 0$ if any two~~

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$$

$$B_1 = F_{23} = \partial_2 A_3 - \partial_3 A_2$$

$$\vdots$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \qquad \phi = -A_0$$

$$m \frac{d^2 x^i}{d\tau^2} = e\, F_{i\rho} \frac{dx^\rho}{d\tau} = e\, F_{ij} \frac{dx^j}{d\tau} + e F_{i0} \frac{dx^0}{d\tau}$$

$$m \frac{d}{d\tau}\left(\frac{dx^i}{d\tau}\right) = e\, E_i \frac{dx^0}{d\tau} + e\left(\partial_i A_j - \partial_j A_i\right)\frac{dx^j}{d\tau}$$

$$p^i/m \qquad = e\, E_i \frac{dx^0}{d\tau} + e\, \epsilon_{ijk} B_k \frac{dx^j}{d\tau}$$

$$\frac{d p^i}{dt} = e\, E_i + e\, \epsilon_{ijk} \frac{dx^j}{dt} B_k$$

$$m \frac{d\vec{p}}{dt} = e\vec{E} + e\, \vec{v} \times \vec{B} \Rightarrow \text{correct equation}$$

Now we shall generalize this to curved space.

A_μ : A ~~contra~~ covariant vector

$$A'_\mu(\vec{x}') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(\vec{x})$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$= D_\mu A_\nu - D_\nu A_\mu$$

→ A covariant rank two tensor.

In locally inertial frame:

$$m \frac{d^2 x'^\mu}{d\tau^2} = e\, \eta^{\mu\nu} F'_{\nu\rho} \frac{dx'^\rho}{d\tau}$$

⇓ A general frame.

⇓

$$\frac{\partial x'^\mu}{\partial x^\nu} m \left[\frac{d^2 x^\nu}{d\tau^2} + \Gamma^\nu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} \right]$$

$$= e\, \eta^{\mu\nu} \frac{\partial x^\sigma}{\partial x'^\nu} \underbrace{\frac{\partial x^\theta}{\partial x'^\rho} \frac{dx'^\rho}{d\tau}}_{\frac{dx^\theta}{d\tau}} F_{\sigma\theta}(\vec{x})$$

$$\times \frac{\partial x^k}{\partial x'^\mu}$$

$$\Rightarrow m \left(\frac{d^2 x^k}{d\tau^2} + \Gamma^k_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} \right)$$

$$= e\, \underbrace{\frac{\partial x^k}{\partial x'^\mu} \eta^{\mu\nu} \frac{\partial x^\sigma}{\partial x'^\nu}}_{g^{k\sigma}} F_{\sigma\theta} \frac{dx^\theta}{d\tau}$$

<u>Check.</u>

$\circ$ $\vec{x}^{(2)}$ $\qquad\qquad$ $\bullet$ $\vec{x}^{(1)}$

A clock sitting at each point.

Proper time elapsed between two successive pulse $= \Delta\tau$.

Clock at $\vec{x}^{(1)}$:

First pulse at x_0; Signal reaches $\vec{x}^{(2)}$ at $x^0 + T$

Second pulse at $x_0 + \Delta x^0$; signal reaches $\vec{x}^{(2)}$ at $x^0 + \Delta x^0 + T$

$$\Delta\tau = \sqrt{-g_{00}(\vec{x}^{(1)})} \cdot \Delta x^0$$

$$\Delta x^0 = \frac{\Delta\tau}{\sqrt{-g_{00}(\vec{x}^{(1)})}}$$

Now calculate the proper time elapsed for the clock at $\vec{x}^{(2)}$ between these two signals:

$$\Delta\tilde{\tau} = \Delta x^0 \cdot \sqrt{-g_{00}(\vec{x}^{(2)})}$$

$$= \Delta\tau \sqrt{\frac{g_{00}(\vec{x}^{(2)})}{g_{00}(\vec{x}^{(1)})}}$$

Thus

$$\frac{\text{Period of clock at } \vec{x}^{(2)}}{\text{Period of clock at } \vec{x}^{(1)}} = \frac{\Delta\tau}{\Delta\tilde{\tau}} = \sqrt{\frac{g_{00}(\vec{x}^{(1)})}{g_{00}(\vec{x}^{(2)})}}$$

These equations tell us how a charged particle moves in a given ~~electric~~ electromagnetic and gravitational field background.

Q. ~~How does a particle produce~~ What are the equations which determine how ~~an~~ electromagnetic and gravitational ~~produce~~ fields are produced in the presence of a charged particle?

Electromagnetic field: Relativistic field equations are known.

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\eta^{\mu\nu}\,\partial_\mu F_{\nu\rho}(x) = J^{\kappa}(x)\,\eta_{\rho\kappa}$$

Four vector current density:

$$(J^0, \vec{J}) = \left(\rho(\vec{x}), \vec{J}(\vec{x})\right)$$

$E_i = -F_{0i}$

$F_{ij} = \epsilon_{ijk} B_k$

$$-\partial_0 F_{0\rho} + \partial_i F_{i\rho} = J^{\kappa}\,\eta_{\rho\kappa}$$

$\Downarrow$

$$\rho = 0: \Rightarrow \; +\vec{\nabla}\cdot\vec{E} = \rho$$

$$\rho = j \Rightarrow \; +\frac{\partial E_j}{\partial t} + \partial_i \epsilon_{ijk} B_k = -J^j$$

$$\vec{\nabla}\times\vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}$$

What is J^μ?

$$\rho(\vec{x},t) = \sum_n e_n \, \delta^3(\vec{x} - \vec{x}_n(t))$$

$\vec{x} = \vec{x}_n(t)$: nth Particle trajectory.

$$J^i(\vec{x},t) = \sum_n e_n \, \delta^3(\vec{x}-\vec{x}(t)) \, \frac{dx_n^i}{dt}$$

$x^\mu = X_n^\mu(u)$: Parametric form of the trajectory.

x^0: monotone increasing fn. of u.

$$J^\mu(\vec{x},t) = \int du \sum_n e_n \, \delta^4(x^\mu - X_n^\mu(u)) \, \frac{dX_n^\mu}{du}$$

Proof:

$$J^0(\vec{x},u) = \int du \sum_n e_n \, \delta^4(x^\mu - X_n^\mu(u)) \, \frac{dX_n^0}{du}$$

$$\| $$

$$\delta^3(x^i - X_n^i(u)) \, \delta(x^0 - X_n^0(u))$$

$$= \sum_n e_n \delta^3(x^i - x_n^i(t)) \left(\frac{dX_n^0}{du}\right)^{-1} \frac{dX_n^0}{du}$$

$$= \sum_n e_n \, \delta^3(x^i - x_n^i(t)) \quad ✓$$

$$J^i(\vec{x},u) = \int du \sum_n e_n \, \delta^4(x^\mu - X_n^\mu(u)) \, \frac{dX_n^i}{du}$$

$$= \sum_n e_n \, \delta^3(x^i - x_n^i(t)) \underbrace{\left(\frac{dX_n^0}{du}\right)^{-1} \frac{dX_n^i}{du}}_{\frac{dx_n^i}{dt}} \quad ✓$$

Thus in the presence of a set of charged particles moving along different trajectories, the electromagnetic field equations take the form:

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\nu F_{\rho\mu} = 0. \rightarrow \text{Bianchi Identities.}$$

$$g^{\mu\nu} \partial_\mu F_{\nu\rho} = -\eta_{\rho\kappa} J^\kappa$$

$$J^\mu = \sum_n e_n \int du \; \delta^{(4)}(x^\nu - X_n^\nu(u)) \frac{dX_n^\mu}{du}$$

$x^\mu = X_n^\mu(u) \Rightarrow$ trajectory of the n-th particle.

$$\eta^{\mu\nu}\partial_\mu F_{\nu\rho}(x) = -\eta_{\rho\kappa} J^\kappa(x)$$

$$\eta^{\rho\kappa}\eta^{\mu\nu}\partial_\mu F_{\nu\rho}(x) = -J^\kappa(x)$$

Consistency of this equation:

$$\underbrace{\eta^{\rho\kappa}\eta^{\mu\nu}\partial_\mu\partial_\kappa F_{\nu\rho}(x)}_{=0} = -\partial_\kappa J^\kappa(x)$$

$\Rightarrow$ we need $\partial_\kappa J^\kappa(x) = 0$

$$\partial_0 J^0 + \partial_i J^i(\vec{x}) = 0.$$

$$\partial_0 \int \underbrace{J^0}_{=\rho} d^3x = -\int \underbrace{\partial_i J^i}_{=\vec{\nabla}\cdot\vec{J}} d^3x$$

$$\underbrace{\frac{d}{dt}\int \rho\, d^3x}_{\text{Change in total charge in a volume.}} = \underbrace{-\int \vec{J}\cdot\vec{ds}}_{\text{Total charge flowing out of the volume.}}$$

Is $\partial_\kappa J^\kappa = 0$?

$$J^\mu = \sum_n e_n \int du\, \delta^{(4)}(x^\nu - X_n^\nu(u)) \frac{dX_n^\mu}{du}$$

$$\delta^{(4)}(x^\nu - X_n^\nu(u)) = \prod_{\nu=1}^{4} \delta(x^\nu - X_n^\nu(u))$$

$$\partial_\mu J^\mu = \sum_n \sum_\mu e_n \int du \left\{ \prod_{\nu \neq \mu} \delta(x^\nu - X_n^\nu) \right\} \delta'(x^\mu - X_n^\mu(u)) \frac{dX_n^\mu}{du}$$

$$= \sum_n \sum_\mu e_n \int du \left\{ \prod_{\nu \neq \mu} \delta(x^\nu - X_n^\nu) \right\} \frac{d}{du} \delta(x^\mu - X_n^\mu) \left(- \frac{dX_n^\mu}{du} \right)^{-1} \frac{dX_n}{du}$$

$$= - \sum_n \int du \frac{d}{du} \left(\prod_\nu \delta(x^\nu - X_n^\nu(u)) \right)$$

$$= - \sum_n \prod_\nu \delta(x^\nu - X_n^\nu(u)) \Big|_{u=-\infty}^{\infty}$$

For any finite x^0, this vanishes as

$X_n^0(\pm\infty) = \pm\infty$

$\Rightarrow \partial_\mu J^\mu = 0.$

Generalization to curved space.

$$\eta^{\mu\nu}\eta^{\kappa\rho}\partial_\mu F_{\nu\rho}(x) = -J^\kappa(x)$$

Principle of equivalence $\Rightarrow$ This is the equation of motion of the electromagnetic field in the locally inertial frame.

(where $g'_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma'^{\kappa}_{\nu\rho} = 0$ at $x'^{\kappa} = x'^{\kappa}_0$)

From this we can get the equations of motion in a general coordinate system by coordinate transformation

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Answer:

$$g^{\mu\nu} g^{\kappa\rho} D_\mu F_{\nu\rho} + J^\kappa(x) = 0.$$

$$J^\kappa(x) = \sum_n e_n \int du\, \delta^{(4)}(x - x_n(u))\, (\det g)^{-1/2} \frac{dx_n^\kappa}{du}$$

Proof: ① The left hand side is a contravariant vector

② In the locally inertial frame the l.h.s. ~~reduces~~ ~~to~~ vanishes by equations of motion.

$\Rightarrow$ It must vanish in all coordinate system.

$$J'^{k}(x_0') = \sum_n e_n \int du\ \delta^{(4)}(x' - X_n'(u))\, du\ \frac{dX_n'^{k}(u)}{du}\Big|_{x'=x_0'}$$

$$= \sum_n e_n \int du \left(\det \frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)^{-1} \delta^{(4)}(x - X_n(u))$$

$$\frac{\partial x'^{k}}{\partial x^{\mu}} \frac{dX_n^{\mu}(u)}{du}\Big|_{x=x_0}$$

$$= \left(\frac{\partial x'^{k}}{\partial x^{\mu}}\right)\left(\det \frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)^{-1} \sum_n e_n \int du\ \delta^{(4)}(x - X_n(u))\ \frac{dX_n^{\mu}(u)}{du}\Big|_{x=x_0}$$

$$g'_{\mu\nu} = g_{\rho\sigma} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}}$$

$$\underset{\overset{\parallel}{-1}}{\det g'} = \det g \cdot \left(\det \frac{\partial x}{\partial x'}\right)^2$$

$$\Rightarrow \left(\det \frac{\partial x}{\partial x'}\right) = \sqrt{-\det g}$$

$$\Rightarrow J'^{k}(x_0') = \frac{\partial x'^{k}}{\partial x^{\mu}} \underbrace{\sqrt{-\det g} \sum_n e_n \int du\ \delta^{(4)}(x - X_n(u)) \frac{dX_n^{\mu}(u)}{du}}_{\overset{|||}{J^{\mu}(u)}}$$

– in curved space.

Ex. Show that:

$$\eta^{\sigma\nu}\eta^{k\rho}\,\partial_{\sigma}' F_{\nu\rho}'(x_0')$$

$$= \frac{\partial x'^{k}}{\partial x^{\mu}} \cdot g^{\sigma\nu} g^{\mu\rho}\, D_{\sigma} F_{\nu\rho}(x_0)$$

$$\Rightarrow g^{\sigma\nu} g^{\mu\rho} D_{\sigma} F_{\nu\rho} + J^{\mu} = 0.$$

A general result about δ-function:

$x^m \quad 1 \le m \le N$: m variables.

$f^m(\vec{x})$: A set of m functions of $x^1, \cdots x^N$.

$$f^m(\vec{x}) = 0 \quad \text{at} \quad x^m = a^m.$$

Then

$$\delta^{(N)}(\vec{f}(x)) \equiv \prod_{m=1}^{N} \delta(f^m(x))$$

$$= \left\{ \left| \det \frac{\partial f^i}{\partial x^j} \right|_{a} \right\}^{-1} \delta^{(N)}(\vec{x} - \vec{a})$$

Proof: $F(\vec{x})$: Any function of $\vec{x}$

$$I = \int d^N x \; F(\vec{x}) \; \delta^{(N)}(\vec{f}(\vec{x}))$$

$$y^m = f^m(x).$$

$$d^N y = \left| \left\{ \det \frac{\partial f^i}{\partial x^j} \right\} \right| d^N x \qquad \left(\text{Assume } \det \frac{\partial f^i}{\partial x^j} > 0 \right)$$

$$I = \int d^N y \left| \left\{ \det \frac{\partial f^i}{\partial x^j} \right\}^{-1} \right| F(\vec{x}) \; \delta^{(N)}(\vec{y})$$

$$= \left| \left\{ \det \frac{\partial f^i}{\partial x^j} \right\}^{-1} \right|_{\vec{x} = \vec{a}} F(\vec{a})$$

$$= \int d^N x \left| \left\{ \det \frac{\partial f^i}{\partial x^j} \right\}^{-1} \right| \delta^{(N)}(\vec{x} - \vec{a}) \; F(\vec{x})$$

True for all $F(\vec{x})$.

$$\Rightarrow \quad \delta^{(N)}(\vec{f}(\vec{x})) = \left| \left\{ \det \frac{\partial f^i}{\partial x^j} \right\}^{-1} \right| \delta^{(N)}(\vec{x} - \vec{a})$$

Single variable: $\delta(f(x)) = (f'(x))^{-1} \, \delta(x - a)$

Maxwell's equations in covariant form:

$$\eta^{\mu\nu}\eta^{\kappa\rho}\partial_\mu F_{\nu\rho} = -J^\kappa$$

$$J^\kappa = \sum_n e_n \int_{-\infty}^{\infty} du \; \delta^{(4)}(x^\mu - X_n^\mu(u)) \, \frac{dX_n^\mu(u)}{du}$$

$$\Rightarrow \quad J^0 = \sum_n e_n \, \delta^{(3)}(\vec{x} - \vec{x}_n(t))$$

$$J^i = \sum_n e_n \, \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \, \frac{dx_n^i}{dt}$$

- Consistency condition:

$$\partial_\mu J^\mu(x) = 0.$$

$\Rightarrow$ Can be explicitly checked.

Physical interpretation:

$$\partial_0 J^0 + \partial_i J^i = 0$$

$$\Rightarrow \partial_0 \int_V d^3x \, J^0 = -\int d^3x \, \underbrace{\partial_i J^i}_{= \vec{\nabla}\cdot\vec{J}}$$

$$= -\int_S \vec{J}\cdot\vec{ds}$$

$$\Rightarrow \underbrace{\frac{d}{dt}\int_V \rho \, d^3x}_{\frac{dQ}{dt}} = -\int_S \vec{J}\cdot\vec{ds}$$

" Net charge flowing out of the surface.

Maxwell's equations in curved space:

$\Rightarrow$ Use principle of equivalence.

$\{x'^\mu\}$: The coordinate system in which

$g'_{\mu\nu}(x'_0) = \eta_{\mu\nu}$, $\Gamma'^\mu_{\nu\rho}(x'_0) = 0$.

$\Rightarrow$ At $x' = x'_0$ the eqs. of motion are:

$$\eta^{\mu\nu}\eta^{\kappa\rho}\partial'_\mu F'_{\nu\rho}(x'_0) = -J'^\kappa(x'_0)$$

Ex. //

$$\frac{\partial x'^\kappa}{\partial x^\sigma} g^{\mu\nu}(x)\, g^{\sigma\kappa}(x)\, D_\mu F_{\nu\rho}(x)\Big|_{x=x_0}$$

$$F_{\nu\rho} \equiv F'_{\sigma\tau}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial x'^\tau}{\partial x^\rho}$$

$$F_{\nu\rho} = \partial_\nu A_\rho - \partial_\rho A_\nu = D_\nu A_\rho - D_\rho A_\nu$$

$$A_\rho = \frac{\partial x'^\sigma}{\partial x^\rho} A'_\sigma$$

What is $J'^\kappa(x'_0)$?

$$J'^{\kappa}(x'_0) = \sum_n e_n \int_{-\infty}^{\infty} du\, \delta^{(4)}(x'^{\mu} - X'^{\mu}_n(u))\, \frac{dX'^{\kappa}_n(u)}{du}\Big|_{x'^{\mu}=x'^{\mu}_0}$$

Particle trajectory in the x^{μ} coordinate:

$$x^{\mu} = X^{\mu}_n(u)$$

$\Rightarrow$ if $x'^{\mu} - X'^{\mu}_n(u) = 0$.

$$\Rightarrow x^{\mu} - X^{\mu}_n(u) = 0.$$

$\delta^{(4)}(x'^{\mu} - X'^{\mu}_n(u)) \to \infty$, $\delta^{(4)}(x^{\mu} - X^{\mu}_n(u))$

what comes here.

$x'^{\mu} = X'^{\mu}_n(u) = f^{\mu}(x)$.

$$\delta^{(4)}(f^{\mu}(x) - \cancel{X'^{\mu}_n(u)}) = \det\left\{\frac{\partial}{\partial x^{\sigma}}(f^{\rho}(x) - \cancel{X'^{\rho}_n(u)})\right\}^{-1} \delta^{(4)}(x^{\mu} - X^{\mu}_n(u))$$

$$= \left\{\det\left(\frac{\partial f^{\rho}(x)}{\partial x^{\sigma}}\right)\right\}^{-1} \delta^{(4)}(x^{\mu} - X^{\mu}_n(u)).$$

$$= \left\{\det\left(\frac{\partial x'^{\rho}}{\partial x^{\sigma}}\right)\right\}^{-1}$$

Generalization of:

$$\delta(f(x)) = \delta(x - x_0)\, (f'(x_0))^{-1}$$

if $f(x) = 0$ at $x = x_0$.

$$\Rightarrow J'^{\kappa}(x'_0) = \sum_n e_n \int_{-\infty}^{\infty} du \left\{\det\left(\frac{\partial x'^{\rho}}{\partial x^{\sigma}}\right)\right\}^{-1}$$

$$\delta^{(4)}(x^{\mu} - X^{\mu}_n(u))\, \frac{\partial x'^{\kappa}}{\partial x^{\nu}}\, \frac{dX^{\nu}_n(u)}{du}\Big|_{\substack{x'=x'_0 \\ \parallel \\ x=x_0}}$$

$=$

$$\cancel{g'_{\mu\nu}(\vec{x}'_0) = g_{\rho\sigma}(\vec{x}_0) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Big|_{\vec{x}_0}}$$

$$g_{\rho\sigma}(\vec{x}_0) = g'_{\mu\nu}(\vec{x}_0) \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} \Big|_{\vec{x}_0}$$

$$\det g = \underset{\underset{-1}{\parallel}}{(\det g')} \left(\det \frac{\partial x'^\mu}{\partial x^\rho}\right)^2$$

$$\Rightarrow \quad \det\left(\frac{\partial x'^\mu}{\partial x^\rho}\right) = (-\det g)^{+1/2}$$

$$\Rightarrow \quad J'^\kappa(x'_0) = \frac{\partial x'^\kappa}{\partial x^\nu} \sum_n e_n \int_{-\infty}^{\infty} du \, (-\det g(x_0))^{-1/2}$$

$$\delta^{(4)}(x^\mu - X_n^\mu(u)) \frac{dX_n^\nu(u)}{du} \Big|_{x = x_0}$$

$$= \frac{\partial x'^\kappa}{\partial x^\nu} J^\nu \Big|_{x = x_0}$$

~~Thus the equation takes the form:~~

$$J^\nu(x) \equiv \sum_n e_n \int_{-\infty}^{\infty} du \, (-\det g(x))^{-1/2}$$

$$\delta^{(4)}(x^\mu - X_n^\mu(u)) \frac{dX_n^\nu(u)}{du}$$

Ex. Check that $J^\nu(x)$ transforms as a covariant vector under an arbitrary coordinate transformation.

G.62

Maxwell's equation in curved space:

$$\frac{\partial x'^k}{\partial x^\sigma} g^{\mu\nu} g^{\sigma k} D_\mu F_{\nu\rho} \Big|_{x=x_0}$$

$$= -\frac{\partial x'^k}{\partial x^\nu} J^\nu(x) \Big|_{x=x_0}$$

Multiply by $\frac{\partial x^\tau}{\partial x'^k} \Big|_{x=x_0}$

$$\Rightarrow \left\{ g^{\mu\nu} g^{\tau\sigma} D_\mu F_{\nu\rho} + J^\tau \right\} \Big|_{x=x_0} = 0.$$

Valid for all points x_0.

$$\Rightarrow \quad g^{\mu\nu} g^{\tau\sigma} D_\mu F_{\nu\rho} + J^\tau = 0.$$

Covariance

+ The fact that it vanishes in locally inertial frame

$\Rightarrow$ it must be true in all frames.

Maxwell's equations in curved space:

$$F_{\mu\nu} - (D_\mu A_\nu - D_\nu A_\mu) = 0 \Rightarrow \text{Rank } (0,2) \text{ tensor}$$

$$g^{\mu\nu} g^{\tau\sigma} D_\mu F_{\nu\rho} + J^\tau = 0 \quad \Rightarrow \text{Rank } (1,0) \text{ tensor}$$

check $D_\tau J^\tau = 0$, $D_\tau (g^{\mu\nu} g^{\tau\sigma} D_\mu F_{\nu\rho}) = 0$

$$J^\tau(x) = \sum_n e_n \int du \, \sqrt{-\det g(x)}^{\,-1} \, \delta^{(4)}(x^\mu - X_n^\mu(u)) \, \frac{dX_n^\tau}{du}$$

Ex. Check that these equations are ~~general~~ invariant under general coordinate transformation

---

General Principle: For a given equation to represent the correct eq. of motion it is sufficient that

---

① the equations are general coordinate invariant

② It reduces to the correct eqs. in the ~~correct~~ Locally inertial frame.

---

e.g. in the Locally inertial frame:

$$g^{\mu\nu} g^{\tau\sigma} D_\mu F_{\nu\rho} + J^\tau$$

$$\to \eta^{\mu\nu} \eta^{\tau\sigma} \partial_\mu F_{\nu\rho} + J^\tau = 0$$

$(g^{\mu\nu} g^{\tau\sigma} D_\mu F_{\nu\rho} + J^\tau)$ transforms as a vector.

If a vector vanishes in one coordinate system, it must vanish in all coordinate systems.

This makes it easy to generalize an equation from its flat space form to curved space.

---

e.g. Klein-Gordon equation:

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi = 0$$

$$\Downarrow$$

$$g^{\mu\nu} D_\mu D_\nu \phi = 0$$

Note: ① $g^{\mu\nu} D_\mu D_\nu \phi$ is a scalar.

---

② In locally inertial frame:

$$g^{\mu\nu} D_\mu D_\nu \phi = \eta^{\mu\nu} \partial_\mu \partial_\nu \phi = 0$$ by principle of equivalence.

$\Rightarrow$ $g^{\mu\nu} D_\mu D_\nu \phi = 0$ in all frames.

A possible modification in presence of particle source:

$$g^{\mu\nu} D_\mu D_\nu \phi = \int d^4 u \left(\sqrt{-\det g}\right)^{-1} \delta^{(4)} (x^\rho - X^\rho_{(u)}) .$$

~ ~~Does not describe gravity~~, ~~but~~ may describe new kinds of forces.

What is the equation that determines $g_{\mu\nu}$?

Principle of equivalence is not applicable.

In locally inertial frame eqs. of motion should look like relativistic eqs. in flat space.

But we do not have equations of motion for metric in flat space!

$\Rightarrow$ We need to find these equations by guesswork.

General form:

(Derivatives of $g_{\mu\nu}$) = (Source)

What is the conserved charge associated with the source?

Clue from non-relativistic limit:

Source for gravitational potential = mass.

In relativistic dynamics mass is not conserved.

$$A \to B + C + \cdots \quad \text{if} \quad m_A > m_B + m_C + \cdots$$

e.g. $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$

Instead Energy is conserved.

But Energy is not Lorentz ~~invariant~~ scalar.

- Instead it is part of a Four vector:

$$p_n^\alpha = m_n \frac{dX_n^\alpha}{d\tau}$$

Define $T^{\alpha 0}$ as the Energy-Momentum density:

$$T^{\alpha 0} = \sum_n p_n^\alpha(t)\, \delta^3(\vec{x} - \vec{x}_n(t))$$

Corresponding current:

$$T^{\alpha i} = \sum_n p_n^\alpha(t)\, \delta^3(\vec{x} - \vec{x}_n(t))\, \frac{dx_n^i(t)}{dt}$$

Lorentz Covariant form:

Ex. $T^{\alpha\mu} = \sum_n \int du\, p_n^\alpha\, \delta^4(x^\nu - X_n^\nu(u))\, \frac{dX_n^\mu}{du}$

Choose $u = \tau$

$$T^{\alpha\mu}(x) = \sum_n m_n \int d\tau\, \delta^4(x^\nu - X_n^\nu(\tau))\, \frac{dX_n^\alpha}{d\tau} \frac{dX_n^\mu}{d\tau}$$

$\Rightarrow$ $T^{\alpha\mu}$ is symmetric in α & μ.

Ex. $T^{\alpha\mu}$ transforms as a contravariant rank 2 tensor under Lorentz transformation

Conservation laws:

$$\partial_\mu T^{\alpha\mu} = \sum_n m_n \int d\tau \left(\partial_\mu \prod_{\nu=0}^{3} \delta(x^\nu - X_n^\nu(\tau))\right) \frac{dX_n^\alpha}{d\tau}\frac{dX_n^\mu}{d\tau}$$

$$= \sum_n m_n \int d\tau \prod_{\nu\neq\mu} \delta(x^\nu - X_n^\nu(\tau))\, \delta'(x^\mu - X_n^\mu(\tau))$$

$$\frac{dX_n^\alpha}{d\tau}\frac{dX_n^\mu}{d\tau}$$

$$= -\sum_n m_n \int d\tau \sum_\mu \prod_{\nu\neq\mu} \delta(x^\nu - X_n^\nu(\tau))\, \frac{d}{d\tau}\delta(x^\mu - X_n^\mu(\tau))$$

$$\frac{dX_n^\alpha}{d\tau}$$

$$= -\sum_n m_n \int d\tau\, \frac{d}{d\tau}\left\{\prod_\nu \delta(x^\nu - X_n^\nu(\tau))\right\} \frac{dX_n^\alpha}{d\tau}$$

$$= +\sum_n m_n \int d\tau \left\{\prod_\nu \delta(x^\nu - X_n^\nu(\tau))\right\} \frac{d^2X_n^\alpha}{d\tau^2}$$

$\frac{d^2X_n^\alpha}{d\tau^2} = 0$ for a free particle.

$\Rightarrow \quad \partial_\mu T^{\alpha\mu} = 0$

Suppose we have a set of particles moving under a background electromagnetic field.

$$m_n \frac{d^2X_n^\alpha}{d\tau^2} = e_n\, \eta^{\alpha\nu} F_{\nu\rho}(X_n)\frac{dX_n^\rho}{d\tau}$$

$$\partial_\mu T^{\alpha\mu} = + \sum_n \int d\tau \prod_\nu \delta(x^\nu - X_n^\nu(\tau))$$

$$e_n\, \eta^{\alpha\nu} F_{\nu\rho}(x) \frac{dX_n^\rho}{d\tau}$$

$$= + \eta^{\alpha\nu} F_{\nu\rho}(x)\, J^\rho$$

$$= \pm \eta^{\alpha\nu} F_{\nu\rho}\, \eta^{\mu\tau} \eta^{\rho\sigma} \partial_\mu F_{\tau\sigma} \underset{\text{Ex.}}{=} -\partial_\mu T^{\alpha\mu}_{em}$$

$$\cancel{= \partial_\mu (\eta^{\alpha\nu} \eta^{\mu\tau} \eta^{\rho\sigma} F}$$

$$\cancel{= \eta^{\alpha\nu} \eta^{\mu\tau} \partial_\mu (F_{\nu\rho} F_{\tau\sigma} \eta^{\rho\sigma}}$$

Define

$$T^{\mu\nu}_{em} = \eta^{\mu\rho} \eta^{\nu\sigma} \eta^{\kappa\tau} F_{\rho\kappa} F_{\sigma\tau}$$

$$- \frac{1}{4} \eta^{\mu\nu} \eta^{\rho\sigma} \eta^{\kappa\tau} F_{\rho\kappa} F_{\sigma\tau}$$

Thus $\partial_\mu (T^{\alpha\mu} + T^{\alpha\mu}_{em}) = 0.$

Interpretation:

$T^{\alpha\mu}_{e.m.}$: Energy momentum tensor of electromagnetic field.

$$T^{00}_{em} = \eta^{\kappa\tau} F_{0\kappa} F_{0\tau} + \frac{1}{4} \eta^{\rho\sigma} \eta^{\kappa\tau} F_{\rho\kappa} F_{\sigma\tau}$$

$$= \eta^{ij} F_{0i} F_{0j} + \frac{1}{4} \eta^{00} \eta^{ij} F_{0i} F_{0j}$$

$$+ \frac{1}{4} \eta^{ij} \eta^{00} F_{i0} F_{j0} + \frac{1}{4} \eta^{ij} \eta^{kl} F_{ik} F_{jl}$$

$$\underset{\text{Ex.}}{=} \cancel{} + \frac{1}{2} \vec{E}^2 + \frac{1}{2} \vec{B}^2$$

$\Rightarrow$ Correct energy density.

We have shown that:

$$T^{\mu\nu} = T^{\mu\nu}_{matter} + T^{\mu\nu}_{e.m.} \quad \text{is conserved.}$$

$$\partial_{\nu} T^{\mu\nu} = 0$$

$$T^{\alpha\mu}_{matter} = \sum_n m_n \int d\tau \; \delta^4(X^\nu - X_n^\nu(\tau)) \frac{dX_n^\alpha}{d\tau} \frac{dX_n^\mu}{d\tau}$$

$$T^{\alpha\mu}_{em} = \eta^{\alpha\rho}\eta^{\mu\sigma}\eta^{\kappa\tau} F_{\rho\kappa} F_{\sigma\tau} - \frac{1}{4}\eta^{\alpha\mu}\eta^{\rho\sigma}\eta^{\kappa\tau} F_{\rho\kappa} F_{\sigma\tau}$$

Ex. $T^{\alpha\mu}_{matter}$ & $T^{\alpha\mu}_{em}$ transform as rank 2 contravariant tensor under a Lorentz transformation.

$T^{\mu\nu}$ should appear on the right hand side of the gravitational equation, but for this we need to generalize the definition to curved space.

Guideline:

① $T^{\mu\nu}$ should transform as a contravariant rank 2 tensor under g.c.t.

② $T^{\mu\nu}$ should reduce to the above expression in the locally inertial frame.

Ex. Show that

$$T^{\alpha\mu}_{matter} = \sum_n m_n \int d\overset{u}{\cancel{\tau}} \left(\sqrt{-\det g}\right)^{-1} \delta^4(X^\nu - X_n^\nu(\tau)) \frac{dX_n^\alpha}{d\tau} \frac{dX_n^\mu}{d\underset{u}{\cancel{\tau}}}$$

$$T^{\alpha\mu}_{em} = g^{\alpha\rho} g^{\mu\sigma} g^{\kappa\tau} F_{\rho\kappa} F_{\sigma\tau} - \frac{1}{4} g^{\alpha\mu} g^{\rho\sigma} g^{\kappa\tau} F_{\rho\kappa} F_{\sigma\tau}$$

satisfy these conditions.

$T^{\mu\nu}$ is unique: If two tensors satisfy this condition then their difference $= 0$ in locally inertial frame.

$D_\mu T^{\alpha\mu} = 0.$

→ can be proved by noting that

① ~~the~~ l.h.s. is a vector.

② It vanishes in the locally inertial frame
⇒ it must vanish in all coordinate system.

Notation: $T_{\mu\nu} = g_{\mu\rho}\, g_{\nu\sigma}\, T^{\rho\sigma}$

Check:

$$g^{\rho\nu} D_\rho T_{\mu\nu} = 0.$$

(Use: $D_\rho\, g_{\mu\nu} = 0$ & that

$$D_\mu (ABC\cdots) = D_\mu A\, BC\cdots + A\, D_\mu B\, C\cdots + \cdots$$)

This fixes the right hand side of the gravitational field eqn.

What about the left hand side?

① Must be a tensor of rank 2.

② Note: $g_{00} \sim -1 - 2\phi$

Newtonian eqn. for ϕ:

$$\nabla^2\phi \propto \text{density}.$$

Two derivatives.

Thus ~~the~~ l.h.s. should contain a term with two derivatives.

→ Made out of linear terms in Riemann tensor (not its covariant derivatives or powers).

$R^{\mu}{}_{\nu\rho\sigma}$: Two indices need to be contracted:

$R_{\mu\nu\rho\sigma} \equiv g_{\mu\kappa} R^{\kappa}{}_{\nu\rho\sigma}$ satisfies:

$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = -R_{\nu\mu\rho\sigma}$

$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$

Thus μ can only be contracted with ρ or σ.

→ same term with opposite sign.

$$\Rightarrow \quad R_{\nu\sigma} = g^{\mu\rho} R_{\mu\nu\rho\sigma}$$

↓

Symmetric in $\nu\sigma$.

The other possible rank 2 tensor is.

$$R\, g_{\nu\sigma}$$

$$g^{\nu\sigma} R_{\nu\rho}$$

⇒ Eqs. of motion:

$$a R_{\mu\nu} + b R g_{\mu\nu} = T_{\mu\nu}.$$

Q: what are a & b?

$$g^{\rho\mu} D_\rho T_{\mu\nu} = 0$$

$$\Rightarrow \quad g^{\rho\mu} D_\rho (a R_{\mu\nu} + b R g_{\mu\nu}) = 0.$$

$$\Rightarrow a g^{\rho\mu} D_\rho R_{\mu\nu} + b D_\nu R = 0$$

→ should be satisfied automatically.

Analogy with electrodynamics (in flat space)

$$\partial_\mu F^{\mu\nu} = - J^\nu$$

$$\Rightarrow \partial_\mu \partial_\nu F^{\mu\nu} = - \partial_\nu J^\nu$$

$$\underset{0}{\parallel} \qquad\qquad \underset{0}{\parallel}$$

due to antisymmetry of μ, ν.

Ex. Check that this also works in curved space:

$$D_\mu F^{\mu\nu} = - J^\nu$$

$$D_\nu D_\mu F^{\mu\nu} = - D_\nu J^\nu$$

$$\underset{0}{\parallel}$$

Recall: We had shown:

$$g^{\rho\mu} D_\rho R_{\mu\nu} - \frac{1}{2} D_\nu R = 0.$$

$$\Rightarrow \quad a : b = 1 : -\frac{1}{2}$$

$$b = -\frac{1}{2} a.$$

$$\Rightarrow \quad a \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = T_{\mu\nu}. \qquad \Rightarrow a \nabla^2 g_{00} = T_{00}$$

Final question what is a?

① Go to the weak static field limit:

$$g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{h_{\mu\nu}}_{\text{small.}}$$

② Find the equation for $h_{00} = -2\phi$ (00 component)

③ Compare with $\nabla^2 \phi = +4\pi G \rho$ ($\rho = T^{00}$) $\rightarrow$ Newton's constant

We shall try to find solutions of the field equations.

~~First~~ Field configuration near a point mass.

$\bullet$

$\Rightarrow$ δ function source.

Best to solve the eqs. without δ function Then identify the source.

Example: Electrostatics around a point charge

$$\nabla^2 \phi = q\, \delta(\vec{x})$$

$$\phi = \frac{c}{r}$$

① First show that $\nabla^2 \phi = 0$ everywhere except at $r=0$.

② Then relate c and q by computing to total electric flux at ∞ and using Gauss law.

$\Rightarrow$ We shall first solve source free Einstein's eqn.:

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 0.$$

$$\Rightarrow \quad g^{\mu\nu}(\ldots) = 0$$

$$\Rightarrow \quad R - 2R = 0 \quad \Rightarrow \quad R = 0.$$

$$\Rightarrow \quad R_{\mu\nu} = 0.$$

Guideline : Spherical symmetry + Time independence

Analogy with electrostatics :

look for $\phi = F(r)$ (independent of θ, ϕ).

Note : ① Here due to the freedom of coordinate transformation the solution is expected to look spherically symmetric only in a special coordinate system.

We can destroy manifest spherical symmetry & time independence by using wrong coordinate system.

---

② $g_{\mu\nu}$ is a tensor.

→ Meaning of spherical symmetry needs to be understood.

$(x^1, x^2, x^3, t) = (\vec{x}, t)$ → coordinates.

---

ds^2 should be a "scalar" fn. of $\vec{x}, \vec{dx}$ and independent of t.

e.g. $$ds^2 = -F_1(r)\,dt^2 + F_2(r)\,dt\,\vec{x}\cdot\vec{dx} + F_3(r)\,\vec{dx}^2 + F_4(r)\,(\vec{x}\cdot\vec{dx})^2$$

$$r^2 = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$$

→ Invariant under

$$x^i \to R_{ij}\, x^j, \quad dx^i \to R_{ij}\, dx^j$$

R : rotation matrix.

Introduce new coordinates r, θ, ϕ:

$$x^3 = r\cos\theta$$
$$x^1 = r\sin\theta\cos\phi$$
$$x^2 = r\sin\theta\sin\phi$$

$$d\vec{x}^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\theta^2.$$

$$\vec{x}\cdot d\vec{x} = x^i dx^i = \tfrac{1}{2} d\sum (x^i)^2 = \tfrac{1}{2} d(r^2) = r\,dr$$

$$\Rightarrow ds^2 = -F_1(r)\,dt^2 + rF_2(r)\,dt\cdot dr + F_3(r)\,(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2) + F_4(r)\,.\,r^2\,dr^2$$

$$= -F_1(r)\,dt^2 + rF_2(r)\,dt\,dr + \left(F_3(r) + r^2F_4(r)\right) dr^2 + r^2F_3(r)\,(d\theta^2 + \sin^2\theta\, d\phi^2).$$

$F_3(r) > 0$ in order to have signature $(-+++)$.

$$t' = t - \phi(r) \quad \Rightarrow \quad dt = dt' + \phi'(r)\,dr$$

$\cancel{r^3 F_3(r) = \rho^3}$

$$ds^2 = -F_1(r)\,(dt'^2 + 2\,dt'\,\phi'(r)\,dr + \phi'(r)^2\,dr^2)$$
$$+ rF_2(r)\,(dt' + \phi'(r)\,dr)\,dr$$
$$+ (F_3(r) + r^2F_4(r))\,dr^2$$
$$+ r^2F_3(r)\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Choose $\phi(r)$ so as to make $dt'\,dr$ term vanish

$$-2F_1(r)\,\phi'(r) + rF_2(r) = 0.$$

$$\Rightarrow \phi'(r) = \frac{1}{2}\frac{rF_2(r)}{F_1(r)} \quad \Rightarrow \phi(r) = \frac{1}{2}\int^r du\,\frac{uF_2(u)}{F_1(u)}$$

$$\Rightarrow ds^2 = -F_1(r)\,dt'^2 + dr^2\left\{-\frac{1}{4}\frac{r^2F_2(r)^2}{F_1(r)} + \frac{1}{2}\frac{r^2F_2(r)^2}{F_1(r)} + F_3(r) + r^2F_4(r)\right\} + r^2F_3(r)\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

(G.76)

Use a new coordinate ρ related to r as:

$$\rho^2 = r^2 F_3(r).$$

$\Rightarrow$ $F_1(r) = \psi(\rho)$ $\quad F_1'(r)\, dr = \psi'(\rho)\, d\rho$

$$\left\{\frac{\psi'(\rho)}{F_1'(r)}\right\}^2 \left\{ + \frac{1}{4} r^2 \frac{F_2(r)^2}{F_1(r)} + F_3(r) + r^2 F_4(r) \right\} = \chi(\rho).$$

Call $t' \to t$.

$$ds^2 = -\psi(\rho)\, dt^2 + \chi(\rho)\, d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

We shall now substitute this in the Einstein's eqn.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

$\times g^{\mu\nu}$ $\Rightarrow R - 2R = 0 \Rightarrow R = 0$

$\Rightarrow R_{\mu\nu} = 0$

Ex Check that

$$R_{\rho\rho} = \frac{\psi''}{2\psi} - \frac{1}{4}\frac{\psi'}{\psi}\left(\frac{\chi'}{\chi} + \frac{\psi'}{\psi}\right) - \frac{1}{\rho}\frac{\chi'}{\chi}$$

$$R_{\theta\theta} = -1 + \frac{\rho}{2\chi}\left(-\frac{\chi'}{\chi} + \frac{\psi'}{\psi}\right) + \frac{1}{\chi}$$

$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$$

$$R_{tt} = -\frac{\psi''}{2\chi} + \frac{1}{4}\frac{\psi'}{\chi}\left(\frac{\chi'}{\chi} + \frac{\psi'}{\psi}\right) - \frac{1}{\rho}\frac{\psi'}{\chi}$$

$R_{\mu\nu} = 0$ for $\mu \neq \nu$.

We need to set each of these to 0.

$$\chi R_{\rho\rho} + \chi R_{tt} = -\frac{1}{\rho\chi}(\psi\chi' + \chi\psi')$$

$$\Rightarrow \frac{\chi'}{\chi} + \frac{\psi'}{\psi} = 0 \Rightarrow \chi\psi = \text{constant} = C_1$$

$t \to at \Rightarrow \psi \to a^{-2} t$

$$\chi\psi \to a^{-2}\chi\psi$$

Adjusting a we can make $\chi\psi = 1$.

$$R_{\theta\theta} = -1 + \frac{\rho}{2\chi}\left(-\frac{\chi'}{\chi} - \frac{\chi'}{\chi}\right) + \frac{1}{\chi} = 0$$

$$\Rightarrow -\chi^2 - \rho\chi' + \chi = 0$$

$$\rho\chi' = \chi(1-\chi) \Rightarrow \frac{1}{\rho} = \frac{\chi'}{\chi(1-\chi)} = \chi'\left(\frac{1}{\chi} + \frac{1}{1-\chi}\right)$$

$$\ln\rho + \ln c = \ln\{\chi/(1-\chi)\}$$

$$\chi/(1-\chi) = c\rho$$

$$\Rightarrow \frac{1}{\chi} - 1 = 1/c\rho \qquad \frac{1}{\chi} = (1 + c\rho)^{-1}$$

$$\chi = \frac{1}{1+(c\rho)^{-1}}$$

$$\psi = (1 + 1/c\rho)$$

$$\Rightarrow ds^2 = -\left(1 + \frac{1}{c\rho}\right)dt^2 + \frac{1}{1 + \frac{1}{c\rho}}d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Check: With this soln.:

$$R_{\rho\rho} = 0, \quad R_{\phi\phi} = 0$$

$\Rightarrow$ All equations are satisfied.

Interpretation of c:

Look at large ρ:

$$g_{00} = -\left(1 + \frac{1}{c\rho}\right)$$

$$\parallel$$

$$-1 - 2\phi \qquad \Rightarrow \phi(\rho) = \frac{1}{2c\rho}$$

For a source of mass M we expect $\phi = -\frac{G_N M}{\rho}$

$$\Rightarrow \quad \frac{1}{2C} = -G_N M.$$

$$\Rightarrow \quad C^{-1} = -2 G_N M.$$

$$\Rightarrow ds^2 = -\left(1 - \frac{2G_N M}{\rho}\right) dt^2 + \frac{1}{1 - \frac{2G_N M}{\rho}} d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

M: Gravitational mass of the object as measured from far.

(Analog of determining the charge as

$$\underset{\text{surface at large distance}}{\int} \vec{E} \cdot \vec{ds} = 4\pi Q.$$
)

Note: The metric seems to be singular at $\rho = 2 G_N M$.

→ shall be discussed later.

~~At~~ present we shall discuss motion of test particle in this background metric.

- Particle moving under the influence of the gravitational field of a point mass:

Choose coordinate system such that:

$$\theta = \frac{\pi}{2}, \quad \frac{d\theta}{du} = 0. \quad \text{for all } u.$$

Eqs. of motion:

$$\frac{d^2\rho}{du^2} + \frac{\chi'(\rho)}{2\chi(\rho)}\left(\frac{d\rho}{du}\right)^2 \cancel{- \frac{\rho}{\chi(\rho)}\left(\frac{d\theta}{du}\right)^2} - \frac{\rho}{\chi(\rho)}\left(\frac{d\phi}{du}\right)^2 + \frac{\psi'(\rho)}{2\chi(\rho)}\left(\frac{dt}{du}\right)^2 = 0$$

$$\frac{d^2\phi}{du^2} + \frac{2}{\rho}\frac{d\phi}{du}\frac{d\rho}{du} = 0.$$

$$\frac{d^2t}{du^2} + \frac{\psi'(\rho)}{\psi(\rho)}\frac{dt}{du}\frac{d\rho}{du} = 0.$$

$$ds^2 = -\psi(\rho)\,dt^2 + \chi(\rho)\,d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

$$\psi(\rho) = \left(1 - \frac{2GM}{\rho}\right). \qquad \chi(\rho) = \left(1 - \frac{2GM}{\rho}\right)^{-1}$$

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{du}\frac{dx^\lambda}{du} = 0.$$

Ex. check $\Gamma^\theta_{\nu\lambda} = 0$ for $\nu, \lambda \neq \theta$

x^μ : ρ, θ, ϕ, t

Eqs. of motion of a test particle in this background:
Assume the mass of the test particle to be small so that it does not affect the metric.

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{du}\frac{dx^\lambda}{du} = 0.$$

$$\frac{d^2\rho}{du^2} + \frac{\chi'(\rho)}{2\chi(\rho)}\left(\frac{d\rho}{du}\right)^2 - \frac{\rho}{\chi(\rho)}\left(\frac{d\theta}{du}\right)^2 - \rho\frac{\sin^2\theta}{\chi(\rho)}\left(\frac{d\phi}{du}\right)^2 + \frac{\psi'(\rho)}{2\chi(\rho)}\left(\frac{dt}{du}\right)^2 = 0. \qquad (1)$$

$$\frac{d^2\theta}{du^2} + \frac{2}{\rho}\frac{d\theta}{du}\frac{d\rho}{du} - \sin\theta\cos\theta\left(\frac{d\phi}{du}\right)^2 = 0 \qquad (2)$$

$$\frac{d^2\phi}{du^2} + \frac{2}{\rho}\frac{d\phi}{du}\frac{d\rho}{du} + 2\cot\theta\frac{d\phi}{du}\frac{d\theta}{du} = 0. \qquad (3)$$

$$\frac{d^2t}{du^2} + \frac{\psi'(\rho)}{\psi(\rho)}\frac{dt}{du}\frac{d\rho}{du} = 0. \qquad (4)$$

Scattering in equatorial plane:

$$\theta = \frac{\pi}{2}, \quad \frac{d\theta}{du} = 0. \Rightarrow \frac{d^2\theta}{du^2} = 0.$$

$\Rightarrow \frac{d\theta}{du}$ remains 0 $\Rightarrow$ θ remains equal to $\frac{\pi}{2}$.

(2) $\to$ trivial.

$\left(\frac{d\phi}{du}\right)^{-1} \times (3)$

$$\Rightarrow \frac{d}{du}\ln\frac{d\phi}{du} + \frac{2}{\rho}\frac{d\rho}{du} = 0 \Rightarrow \ln\frac{d\phi}{du} + \ln\rho^2 = \ln K$$

$$\Rightarrow \rho^2\frac{d\phi}{du} = K$$

$\left(\frac{dt}{du}\right)^{-1} \times$ (4)

$\Rightarrow \quad \ln \frac{dt}{du} + \ln \psi = L$

$\frac{dt}{du} = \frac{L}{\psi}$ $\quad L =$ constant $= 1$ by scaling u.

$$\Rightarrow \frac{d^2\rho}{du^2} + \frac{\chi'(\rho)}{2\chi(\rho)}\left(\frac{d\rho}{du}\right)^2 - \frac{\rho}{\chi(\rho)}\frac{K^2}{\rho^4} + \frac{\psi'(\rho)}{2\chi(\rho)}\frac{1}{\psi^2} = 0$$

$\times \; 2\chi(\rho)\frac{d\rho}{du}$

$$\frac{d}{d\rho}\left\{\chi(\rho)\left(\frac{d\rho}{du}\right)^2 + \frac{K^2}{\rho^2} - \frac{1}{\psi}\right\} = 0.$$

$$\Rightarrow \chi(\rho)\left(\frac{d\rho}{du}\right)^2 + \frac{K^2}{\rho^2} - \frac{1}{\psi} = \textbf{constant} = N$$

Eliminate u using $\frac{dt}{du} = \frac{1}{\psi}$.

$$\boxed{\begin{aligned} &\frac{\chi(\rho)}{\psi(\rho)^2}\left(\frac{d\rho}{dt}\right)^2 + \frac{K^2}{\rho^2} - \frac{1}{\psi} = N. \\ &\rho^2 \frac{d\phi}{dt} = K\psi \end{aligned}}$$

Note: $\left(\frac{d\tau}{du}\right)^2 = -\left(\frac{ds}{du}\right)^2 = -\left[-\psi\left(\frac{dt}{du}\right)^2 + \chi\left(\frac{d\rho}{du}\right)^2 + \rho^2\left(\frac{d\phi}{du}\right)^2\right]$

$$= -\left[\chi\left(\frac{d\rho}{du}\right)^2 + \rho^2 \frac{K^2}{\rho^4} - \frac{1}{\psi}\right] = -N$$

Thus $N < 0$ for massive particles

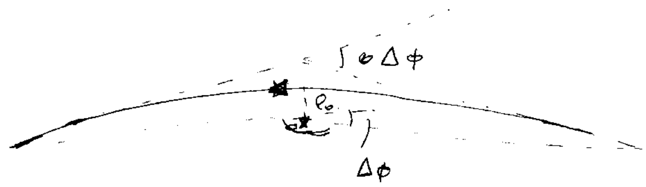
$N = 0$ for massless particles.

Particle trajectory: obtained by eliminating t:

$$\frac{\chi}{\psi^2}\left(\frac{d\rho}{d\phi}\right)^2 \cdot \frac{K^2\psi^2}{\rho^4} + \frac{K^2}{\rho^2} - \frac{1}{\psi} = N \;\Rightarrow\; \frac{K^2\chi}{\rho^4}\left(\frac{d\rho}{d\phi}\right)^2 + \frac{K^2}{\rho^2} - \frac{1}{\psi} = N$$

$$d\phi = \frac{K\sqrt{\chi}}{\rho^2}\, d\rho \; \frac{1}{\left\{N - \frac{K^2}{\rho^2} + \frac{1}{\psi}\right\}^{1/2}}$$

Bending of light near a massive object:



$N = 0$

$\cancel{d\phi = \frac{K}{\rho}}$ ρ_0 : Minimum value of ρ.

$\Rightarrow \dfrac{d\rho}{d\phi} = 0$ at $\rho = \rho_0$.

$$\Rightarrow \frac{K^2}{\rho_0^2} - \frac{1}{\psi(\rho_0)} = 0 \Rightarrow K^2 = \frac{\rho_0^2}{\psi(\rho_0)}.$$

$$\Rightarrow d\phi = \sqrt{\frac{\rho_0^2}{\psi(\rho_0)}}\, \sqrt{\chi(\rho)}\, d\rho \; \frac{1}{\left\{\frac{1}{\psi} - \frac{\rho_0^2}{\psi(\rho_0)\rho^2}\right\}^{1/2}}$$

Note: $\dfrac{d\phi}{dt} = \dfrac{K\psi}{\rho^2} > 0$ if $K > 0$.

$$= \rho_0 \sqrt{\chi(\rho)}\, \frac{d\rho}{\rho^2} \; \frac{1}{\left\{\frac{\psi(\rho_0)}{\psi(\rho)} - \frac{\rho_0^2}{\rho^2}\right\}^{1/2}} = \rho_0 \sqrt{\chi(\rho)}\, \frac{d\rho}{\rho^2}\, \rho_0^{-1}\, \psi(\rho)^{1/2} \left\{\frac{\psi(\rho_0)}{\rho_0^2} - \frac{\psi(\rho)}{\rho^2}\right\}^{-1/2}$$

$$\chi(\rho) = \left(1 - \frac{2GM}{\rho}\right)^{-1} \simeq 1 + \frac{2GM}{\rho}$$

$$\psi(\rho) = 1 - \frac{2GM}{\rho}$$

$$C = -2MG\left(\frac{1}{\rho_+}-\frac{1}{\rho_-}\right)\left(\frac{1}{\rho_+^2}-\frac{1}{\rho_-^2}\right)^{-1} + 4M^2G^2 + 4M^2G^2$$

$$= -2MG\left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right)^{-1}$$

$$d\phi = \frac{K}{\rho^2}\,d\rho\,\left(1+\frac{MG}{\rho}\right)\left[2MG\left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right)^{-1}\left(\frac{1}{\rho}-\frac{1}{\rho_+}\right)\left(\frac{1}{\rho_-}-\frac{1}{\rho}\right)\right]^{-1/2}$$

$$K^2 = \left(\frac{1}{\rho_+^2}-\frac{1}{\rho_-^2}\right)^{-1}\left\{1+\frac{2MG}{\rho_+}+\frac{4M^2G^2}{\rho_+^2}-1-\frac{2MG}{\rho_-}-\frac{4M^2G^2}{\rho_-^2}\right\}$$

$$= \left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right)^{-1}\left\{2MG+4M^2G^2\left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right)\right.$$

$$= \left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right)^{-1}2MG\left\{1+2MG\left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right)\right\}$$

$$\Rightarrow d\phi = \frac{d\rho}{\rho^2}\left(1+\frac{MG}{\rho}\right)\left[\left(\frac{1}{\rho}-\frac{1}{\rho_+}\right)\left(\frac{1}{\rho_-}-\frac{1}{\rho}\right)\right]^{-1/2}\left\{1+MG\left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right)\right\}$$

$$\frac{1}{\rho} = x$$

$$d\phi = -dx\,(1+MGx)\left[\left(x-\frac{1}{\rho_+}\right)\left(\frac{1}{\rho_-}-x\right)\right]^{-1/2}\left\{1+MG\left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right)\right\}$$

$$\left(x-\frac{1}{\rho_+}\right)\left(\frac{1}{\rho_-}-x\right) = \frac{1}{4}\left(\frac{1}{\rho_+}-\frac{1}{\rho_-}\right)^2-\left\{x-\frac{1}{2}\left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right)\right\}^2$$

$$x-\frac{1}{2}\left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right) = \frac{1}{2}\left(\frac{1}{\rho_-}-\frac{1}{\rho_+}\right)\sin\alpha$$

$$d\phi = -\frac{1}{2}\left(\frac{1}{\rho_-}-\frac{1}{\rho_+}\right)\cos\alpha\,d\alpha\,\left\{1+\frac{MG}{2}\left(\frac{1}{\rho_-}-\frac{1}{\rho_+}\right)\sin\alpha\right\}$$

$$\left\{\frac{1}{2}\left(\frac{1}{\rho_-}-\frac{1}{\rho_+}\right)\right\}^{-1}\frac{1}{\cos\alpha}\left\{1+MG\left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right)\right\}$$

$$= -\left(1+MG\left(\frac{1}{\rho_+}+\frac{1}{\rho_-}\right)\right)d\alpha\,\left\{1+\frac{MG}{2}\left(\frac{1}{\rho_-}-\frac{1}{\rho_+}\right)\sin\alpha\right\}$$

$\rho = \rho_- \Rightarrow x = \frac{1}{\rho_-} \Rightarrow \sin\alpha = 1 \Rightarrow \alpha = \frac{\pi}{2}$

$\rho = \rho_+ \Rightarrow x = \frac{1}{\rho_+} \Rightarrow \sin\alpha = -1 \Rightarrow \alpha = -\frac{\pi}{2}$

$$\phi(\rho_+) - \phi(\rho_-) = -\int_{\pi/2}^{-\pi/2} \left\{1 + MG\left(\frac{1}{\rho_+} + \frac{1}{\rho_-}\right)\right\} d\alpha \left\{1 + \frac{MG}{2}\left(\frac{1}{\rho_-} - \frac{1}{\rho_+}\right)\sin\alpha + \frac{MG}{2}\left(\frac{1}{\rho_+} + \frac{1}{\rho_-}\right)\right\}$$

$$= \left\{1 + MG\left(\frac{1}{\rho_+} + \frac{1}{\rho_-}\right)\right\} \left[\pi + \frac{MG}{2}\left(\frac{1}{\rho_-} - \frac{1}{\rho_+}\right) \cdot 0 + \pi \cdot \frac{MG}{2}\left(\frac{1}{\rho_+} + \frac{1}{\rho_-}\right)\right]$$

$$= \pi \left\{1 + \frac{3}{2} MG\left(\frac{1}{\rho_+} + \frac{1}{\rho_-}\right)\right\}$$

Total change in ϕ over a period:

$$\Delta\phi = 2\pi + 3\pi MG\left(\frac{1}{\rho_+} + \frac{1}{\rho_-}\right)$$

$\Rightarrow$ Precession of the perihelion.

The orbit comes back to the original location after

$$\frac{2\pi}{3\pi MG\left(\frac{1}{\rho_+} + \frac{1}{\rho_-}\right)}$$

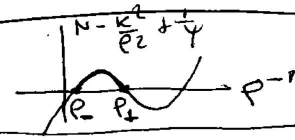
revolutions.

$\Rightarrow$ A coordinate independent statement.

Bound orbit

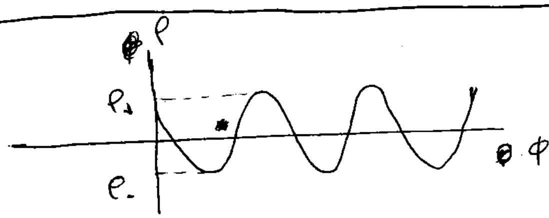
$$d\phi = \frac{K\sqrt{x}}{\rho^2}\, d\rho \frac{1}{\left\{N - \frac{K^2}{\rho^2} + \frac{1}{\psi}\right\}^{1/2}}$$

$$\frac{d\rho}{d\phi} = \frac{\rho^2}{K\sqrt{x}} \left\{N - \frac{K^2}{\rho^2} + \frac{1}{\psi}\right\}^{1/2}$$



Note: $\frac{d\phi}{dt} > 0$ for all t if $K > 0$

since $\frac{d\phi}{dt} = \frac{K\psi}{\rho^2}$



$\frac{d\phi}{d\rho} = 0$ at $\rho = \rho_\pm$

$$N - \frac{K^2}{\rho_\pm^2} + \frac{1}{\psi(\rho_\pm)} = 0.$$

$$\Rightarrow \quad K^2\left(\frac{1}{\rho_+^2} - \frac{1}{\rho_-^2}\right) = \frac{1}{\psi(\rho_+)} - \frac{1}{\psi(\rho_-)}$$

$$K^2 = \left(\frac{1}{\rho_+^2} - \frac{1}{\rho_-^2}\right)^{-1} \left\{\frac{1}{\psi(\rho_+)} - \frac{1}{\psi(\rho_-)}\right\}$$

$$N(\rho_+^2 - \rho_-^2) = \frac{\rho_-^2}{\psi(\rho_-)} - \frac{\rho_+^2}{\psi(\rho_+)}$$

$$\Rightarrow \quad N = \frac{1}{\rho_+^2 - \rho_-^2}\left\{\frac{\rho_-^2}{\psi(\rho_-)} - \frac{\rho_+^2}{\psi(\rho_+)}\right\}.$$

G.82

$$d\phi = \rho_0 \frac{d\rho}{\rho^2}\left(1+\frac{GM}{\rho}\right)\left\{1-\frac{2GM}{\rho_0}+\frac{2GM}{\rho}-\frac{\rho_0^2}{\rho^2}\right\}^{-1/2}$$

$$\simeq \rho_0 \frac{d\rho}{\sqrt{\rho^2-\rho_0^2}}\left(1+\frac{GM}{\rho}\right)\left\{1+\frac{\rho^2}{\rho^2-\rho_0^2}\cdot GM \cdot \frac{\rho-\rho_0}{\rho\rho_0}\right\}$$

$$\simeq \frac{\rho_0\, d\rho}{\rho\sqrt{\rho^2-\rho_0^2}}\left(1+\frac{GM}{\rho}+\frac{GM\,\rho}{(\rho+\rho_0)\rho_0}\right) \qquad \rho = \rho_0 \sec\theta$$

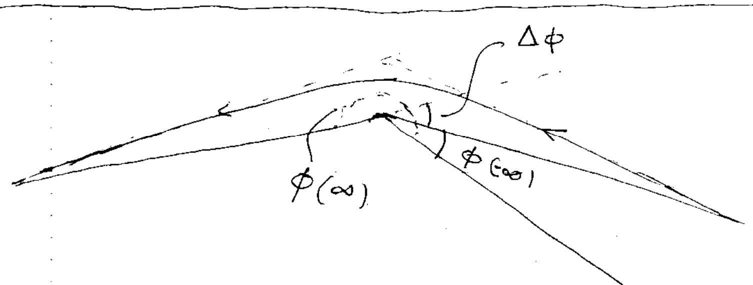
$$\cancel{\frac{d\rho}{\rho\sqrt{\rho^2-\rho_0^2}}\left(1+\frac{GM\rho\rho_0}{\rho\rho_0(\rho+\rho_0)}\right) = \frac{\rho\, d\rho}{\sqrt{\rho^2-\rho_0^2}}\left(1+\frac{GM}{\rho+\rho_0}\right)}$$

$$\phi(\infty)-\phi(\rho_0) = \rho_0\int_{\rho_0}^{\infty}\frac{d\rho}{\rho\sqrt{\rho^2-\rho_0^2}}\left(1+\frac{MG}{\rho}+\frac{GM\rho}{\rho_0(\rho+\rho_0)}\right)$$

$$= \frac{\pi}{2}+\frac{2MG}{\rho_0}$$

$$\phi(\rho_0)-\phi(-\infty) = \frac{\pi}{2}+\frac{2MG}{\rho_0}$$

$$\Rightarrow \quad \phi(\infty)-\phi(-\infty) = \pi+\frac{4MG}{\rho_0}$$



$$\phi(\infty)-\phi(-\infty) = \pi+\Delta\phi$$

$\Rightarrow \Delta\phi = \dfrac{4MG}{\rho_0}$ → Distance of closest approach to the sun.

General relativistic effects small.

G.84

$$d\phi = \frac{K}{\rho^2}\sqrt{\chi(\rho)}\, d\rho \Big[\frac{1}{\rho_+^2-\rho_-^2}\left\{ \frac{\rho_-^2}{\psi(\rho_-)} - \frac{\rho_+^2}{\psi(\rho_+)} \right\}$$

$$- \frac{1}{\rho^2}\left(\frac{1}{\rho_+^2} - \frac{1}{\rho_-^2}\right)^{-1}\left\{ \frac{1}{\psi(\rho_+)} - \frac{1}{\psi(\rho_-)} \right\} + \frac{1}{\psi(\rho)} \Big]^{-1/2}$$

$$\chi = \left(1 - \frac{2MG}{\rho}\right)^{-1} \qquad \psi = \left(1 - \frac{2MG}{\rho}\right).$$

$$(\chi(\rho))^{1/2} \simeq \left(1 + \frac{MG}{\rho}\right)$$

[ ] :

$G(1)$ term: $\frac{1}{\rho_+^2-\rho_-^2}(\rho_-^2 - \rho_+^2) - \frac{1}{\rho^2}\left(\frac{1}{\rho_+^2} - \frac{1}{\rho_-^2}\right)(1-1)$

$$+ 1 = -1 + 1 = 0.$$

Thus we need to keep $G(\frac{GM}{\rho})$ as well as $G((\frac{GM}{\rho})^2)$ term if we want to compute first order correction

---

$$\frac{1}{\psi(\rho)} = \left(1 - \frac{2MG}{\rho}\right)^{-1} = 1 + \frac{2MG}{\rho} + \frac{4M^2G^2}{\rho^2}$$

$$\frac{1}{\psi(\rho_+)} = 1 + \frac{2MG}{\rho_+} + \frac{4M^2G^2}{\rho_+^2}$$

$$\frac{1}{\psi(\rho_-)} = 1 + \frac{2MG}{\rho_-} + \frac{4M^2G^2}{\rho_-^2}$$

$$[\ \ ] = c\left(\frac{1}{\rho} - \alpha\right)\left(\frac{1}{\rho} - \beta\right)$$

Note [ ] = 0 at $\rho = \rho_\pm$.

$$\Rightarrow [\ \ ] = c\left(\frac{1}{\rho} - \frac{1}{\rho_+}\right)\left(\frac{1}{\rho} - \frac{1}{\rho_-}\right)$$

$\frac{1}{\rho^2}$ term in [ ] $= -\frac{1}{\rho^2}\left(\frac{1}{\rho_+^2} - \frac{1}{\rho_-^2}\right)^{-1}\left\{2MG\left(\frac{1}{\rho_+} - \frac{1}{\rho_-}\right) + 4M^2G^2\left(\frac{1}{\rho_+^2} - \frac{1}{\rho_-^2}\right)\right\}$

$+ 4M^2G^2/\rho^2$

$$d\ell = \rho_0 \sec\theta \tan\theta \, d\theta$$

$$\rho_0 \int_0^{\pi/2} \frac{\rho_0 \sin\theta \cancel{\tan\theta}\, d\theta}{\rho_0 \sec\theta \cdot \rho_0 \tan\theta} \left(1 + \frac{GM}{\rho_0 \sec\theta} + \frac{GM \sec\theta}{\rho_0 (1 + \sec\theta)} \right)$$

$$= \cancel{\rho_0} \int_0^{\pi/2} d\theta \left(1 + \frac{GM}{\rho_0} \cos\theta + \frac{GM}{\rho_0 (1 + \cos\theta)} \right)$$

$$= \frac{\pi}{2} + \frac{GM}{\rho_0} \cdot 1 + \frac{GM}{\rho_0} \int_0^{\pi/2} \frac{d\theta}{\underbrace{1 + \cos\theta}_{\parallel \; 2\cos^2\frac{\theta}{2}}}$$

$$= \frac{\pi}{2} + \frac{GM}{\rho_0} + \frac{GM}{2\rho_0} \cdot \int_0^{\pi/2} \sec^2\frac{\theta}{2} \, d\theta$$

$$= \frac{\pi}{2} + \frac{GM}{\rho_0} + \frac{GM}{2\rho_0} \, 2 \tan\frac{\theta}{2} \Big|_0^{\pi/2}$$

$$= \frac{\pi}{2} + \frac{GM}{\rho_0} + \frac{GM}{\rho_0} \cdot 1$$

$$= \frac{\pi}{2} + \frac{2GM}{\rho_0}.$$

Apparent singularity at $r = 2GM$:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$r'^2 - t'^2 = T^2\left(\frac{r}{2GM} - 1\right)\exp\left(\frac{r}{2GM}\right) \to$$

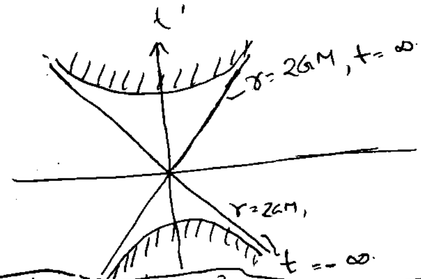
$$\frac{2r't'}{r'^2 + t'^2} = \tanh\left(\frac{t}{2MG}\right) \longrightarrow$$

Monotone ~~increasing~~ for $\cancel{r > 2GM}$, $-\infty < t < \infty$. $\Rightarrow$ ~~unique~~ soln. for $r'^2 - t'^2 > 0$ $\to$ ~~(r', t')~~ ~~$(-r', -t')$~~

Ex. $\Rightarrow$

$$ds^2 = -\frac{32G^3M^3}{rT^2}\exp\left(-\frac{r}{2GM}\right)(dt'^2 - dr'^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Note: $r = 2GM \Rightarrow r'^2 - t'^2 = 0 \Rightarrow r' = \pm t'$



outside horizon:

Physical region: $r \geq 2GM$

$\Rightarrow r'^2 \geq t'^2 \Rightarrow$ ~~$r' > t' \to \infty$~~

$r' > 0$ or $r' < 0$: Two isolated regions outside.

Take $r' > 0$ to be the physical region.

$1 \leftrightarrow 1$ correspondence with (r, t) for $r \geq 2GM$

$r' = -t' \Rightarrow \tanh\left(\frac{t}{2MG}\right) = -1 \Rightarrow t \to -\infty.$

$r = 0 \Rightarrow r'^2 - t'^2 = -T^2 \Rightarrow$ singularity

Note: $\left(\frac{r}{2GM}-1\right) \exp\left(\frac{r}{2GM}\right) \to$ Monotone increasing for $r > 0$

$\tanh \frac{t}{2MG} \to$ Monotone decreasing. for $-\infty < t < \infty$.

$\Rightarrow$ Give r', t'; r & t are uniquely determined

Given r, t ; $r > 2GM$.

$r'^2 - t'^2 = a > 0$.

$\frac{2r't'}{r'^2+t'^2} = b \qquad -1 < b < 1$

$\Rightarrow \quad r'^2 + t^2 = \frac{2}{b} r't'$

$\Rightarrow 2r'^2 = a + \frac{2}{b} r't'$

$\Rightarrow \quad t' = \frac{b}{2r'}(2r'^2 - a) = br' - \frac{ab}{2r'}$

$r'^2 - \left(br' - \frac{ab}{2r'}\right)^2 = a$

$\Rightarrow \quad r'^2(1-b) + ab^2 - \frac{a^2b^2}{4r'^2} = a$

$4r'^4(1-b) + 4r'^2 a(b^2-1) - a^2b^2 = 0$.

$\Rightarrow$ Two roots. Product < 0.

$\Rightarrow$ one +ve root. r'^2 is unique.

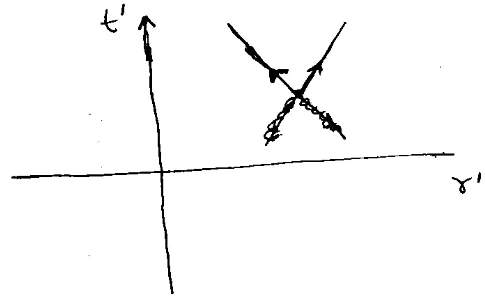
$r' = \pm(--) \qquad t' = \pm \cdots \quad (r', t'), (-r', t') \to$ same r, t.

Light cone:

. $ds^2 \leq 0$ for physical particle.

$\Rightarrow \quad dt'^2 > dr'^2$

Tangent to a physical trajectory must lie within a 45° cone about t' axis.



Future light cone: Tangent along which time increases.

Notion of increasing time: borrowed from a particle falling from $r = \infty$.

Increasing $t \Rightarrow$ Increasing t'.

Future light cone of a space-time point inside the horizon.

① Lies inside the horizon.

② Hits the singularity. the event horizon of

$\Rightarrow$ Once a particle is inside a black hole, it must hit the singularity in future.

(G.90)

In order to find the most general homogeneous isotropic metric we need to define ~~sym~~ symmetries of a metric in a more general context.

$g_{\mu\nu}(x)$: A metric.

$x^\mu \to x'^\mu = f^\mu(x)$: A coordinate trs.

$$g'_{\mu\nu}(x') = g_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu}$$

The coordinate transformation is called an isometry if

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x'). \Rightarrow g_{\mu\nu}(x') = g_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu}$$

Example: Minkowski space has Poincare trs. as isometry group → Active & Passive. (G.90a)

$$x'^\mu = x^\mu + \epsilon\, \xi^\mu(x).$$

ϵ : small.

$$x^\mu \simeq x'^\mu - \epsilon\, \xi^\mu(x').$$

$$\frac{\partial x^\rho}{\partial x'^\mu} = \delta^\rho_\mu - \epsilon\, \partial_\mu \xi^\rho(x') \simeq \delta^\rho_\mu - \epsilon\, \partial_\mu \xi^\rho(x)$$

$$\Rightarrow g_{\mu\nu}(x') \simeq g_{\mu\nu}(x) + \epsilon\, \xi^\rho \partial_\rho g_{\mu\nu}(x).$$

$$\Rightarrow g_{\mu\nu}(x) + \epsilon\, \xi^\rho \partial_\rho g_{\mu\nu} = g_{\rho\sigma} \{ \delta^\rho_\mu - \epsilon\, \partial_\mu \xi^\rho(x) \} \{ \delta^\sigma_\nu - \epsilon\, \partial_\nu \xi^\sigma(x) \}.$$

$$= g_{\mu\nu}(x) - \epsilon\, g_{\rho\nu} \partial_\mu \xi^\rho(x) - \epsilon\, g_{\mu\sigma} \partial_\nu \xi^\sigma(x)$$

G.90a

Suppose P & Q are two points in the space with coordinates $\{x^\mu_{(1)}\}$ & $\{x^\mu_{(2)}\}$ in the x coordinate system, such that

$$x^\mu_{(2)} = f^\mu(x_{(1)})$$

---

f^μ is an isometry.

The metric around Q in the x coordinate system is $g_{\mu\nu}(x)$ around $x^\mu = x^\mu_{(2)}$.

Now consider the coordinate system

$$x'^\mu = f^\mu(x)$$

In this coordinate system P is located at

$$x'^\mu = f^\mu(x_{(1)}) = x^\mu_{(2)}$$

Metric:

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x')$$ since f is an isometry.

$\Rightarrow$ The metric in x' coordinate system near P is identical to the metric in x coordinate system near Q. ← All tensors are identical.

Active viewpoint: $f^\mu(x)$ takes P to Q.

~~Similarly~~ Given any point P in space, f^μ takes it to another point Q such that space around P & Q look the same once we make appropriate coordinate transformation.

Use $D_\rho g_{\mu\nu} = 0$

$$\Downarrow$$

$$\partial_\rho g_{\mu\nu} - \Gamma^\tau_{\rho\mu} g_{\tau\nu} - \Gamma^\tau_{\rho\nu} g_{\tau\mu} = 0.$$

$$\Rightarrow \quad \xi^\rho \Gamma^\tau_{\rho\mu} g_{\tau\nu} + \xi^\rho \Gamma^\tau_{\rho\nu} g_{\tau\mu} + g_{\rho\nu} \partial_\mu \xi^\rho + g_{\mu\sigma} \partial_\nu \xi^\sigma = 0$$

$$\Rightarrow \quad g_{\rho\nu} D_\mu \xi^\rho + g_{\mu\sigma} D_\nu \xi^\sigma = 0$$

$$\Rightarrow \quad D_\mu (g_{\rho\nu} \xi^\rho) + D_\nu (g_{\mu\sigma} \xi^\sigma) = 0.$$

$$\Rightarrow \quad D_\mu \xi_\nu + D_\nu \xi_\mu = 0.$$

A $\xi_\mu(x)$ satisfying this equation is known as a killing vector.

---

* Static metric $\Leftrightarrow$ A time-like killing vector in asymptotic space-time.

---

* Note: Set of isometries form a group.

$x^\mu \to x'^\mu = f^\mu(x) \quad \Rightarrow \quad \cancel{g'_{\mu\nu}(x') = g_{\mu\nu}(x')}$

$$\Rightarrow \quad g_{\mu\nu}(f(x)) \frac{\partial f^\mu}{\partial x^\sigma} \frac{\partial f^\nu}{\partial x^\rho} = g_{\sigma\rho}(x).$$

$x^\mu \to x'^\mu = \phi^\mu(x).$

$$\Rightarrow \quad g_{\mu\nu}(\phi(x)) \frac{\partial \phi^\mu}{\partial x^\rho} \frac{\partial \phi^\nu}{\partial x^\sigma} = g_{\rho\sigma}(x)$$

$F^\mu(x) = f^\mu(\phi(x))$

Define $y^\mu = \phi^\mu(x)$. $\Rightarrow f^\mu(y) = F^\mu(x)$

$\Rightarrow \dfrac{\partial F^\mu}{\partial x^\rho} = \dfrac{\partial f^\mu}{\partial y^\alpha}\bigg|_{y^\alpha = \phi^\alpha(x)} \dfrac{\partial \phi^\alpha(x)}{\partial x^\rho}$

$$g_{\mu\nu}(F(x)) \cdot \frac{\partial F^\mu}{\partial x^\rho} \frac{\partial F^\nu}{\partial x^\sigma}$$

$$= g_{\mu\nu}(f(y)) \frac{\partial f^\mu}{\partial y^\alpha} \frac{\partial \phi^\alpha(x)}{\partial x^\rho} \frac{\partial f^\nu}{\partial y^\beta} \frac{\partial \phi^\beta(x)}{\partial x^\sigma}$$

$$= g_{\alpha\beta}(\phi(x)) \frac{\partial \phi^\alpha}{\partial x^\rho} \frac{\partial \phi^\beta}{\partial x^\sigma} = g_{\rho\sigma}(x)$$

Thus if $f^\mu(x)$ & $\phi^\mu(x)$ are isometries, then so is $F^\mu(x) = f^\mu(\phi(x))$

Denote F by $f \circ \phi$

$\Rightarrow$ Set of isometries form a group.

A metric has spherical symmetry if it has an isometry group $SO(3)$.

Static & spherically symmetric:

① An isometry corresponding to time translation

$\Rightarrow$ Time like killing vector (in asymptotic spacetime

② ~~There~~ A set of isometries forming the group $SO(3)$ and commuting with the isometry generated by time translation.

Space-time describing the whole universe:

→ Impossible to describe exactly.

→ need approximation.

---

① Homogeneous → The space-time looks the same from every point in space at a given 'time'.

⇒ Assumes the notion of a 'cosmic time' coordinate t & space coordinates

⇒ Certainly not ~~true~~ over galactic distance scale but could be true on averaging over large distance scale.

Like a gas is not homogeneous over ~~[illegible]~~ molecular distance scale but is homogeneous over larger distance scale.

---

② ~~[illegible]~~ Isotropic: The space ~~[illegible]~~ should look the same in all space directions at a given space-time point.

Example of such a metric:

$$- f_1(t)^2 \, dt^2 + (f_2(t))^2 \, (dx^2 + dy^2 + dz^2).$$

→ Symmetries:

i) $(x, y, z) \to (x+a, y+b, z+c)$ ⇒ Homogeneous.

---

ii) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to S \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. $\quad S S^T = I$

↳ rotation matrix.

Ex. Consider a metric $g_{\mu\nu}(x)$ with a set of isometries $x^\mu \to f^\mu_{k}(x, \lambda_1, \dots \lambda_n)$ ~~1, 2, — n.~~

Now consider a new coordinate system $y^\mu = \phi^\mu(x)$.

Show that in the y coordinate system the metric still has a set of n isometries with the same isometry group.

- Back to cosmology:

How do we describe a spatially homogeneous, isotropic universe.

- Metric:

$$ds^2 = -f_1(t, \vec{x})\, dt^2 + f_{2i}(t, \vec{x})\, dt\, dx^i + f_{3ij}(\vec{x}, t)\, dx^i dx^j$$

(x^1, x^2, x^3) : spatial coordinates.

~~f_{2i} can be set to zero by $x^i \to f^i(\vec{x}, t)$~~

Homogeneous: There must be an isometry $x^\mu \to f^\mu(x)$ ~~this isometry~~ such that given two space-time points $P = (t, \vec{x}_{(1)})$ and $Q = (t, \vec{x}_{(2)})$,

$$\left(f^0(t, \vec{x}_{(1)}),\ f^i(t, \vec{x}_{(1)})\right) = (t, \vec{x}_{(2)}).$$

In that case by a coordinate trs. the metric around P & Q may be made to look identical.

In $\vec{x}$ coordinate, metric ~~shown~~ is

$g_{\mu\nu}(t, \vec{x})$

→ Looks different around $P(t, \vec{x} = \vec{x}_{(1)})$ & $Q(t, \vec{x} = \vec{x}_{(2)})$.

After coordinate trs. to $y^\mu = f^\mu(\vec{x})$, the metric around P is

$g_{\mu\nu}(t, \vec{y})$

Point P: $(t, \vec{x}_{(2)})$.

→ Metric around P in y coordinate system is the same as the metric around Q in the x coordinate system.

→ Use coord. trs. to set $f_{2i} = 0$. + Isotropy. G-95a use $x^i \to f^i(\vec{x}, t)$

Note: The coordinate trs. is of the form:

$$t' = f^0(t, \vec{x}) = t$$
$$x'^i = f^i(t, \vec{x})$$

Ex. Take f_2 small. Work out the coordinate trs. to first order.

In order that it is an isometry, the metric should look the same.

$$\Rightarrow ds^2 = -f_1(t', \vec{x}')\, dt'^2 + f_{3ij}(\vec{x}', t')\, dx'^i dx'^j$$

$$= -f_1(t, \vec{f}(t, \vec{x}))\, dt^2 + f_{3ij}(\vec{f}(t, \vec{x}), t) \left\{ \frac{\partial f^i}{\partial t} dt + \frac{\partial f^i}{\partial x^k} dx^k \right\} \left\{ \frac{\partial f^j}{\partial t} dt + \frac{\partial f^j}{\partial x^l} dx^l \right\}$$

should be

$$= -f_1(t, \vec{x})\, dt^2 + f_{3ij}\, dx^i dx^j$$

G.95a

Isotropy: For any point $(t, \vec{x}_0)$ and a pair of nearby points $(t, \vec{x}_0 + \vec{\delta x})$, $(t, \vec{x}_0 + \vec{\delta y})$, ~~there must be a transformation~~ such that

$$g_{\mu\nu}(\vec{x}_0, t)\, \delta x^\mu\, \delta x^\nu = g_{\mu\nu}(\vec{x}_0, t)\, \delta y^\mu\, \delta y^\nu$$

there must be an isometry $x'^\mu = f^\mu(t, \vec{x})$ such that

(i) $f^0(t, \vec{x}) = t$ for all $\vec{x}$,

$f^i(t, \vec{x}_0) = x_0^i$

&

(ii) $f^i(t, \vec{x}_0 + \vec{\delta x}) = x_0^i + \delta y^i$

$\Rightarrow$

$$f_1(t, \vec{f}(t,\vec{x})) = f_1(t,\vec{x}).$$

$$f_{3ij} \frac{\partial f^i}{\partial x^k} \frac{\partial f^j}{\partial t} = 0.$$

$$f_{3ij} \frac{\partial f^i}{\partial x^k} \frac{\partial f^j}{\partial x^l} = f_{3ij}$$

$\Rightarrow$ $\frac{\partial f^j}{\partial t} = 0$, $f_1(t,\vec{x})$ is independent of $\vec{x}$.

and $x^i \to f^i(\vec{x})$ is an isometry of the 3-dimensional metric $f_{3ij}(\vec{x}, t)$

$\Rightarrow$ $f_{3ij}(\vec{x}, t) = \hat{f}(t)\, \tilde{f}_{3ij}(\vec{x})$. ← not needed.

Thus what we need to find is the general 3-dimensional metric $\hat{f}_{3ij}$ which is homogeneous and isotropic.

At fixed t we have
① Flat space
② Sphere
③ Constant -ve curvature

(i) Flat space:

$$f_{3ij}\, dx^i dx^j = \hat{f}(t)\, dx^i dx^i$$

(ii) Surface of a 3-dimensional sphere:

$$\sum_{i=1}^{4} z_i^2 = R^2 \qquad ds^2 = \Sigma (dz_i)^2$$

This is homogeneous:

Given $\vec{z}, \vec{z}'$ on sphere, there is an isometry (SO(3) rotation) which

E.g. take $z_4 = 0$ point. Rotation in z_1, z_2, z_3 keeping the point fixed.

i) Preserve the metric
ii) Maps the sphere onto itself.
iii) Maps $\vec{z}$ to $\vec{z}'$.

This is also isotropic.

Given any point on the sphere $(0,0,0,R)$, there is a group of isometries which

i) Leave the point fixed.

ii) Maps neighbouring points which are equidistant from $(0,0,0,R)$ into each other.

Rotation among 1, 2, 3.

Metric:

$$z_4 = \rho\cos\psi$$
$$z_3 = \rho\sin\psi\cos\theta$$
$$z_1 = \rho\sin\psi\sin\theta\cos\phi$$
$$z_2 = \rho\sin\psi\sin\theta\sin\phi$$

Ex. $ds^2 = d\rho^2 + \rho^2\{d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\}$.

$\rho = R$, $d\rho = 0$

$r = \sin\psi$

$$d\psi^2 = \frac{dr^2}{\cos^2\psi} = \frac{dr^2}{1-r^2}$$

$$\Rightarrow \quad ds^2 = R^2\left\{\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right\}.$$

iii) A 'hyperboloid':

$$z_4^2 - (z_1^2 + z_2^2 + z_3^2) = R^2 \Rightarrow$$

$z_4^2 \geq R^2 \Rightarrow |z_4| > R$.
$\Rightarrow$ Two branches
$z_4 > R$, $z_4 < -R$.
Concentrate on $z_4 > R$ branch.

$$ds^2 = -(dz_4)^2 + \sum_{i=1}^{3} dz_i^2.$$

Metric invariant under:

$$\begin{pmatrix} z_1 \\ \vdots \\ z_4 \end{pmatrix} \to \Lambda \begin{pmatrix} z_1 \\ \vdots \\ z_4 \end{pmatrix}$$

Λ : (3+1) dimensional Lorentz trs.

Any two points $(z_1, \ldots z_4)$ & $(z_1', \ldots z_4')$ on the surface $(z_4 > R,\ z_4' > R)$ can be related by a Lorentz trs.

$\Rightarrow$ Homogeneous.

~~Rotation about~~ Take a specific point on the surface : $\begin{pmatrix} 0 \\ 0 \\ 0 \\ R \end{pmatrix}$.

$\Rightarrow$ Invariant under rotation in (1,2,3)

$\Rightarrow$ Isotropic.

G.99

$$Z_4 = \rho \cosh\psi$$
$$Z_3 = \rho \sinh\psi \cos\theta$$
$$Z_1 = \rho \sinh\psi \sin\theta \cos\phi$$
$$Z_2 = \rho \sinh\psi \sin\theta \sin\phi.$$

$$ds^2 = -d\rho^2 + \rho^2 d\psi^2 + \rho^2 \sinh^2\psi \{ d\theta^2 + \sin^2\theta\, d\phi^2 \}$$

$$\rho = R \Rightarrow d\rho = 0.$$

$$r = \sinh\psi.$$

$$\Rightarrow d\psi = \frac{dr}{\cosh\psi} = \frac{dr}{\sqrt{1+r^2}}$$

$$ds^2 = R^2 \left\{ \frac{dr^2}{1+r^2} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \right\}$$

Thus the most general ~~3-d~~ Homogeneous & isotropic 3-d metric is:

$$ds^2 = -f_1(t)^2 dt^2 + f_3(t)^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \right\}$$

$k = 0$: Flat 3-d space

$k = 1$: sphere

$k = -1$: Hyperboloid.

$$t \to f(t) \Rightarrow f_1(t) = 1. \qquad f_3(t) = \lambda(t)$$

$$ds^2 = -dt^2 + \lambda(t)^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \right\}.$$

$$ds^2 = -dt^2 + (\lambda(t))^2 \left\{ \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \right\}$$

$k = 1, 0, -1$

Homogeneity $\Rightarrow$ all space-time around points (r, θ, ϕ) look the same

Prove that the trajectory of a particle is a geodesic in 3d.

Consider a light ray travelling from (a, θ_0, ϕ_0) at $t = t_1$ to $(0, \theta_0, \phi_0)$ at $t = t_2$.

The spatial distance is the only invariant.

$ds^2 = 0.$

$\Rightarrow \quad dt = -\lambda(t) \dfrac{dr}{1-kr^2} \qquad$ (since $dr < 0$).

$$\int_{t_1}^{t_2} \frac{dt}{\lambda(t)} = \int_0^a \frac{dr}{1-kr^2}$$

Source : Fixed at (a, θ_0, ϕ_0) at all t. → Star

Observer : Fixed at $(0, \theta_0, \phi_0)$ at all t. → Earth

$\Rightarrow ds^2 = -dt^2$, $d\tau^2 = dt^2$

Suppose the source ~~emits~~ has rest frequency ν measured in its local inertial frame.

$\Rightarrow$ period $\Delta\tau = 1/\nu$

If t_1 and $t_1 + \Delta t_1$ are the times when the outgoing wave is at its peak then

$\Delta t_1 = \Delta\tau_1 = 1/\nu.$

Suppose the peaks reach the detector at t_2 and $t_2 + \Delta t_2$.

$$\int_{t_1+\Delta t_1}^{t_2+\Delta t_2} \frac{dt}{\lambda(t)} = \int_0^a \frac{dr}{1-kr^2}$$

$\Rightarrow \quad \dfrac{\Delta t_2}{\lambda(t_2)} - \dfrac{\Delta t_1}{\lambda(t_1)} = 0 \qquad \Rightarrow \quad \Delta t_2 = \dfrac{\lambda(t_2)}{\lambda(t_1)} \Delta t_1 = \dfrac{\lambda(t_2)}{\lambda(t_1)} \cdot 1/\nu.$

The lapse of proper time at the detector between two succesive ~~dat~~ peaks

$$\Delta\tau_2 = \Delta t_2 = \frac{\lambda(t_2)}{\lambda(t_1)} \frac{1}{\nu}.$$

$\Rightarrow$ Apparent frequency as observed at the detector

$$\nu' = \frac{1}{\Delta\tau_2} = \frac{\lambda(t_1)}{\lambda(t_2)} \nu.$$

On the other hand if the same source is placed at $r=0$, the observed frequency will be ν.

observation : $\nu' < \nu$.

$\Rightarrow$ Red shift in spectral lines

Thus $\lambda(t_1) < \lambda(t_2)$.

Since $t_2 > t_1$ this means that $\lambda(t)$ is increasing.

$\Rightarrow$ Proper Distance between two points $(t, r_1, \theta_1, \phi_1)$ and $(t, r_2, \theta_2, \phi_2)$ is increasing with time.

$\Rightarrow$ Universe is expanding.

Note: When a is small so that $\lambda(t_1)/\lambda(t_2) \simeq 1$, then the red shift can also be interpreted as due to the doppler shift.

$(t_2 - t_1) \equiv t_0 - t_1$ The "distance" between the source and the observer in light years.

t_2 : Present time $\to$ Choose it to be t_0 $\Rightarrow t_1 < t_0$

~~$\lambda(t_1)/\lambda(t_2)$~~ Some definitions:

$$z(t_1, t_0) \equiv \frac{\lambda(t_0)}{\lambda(t_1)} - 1 \to \text{red-shift parameter.}$$

For stars which are close to us, the red-shift parameter is small.

For stars which are far away from us, the red-shift parameter is large.

$$\frac{\lambda(t)}{\lambda(t_0)} \equiv 1 + (t - t_0)\, \mathcal{H}_0(t_0) + \frac{1}{2}\, q_0(t_0)\, \mathcal{H}_0(t_0)^2\, (t - t_0)^2 + \cdots$$

$\mathcal{H}_0(t_0)$: Hubble constant at time t_0.

Interpretation: When $t_1 - t_0$ is small

$$z(t_1, t_0) = \frac{1}{1 + (t_1 - t_0)\, \mathcal{H}_0} - 1 \simeq \mathcal{H}_0(t_0)\,(t_0 - t_1)$$

$\Rightarrow$ z is proportional to the distance of the star.

$\Rightarrow$ Proportionality constant: Hubble constant.

$q_0(t_0)$: Deceleration parameter at time t_0

$\Rightarrow$ Tells us how quickly the expansion of the universe is slowing down.

We can ~~compute~~ measure these parameters if we know Z as a function of $(t_0 - t_1)$.

Q. In the derivation we have assumed that ~~to~~ the earth and the stars are at "rest" in (r, θ, ϕ, t) coordinate system.

To what extent are these assumptions correct?

For this we need to examine the ~~behavior~~ form of the energy-momentum tensor

---

⇒ Will tell us something about the motion of the ~~particles~~ matter ~~stars~~ in the universe.

G.10

The functional form of $\hat{a}(t)$ depends on what kind of matter is present in the universe.

$T_{\mu\nu}$ must also remain form invariant under the isometries of space-time.

$$\Rightarrow \quad \underset{T_{\mu\nu}(x')''}{T'_{\mu\nu}}(x') \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} = T_{\sigma\rho}(x)$$

The isometries here are of the form:

$$t' = t$$
$$x'^{i} = f^{i}(\vec{x})$$

$$\Rightarrow \quad \frac{\partial x'^{\mu}}{\partial x^{\rho}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \frac{\partial f^{i}}{\partial x^{j}} & \\ 0 & & & \end{pmatrix}.$$

$$T_{00}(x) = T'_{\mu\nu} \frac{\partial x'^{\mu}}{\partial x^{0}} \frac{\partial x'^{\nu}}{\partial x^{0}} = T'_{00}(x') = T_{00}(\vec{x}')$$

$$\Rightarrow \quad T_{00}(t, \vec{x}) = T_{00}(t, \vec{x}').$$

$$\Rightarrow \quad T_{00} = \rho(t) \qquad \text{No } \vec{x} \text{ dependence.}$$

$$T_{0i}(\vec{x}, t) = T'_{\mu\nu}(\vec{x}', t) \frac{\partial x'^{\mu}}{\partial x^{0}} \frac{\partial x'^{\nu}}{\partial x^{i}}$$
$$= T'_{0j}(\vec{x}', t) \frac{\partial x'^{j}}{\partial x^{i}} = T_{0j}(\vec{x}', t) \frac{\partial x'^{j}}{\partial x^{i}}$$

$\Rightarrow$ $T_{0i}(\vec{x}, t)$ is a form invariant rank $(0,1)$ tensor in the 3-dimensional space.

(G.105)

$$T'_{ij}(\vec{x},t) = T'_{\mu\nu}(\vec{x}',t)\,\frac{\partial x'^{\mu}}{\partial x^i}\,\frac{\partial x'^{\nu}}{\partial x^j}$$

$$= T'_{kl}(\vec{x}',t)\,\frac{\partial x'^k}{\partial x^i}\,\frac{\partial x'^l}{\partial x^j} = T_{kl}(\vec{x}',t)\,\frac{\partial x'^k}{\partial x^i}\,\frac{\partial x'^l}{\partial x^j}$$

$\Rightarrow$ $T_{ij}(\vec{x},t)$ is a form invariant tensor in 3-dimensions.

Result: In a isotropic, homogeneous space there is no form invariant vector, and the only form invariant rank (0,2) tensor is the metric.

$\Rightarrow$ $T_{0i}(\vec{x},t) = 0.$

$T'_{ij}(\vec{x},t) = p(t)\,g_{ij}(\vec{x})$.

$T_{00}(\vec{x},t) = +\rho(t)$

Define : $U^{\mu} = (1,0,0,0)$.

$$T_{\mu\nu} = (\rho+p)\,U_{\mu}U_{\nu} + p\,g_{\mu\nu}.$$

$\Rightarrow$ $T_{00} = (p+\rho) - p = \rho$

$T_{ij} = p\,g_{ij}$ $\qquad T_{0i} = 0 \rightarrow \frac{dx^i}{d\tau} = 0$ on average

$\Rightarrow$ Einstein's eqn.:

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = (\rho+p)\,U_{\mu}U_{\nu} + p\,g_{\mu\nu}.$$

3 unknown functions of time.

$\lambda(t)$, $\rho(t)$, $p(t)$.

G.106.a

$T^{00}(\vec{x},t) = \sum_n \delta^{(3)}(\vec{x}-\vec{x}_n(t))\, p_n^0 \qquad m_n \frac{dx_n^0}{d\tau} \to$ Energy.

$T^{i0} = \sum_n \delta^{(3)}(\vec{x}-\vec{x}_n(t))\, p_n^i$

$T^{ij} = \sum_n \delta^{(3)}(\vec{x}-\vec{x}_n(t))\, p_n^i(t) \frac{dx_n^j}{dt} \qquad p_n^\alpha = m_n \frac{dx_n^\alpha}{d\tau}$

$T^{i0} = 0 \Rightarrow$ Average $\vec{p}_n$ is zero.

$\Rightarrow$ Stars & galaxies have zero average velocity in the (r,θ,ϕ,t) coordinate system.

T^{00} = Energy density $\equiv \rho$

$T_{00} = \rho$

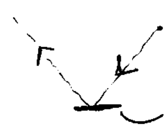
What about T^{ij}?

Note : $T^{ij} = p\, g^{ij}$

$\Rightarrow \sum_n \delta^{(3)}(\vec{x}-\vec{x}_n(t))\, p_n^i(t) \frac{dx_n^j}{dt} = p\, g^{ij}$

$p_n^i = m_n \frac{dx_n^i}{d\tau}$

p = pressure.



surface area δA per unit time.

Calculate the total momentum transfer per unit time to A in a locally inertial frame.

Show that it has the same expression as $p\,\delta A$.

Recall the interpretation of $T^{\alpha\mu}$

$T^{\alpha 0}$ = Four momentum density.

$T^{\alpha i}$ = Four momentum flux.

$\Rightarrow \int_V d^3x \, T^{\alpha 0}$ = Net four momentum of all particles inside a volume δv.

$\int_A T^{\alpha i} n^i \, d^2x$ = Net flux of Four momentum through the area A.

$$T_{ij} = p(t) \, g_{ij}$$

Use ~~locally inertial frame~~ coordinate system in which $g_{ij} = \delta_{ij}$ at $\vec{x}_0$.

$$T_{ij}(\vec{x}_0) = \cancel{p(t)\, g_{ij}(\vec{x}_0)} = p(t) \, \delta_{ij}$$

δA in 1-2 plane.

Total p^1 flowing through δA / unit time

$$= \delta A \cdot T^{11} = p(t) \, \delta A.$$

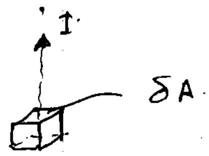
Note: +ve p^1 : Flows along x^1

-ve p^1 : Flows along $-x^1$

$\Rightarrow$ Both contribute +ve amount.

G.10088

Now imagine placing a small empty cube



Nothing hits it from inside.

Total incident momentum on δA

$= \frac{1}{2} p(t).\ \delta A$

Elastic collision $\Rightarrow$ Goes back with same momentum.

Total Momentum transfer/time $= 2\cdot\frac{1}{2} p(t)\,\delta A$

$\parallel$

Force.

$\text{Force}/\delta A = p(t)$

$\parallel\!\!\mid$

Pressure.

Note : $T^{12} = 0$. Why ?

+ve p^2 : Flows both along x^1 & $-x^1$

-ve p^2 : Flows both along x^1 & $-x^1$

$\Rightarrow$ Cancels in a homogeneous & isotropic universe.

G.109

$$T_{00} = \rho(t)$$

$$T_{ij} = p(t)\, g_{ij}$$

Now substitute into Einsteins eqn.

$$ds^2 = -dt^2 + (\lambda(t))^2 \underbrace{\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2+\sin^2\theta\, d\phi^2)\right\}}_{\tilde{g}_{ij}\, dx^i\, dx^j}$$

Ex. check that

$$R_{00} = 3\ddot{\lambda}/\lambda \qquad R_{0i} = 0.$$

$$R_{ij} = -(\lambda\ddot{\lambda} + 2\dot{\lambda}^2 + 2k)\, \tilde{g}_{ij}$$

$$\Rightarrow \quad R = -\frac{6\ddot{\lambda}}{\lambda} - 6\left(\frac{\dot{\lambda}}{\lambda}\right)^2 - \frac{6k}{\lambda^2}$$

00:

$$-8\pi G\rho = -3\left(\frac{\dot{\lambda}}{\lambda}\right)^2 - 3\frac{k}{\lambda^2} \qquad \text{---- (1)}$$

$$-8\pi G p = 2\frac{\ddot{\lambda}}{\lambda} + \left(\frac{\dot{\lambda}}{\lambda}\right)^2 + \frac{k}{\lambda^2} \qquad \text{---- (2)}$$

From (1):

$$\frac{d}{dt}\left(\frac{8\pi G}{3}\rho\lambda^3\right) = \frac{d}{dt}\left(\lambda\dot{\lambda}^2 + k\lambda\right)$$

$$= \lambda^2\dot{\lambda}\underbrace{\left\{\frac{2\ddot{\lambda}}{\lambda} + \left(\frac{\dot{\lambda}}{\lambda}\right)^2 + \frac{k}{\lambda^2}\right\}}_{=\,-8\pi G p}$$

$$\Rightarrow \quad \frac{d}{dt}(\rho\lambda^3) = -3p\lambda^2\dot{\lambda} \qquad \text{------ (3)}$$

G.110

check: 1 + 3 $\Rightarrow$ 2.

Puzzle: 2 eqn.s but 3 unknowns –

$\rho(t), \; p(t), \; \lambda(t)$.

We need one more eqn..

$\Rightarrow$ Equation of state – relates ρ & p.

Radiation: $p = \frac{\rho}{3}$.

Non-relativistic matter: $p \ll \rho$.

Cosmological constant: $p = -\rho$

$\Rightarrow p \simeq 0$

Note:

$$p \sim \frac{dx_n^i}{dt} \frac{dx_n^j}{dt}$$

$$\rho \sim \frac{dx_n^0}{dt} \frac{dx_n^0}{dt} \gg \left(\frac{dx_n^i}{dt}\right)\left(\frac{dx_n^j}{dt}\right)$$

Momentum << Energy.

Substitute this & analyze.

$$T_{em}^{\alpha\mu} = g^{\alpha\rho} g^{\mu\sigma} g^{\kappa\tau} F_{\rho\kappa} F_{\sigma\tau} - \frac{1}{4} g^{\alpha\mu} g^{\rho\sigma} g^{\kappa\tau} F_{\rho\kappa} F_{\sigma\tau}$$

$$(T_{em})_{ij} = g^{\kappa\tau} F_{i\kappa} F_{j\tau} - \frac{1}{4} g_{ij} \, g^{\rho\sigma} g^{\kappa\tau} F_{\rho\kappa} F_{\sigma\tau}$$

$$= -E_i E_j + g^{kl} \epsilon_{ikm} \epsilon_{jln} B_m B_n - \frac{1}{4} g_{kl} \times 2$$

$$(g^{ij} E_i E_j + \frac{1}{2} g^{ik} g^{jl} F_{ij} F_{kl})$$

ρ & p are not independent.

Interpretation of p: pressure.

$\Rightarrow$ There is an eqn. of state relating ρ & p.

e.g. $p = \frac{\rho}{3}$ for a gas of massless particles

$p = 0$ for non-relativistic matter at zero temperature.

etc.

$\Rightarrow$ We have two unknowns.

Energy momentum conservation:

$$R^3(t)\,\frac{dp}{dt} = \frac{d}{dt}\left\{ R^3(t)\left(\rho(t) + p(t)\right)\right\}$$

$\rightarrow$ One relation between R, ρ, & p.

Finally Einstein's eqns has one more independent component which

~~one component~~:

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = -8\pi G\, T_{\mu\nu}$$

$$\times g^{\mu\nu} \Rightarrow -R = -8\pi G\, T_{\mu\nu}\, g^{\mu\nu}$$

$$\Rightarrow R_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, g^{\rho\sigma} T_{\rho\sigma}\right)$$

00 - component:

$$3\ddot{\lambda} = -4\pi G(\rho + 3p)\,\lambda$$

$\Rightarrow$ determines $\lambda(t)$.

Eqs. involving ρ, p and λ:

$$\left(\frac{\dot\lambda}{\lambda}\right)^2 + \frac{k}{\lambda^2} = \frac{8\pi G}{3}\rho$$

$$\frac{d}{dt}(\rho\lambda^3) = -3p\lambda^2\dot\lambda$$

Eqn of state: Non relativistic matter

$$p = 0.$$

$$\Rightarrow \rho\lambda^3 = \underbrace{C}_{\text{constant}}$$

$$\rho = \frac{C}{\lambda^3} \qquad C = \rho_0\lambda_0^3 > 0$$

Present density Present scale factor

$$\dot\lambda^2 + k = \frac{8\pi G}{3}\frac{C}{\lambda^3}\cdot\lambda^2$$

$$\frac{1}{2}\dot\lambda^2 - \frac{8\pi G}{3}\frac{C}{\lambda} = -k \Rightarrow t - t_0 = \int_{\lambda_0}^{\lambda}\frac{d\lambda}{\sqrt{\frac{8\pi G}{3}\frac{C}{\lambda} - k}}$$

Motion of a particle of unit mass in a potential:

$$V(\lambda) = -\frac{\cancel{8}^{4}\pi G}{3}\frac{C}{\lambda} \Rightarrow \text{Attractive potential}$$

$$E = -\frac{k}{2}.$$

Radial motion of a particle moving in an attractive Coulomb potential.

G.113

Different cases:

① $k = -1 \Rightarrow E > 0.$

$\Rightarrow$ Particle goes to ∞ as $t \to \infty$.

$\lambda \to \infty$ as $t \to \infty$.

② $k = 0. \quad E = 0. \quad \dot{\lambda} > 0$ at present.

Particle goes to ∞ as $t \to \infty$.

$\lambda \to \infty$ as $t \to \infty$.

③ $k = +1 \Rightarrow E = -1$

$\Rightarrow$ Particle must turn back and fall into $\lambda = 0$ point at finite time.

$\Rightarrow$ Big crunch.

Approach to ∞:

① $k = -1$:

For large time $\lambda \to \infty$.

$\Rightarrow \dot{\lambda}^2 \simeq -k = 1$

$\Rightarrow \lambda \simeq t \Rightarrow$ Linear increase with t.

G.114

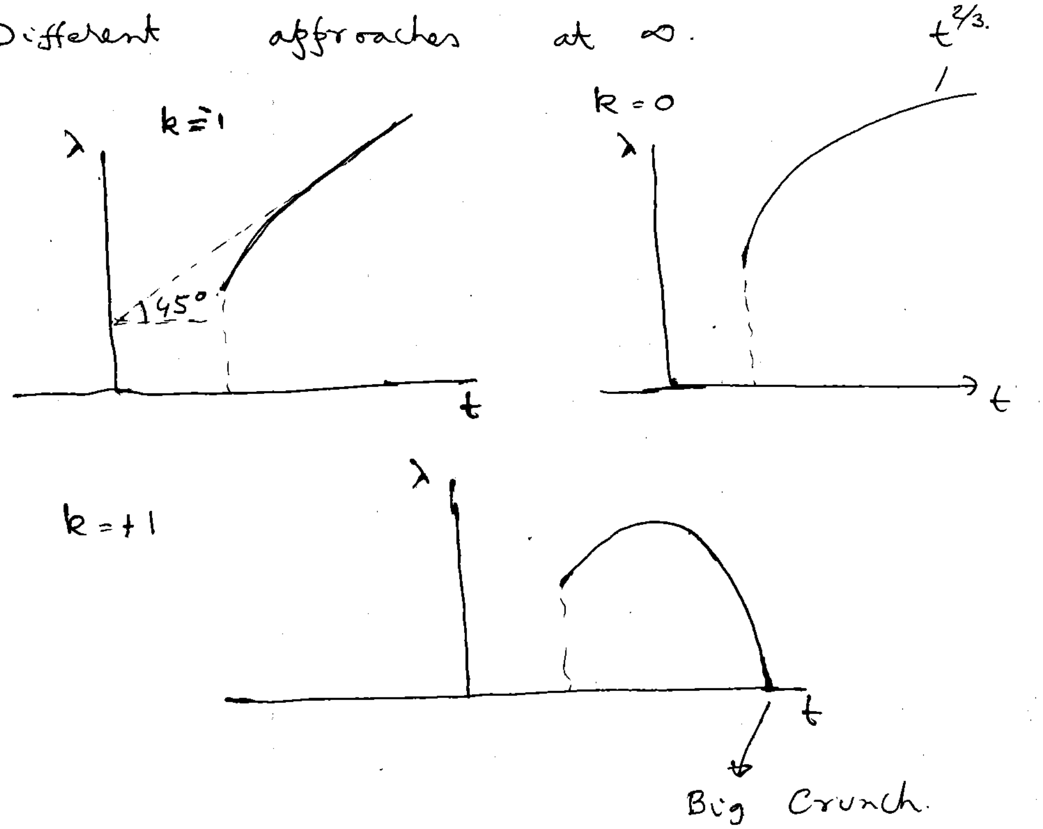
(2) $k = 0$:

$$\frac{d\lambda}{dt} = \left(\frac{8\pi G}{3} \, \frac{C}{\lambda}\right)^{1/2}$$

$$\Rightarrow \frac{2}{3}\lambda^{3/2} \simeq \left(\frac{8\pi G C}{3}\right)^{1/2} t + \text{Constant}$$

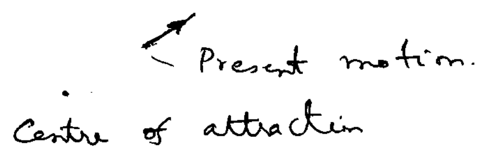
$$\lambda \simeq \left(\frac{3}{2}\right)^{2/3}\left(\frac{8\pi G C}{3}\right)^{1/3} t^{2/3}$$

Different approaches at ∞.



In all three cases when we extrapolate backwards in time:

$\lambda \to 0$ at finite time.



In the past it must have started from $\lambda = 0$.

$\Rightarrow$ Big bang.

Of course when λ becomes small, temperature goes up.

$\Rightarrow$ Eqn. of state changes.

But ~~in~~ the singularity in the past still persists.

With some general assumptions about the energy momentum tensor one can show that there is a singularity in the past.

Determination of the constants:

~~[illegible]~~ Two first order eqs. involving ρ & λ.

$\Rightarrow$ We need to know the present values of ρ and λ.

$\Rightarrow$ ρ_0 and λ_0.

$\rightarrow$ should also tell us about k.

$$k = \frac{8\pi G}{3} \frac{c}{\lambda} - \dot{\lambda}^2$$

$$= \frac{8\pi G}{3} \frac{\rho_0 \lambda_0^3}{\lambda_0} - \dot{\lambda}_0^2$$

Recall the definitions of the Hubble constant and the deceleration parameter:

$$\lambda(t) = \lambda(t_0)[1 + H_0 (t-t_0) - \tfrac{1}{2} q_0 H_0^2 (t-t_0)^2]$$

λ''

$$k = \frac{8\pi G}{3} \rho_0 \lambda_0^2 - H_0^2 \lambda_0^2$$

$$= \left\{ \frac{8\pi G}{3} \rho_0 - H_0^2 \right\} \lambda_0^2$$

$$\rho_c = \frac{3H_0^2}{8\pi G}$$

$\rightarrow$ critical density.

ρ_0 & H_0 are measurable.

$$\frac{k}{\lambda_0^2} = H_0^2 (\Omega_{tot} - 1)$$

$$\Omega_m = \frac{\rho_{m0}}{\rho_c}$$

$$\Omega_\Lambda = \frac{\rho_{\Lambda 0}}{\rho_c}$$

$$\Omega_\lambda = \frac{\rho_{\lambda 0}}{\rho_c}$$

G.117

Given ρ_0 & H_0, we ~~...~~ get:

① $\frac{8\pi G}{3}\rho_0 - H_0^2 > 0 \Rightarrow k = 1$, $\lambda_0 = \frac{1}{\sqrt{\frac{8\pi G}{3}\rho_0 - H_0^2}}$

② $\frac{8\pi G}{3}\rho_0 - H_0^2 < 0 \Rightarrow k = -1$, $\lambda_0 = \frac{1}{\sqrt{H_0^2 - \frac{8\pi G}{3}\rho_0}}$

③ $\frac{8\pi G}{3}\rho_0 - H_0^2 = 0. \Rightarrow k = 0.$

λ_0 undetermined.

Note if $k=0$,

$$ds^2 = -dt^2 + \lambda^2(t)\,(dx^2 + dy^2 + dz^2)$$

New coordinates:

$x' = ax$, $y' = ay$, $z' = az$

$$\Rightarrow ds^2 = -dt^2 + \frac{1}{a^2}\lambda^2(t)\,(dx'^2 + dy'^2 + dz'^2)$$

$\Rightarrow$ Overall multiplicative constant in λ is irrelevant.

If we can measure ρ_0 and H_0, we have all the information we need to determine our future.

It is difficult to determine ρ_0.

Not only need to know the contribution from stars & galaxies but also dark matter (elementary particles, big planet like objects etc.).

Another method :

Note : $\ddot{\lambda}(t_0) = - q_0 \lambda_0 H_0^2$

Can be found by knowing z vs. t_s relation.

From Einstein's eqn.:

$$2\frac{\ddot{\lambda}}{\lambda} + \left(\frac{\dot{\lambda}}{\lambda}\right)^2 + \frac{k}{\lambda^2} = 0.$$

At $t = t_0$

$$-2 q_0 H_0^2 + H_0^2 + \frac{k}{\lambda_0^2} = 0.$$

$$\Rightarrow \left(-H_0^2 + 2q_0 H_0^2\right) = \frac{k}{\lambda_0^2}$$

If $q_0 > \frac{1}{2}$, $k = 1$, $\lambda_0 = \dfrac{1}{H_0\sqrt{2q_0 - 1}}$

$q_0 < \frac{1}{2}$, $k = -1$, $\lambda_0 = \dfrac{1}{H_0\sqrt{1 - 2q_0}}$

$q_0 = \frac{1}{2}$, $k = 0$, λ_0 arbitrary.

$$\left(\frac{\dot\lambda}{\lambda}\right)^2 + \frac{k}{\lambda^2} = \frac{8\pi G}{3}\rho$$

Cosmological constant

$$\frac{d}{dt}(\rho_s \lambda^3) = -3 p_s \lambda^2 \dot\lambda \qquad s = m, \Lambda, r$$

matter, radiation

Need $\lambda_0(t_0)$, $\rho_s(t_0)$ for solving the eqs.
$\lambda_0(t_0) = \lambda_0$, $\rho_s(t_0) = \rho_{s0}$

t_0 : present time

$\mathcal{H}_0 = 60\text{-}70\,\text{Km/sec/Megaparsec}$
parsec $= 3.086\times10^{18}$ cm
Megaparsec $= 10^6$ parsec

$$\rho_m(t) = \rho_{m0}\frac{\lambda_0^3}{\lambda^3}$$

$$\rho_\Lambda(t) = \rho_{\Lambda 0}, \qquad \rho_r(t) = \rho_{r0}\frac{\lambda_0^4}{\lambda^4}$$

$$\mathcal{H}_0^2 + \frac{k}{\lambda_0^2} = \frac{8\pi G}{3}\rho_0 \qquad \rho_0 = \rho_{m0} + \rho_{\Lambda 0} + \rho_{r0}$$

Define $\frac{3\mathcal{H}_0^2}{8\pi G} = \rho_c \rightarrow$ critical density.
$\simeq 8\times10^{-30}$ gm/cc.

$$\frac{8\pi G}{3}\rho_c + \frac{k}{\lambda_0^2} = \frac{8\pi G}{3}\rho_0$$

$\rho_0 > \rho_c \Rightarrow k = +1, \quad \lambda_0 = \left\{\frac{8\pi G}{3}(\rho_0 - \rho_c)\right\}^{-1/2}$

$\rho_0 < \rho_c \Rightarrow k = -1, \quad \lambda_0 = \left\{\frac{8\pi G}{3}(\rho_c - \rho_0)\right\}^{-1/2}$

$\rho_0 = \rho_c \Rightarrow k = 0, \quad \lambda_0$ undetermined.

$$ds^2 = -dt^2 + \lambda(t)^2(dx^2 + dy^2 + dz^2)$$

$$(x, y, z) \to a(x, y, z) \Rightarrow \lambda(t) \to a\lambda(t)$$

Thus λ_0 can be changed using coordinate trs. (not physical)

Lesson: Knowing $\mathcal{H}_0, \rho_{m0}, \rho_{\Lambda 0}, \rho_{r0}$ we know the relevant initial conditions.

(G.121)

Deceleration parameter:

$$\dot{\lambda}^2 = -k + \frac{8\pi G}{3}\rho\lambda^2$$

$$= -k + \frac{8\pi G}{3}\left(\rho_{m0}\frac{\lambda_0^3}{\lambda} + \rho_{\Lambda 0}\lambda^2 + \rho_{r0}\frac{\lambda_0^4}{\lambda^2}\right)$$

$$2\dot{\lambda}\ddot{\lambda} = \frac{8\pi G}{3}\left(-\rho_{m0}\frac{\lambda_0^3}{\lambda^2} + 2\rho_{\Lambda 0}\lambda - 2\rho_{r0}\frac{\lambda_0^4}{\lambda^3}\right)\dot{\lambda}$$

$$2\ddot{\lambda} = \frac{8\pi G}{3}\left(-\rho_{m0}\frac{\lambda_0^3}{\lambda^2} + 2\rho_{\Lambda 0}\lambda - 2\rho_{r0}\frac{\lambda_0^4}{\lambda^3}\right)$$

$$\frac{\lambda(t)}{\lambda_0} = 1 + (t-t_0)H_0 - \frac{1}{2}q_0 H_0^2 (t-t_0)^2 + \cdots$$

$\ddot{\lambda} = -q_0 H_0^2 \lambda_0$ at $t = t_0$.

$$-2q_0 H_0^2 = \frac{8\pi G}{3}\left(-\rho_{m0} + 2\rho_{\Lambda 0} - 2\rho_{r0}\right)$$

$$2q_0 = (+\Omega_m - 2\Omega_\Lambda + 2\Omega_r)$$

Knowing $q_0 \Rightarrow$ information about this particular combination.

$q_0 < 0 \Rightarrow$ Ω_Λ must be present.

Determination of individual Ω's:

Ω_r can be calculated from known temperature of microwave blackbody radiation ($T = 2.73°$ K).

$$\rho_{r0} = aT^4 \qquad \Omega_r = \frac{\rho_{r0}}{\rho_c} \simeq 5\times 10^{-5}$$

$$\Omega_r \ll 1$$

Define $\Omega_m = \frac{\rho_{m0}}{\rho_c}$, $\Omega_\Lambda = \frac{\rho_{\Lambda 0}}{\rho_c}$, $\Omega_r = \frac{\rho_{r0}}{\rho_c}$

$\Rightarrow$ Need to know $H_0, \Omega_m, \Omega_\Lambda, \Omega_r$.

H_0 : determined from redshift of distant galaxies.

$$\nu' = \nu \frac{\lambda(t_1)}{\lambda_0(t_0)} = \nu \frac{\lambda(t_1)}{\lambda_0}$$

$$\frac{\lambda(t_1)}{\lambda_0} = \frac{\nu'}{\nu}$$

t_1 : Time at which light was emitted from the source

$t_0 - t_1$: distance of the source in light years

$$\frac{\lambda(t_1)}{\lambda_0} = \frac{\lambda_0 + (t_1 - t_0)\dot{\lambda}(t_0) + \cdots}{\lambda_0}$$

$$= 1 - H_0 (t_0 - t_1)$$

$$\Rightarrow \quad H_0 = \frac{1}{t_0 - t_1} \left\{ 1 - \frac{\nu'}{\nu} \right\}$$

Q. How to measure the distance $t_0 - t_1$?

Need 'standard candles' whose absolute luminosity is known.

e.g. type Ia supernova whose theory is well understood.

Comparing the observed luminosity with the absolute luminosity we can find out how far the source is

Ex 122

Ω_m: Estimate the total mass of all the stars in the nearby galaxies and galaxy clusters and find the average

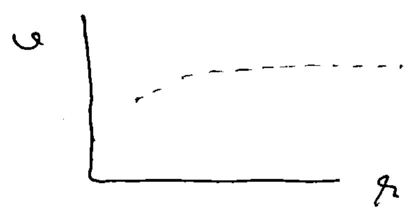
$\Rightarrow \Omega_m \simeq .04$

However this is not consistent with observation on gravitational mass.

Rotation curve of stars in spiral galaxies:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$v = \sqrt{\frac{GM}{r}}$$



$v \to$ constant.

$\Rightarrow$ There is more matter in the outer part of the galaxy than what is observed. $\rightarrow$ dark matter

~~Rotation~~ Various other evidence

$\leadsto \Omega_m \sim .3 \qquad \Omega_\Lambda \sim .7$

Independent tests:

① Age of the universe:

$$\frac{\dot\lambda}{\lambda} = \sqrt{\frac{8\pi G}{3}\rho - \frac{k}{\lambda^2}}$$

$$\int_0^{t_0} dt = \int_0^{\lambda_0} \frac{d\lambda}{\lambda\sqrt{\frac{8\pi G}{3}\left(\rho_{m0}\frac{\lambda_0^3}{\lambda^3} + \rho_{\Lambda 0} + \rho_{r0}\frac{\lambda_0^4}{\lambda^4}\right) - \frac{k}{\lambda^2}}}$$

$$\lambda = \lambda_0 u$$

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{du}{\sqrt{\frac{\Omega_m}{u} + \Omega_\Lambda u^2 + \frac{\Omega_r}{u^2} - \frac{k}{H_0^2\lambda_0^2}}}$$

$t_0 >$ the age of the oldest object in the universe. $\Rightarrow$ bounds on the parameters.

R.1

Cosmological parameters (For ~~from~~ Homogeneous & isotropic)

$H_0, \Omega_m, \Omega_\Lambda, \Omega_R$.

$$H_0 = \frac{\dot{\lambda}(t)}{\lambda(t)}\Big|_{t=t_0} \Rightarrow \rho_c = \frac{3H_0^2}{8\pi G}$$

$$\Omega_m = \frac{\rho_m}{\rho_c}\Big|_{t_0}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}\Big|_{t_0}, \quad \Omega_R = \frac{\rho_R}{\rho_c}\Big|_{t_0}$$

H_0: Calculated from distance - redshift relation.

$$\frac{\lambda(t_1)}{\lambda_0} = 1 - \underbrace{(t_0 - t_1)}_{\text{Measured from apparent luminosity}} H_0 + O(t_0-t_1)^2$$

~~Known~~ measured from redshift

ρ_m: Measured from estimate of ~~m~~ total mass in galaxies from motion of stars and galaxies.

ρ_R: Measured from temperature of microwave background.

Ω_Λ: measured from.

$$2q_0 = (\Omega_m - 2\Omega_\Lambda + 2\Omega_R)$$

deceleration parameter.

$$\frac{\lambda(t_1)}{\lambda_0} = 1 + (t_1 - t_0) H_0 - \frac{1}{2} q_0 H_0^2 (t_1 - t_0)^2 + \cdots$$

q_0 can be measured from more detailed knowledge of distance - redshift (time $(t_1 - t_0)$) relation.

In actual cosmological measurements. measurement of the distance itself is affected by the cosmological parameters and we get complicated equation involving the cosmological parameters.

Another point:

$$ds^2 = -dt^2 + (\lambda(t))^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right)$$

$r \to \rho/\lambda_0$

$$\Rightarrow ds^2 = -dt^2 + \left(\frac{\lambda(t)}{\lambda_0}\right)^2 \left(\frac{d\rho^2}{1 - \frac{k}{\lambda_0^2}\rho^2} + \rho^2 d\theta^2 + \rho^2 \sin^2\theta \, d\phi^2 \right)$$

$$\underset{a(t)}{\parallel} \qquad a(t_0) = 1.$$

Physical quantity : $\frac{k}{\lambda_0^2} \to$ given in terms of $H_0, \Omega_m, \Omega_\Lambda, \Omega_R$

Distance measurement:

$P = (r=0,\ \theta=0,\ \phi=0)$

$\dot{P}$ (source) $\cdot Q$ (observer) $Q: (r=r_0,\ \theta=0,\ \phi=0)$

Geodesic along $\theta = \phi = 0$. (Check)

$$dt^2 = (\lambda(t))^2 \left(\frac{dr^2}{1-kr^2}\right)$$

$$\int_{t_1}^{t_0} \frac{dt}{\lambda(t)} = \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}} \begin{cases} = r_0 & \text{for } k=0 \\ = \sin^{-1} r_0 & \text{for } k=1 \\ = \sinh^{-1} r_0 & \text{for } k=-1 \end{cases}$$

$\Rightarrow$ gives $t_1 = f(r_0, H_0, \Omega_m, \Omega_r, \Omega_\Lambda)$

Suppose N photons of total energy E_0 come out of the source. P

By the time the photons reach Q each photon energy is scaled down by $\frac{\lambda(t_1)}{\lambda(t_0)}$

$\Rightarrow$ Total energy then at $Q = E_0 \frac{\lambda(t_1)}{\lambda(t_0)}$

Area of the sphere at r_0:

$$\lambda_0^2 r_0^2 \int d\theta\, d\phi \sin\theta = 4\pi r_0^2 \lambda_0^2$$

$\Rightarrow$ Energy / area

$$F = \frac{1}{4\pi r_0^2 \lambda_0^2} E_0 \frac{\lambda(t_1)}{\lambda(t_0)}$$

F is observable.

E_0 is calculable for a standard candle.

$\Rightarrow$ $r_0 = g(t_1, H_0, \Omega_m, \Omega_\lambda, \Omega_\Lambda)$

$\Rightarrow$ Solving this two eqs.

we get t_1 as a fn. of the

cosmological parameters.

$$\frac{\lambda(t_1)}{\lambda(t_0)} = 1 - H_0(t_0 - t_1) + \frac{1}{2} q_0 H_0^2 (t_0 - t_1)^2 + \cdots$$

Knowing $\frac{\lambda(t_1)}{\lambda(t_0)}$ we get, for each

observation, one relation involving the

cosmological parameters.

For $\mathcal{H}_0$ we need observation at low red-shift.

$\Rightarrow t_0 - t_1$ small $\Rightarrow r_0$ small.

$t_0 - t_1 \simeq \lambda_0 r_0$

$$F = \frac{1}{4\pi r_0^2 \lambda_0^2} E_0 \Rightarrow r_0 \lambda_0 = \sqrt{\frac{E_0}{4\pi F}} \Rightarrow t_0 - t_1 = \sqrt{\frac{E_0}{4\pi F}}$$

$$\frac{\lambda(t_1)}{\lambda(t_0)} = 1 - \mathcal{H}_0 (t_0 - t_1) \Rightarrow \mathcal{H}_0 \text{ directly}$$

For q_0, which requires observation at larger redshift, we need to keep corrections of order $t_0 - t_1$ in each equation.

Ex. ~~Examine~~ ~~how these~~ Find t_0 in terms of $\frac{E_0}{F}$ and the cosmological parameters to next order.

Other measurements

(1) Age of the universe:

(2) Fluctuations in the microwave background.

Cosmological Horizon (in comoving coordinates)

$$\left(\frac{\dot{\lambda}}{\lambda}\right)^2 + \frac{k}{\lambda^2} = \frac{8\pi G}{3}\rho$$

$$\rho = \rho_m + \rho_r + \rho_\Lambda$$

$$\frac{C_m}{\lambda^4} \quad \frac{C_r}{\lambda^3} \quad C_\Lambda$$

Consider early universe where $\frac{8\pi G}{3}\rho \gg \left|\frac{k}{\lambda^2}\right|$

$$\rho = C\lambda^{-\alpha}$$

$\alpha =$ 3 for matter
4 for radiation
0 for cosmological constant

$$\frac{\dot{\lambda}^2}{\lambda^2} \cong \frac{8\pi G}{3} C \lambda^{-\alpha}$$

$$\lambda^{\frac{\alpha-2}{2}}\dot{\lambda} = \sqrt{\frac{8\pi G C}{3}}$$

$$\frac{2}{\alpha}\lambda^{\frac{\alpha}{2}} = (t-K)\sqrt{\frac{8\pi G C}{3}}$$

$$\lambda = \left\{\frac{\alpha}{2}\sqrt{\frac{8\pi G C}{3}}\right\}^{2/\alpha}(t-K)^{2/\alpha}$$

$$d(t) = \int_0^t \frac{dt'}{\lambda(t')}$$

$$d(t_f) - d(t_i) = \int_{t_i}^{t_f} \frac{dt'\,(t'-K)^{-2/\alpha}}{\left\{\frac{\alpha}{2}\sqrt{\frac{8\pi G C}{3}}\right\}^{-2/\alpha}}$$

$$= \left\{\frac{\alpha}{2}\sqrt{\frac{8\pi G C}{3}}\right\}^{-2/\alpha}\left\{(t_f - K)^{1-\frac{2}{\alpha}} - (t_i - K)^{1-\frac{2}{\alpha}}\right\}\frac{\alpha}{\alpha-2}$$

$$= \left\{\frac{\alpha}{2}\sqrt{\frac{8\pi G C}{3}}\right\}^{-2/\alpha}\left[\left\{\frac{2}{\alpha}\sqrt{\frac{8\pi G C}{3}}\ \lambda_f^{\frac{\alpha}{2}}\right\}^{1-\frac{2}{\alpha}} - \left\{\frac{2}{\alpha}\sqrt{\frac{8\pi G C}{3}}\ \lambda_i^{\alpha/2}\right\}^{1-\frac{2}{\alpha}}\right]\frac{\alpha}{2-\alpha}$$

$$= \frac{2}{\alpha}\left(\sqrt{\frac{8\pi G C}{3}}\right)^{-1}\frac{\alpha}{-2+\alpha}\ \lambda_f^{\frac{\alpha-2}{2}}\left(1 - \left(\frac{\lambda_i}{\lambda_f}\right)^{\frac{\alpha-2}{2}}\right)$$

$$= \frac{2}{\alpha-2}\left(\sqrt{\frac{8\pi G C}{3}}\right)^{-1}\lambda_f^{\frac{\alpha-2}{2}}\left\{1 - \left(\frac{\lambda_i}{\lambda_f}\right)^{\frac{\alpha-2}{2}}\right\}$$

$$d(t_f) - d(t_i) = \frac{2}{\alpha-2}\sqrt{\frac{3}{8\pi G C}}\ \lambda_f^{\frac{\alpha-2}{2}}\left\{1 - \left(\frac{\lambda_i}{\lambda_f}\right)^{\frac{\alpha-2}{2}}\right\}$$

t_{eq}: Matter-radiation equality.

$$\rho_{matter} = \rho_{rad.}$$

$$d_{eq} \equiv d(t_{eq})$$

$\Rightarrow$ $0 \le t \le t_{eq}$: radiation dominated. $\alpha = 4$

$$d_{eq} = \sqrt{\frac{3}{8\pi G C_r}}\ \lambda_{eq}$$

$t > t_{eq}$ matter dominated $\alpha = 3$.

$$d(t) - d_{eq} = 2\sqrt{\frac{3}{8\pi G C_m}}\left(\lambda(t)\right)^{1/2}\left\{1 - \left(\frac{\lambda_{eq}}{\lambda(t)}\right)^{1/2}\right\}$$

$$\frac{C_m}{\lambda_{eq}^3} = \frac{C_r}{\lambda_{eq}^4} \Rightarrow C_r = C_m \cdot \lambda_{eq}.$$

G.125

$$\cancel{d(t_{eq})}\ d_{eq} = \sqrt{\frac{3}{8\pi G C_m}}\ \lambda_{eq}^{1/2}$$

$$d_{rec} - d_{eq} = 2\sqrt{\frac{3}{8\pi G C_m}}\ (\lambda_{rec})^{1/2} \left\{1 - \left(\frac{\lambda_{eq}}{\lambda_{rec}}\right)^{1/2}\right\}$$

$$d_{rec} \simeq 2\sqrt{\frac{3}{8\pi G C_m}} \left\{(\lambda_{rec})^{1/2} - \frac{1}{2}(\lambda_{eq})^{1/2}\right\} \simeq 2\sqrt{\frac{3}{8\pi G C_m}}\ \lambda_{rec}^{1/2}$$

$$d(t_0) - d_{eq} = 2\sqrt{\frac{3}{8\pi G C_m}}\ (\underset{\lambda(t_0)}{\underset{\parallel}{\lambda_0}})^{1/2} \left(1 - \left(\frac{\lambda_{eq}}{\lambda_0}\right)^{1/2}\right)$$

$$\simeq 2\sqrt{\frac{3}{8\pi G C_m}}\ (\lambda_0)^{1/2}$$

~~separation between the diametrically opposite points in the mid~~

$$d(t_0) - d_{rec} \simeq 2\sqrt{\frac{3}{8\pi G C_m}}\ (\lambda_0)^{1/2}$$

$\parallel$

comoving distance of a point from which CMBR is emitted.

$\Rightarrow$ Comoving distance between diametrically opposite points of microwave emission

$$\Delta = 2 \times 2\sqrt{\frac{3}{8\pi G C_m}}\ (\lambda_0)^{1/2}$$

$$d_{rec} \simeq 2\sqrt{\frac{3}{8\pi G C_m}}\ \lambda_{rec}^{1/2}$$

$$\frac{\Delta}{d_{rec}} \sim \left(\frac{\lambda_0}{\lambda_{rec}}\right)^{1/2} >> 1$$

G.126

$$d_{eq} = \sqrt{\frac{3}{8\pi G C_m}}\,\lambda_{eq}^{1/2} + \delta$$

some unknown piece which originated before the radiation dominated era began.

$$d_{rec} \simeq 2\sqrt{\frac{3}{8\pi G C_m}}\,(\lambda_{rec})^{1/2} + \delta$$

$$\frac{\Delta}{d_{rec}} = \frac{2 \times 2\sqrt{\frac{3}{8\pi G C_m}}\,(\lambda_0)^{1/2}}{2\sqrt{\frac{3}{8\pi G C_m}}\,\lambda_{rec}^{1/2} + \delta}$$

~ could be smaller than 1 if

$$\delta > 4\sqrt{\frac{3}{8\pi G C_m}}\,(\lambda_0)^{1/2}.$$

$$\rho_{mo} = \frac{C_m}{\lambda_0^3} \Rightarrow C_m = \rho_{mo}\,\lambda_0^3$$

$$\Rightarrow \delta > 4\sqrt{\frac{3}{8\pi G \rho_{mo}}}\,\frac{1}{\lambda_0}$$

G.127

Origin of the unknown piece (inflation)

~~Bt~~ ~~the~~ Before the radiation dominated era the Universe was dominated by a large cosmological constant.

$$\left(\frac{\dot{\lambda}}{\lambda}\right)^2 = \frac{8\pi G}{3} c_0$$

$$\dot{\lambda} = \sqrt{\frac{8\pi G}{3} c_0}\;\lambda$$

$$\lambda = K_0\, e^{\sqrt{\frac{8\pi G c_0}{3}}\, t}$$

$$\lambda(t_f) = \lambda(t_i)\, e^{\sqrt{\frac{8\pi G c_0}{3}}\,(t_f - t_i)}$$

$$d(t_f) - d(t_i) = \int_{t_i}^{t_f} \frac{dt'}{K_0\, e^{\sqrt{\frac{8\pi G c_0}{3}}\, t'}}$$

$$= -\frac{1}{K_0}\, \frac{e^{-\sqrt{\frac{8\pi G c_0}{3}}\, t'}}{\sqrt{\frac{8\pi G c_0}{3}}}\Bigg|_{t_i}^{t_f}$$

$$= \frac{1}{K_0} \sqrt{\frac{3}{8\pi G c_0}}\; e^{-\sqrt{\frac{8\pi G c_0}{3}}\, t_f} \left(e^{\sqrt{\frac{8\pi G c_0}{3}}\,(t_i - t_f)} - 1 \right)$$

$$\simeq \frac{1}{\lambda(t_f)} \sqrt{\frac{3}{8\pi G c_0}}\; \underbrace{e^{\sqrt{\frac{8\pi G c_0}{3}}\,(t_i - t_f)}}_{\text{Could be a large number.}}$$

$$= e^N$$

At t_f, the energy density gets converted to energy density of radiation.

→ serves as the initial condition for the radiation dominated universe.

$$\lambda_{in} = \lambda(t_f)\Big|_{cc \text{ dominated}}$$

$$\frac{C_r}{\lambda(t_f)^4} = C_0 \,, \quad d$$

⇒ From the point of view of the radiation dominated era

$$\frac{C_r}{\lambda_{in}^4} = C_0$$

$$d_{in} = \frac{1}{\lambda_{in}} \sqrt{\frac{3}{8\pi G C_0}}\; e^N \equiv \delta$$

↘ Large number.

By taking N to be sufficiently large we can make d_{in} big and hence solve the problem.

Gravitational action principle

Look for an action S such that extremizing S we get Einstein's eq. + other possible field eqs.

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = -8\pi G\, T_{\mu\nu}.$$

$$\delta S \propto \int d^4x \sqrt{-\det g}\, \Big[\delta g_{\mu'\nu'}(x)\, g^{\mu\mu'}_{(x)} g^{\nu\nu'}_{(x)}$$

$$\left(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + 8\pi G\, T_{\mu\nu} \right)$$

$$+ \text{other eqs. of motion}$$

$$\times\ \delta(\text{other dynamical variables}) \Big]$$

First concentrate on the other eqs. of motion:

Particle motion

Eq. of motion $\to$ geodesic eq.

$$S_{particle} = c \int_0^1 ds \sqrt{g_{\mu\nu}(X(s)) \frac{dX^\mu}{ds} \frac{dX^\nu}{ds}}$$

$$\delta S = c \int_0^1 ds.\ \delta X^\mu(s) \times (\text{geodesic eq.})$$

How about variation of S with respect to $g_{\mu\nu}(x)$.

Change $g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$

$\delta S_{particle} + S_{particle}$

$$= c\int_0^1 ds \sqrt{g_{\mu\nu}(x)\frac{dX^\mu}{ds}\frac{dX^\nu}{ds} + \delta g_{\mu\nu}(x)\frac{dX^\mu}{ds}\frac{dX^\nu}{ds}}$$

$$= c\int_0^1 ds \sqrt{g_{\mu\nu}\frac{dX^\mu}{ds}\frac{dX^\nu}{ds}}$$

$$\left\{1 + \frac{1}{2}\frac{\delta g_{\rho\sigma}(X(s))\frac{dX^\rho}{ds}\frac{dX^\sigma}{ds}}{g_{\kappa\sigma}(x)\frac{dX^\kappa}{ds}\frac{dX^\sigma}{ds}}\right\}$$

$$\delta S = \frac{c}{2}\int d^4x \, \sqrt{-\det g(x)}\, \delta g_{\rho\sigma}(x) \int_0^1 ds \left(\sqrt{-\det g(x)}\right)^{-1} \prod_{\mu=0}^{3}\delta(x^\mu - X^\mu(s))$$

$$\left(g_{\rho\sigma}(x)\frac{dX^\rho}{ds}\frac{dX^\sigma}{ds}\right)^{-1/2}\frac{dX^\rho}{ds}\frac{dX^\sigma}{ds}$$

$$= \frac{c}{2}\int d^4x \, \delta g_{\rho\sigma}(x) \sqrt{-\det g(x)}$$

$$\int_0^1 ds \left(\sqrt{-\det g(X(s))}\right)^{-1} \delta^{(4)}(x - X(s))$$

$$\left(\frac{d\tau}{ds}\right)^{-1}\frac{dX^\rho}{ds}\frac{dX^\sigma}{ds}$$

$$= \frac{1}{2m}\int d^4x \; \delta g_{\rho\sigma}(x) \sqrt{-\det g(x)} \; T^{\rho\sigma}(x)$$

Take $S_{particle} = m \int_0^1 ds \sqrt{g_{\mu\nu}(x) \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}$

~~Then~~ $\delta S_{particle} = \frac{1}{2}\int d^4x \; \delta g_{\mu\nu}(x) \; T^{\mu\nu}(x)$

Maxwell eq:

$$D_\mu F^{\mu\nu} = 0.$$

In free space it is obtained by varying the action:

$$S_{maxwell} = -\frac{1}{4}\int d^4x \; F_{\mu\nu} F_{\rho\sigma} \eta^{\mu\rho}\eta^{\nu\sigma}$$

$$\delta S_{maxwell} \propto \int d^4x \; \partial_\mu F^{\mu\rho}(x) \; \delta A_\rho(x)$$

~~What~~ What action gives the curved space Maxwell's eq:

Ans. $S_{maxwell} = -\frac{1}{4}\int d^4x \sqrt{-\det g} \; g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho}(x) F_{\nu\sigma}(x)$

Ex. Check that:

$$\delta S_{maxwell} \propto \int D_\mu F^{\mu\nu}(x) \; \delta A_\nu(x) \sqrt{-\det g(x)}$$

under $A_\mu \to A_\mu + \delta A_\mu$.

Q. What is the $\delta S_{maxwell}$ under

$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$.

$g^{\mu\nu}$: Matrix elements of g^{-1}.

$$\delta g^{-1} = - g^{-1}\, \delta g\, g^{-1}$$

$$\delta g^{\mu\nu} = - g^{\mu\rho}\, \delta g_{\rho\sigma}\, g^{\sigma\nu}$$

$$= \delta\left(\sqrt{-\det g}\right)$$

$$= + \frac{1}{2}\left(\sqrt{-\det g}\right)^{-1} \delta\left(-\det g\right)$$

$$-\det\left(g + \delta g\right) = -\det g \left(1 + g^{-1}\delta g\right)$$

$$= -\det g\left(1 + g^{\mu\nu}\delta g_{\mu\nu}\right)$$

$$\Rightarrow -\delta \det(g) = -(\det g)\, g^{\mu\nu}\,\delta g_{\mu\nu}$$

$$\Rightarrow \delta\left(\sqrt{-\det g}\right) = \frac{1}{2}\sqrt{-\det g}\; g^{\mu\nu}\,\delta g_{\mu\nu}$$

Ex. check that

$$\delta S_{maxwell} = \frac{1}{2}\int d^4x \sqrt{\det g}\, \delta g_{\mu\nu}(x)\, T^{\mu\nu}_{maxwell}(x)$$

$$\delta S_{\text{matter}} = \frac{1}{2}\int d^4x \; \delta g_{\mu\nu}(x) \sqrt{-\det g(x)} \; T^{\mu\nu}_{\text{matter}}$$

Grav. eq. of motion:

$$(R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}) + 8\pi G \, T_{\mu\nu} = 0.$$

$$R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} + 8\pi G \, T^{\mu\nu} = 0.$$

Need $S_{\text{grav.}}$ such that

$$\delta S_{\text{grav.}} = + \frac{1}{16\pi G} (R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu})$$

Then $\delta S_{\text{grav}} + \delta S_{\text{matter}}$

$$= \frac{1}{16\pi G}\left\{ R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} + 8\pi G \, T^{\mu\nu}_{\text{matter}} \right\}$$

Show that

$$\delta \int \sqrt{-\det g} \, R = \int d^4x \sqrt{-\det g} \; (R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu}) \, \delta g_{\mu\nu}(x)$$

$$\Rightarrow \; S_{\text{grav.}} = -\frac{1}{16\pi G}\int \sqrt{-\det g} \; R.$$

All the equations of motion can be derived by varying $S_{\text{grav}} + S_{\text{matter}}$ with respect to the dynamical variables.

Gravitational radiation:

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$R = 8\pi G T^\lambda{}_\lambda$

$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = -8\pi G\, T_{\mu\nu} \Rightarrow R_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2} T^\lambda{}_\lambda g_{\mu\nu}\right)$

Linearized equation: $\Box \equiv g^{\mu\nu} D_\mu D_\nu$

$$\underbrace{\left(\Box h_{\mu\nu} - \frac{\partial^2}{\partial x^\lambda \partial x^\mu} h^\lambda{}_\nu - \frac{\partial^2}{\partial x^\lambda \partial x^\nu} h^\lambda{}_\mu + \frac{\partial^2}{\partial x^\mu \partial x^\nu} h^\lambda{}_\lambda\right)}_{R^{(1)}_{\mu\nu}} = -16\pi G\, S_{\mu\nu}$$

$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^\lambda{}_\lambda \Rightarrow T_{\mu\nu} = S_{\mu\nu} - \frac{1}{2} S^\lambda{}_\lambda \eta_{\mu\nu}$

$g_{\mu\nu} \to \eta_{\mu\nu}$ since $T_{\mu\nu}$ is small.

$\partial_\mu T^{\mu\nu} = 0$ to this order.

$\Rightarrow \partial_\mu S^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\partial_\mu S^\lambda{}_\lambda = 0.$

Ex. check that the l.h.s. satisfies this condition.

$$\partial_\mu R^{(1)\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\partial_\mu R^{(1)\lambda}{}_\lambda = 0.$$

Gauge freedom:

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

$$x'^\mu = x^\mu + \epsilon^\mu(x).$$

$$\delta g_{\mu\nu}(x) = g_{\mu\nu}(x') - g_{\mu\nu}(x)$$

$$= \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu.$$

Weak field : $\delta h_{\mu\nu} = \partial_\mu h_\nu + \partial_\nu h_\mu$

$\sim$ analogous to $\delta A_\mu = \partial_\mu \epsilon$ in QED.

We need to fix a gauge.

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0.$$

$\Rightarrow \frac{\partial}{\partial x^\mu} h^\mu{}_\nu = \frac{1}{2}\frac{\partial}{\partial x^\nu} h^\mu{}_\mu$ in the weak field approximation.

Ex. show that in the weak field approximation if $h_{\mu\nu}(x)$ does not satisfy this gauge condition then we can find an $\epsilon_\mu(x)$ s.t. $h_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ satisfies the gauge condition.

In this gauge:

$$\Box h_{\mu\nu} = -16\pi G S_{\mu\nu}$$

First we shall consider source free Einstein's equations:

$$\Box h_{\mu\nu} = 0.$$

complex

Plane wave: $h_{\mu\nu} = e_{\mu\nu}\, e^{i k_\mu x^\mu} + c.c.$, $e_{\mu\nu} = e_{\nu\mu}$

$$\Box h_{\mu\nu} = -e_{\mu\nu}\, \eta^{\rho\sigma} k_\rho k_\sigma\, e^{ik\cdot x}$$

$$\Rightarrow \cancel{k}\; k^2 \equiv \eta^{\rho\sigma} k_\rho k_\sigma = 0.$$

Gauge condition:

$$\partial_\mu h^\mu{}_\nu = \frac{1}{2} \partial_\nu h^\lambda{}_\lambda$$

$$\Rightarrow k^\mu e_{\mu\nu} = \frac{1}{2} k_\nu e^\lambda{}_\lambda$$

Gauge trs:

$$\delta h_{\mu\nu} = \partial_\mu h_\nu + \partial_\nu h_\mu$$

$$h_\mu = i \underbrace{\xi_\mu}_{\text{complex}} e^{ik\cdot x} + c.c. \text{ (say)}$$

$$e_{\mu\nu} \to \cancel{k}\; e_{\mu\nu} - k_\mu \xi_\nu - k_\nu \xi_\mu = e'_{\mu\nu}$$

$$k^\mu e'_{\mu\nu} = k^\mu e_{\mu\nu} - k^2 \xi_\nu - k_\nu k\cdot\xi = k^\mu e_{\mu\nu} - k_\nu k\cdot\xi$$

$\frac{1}{2} k_\nu e'^{\nu}{}_{\lambda} = \frac{1}{2} k_\nu e^{\nu}{}_{\lambda} + \frac{1}{2} k_\nu (-2 k \cdot \xi)$

$\Rightarrow \quad k^\mu e'_{\mu\nu} = k^\mu e_{\mu\nu}$

~~The~~ To begin with $e_{\mu\nu}$ has 10 components.

Gauge condition removes 4 components.

Gauge freedom : removes 4 components.

$\Rightarrow \quad 10 - 4 - 4 = 2$ independent components.

Example: Take $\vec{k} = (0, 0, k^3) \quad k^0 = k^3$

Ex. Show that we can take e_{11} and e_{12} as independent components. $\left\{ \begin{array}{l} e_{01} = -e_{31} \\ e_{02} = -e_{32} \\ e_{22} = -e_{11} \\ e_{03} = -\frac{1}{2}(e_{00} + e_{33}) \end{array} \right.$

Other components are either determined in terms of e_{11} and e_{12} or can be set to zero by gauge transformation.

$\Rightarrow$ Two polarizations of gravity wave just like electromagnetic wave has two polarizations.

Circularity polarized gravity wave:

$e \to e^{ih\theta} e$ under a rotation θ

about the ~~axes~~ direction of propagation

h: helicity.

Ex. check that $e_\pm = e_{11} \mp i e_{12} = -e_{22} \mp i e_{12}$

describe circularly polarized gravity

wave of ~~hel~~ helicity 2.

Energy momentum tensor of gravity

waves:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu}.$$

Involves contribution from matter sector only.

$$D_\mu T^{\mu\nu} = 0.$$

However there is another way to define energy momentum tensor when gravity is weak.

(11)

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = R^{(1)}_{\mu\nu} - \tfrac{1}{2} R^{(1)} \eta_{\mu\nu} + K_{\mu\nu}$$

$\cancel{R_{\mu\nu} = R^{(1)}_{\mu\nu} + R_{\mu\nu}}$ $\cancel{R = R^{(1)} + R}$

rest.

part linear in $h_{\sigma\rho}$

$$R^{(1)}_{\mu\nu} - \tfrac{1}{2} R^{(1)} \eta_{\mu\nu} = -8\pi G \left[T_{\mu\nu} + \frac{1}{8\pi G} K_{\mu\nu}\right]$$

Gravitational energy momentum tensor.

$$\eta^{\mu\rho} \partial_\rho \left(R^{(1)}_{\mu\nu} - \tfrac{1}{2} R^{(1)} \eta_{\mu\nu}\right) = 0.$$

$\Rightarrow$

$$\eta^{\mu\rho} \partial_\rho \left(T_{\mu\nu} + \frac{1}{8\pi G} K_{\mu\nu}\right) = 0$$

$\Downarrow$

Flat space conservation laws.

Ex. check that to quadratic order in $h_{\mu\nu}$,

$$\langle t_{\mu\nu} \rangle = \frac{k_\mu k_\nu}{8\pi G} \left(|e_{11}|^2 + |e_{12}|^2\right)$$

$$= \frac{k_\mu k_\nu}{16\pi G} \left(|e_+|^2 + |e_-|^2\right)$$

Averaging over space or time (over a period) presence of source

⇒ produces gravitational radiation in manner analogous to electromagnetic radiation by solving:

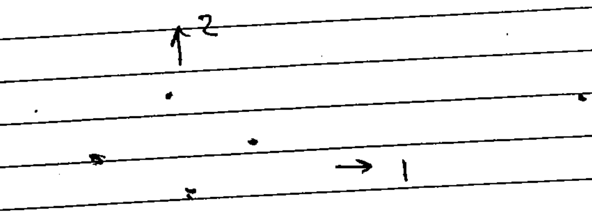
$$\Box h_{\mu\nu} = -16\pi G\, S_{\mu\nu}$$

⇒ the source loses energy.

Detection of gravitational waves:

① Direct detection: not yet but attempts are in progress.

$e_{11} = -e_{22}$



If a gravity wave passes through the distance along 1 will oscillate relative to distance along 2.

⇒ in principle measurable by interferometer

LIGO: ~~Space~~ Earth based experiment

LISA: Space based experiment

(2) Indirect detection: Binary pulsars.

Two heavy objects rotating about each other.

Period of rotation: observable from earth.

As the body loses energy its period of oscillation changes. Period will decrease.

→ observable from earth.

Tetrad (Vierbein formalism).

Coupling of gravity to classical field theories: $\eta_{\mu\nu} \to g_{\mu\nu}$, $d^4x \to d^4x\sqrt{-\det g}$, $\partial_\mu \to D_\mu$

e.g. ϕ^4 theory

$$S = \int d^4x \sqrt{-\det g}\left[-\frac{1}{2} g^{\mu\nu} D_\mu\phi\, D_\nu\phi - \frac{1}{2} m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right]$$

Maxwell theory:

$$S = -\frac{1}{4}\int d^4x \sqrt{-\det g}\; g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$$

Yang-Mills theory

$$S = -\frac{1}{2g^2}\int d^4x \sqrt{-\det g}\; g^{\mu\rho} g^{\nu\sigma}\, \mathrm{Tr}(F_{\mu\nu} F_{\rho\sigma})$$

What about fermions?

$$S_\psi = \int \bar{\Psi}(\gamma^\mu \partial_\mu - m)\,\psi \; d^4x$$

$\partial_\mu \overset{?}{\to} D_\mu$

But ψ is not a conventional tensor.

How do we define $D_\mu \psi$?

treat ψ's as scalars and
If we do not ~~replace~~ D_μ modify the action then we do not have general coordinate invariance.

$$\partial'_\mu \psi(x') = \frac{\partial x^\rho}{\partial x'^\mu} \partial_\rho \psi(x)$$

γ^μ's are fixed matrices and hence do not transform.

$$\Rightarrow \bar\psi(x')\gamma^\mu \partial'_\mu \psi(x') = \bar\psi(x)\gamma^\mu \underbrace{\frac{\partial x^\rho}{\partial x'^\mu}}_{\text{extra term.}} \partial_\rho \psi(x)$$

$\rightarrow$ Requires a new formalism for describing gravity.

Tetrad / Vier bein formalism.

Consider a metric $g_{\mu\nu}(x)$.

y: some particular point on the manifold.

There is a coordinate system $\xi^\mu_y(x;y)$ in which the metric at y is $\eta_{\alpha\beta}$.

$$g_{\mu\nu}(y) = \left[\frac{\partial \xi^a(x;y)}{\partial x^\mu} \frac{\partial \xi^b(x;y)}{\partial x^\nu} \eta_{ab} \right]_{x=y}$$

$$= \frac{\partial \xi^a(x;y)}{\partial x^\mu}\Big|_{x=y} \frac{\partial \xi^b(x;y)}{\partial x^\nu}\Big|_{x=y} \cdot \eta_{ab}$$

$$e^a_\mu(y) = \frac{\partial \xi^a(x;y)}{\partial x^\mu}\Big|_{x=y}$$

$$g_{\mu\nu}(y) = e^a_\mu(y)\, e^b_\nu(y)\, \eta_{ab}$$

This can be done at every point y.

$e^a_\mu(y)$: tetrad / vierbein

→ a collection of four covariant vectors.

Under a change of coordinates

$x \to x'$, $y \to y'$

$$e'^a_\mu(y') = \frac{\partial \xi^a(x';y')}{\partial x'^\mu}\Big|_{x'=y'}$$

$$= \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial \xi^a(x;y)}{\partial x^\rho}\Big|_{x=y} = \frac{\partial y^\rho}{\partial y'^\mu} e^a_\rho(y)$$

Note: Given $e^a_\mu(y)$, the metric $g_{\mu\nu}(y)$ is completely determined.

Thus ~~if $e^a_\mu(y)$ are given~~ instead of $g_{\mu\nu}(x)$, we can use $e^a_\mu(x)$ as the dynamical variables of gravity.

However there is one problem. $g_{\mu\nu}$ is symmetric in μ,ν.

$\Rightarrow$ 10 independent components.

e^a_μ has no symmetry $\Rightarrow$ 16 independent components.

$\Rightarrow$ Given $g_{\mu\nu}$, e^a_μ is not completely fixed.

Origin of the ambiguity:

Given one set of $\xi^a(x;y)$, we can make a Lorentz trs. to get another $\xi'^a(x;y)$

$\xi^a(x;y) \to \Lambda^a{}_b\, \xi^b(x;y)$

$$\Lambda^a{}_c \Lambda^b{}_d \eta_{ab} = \eta_{cd}$$

$\Lambda^a{}_b$ at different y's could be different.

$$e^a{}_\mu(y) = \frac{\partial \xi^a(x;y)}{\partial x^\mu}$$

$$\to \frac{\partial}{\partial x^\mu}\left(\Lambda^a{}_b(y)\, \xi^b(x;y)\right)\Big|_{x=y}$$

$$= \Lambda^a{}_b(y)\, e^b{}_\mu(y).$$

$$g_{\mu\nu}(y) = e^a{}_\mu(y)\, e^b{}_\nu(y)\, \eta_{ab}$$

$$\to \Lambda^a{}_c(y)\, e^c{}_\mu(y)\, \Lambda^b{}_d(y)\, e^d{}_\nu(y)\, \eta_{ab}$$

$$= \eta_{cd}\, e^c{}_\mu(y)\, e^d{}_\nu(y) = g_{\mu\nu}(y)$$

Thus ~~a theory~~ the usual theories of gravity written in terms of $e^a{}_\mu$ should have a gauge invariance under $e^a{}_\mu \to \Lambda^a{}_b(y)\, e^b{}_\mu(y)$

6 Lorentz generators → ~~...~~ fixes 6.
of the components of $e_\mu{}^a$.
⇒ 16 − 6 = 10 components
→ same as that of $g_{\mu\nu}$.

Define $E^\mu{}_a$: matrix inverse of e

$$E^\mu{}_a e_\nu{}^a = \delta^\mu{}_\nu , \quad E^\mu{}_a e_\mu{}^b = \delta_a{}^b$$

Any field which has tensorial trs. under general coordinate trs. can be made into scalars by contracting with the vierbeins.

$$\hat{A}_a = E_a{}^\mu A_\mu , \quad A_\mu = e_\mu{}^a \hat{A}_a$$

$$\hat{A}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m} = e_{\mu_1}{}^{a_1} \cdots e_{\mu_n}{}^{a_n}$$

$$E_{b_1}{}^{\nu_1} \cdots E_{b_m}{}^{\nu_m} A^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m}$$

What happens to covariant derivatives?

$$D_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho$$

$$= \partial_\mu (e_\nu{}^a \hat{A}_a) - \Gamma^\rho_{\mu\nu} e_\rho{}^a \hat{A}_a$$

$$= (\partial_\mu e_\nu{}^a)\, \hat{A}_a + e_\nu{}^a \partial_\mu \hat{A}_a - \Gamma^\rho_{\mu\nu} e_\rho{}^a \hat{A}_a$$

$$E_b{}^\nu D_\mu A_\nu = (E_b{}^\nu \partial_\mu e_\nu{}^a)\, \hat{A}_a$$

$$+ \partial_\mu \hat{A}_b - E_b{}^\nu \Gamma^\rho_{\mu\nu} e_\rho{}^a \hat{A}_a$$

$$= \partial_\mu \hat{A}_b + \hat{\omega}_{\mu b}{}^a \hat{A}_a = \mathcal{D}_\mu \hat{A}_b$$

$$\omega_{\mu b}{}^a = E_b{}^\nu \partial_\mu e_\nu{}^a - E_b{}^\nu \Gamma^\rho_{\mu\nu} e_\rho{}^a$$

$$D_\mu B^\nu = \partial_\mu B^\nu + \Gamma^\nu_{\mu\rho} B^\rho$$

$$= \partial_\mu (E_a{}^\nu B^a) + \Gamma^\nu_{\mu\rho} E_a{}^\rho B^a$$

$$= E_a{}^\nu \partial_\mu B^a + (\partial_\mu E_a{}^\nu) B^a$$

$$+ \Gamma^\nu_{\mu\rho} E_a{}^\rho B^a$$

$$e_\nu{}^b D_\mu B^\nu = \partial_\mu B^b + (e_\nu{}^b \partial_\mu E_a{}^\nu) B^a$$

$$+ \Gamma^\nu_{\mu\rho} e_\nu{}^b E_a{}^\rho B^a$$

$$\nabla_\nu B^b = \partial_\nu B^b + \omega_\nu{}^b{}_a B^a$$

$$\omega_\mu{}^b{}_a = e^b{}_\nu \partial_\mu E^\nu{}_a + \Gamma^\nu_{\mu\rho} e_\nu{}^b E^\rho{}_a$$

$$\nabla_\mu (A_b B^b) = \partial_\mu (A_b B^b)$$

$$\Rightarrow \text{Ex}: \quad \hat{\omega}_\mu{}^{ba} = \omega_\mu{}^{ba}$$

$$\omega_\mu{}^{ab} = - \omega_\mu{}^{ba}$$

$$\nabla_\mu (\hat{A}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m})$$

$$= \partial_\mu \hat{A}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m}$$
$$+ \omega_\mu{}^{a_1}{}_{c_1} \hat{A}^{c_1 a_2 \cdots a_n}{}_{b_1 \cdots b_m}$$
$$+ \omega_\mu{}^{a_2}{}_{c_2} \hat{A}^{a_1 c_2 a_3 \cdots a_n}{}_{b_1 \cdots b_m}$$
$$+ \cdots + \omega_\mu{}^{a_n}{}_{c_n} \hat{A}^{a_1 \cdots a_{n-1} c_n}{}_{b_1 \cdots b_m}$$
$$+ \omega_{\mu b_1}{}^{d_1} \hat{A}^{a_1 \cdots a_n}{}_{d_1 b_2 \cdots b_m} + \cdots$$
$$+ \omega_{\mu b_m}{}^{d_m} \hat{A}^{a_1 \cdots a_n}{}_{b_1 \cdots b_{m-1} d_m}$$

$$\mathcal{D}_\mu \left(\hat{A}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m} \hat{B}^{c_1 \cdots c_k}{}_{d_1 \cdots d_l} \right)$$

$$= \mathcal{D}_\mu \left(\hat{A}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m} \right) \hat{B}^{c_1 \cdots c_k}{}_{d_1 \cdots d_l}$$
$$+ \hat{A}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m} \hat{B}^{c_1 \cdots c_k}{}_{d_1 \cdots d_l}.$$

Role of $\mathcal{D}_\mu$:

$$e^{a_1}_{\mu_1} \cdots e^{a_n}_{\mu_n} E^{\nu_1}_{b_1} \cdots E^{\nu_m}_{b_m} D_\mu \hat{A}^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m}$$

$$= \mathcal{D}_\mu \hat{A}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m}.$$

$\hat{A}_a$ is a scalar under general coordinate trs.

What about local Lorentz trs.?

$$\hat{A}^a = e^a_\mu A^\mu \to \Lambda^a{}_b(x)\, e^b_\mu(x) A^\mu(x)$$

$$= \Lambda^a{}_b(x)\, A^b(x)$$

$\Rightarrow A^a(x)$ transforms like a Lorentz contravariant vector.

Note: $\Lambda^a{}_b(x)$ ... x dependent.

$$\omega_\mu{}^a{}_b = e_\nu{}^a \partial_\mu E^\nu{}_b + \Gamma^\mu_{\nu\rho}\, e_\nu{}^a E^\rho{}_b$$

$$e_\mu{}^a \to \Lambda^a{}_b\, e_\mu{}^b$$

$$\Lambda^a{}_b \Lambda^c{}_d\, \eta^{bd} = \eta^{ac}$$

$$\Lambda \eta \Lambda^T = \eta \Rightarrow \Lambda^{-1} = \eta \Lambda^T \eta$$

$$e \to \Lambda e \qquad E = e^{-1} \to e^{-1}\Lambda^{-1} = E \eta \Lambda^T \eta$$

$$\cancel{E'^a_\mu = E_\mu^b\, \eta_{bc}\, \Lambda}$$

$$E'^\mu{}_a = E^\mu{}_{b'} \underbrace{\eta^{bc}\Lambda^d{}_c\, \eta_{da}}_{(\Lambda^{-1})^{b'}{}_a}$$

$$\omega'_\mu{}^a{}_b = \Lambda^a{}_c\, e_\nu{}^c\, \partial_\mu \left(E^\nu{}_{b'}\, \eta^{b'c'} \Lambda^{d'}{}_{c'}\, \eta_{d'b}\right)$$
$$+ \Gamma^\mu_{\nu\rho}\, \Lambda^a{}_b\, e_\nu{}^b\, E^\rho{}_{b'}\, \eta^{b'c'} \Lambda^{d'}{}_{c'}\, \eta_{d'b}$$

$$= \left(e_\nu{}^c\, \partial_\mu E^\nu{}_{b'}\right) \Lambda^a{}_c \underbrace{\eta^{b'c'} \Lambda^{d'}{}_{c'}}_{(\Lambda^{-1})^{b'}{}_b}$$

$$+ \Lambda^a{}_c\, \partial_\mu (\Lambda^{-1})^{b'}{}_b\, e_\nu{}^c\, E^\nu{}_{b'}$$

$$+ \Gamma^\mu_{\nu\rho}\, \Lambda^a{}_b\, e_\nu{}^b\, E^\rho{}_{b'} (\Lambda^{-1})^{b'}{}_b$$

$$= \left(\Lambda\, \partial_\mu \Lambda^{-1}\right)^a{}_b + \left(\Lambda\, \omega_\mu\, \Lambda^{-1}\right)^a{}_b$$

$\Rightarrow$ Under local Lorentz trs. $(\omega_\mu)^a{}_b$

transforms as a gauge field.

Ex. check that $\omega_\mu{}^a{}_b$ is a

covariant vector field under g.c.t.

Thus:

$\mathcal{D}_\mu \hat{A}^a \to \Lambda^a{}_b \mathcal{D}_\mu \hat{A}^b$ under L.L.T.

etc.

Not surprising since

$\mathcal{D}_\mu \hat{A}^a = e^a_\nu (D_\mu A^\nu)$

Thus using $\omega_\mu{}^{ab}$ we can

now construct a general coordinate

invariant and local Lorentz

invariant action.

Use only $\hat{A}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m}$ fields.

Use $\mathcal{D}_\mu$ always.

Contract μ, ν with $g^{\mu\nu}$, a, b with η.

Dirac action:

$\bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$

$\Rightarrow \bar{\Psi}(-i\gamma^a E^\mu_a \partial_\mu - m)\Psi$

γ^a : Usual γ-matrices.

→ This is manifestly general coordinate invariant.

However this is not manifestly local Lorentz invariant.

$E^\mu_a \to E^\mu_b (\Lambda^{-1})^b{}_a$

$\gamma^a \to \gamma^a$

Strategy: Postulate a trs. of Ψ.

$\Psi \to R(\Lambda)\Psi$

↳ 4×4 matrix.

$\Psi_\alpha \to R(\Lambda)_{\alpha\beta}\Psi_\beta$

↳ Spinor Representation of the Lorentz trs. matrix Λ

$\bar{\Psi}(-i\gamma^a E_a^\mu - m)\Psi$

$= \Psi^\dagger R(\Lambda)^\dagger \gamma^0 (-i\gamma^a E_a^\mu \partial_\mu - m) R(\Lambda)\Psi$

Recall that ordinary Dirac Lagrangian is Lorentz invariant.

$$\boxed{R(\Lambda)^\dagger \gamma^0 R(\Lambda) = \gamma^0}$$

$\Rightarrow R(\Lambda)^\dagger \gamma^0 = \gamma^0 (R(\Lambda))^{-1}$

Furthermore, under an ordinary Rigid Lorentz tr

~~$R(\Lambda)^\dagger \gamma^0 \gamma^\mu \partial_\mu$~~

$\Psi'(x') = R(\Lambda)\Psi(x)$

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$\Psi'(x) = R(\Lambda)\Psi(\Lambda^{-1}x)$

$\int d^4x \bar{\Psi} i\gamma^\mu \partial_\mu \Psi(x)$

$\to \int d^4x \bar{\Psi}'(x) i\gamma^\mu \partial_\mu \Psi'(x)$

$= i\int d^4x \Psi^\dagger(\Lambda^{-1}x) R(\Lambda)^\dagger \gamma^0 \gamma^\mu \partial_\mu R(\Lambda)\Psi(\Lambda^{-1}x)$

$y = \Lambda^{-1}x \quad y^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu$

$$\frac{\partial}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu} = (\Lambda^{-1})^\nu{}_\mu \frac{\partial}{\partial y^\nu} \qquad d^4x = d^4y$$

$$\Rightarrow S \to i\int d^4y\, \psi^\dagger(y)\, R(\Lambda)^\dagger \gamma^0 \gamma^\mu (\Lambda^{-1})^\nu{}_\mu \frac{\partial}{\partial y^\nu}\, R(\Lambda)\, \psi(y)$$

$$\Rightarrow \boxed{R(\Lambda)^\dagger \gamma^0 \gamma^\mu (\Lambda^{-1})^\nu{}_\mu R(\Lambda) = \gamma^0 \gamma^\nu.}$$

$$\Rightarrow \bar\psi(-i\gamma^a E_\mu^a - m)\psi \quad (\Lambda^{-1})^a{}_b$$

$$\to \psi^\dagger R(\Lambda)^\dagger \gamma^0 (-i\gamma^b E^\mu_a \partial_\mu - m) R(\Lambda)\psi$$

$$= \psi^\dagger R(\Lambda)^\dagger \gamma^0 (-i\gamma^b E^\mu_a \partial_\mu R(\Lambda))\, \psi\, (\Lambda^{-1})^a{}_b$$

$$+ \psi^\dagger R(\Lambda)^\dagger \gamma^0 (\gamma^b) E^\mu_a (\Lambda^{-1})^a{}_b R(\Lambda)(-i\partial_\mu \psi)$$

$$- m\, \psi^\dagger R(\Lambda)^\dagger \gamma^0 R(\Lambda)\psi$$

$$= \psi^\dagger R(\Lambda)^\dagger \gamma^0 (-i\gamma^b (\Lambda^{-1})^a{}_b E^\mu_a \partial_\mu R(\Lambda)\psi)$$

$$+ \psi^\dagger \gamma^0 (-i\partial_\mu \psi) - m\psi^\dagger \gamma^0 \psi$$

$\Rightarrow$ We are left with an extra term $\propto \partial_\mu R(\Lambda)$.

Q. How can we get rid of it?

Extra term:

$$\psi^\dagger (R(\Lambda))^\dagger \gamma^0 (-i\gamma^b) E^\mu_a \, \partial_\mu R(\Lambda) \psi \, (\Lambda^{-1})^a{}_b$$

$$= \psi^\dagger \gamma^0 \cdot (-i\gamma^a E^\mu_a) R(\Lambda)^{-1} \partial_\mu R(\Lambda) \psi$$

Need to add:

$$\bar{\psi} \gamma^0 \gamma^b E^\mu_b \Omega_\mu \psi$$

$$\rightarrow \bar{\psi} R(\Lambda)^\dagger \gamma^0 \gamma^b \Lambda^a{}_b E^\mu_b \Omega'_\mu R(\Lambda) \psi$$

$$= \bar{\psi} \gamma^0 \gamma^a E^\mu_a R(\Lambda)^{-1} \Omega'_\mu R(\Lambda) \psi$$

$$= \bar{\psi} \gamma^0 \gamma^a E^\mu_a \Omega_\mu \psi$$

Need

$$+i \bar{\psi} \gamma^0 \gamma^a E^\mu_a (R(\Lambda))^{-1} \partial_\mu R(\Lambda) \psi$$

$$\Rightarrow R(\Lambda)^{-1} \Omega'_\mu R(\Lambda)$$

$$= \Omega_\mu + i R(\Lambda)^{-1} \partial_\mu R(\Lambda)$$

$$\boxed{\Omega'_\mu = R(\Lambda) \Omega_\mu R(\Lambda)^{-1} + i \partial_\mu R(\Lambda) (R(\Lambda))^{-1}}$$

However we do not want to introduce a new gauge field.

$\mathcal{D}_\mu B^b = \partial_\mu B^b + \omega_\mu{}^b{}_a B^a$

Compare with

$D_\mu \phi = \partial_\mu \phi + i A_\mu \phi$

$\phi \to e^{i\alpha(x)} \phi \quad A_\mu \to A_\mu - \partial_\mu \alpha$

or more generally for non-abelian case

$\phi \to R\phi, \quad \underset{\sim}{A_\mu} \to R \underset{\sim}{A_\mu} R^{-1} - iR\,\partial_\mu R^{-1}$

$A_\mu = A_\mu^k T^k$

Can we compare these two cases?

$\omega_\mu{}^b{}_a = \sum_{c<d} \omega_\mu^{cd} (\Lambda_{cd})^a{}_b$

$(\Lambda_{cd})^a{}_b = \delta_c{}^a \eta_{db} - \delta_d{}^a \eta_{cb}$.

$\mathcal{D}_\mu B^b = \partial_\mu B^b + i \sum_{c<d} \omega^{cd} \underbrace{(-i(\Lambda_{cd})^a{}_b)}_{(T^k)^a{}_b} B^b$

analog of gauge index k → runs over 6 values.

→ generators of the Lorentz group.

Ex. Calculate the commutators

$$[T^{cd}, T^{ef}] = i \underbrace{f^{cdefgh}}_{\text{structure constants.}} T^{gh}$$

Find $\Sigma^{ab} \propto [\gamma^a, \gamma^b]$ such that

$$[\Sigma^{cd}, \Sigma^{ef}] = i f^{cdefgh} \Sigma^{gh}$$

same constants.

Define

$$\mathcal{D}_\mu \psi = \partial_\mu \psi + \underbrace{i \sum_{c<d} \omega^{cd} \Sigma^{cd}}_{\Omega_\mu} \psi$$

$$\mathcal{D}_\mu \psi \to R(\Lambda) \mathcal{D}_\mu \psi$$

$$\Rightarrow \Omega'_\mu = R(\Lambda) \Omega_\mu R(\Lambda)^{-1}$$
$$+ 2 \partial_\mu R(\Lambda) (R(\Lambda))^{-1}$$

General Relativity Examination: December, 1998

1. The Schwarzschild metric is given by:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{1}$$

Consider a spaceship which starts from rest at $r = r_0$, and moves radially outwards in such a way that an observer sitting inside the spaceship feels a constant acceleration a. Find a differential equation which will determine the radial coordinate r of the spaceship in terms of the proper time τ of the observer inside the spaceship.

2. Let us assume that the density of matter in the universe is critical so that the Robertson-Walker metric is given by:

$$ds^2 = -dt^2 + (\lambda(t))^2(dx^2 + dy^2 + dz^2) \tag{2}$$

The energy density ρ and pressure p of a gas of photon at temperature T is given by

$$\rho = 3p = CT^4\,, \tag{3}$$

where C is a known constant. Let t_0 denote the present instant of time, H_0 denote the Hubble constant, and T_0 denote the temperature of the microwave background radiation at $t = t_0$. Let t_1 denote the particular instant of time (in the past) when the energy density due to matter was exactly equal to that due to radiation.

(a) Find $t_0 - t_1$ in terms of H_0, T_0, C and the Newton's constant G.

- *You can make the simplified assumption that during the period between t_1 and t_0, the expansion of the universe is matter dominated; i.e. in the Einstein's equation the contribution to the energy momentum tensor from the radiation can be ignored.*
- *You should not assume that the big bang occured at $t = 0$.*

(b) Let us make the simplified assumption that for $t \leq t_1$ the expansion of the universe is radiation dominated, *i.e.* in the Einstein's equation we can take the energy momentum tensor to be the one given in eq.(3). Let P and Q be two space-time points at time $t = t_1$. What is the minimum distance between P and Q such that their past light cones never intersect in the past?

You should express your answer in terms of the constant C, the temperature T_1 of the radiation at time t_1, and the Newton's constant G.

- *Hint: For minimum distance between P and Q satisfying this property, the past light-cones of P and Q will intersect exactly at the time of the big bang.*