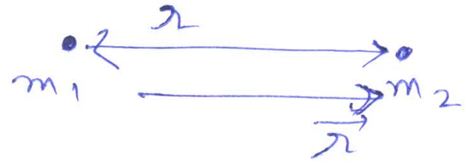
16/7/07

General Relativity

Find a theory of gravity.

Newton's theory :— (→ we already have this theory)



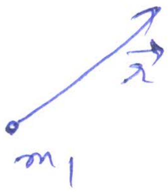
$$\vec{F} = -\frac{G m_1 m_2}{r^2}\hat{r}$$

(Divide the problem in 2 parts.) →

Introduce Gravitational potential $\phi(\vec{r})$

A point mass m_1 produces:

$$\phi = -\frac{Gm}{r}$$



(Then we say that) →

Force of a point mass m_2 at $\vec{r}$

is $\vec{F} = -(\vec{\nabla}\phi)\, m_2$

(What is wrong with Newtonian gravity?)

<u>Theoretical reason</u> :— Not compatible with special theory of relativity.

(no signal faster than c possible) (exp. reason :— mercury perihelion)

(to send a signal in newtonian picture, just move m_1 a little bit — force on m_2 changes instantaneously — grav. field changes instantaneously)

Within the framework of GTR, it will be seen intrinsically that there can't be 2 kinds of charges as in EM — it can't be seen from Newtonian gravity.

The reverse case :-

$\partial x/\partial\theta = a\cos\theta\cos\phi$, $\partial x/\partial\phi = -a\sin\theta\sin\phi$
$\partial y/\partial\theta = a\cos\theta\sin\phi$, $\partial y/\partial\phi = a\sin\theta\cos\phi$

$$\tilde{g}_{\theta\theta} = g_{xx}\frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\theta} + g_{yy}\frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\theta} + 2g_{xy}\frac{\partial x}{\partial\theta}\frac{\partial y}{\partial\theta}$$

$$= \left(1+\frac{x^2}{a^2-x^2-y^2}\right)a^2\cos^2\theta\cos^2\phi$$

$$+\left(1+\frac{y^2}{a^2-x^2-y^2}\right)a^2\cos^2\theta\sin^2\phi$$

$$+2\frac{xy}{a^2-x^2-y^2}a^2\cos^2\theta\sin\phi\cos\phi$$

$$= a^2\cos^2\theta + a^2\cos^2\theta\,\frac{x^2\cos^2\phi+y^2\sin^2\phi+2xy\sin\phi\cos\phi}{a^2-x^2-y^2}$$

$$= a^2\cos^2\theta\left[1+\frac{(x\cos\phi+y\sin\phi)^2}{a^2-x^2-y^2}\right]$$

$$\cancel{= a^2\cos^2\theta \quad a^2 - x^2(1-\cos^2\phi)}$$

$$\tilde{g}_{\theta\theta} = a^2\cos^2\theta\left[1+\frac{(a\sin\theta\cos^2\phi + a\sin\theta\sin^2\phi)^2}{a^2-a^2\sin^2\theta}\right]$$

$$= a^2\cos^2\theta\left[1+\frac{\cancel{a^2}\sin^2\theta}{\cancel{a^2}\cos^2\theta}\right]$$

$$= a^2\cos^2\theta + a^2\sin^2\theta = a^2$$

$$\Rightarrow \tilde{g}_{\theta\theta} = a^2$$

Again, $$\tilde{g}_{\phi\phi} = \left(1+\frac{x^2}{a^2-x^2-y^2}\right)a^2\sin^2\theta\sin^2\phi$$

$$+\left(1+\frac{y^2}{a^2-x^2-y^2}\right)a^2\sin^2\theta\cos^2\phi$$

$$-\frac{2xy}{a^2-x^2-y^2}a^2\sin^2\theta\sin\phi\cos\phi$$

$$= a^2\sin^2\theta + \frac{a^2\sin^2\theta}{a^2-x^2-y^2}\left(x^2\sin^2\phi+y^2\cos^2\phi-2xy\sin\phi\cos\phi\right)$$

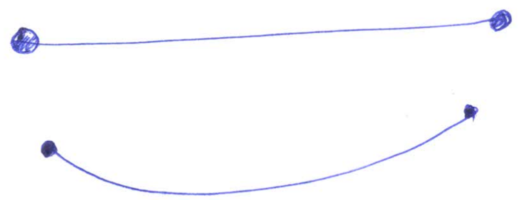
$$= a^2\sin^2\theta\left[1+\frac{(x\sin\phi-y\cos\phi)^2}{a^2\cos^2\theta}\right]$$

$$\cancel{= a^2\sin^2\theta\left[1+\frac{a^2(\sin^2\phi-\cos^2\phi)^2}{a^2\cos^2\theta}\right]}$$

to use retarded potential.

It's necessary that we modify the laws itself – the way we have formulated it – the potential we have written is not ~~entirely correct~~ sufficient

Geometry



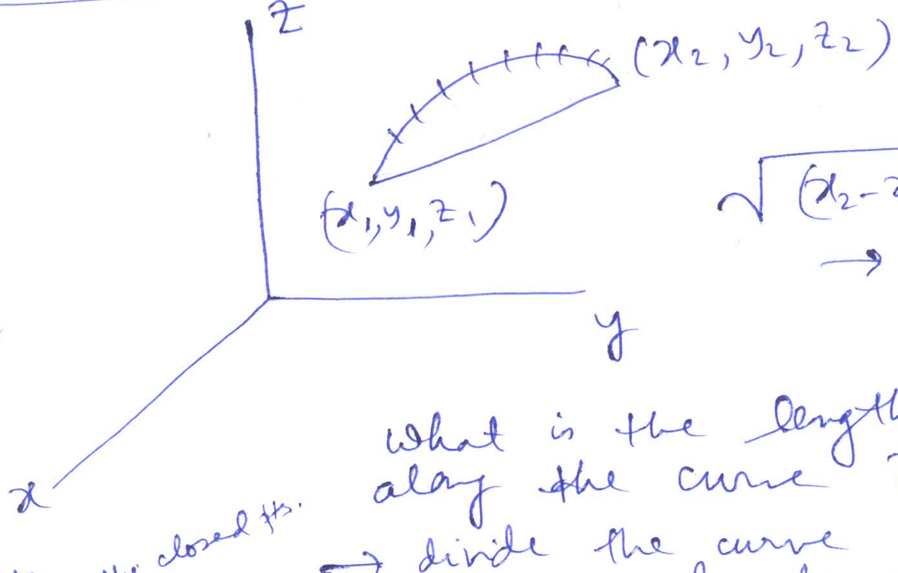
Given 2 pts., distance betn. them is not a well-defined notion – it's a path-dep. concept

So we will talk about the distance along a path – gives info about the spacial geometry

There is a special kind of geom called Riemannian geom. which doesn't require specifying the notion of distance along each distinct curve

→ this is useful

Euclidean geometry



$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

→ along a st. line

there can be many ways of joining them

What is the length measured along the curve?

→ divide the curve into infinitesimal elements .....

st. line betn. 2 infinitesimally closed pts.

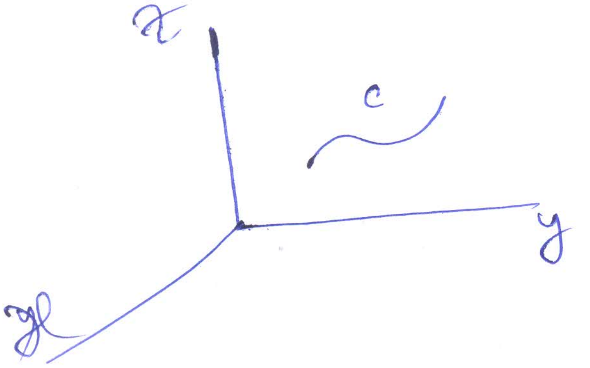
Minimum distance between two points (x, y, z) & $(x+dx, y+dy, z+dz)$

(which is of course the st. line distance).

Length of a curve C :-

$$\int_C \sqrt{(dx)^2+(dy)^2+(dz)^2}$$

$(dx)^2$ under sq. root is somewhat peculiar



$$= \int_0^1 ds \sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2}$$

Suppose C corresponds to (parametrizing)

$$x = f_1(s)$$
$$y = f_2(s) \qquad 0 \le s \le 1$$
$$z = f_3(s)$$

(2 eqns are needed — but here s is arbitrary & 3 eqns can be written for convenience — one is redundant)

(In order to encode the geom. of space, we need to know this infinitesimal st. line distance.)

Geometry can be encoded by specifying the distance between infinitesimally close points.

But what do we mean by 2 infinitesimally close pts. before introducing the notion of distance? here to introduce the coord. system. even before introducing the notion of distance - coord. are real nos. & so close pts. means their coord. are close.

First thing to define any space is introdn. of coord. system — this is the topology

We assume all this is given to us — now introduce the notion of distance.

Suppose we have an n-dimensional space & Suppose we have a coordinate system (coordinate patch — x^i's may not have infinite range)

$$(x^1, x^2, \ldots, x^N)$$

describing points & some region of this space.

$(x^1, x^2, \ldots, x^N)$ & $(x^1+dx^1, x^2+dx^2, \ldots x^N+dx^N)$

(are 2 infinitesimally separated pts. — each comp. differs infinitesimally.)

Distance between these two points encodes information about geometry.

<u>Riemannian geometry</u> :-

$$ds = \sqrt{\sum_{i,j=1}^{N} g_{ij}\, dx^i\, dx^j}$$

g_{ij} → given functions of $(x^1, \ldots, x^N)$

(so no need to specify the length of all possible fns. — only we need to know the fns $g_{\mu\nu}(x)$)

We can choose $g_{\mu\nu}(x)$ to be symmetric

$$\Longrightarrow \frac{N(N+1)}{2} \text{ functions}$$

In 4D, the 10 fns. $g_{\mu\nu}(x)$ will play the role of <u>grav.' potentials</u> — 9 of them won't be useful in the Newtonian grav. limit

---

If $g_{ij}(x) = \delta_{ij}$, this gives the Euclidean geometry.

$g_{ij}(x)$

$\longrightarrow$ different geometries

except when they can be related by a transformation of coordinates.

(but this is not always true — this happens when we can have diff. choices of coord. sys.)

Effect of change of coordinates :-

$(x^1, \ldots, x^N)$

$x'^1 = f^1(x)$

$\vdots \qquad \vdots$

$x'^N = f^N(x)$

Inverse transformation

$x^1 = \phi^1(x')$

$x^2 = \phi^2(x')$

$\vdots$

$x^N = \phi^N(x')$

for good coord. sys. / the corr. should be one-to-one — 2 pts. should differ in each system

---

Suppose we have a given geometry :-

$$ds^2 = \sum_{i,j=1}^{N} g_{ij}(x)\, dx^i\, dx^j$$

$\longrightarrow$ metric in the x-coordinate

$$= \sum_{i,j=1}^{N} g'_{ij}(x')\, (dx')^i\, (dx')^j$$

$\longrightarrow$ metric in the x'-coordinates

$$dx^i = \sum_{k=1}^{N} \frac{\partial x^i}{\partial x'^k}\, dx'^k = \sum_{k=1}^{N} \frac{\partial \phi^i(x')}{\partial x'^k}\, dx'^k$$

$$dx^j = \sum_{l=1}^{N} \frac{\partial x^j}{\partial x'^l}\, dx'^l = \sum_{l=1}^{N} \frac{\partial \phi^j(x')}{\partial x'^l}\, dx'^l$$

$$ds^2 = \sum_{i,j} g_{ij}(x)\, dx^i\, dx^j$$

$$= \sum_{i,j,k,l} g_{ij}(x)\, \frac{\partial x^i}{\partial x'^k}\, \frac{\partial x^j}{\partial x'^l}\, dx'^k\, dx'^l$$

On the other hand,

$$ds^2 = \sum_{k,l} g'_{kl}(x') \, dx'^k \, dx'^l$$

Hence,

$$\boxed{g'_{kl}(x') = \sum_{i,j=1}^{N} g_{ij}(x) \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l}}$$

<u>Example</u> :- 3-dimensional Euclidean geometry :-

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

Introduce spherical polar coordinates:

$$x^1 = r \sin\theta \cos\phi$$
$$x^2 = r \sin\theta \sin\phi$$
$$x^3 = r \cos\theta$$

$$dx^1 = dr \sin\theta \cos\phi + d\theta \, r \cos\theta \cos\phi - d\phi \, r \sin\theta \sin\phi$$

$$dx^2 = dr \sin\theta \sin\phi + d\theta \, r \cos\theta \sin\phi + d\phi \, r \sin\theta \cos\phi$$

$$dx^3 = dr \cos\theta - d\theta \, r \sin\theta$$

$$\therefore ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

Hence, $g'_{rr} = 1$, $g'_{\theta\theta} = r^2$, $g'_{\phi\phi} = r^2 \sin^2\theta$

$g'_{r\theta} =$ —— $= 0$

$$x^2 = x'^2$$
$$x^3 = x'^3$$

$$dx^1 = dx'^1 + 2x'^2\, dx'^2$$
$$dx^2 = dx'^2$$
$$dx^3 = dx'^3$$

$$ds^2 = (dx'^1)^2 + 4(x'^2)\, dx'^1 dx'^2 + (dx'^2)^2 \left(4(x'^2)^2+1\right) + (dx'^3)^2$$

Given 2 metrics, it's not easy to ~~prove~~ see whether one is the coord. trs. of the other — what is the intrinsic prop. of geom. which is insensitive to the coord. choice?

Example of a non-euclidean geometry :-

Surface of a 2-dimensional sphere.

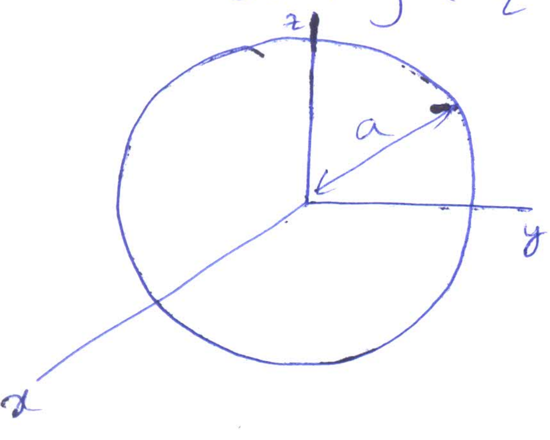
(best way → embedding it in 3D with the help of an eqn.)

$$x^2 + y^2 + z^2 = a^2$$

(2 coord. needed to describe it)

choose

(x, y) Cooordinates

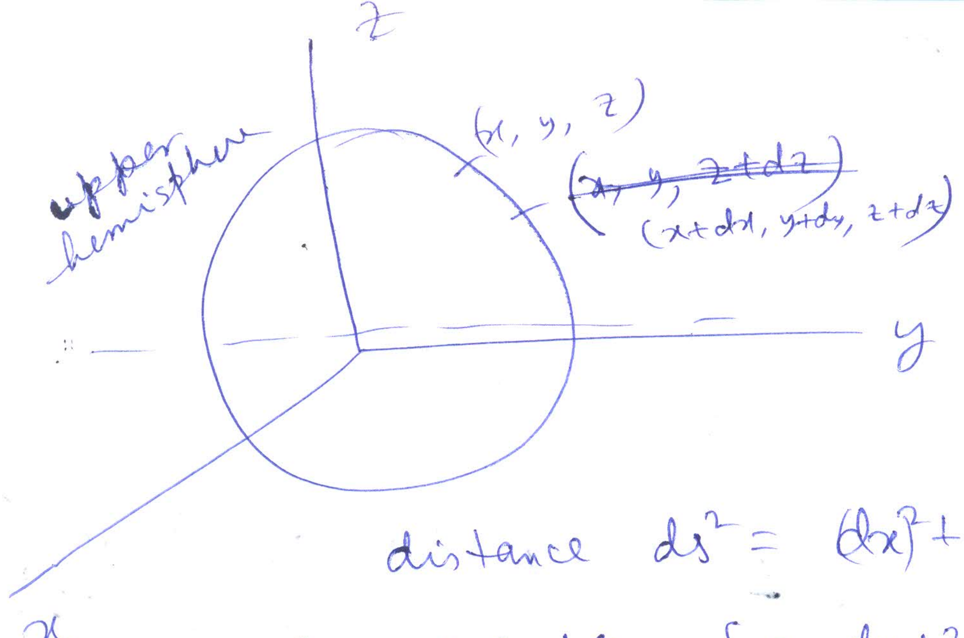


Then $z = \pm\sqrt{a^2 - x^2 - y^2}$

2 values ⇒ so (x, y) isn't a good coord. system

A good coord. system should be single valued.

But locally it is a good coord. sys. — say on the upper hemisphere.



Choose $z = \sqrt{a^2 - x^2 - y^2}$

(only for infinitesimally close pts.)

distance $ds^2 = (dx)^2 + (dy)^2 + (dz)^2$

$$= dx^2 + dy^2 + \left\{\frac{2x\,dx + 2y\,dy}{2\sqrt{a^2-x^2-y^2}}\right\}^2$$

but z is no longer indep.

$$= dx^2\left\{1 + \frac{x^2}{a^2-x^2-y^2}\right\} + dy^2\left\{1 + \frac{y^2}{a^2-x^2-y^2}\right\} + \frac{2xy\,dx\,dy}{a^2-x^2-y^2}$$

A simpler choice of coord. :- Spherical polar coordinates in 3-dim.

$$(ds^2)_{3\text{-dim}} = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

$r = a$ describes the same sphere

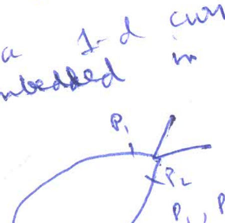
$$(ds^2)_{2\text{-dim}} = a^2 d\theta^2 + a^2 \sin^2\theta\, d\phi^2$$

in the (θ, ϕ) coordinate system.

the sphere is a good embedding

α isn't a good embedding

a 1-d curve embedded in 2-d



P_1, P_2 seem to be close but actually you have to go round — so not close

$$x = a \sin\theta \cos\phi$$
$$y = a \sin\theta \sin\phi$$
$$z = a \cos\theta$$

Ex. Check that under this trs., the metric in (x, y) coordinate becomes the metric in (θ, ϕ) coordinates.

(For $a = 1$ & $a = 2$, the 2 metrics aren't related by a coord. trs. — for euclidean geom. we would have got just a multiplying factor of 2 — it's not so here)

Construct quantities out of the metric & its derivatives which do not change under coordinate transformation.

H/W

$$x = a \sin\theta \cos\phi$$
$$y = a \sin\theta \sin\phi$$

$\therefore \quad \tan\phi = y/x \Rightarrow \phi = \tan^{-1}(y/x)$

Again, $\quad x^2 + y^2 = a^2 \sin^2\theta$

$\Rightarrow \quad \theta = \sin^{-1}\left(\sqrt{\dfrac{x^2+y^2}{a^2}}\right)$ in the upper hemisphere

$$\frac{\partial\phi}{\partial x} = \frac{-y/x^2}{1 + y^2/x^2} = -\frac{y}{x^2+y^2}$$

$$\frac{\partial\phi}{\partial y} = \frac{1/x}{1+y^2/x^2} = \frac{x}{x^2+y^2}$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2+y^2}{a^2}}} \frac{1}{a} \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}\sqrt{a^2-x^2-y^2}}$$

$$\frac{\partial\theta}{\partial y} = \frac{y}{\sqrt{x^2+y^2}\sqrt{a^2-x^2-y^2}}$$

Now, $g_{xx} = \tilde{g}_{\theta\theta} \dfrac{\partial\theta}{\partial x}\dfrac{\partial\theta}{\partial x} + \tilde{g}_{\phi\phi}\dfrac{\partial\phi}{\partial x}\dfrac{\partial\phi}{\partial x}$

$$= a^2 \frac{x^2}{(x^2+y^2)(a^2-x^2-y^2)} + a^2\sin^2\theta \frac{y^2}{(x^2+y^2)^2}$$

$$= \frac{a^2x^2}{(x^2+y^2)(a^2-x^2-y^2)} + \frac{y^2}{x^2+y^2}$$

$$= \frac{a^2x^2 + a^2y^2 - (x^2+y^2)y^2}{(x^2+y^2)(a^2-x^2-y^2)}$$

$$\Rightarrow g_{xx} = \frac{\cancel{(x^2+y^2)}(a^2-y^2)}{\cancel{(x^2+y^2)}(a^2-x^2-y^2)} = 1 + \frac{x^2}{a^2-x^2-y^2}$$

Again, $g_{yy} = \tilde{g}_{\theta\theta} \frac{\partial \theta}{\partial y} \frac{\partial \theta}{\partial y} + \tilde{g}_{\phi\phi} \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial y}$

$$= a^2 \frac{y^2}{(x^2+y^2)(a^2-x^2-y^2)} + \cancel{a^2 \sin^2\theta} \frac{x^2}{\cancel{(x^2+y^2)^2}}$$

$$= 1 + \frac{y^2}{a^2-x^2-y^2}$$

Lastly, $g_{xy} = \tilde{g}_{\theta\theta} \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} + \tilde{g}_{\phi\phi} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y}$

$$= a^2 \frac{x}{\sqrt{x^2+y^2}\sqrt{a^2-x^2-y^2}} \frac{y}{\sqrt{x^2+y^2}\sqrt{a^2-x^2-y^2}}$$

$$+ \cancel{a^2 \sin^2\theta} \left(\frac{-y}{\cancel{x^2+y^2}}\right)\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{a^2 xy}{(x^2+y^2)(a^2-x^2-y^2)} - \frac{xy}{x^2+y^2}$$

$$= \frac{\cancel{a^2 xy} - \cancel{a^2 xy} + (x^2+y^2)xy}{(x^2+y^2)(a^2-x^2-y^2)}$$

$$= + \frac{xy \cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}(a^2-x^2-y^2)}$$

$$\Rightarrow \boxed{g_{xy} = \frac{xy}{a^2-x^2-y^2} = g_{yx}}$$

$$\tilde{g}_{\phi\phi} = a^2\sin^2\theta\left[1 + \cancelto{0}{\frac{a^2\sin^2\theta\,(\sin\phi\cos\phi - \cos\phi\sin\phi)}{a^2\cos^2\theta}}\right]$$

$$\Rightarrow \boxed{\tilde{g}_{\phi\phi} = a^2\sin^2\theta}$$

Lastly, $\tilde{g}_{\theta\phi} = \left(1+\frac{x^2}{a^2-x^2-y^2}\right)(-a^2)\sin\theta\cos\theta\sin\phi\cos\phi$

$$+\left(1+\frac{y^2}{a^2-x^2-y^2}\right)a^2\sin\theta\cos\theta\sin\phi\cos\phi$$

$$+\frac{2xy}{a^2-x^2-y^2}\,a^2\cos\theta\cos\phi\sin\theta\cos\phi$$

$$= \frac{a^2\sin\theta\cos\theta}{a^2-x^2-y^2}\left[(-x^2+y^2)\sin\phi\cos\phi + 2xy\cos^2\phi\right]$$

$$= \frac{\cancel{a^2}\sin\theta\,\cancel{\cos\theta}}{\cancel{a^2}\cos^{\cancel{2}}\theta}\cos\phi\,a^2\left[\sin\phi\left(-\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi\right) + 2\cos\phi\sin^2\theta\sin\phi\cos\phi\right]$$

$$= a^2\frac{\sin\theta}{\cos\theta}\cos\phi\sin^2\theta\sin\phi\left[-\cos^2\phi + \sin^2\phi + 2\cos^2\phi\right.$$

$$\tilde{g}_{\theta\phi} = g_{xx}\frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\phi} + g_{yy}\frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\phi} + 2g_{xy}\frac{\partial x}{\partial\theta}\frac{\partial y}{\partial\phi}$$

$$= \left(1+\frac{x^2}{a^2-x^2-y^2}\right)(-a^2)\sin\theta\cos\theta\sin\phi\cos\phi$$

$$+\left(1+\frac{y^2}{a^2-x^2-y^2}\right)a^2\sin\theta\cos\theta\sin\phi\cos\phi$$

$$+\frac{2xy}{a^2-x^2-y^2}\,a^2\sin\theta\cos\theta\cos^2\phi$$

$$= -\frac{\cancel{a^2}}{2}\sin 2\theta\,\frac{x^2\sin\phi\cos\phi}{\cancel{a^2}\cos^2\theta} + \frac{\cancel{a^2}}{2}\sin 2\theta\,\frac{y^2\sin\phi\cos\phi}{\cancel{a^2}\cos^2\theta}$$

$$+\frac{xy\,\cancel{a^2}}{\cancel{a^2}\cos^2\theta}\sin 2\theta\cos^2\phi$$

$$= \frac{\sin 2\theta}{\cos^2\theta}\left[-\frac{x^2}{2}\sin\phi\cos\phi + \frac{y^2}{2}\sin\phi\cos\phi + xy\cos^2\phi\right]$$

$$\tilde{g}_{\theta\phi} = 2a^2 \tan\theta \left[-\frac{1}{2}\sin^2\theta \cos^2\phi \sin\phi\cos\phi + \frac{1}{2}\sin^2\theta \sin^2\phi \sin\phi\cos\phi + \cos^2\phi \sin^2\theta \sin\phi\cos\phi \right]$$

$$= 2a^2 \tan\theta \sin^2\theta \sin\phi\cos\phi \left[-\frac{1}{2}\cos^2\phi + \frac{1}{2}\sin^2\phi + \cos^2\phi \right]$$

$$\tilde{g}_{\theta\phi} = g_{xx}\frac{\partial x}{\partial\theta}\frac{\partial x}{\partial\phi} + g_{yy}\frac{\partial y}{\partial\theta}\frac{\partial y}{\partial\phi} + g_{xy}\frac{\partial x}{\partial\theta}\frac{\partial y}{\partial\phi} + g_{yx}\frac{\partial y}{\partial\theta}\frac{\partial x}{\partial\phi}$$

$$= \left(1 + \frac{x^2}{a^2\cos^2\theta}\right)(-a^2)\sin\theta \sin\phi\cos\phi\cos\theta + \left(1+\frac{y^2}{a^2\cos^2\theta}\right) a^2 \sin\theta\cos\theta\sin\phi\cos\phi + \frac{xy}{\cancel{a^2\cos^2\theta}}\left[\cancel{a^2}\sin\theta\cos\theta\cos^2\phi - \cancel{a^2}\sin\theta\cos\theta\sin^2\phi\right]$$

$$= \frac{-x^2+y^2}{\cos\theta}\sin\theta\sin\phi\cos\phi + \frac{xy}{\cos\theta}\sin\theta\left(\cos^2\phi - \sin^2\phi\right)$$

$$= a^2\tan\theta\left[\left(-\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi\right)\sin\phi\cos\phi + \sin^2\theta\sin\phi\cos\phi\left(\cos^2\phi - \sin^2\phi\right)\right]$$

$$= a^2\sin^2\theta\sin\phi\cos\phi\tan\theta\left[\left(-\cos^2\phi+\sin^2\phi\right)+\left(\cos^2\phi-\sin^2\phi\right)\right]$$

$$= 0$$

---

in this coordinate system.

Since T is a tensor (its trs. law is homogeneous)

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(x_{(0)}) = 0 \text{ in all}$$

coordinate systems.

~~But it~~ (Though it is true for a particular choice $x_{(0)}$, $x_{(0)}$ is completely arbitrary $\Rightarrow T$ vanishes everywhere)

Since $x_{(0)}$ can be chosen to be whatever point we like,

$$\therefore \quad T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}(x) = 0 \text{ everywhere}$$

(i.e., T is trivial).

(Now the proof boils down to showing that such a coord. system can be chosen for every $x_{(0)}$)

Suppose to begin with

$$\partial_k g_{ij}(x_{(0)}) \neq 0$$

$\Downarrow$ using this we calculate

$$\Gamma^i_{jk}(x_{(0)}) \longrightarrow \text{not zero in general}$$

Choose a new coordinate system $\{x'\}$ such that

$$x^i = x^i_{(0)} + x'^i - x'^i_{(0)} + \frac{1}{2} A^i_{jk} (x'^j - x'^j_{(0)})(x'^k - x'^k_{(0)})$$

[Note that for $x^i = x^i_{(0)}$, $x'^i = x'^i_{(0)}$]

Here, choice of coord. system means adjusting the consts. $x^i_{(0)}$ & $x'^i_{(0)}$ & the coefficients A^i_{jk}.

Choosing $x_{(0)}$ & $x'_{(0)}$ isn't particularly useful

Ex: Choose $A^i_{jk} = -\Gamma^i_{jk}(x_{(0)})$ then

$$\partial'_i g'_{jk} = 0 \quad \text{at } x' = x'_{(0)}$$

(which is the same as $x = x_{(0)}$)

$\underline{\text{l.h.s.}}$:- $\dfrac{\partial}{\partial x'^{i_1}}\left\{\dfrac{\partial x^{j_2}}{\partial x'^{i_2}}\,\dfrac{\partial x^{j_3}}{\partial x'^{i_3}}\,g_{j_2 j_3}(x)\right\}$

$$= \frac{\partial x^{j_2}}{\partial x'^{i_2}}\,\frac{\partial x^{j_3}}{\partial x'^{i_3}}\,\underbrace{\frac{\partial}{\partial x'^{i_1}}\left(g_{j_2 j_3}(x)\right)}_{\frac{\partial x^{j_1}}{\partial x'^{i_1}}\frac{\partial}{\partial x^{j_1}}\left(g_{j_2 j_3}(x)\right)}$$

$$+ \frac{\partial^2 x^{j_2}}{\partial x'^{i_1}\partial x'^{i_2}}\,\frac{\partial x^{j_3}}{\partial x'^{i_3}}\,g_{j_2 j_3}(x)$$

$$+ \frac{\partial x^{j_2}}{\partial x'^{i_2}}\,\frac{\partial^2 x^{j_3}}{\partial x'^{i_1}\partial x'^{i_3}}\,g_{j_2 j_3}(x)$$

The first term is exactly what we want — however, there are 2 extra terms, not zero in general.

$\therefore$ the given quantity isn't a rank 3 covariant tensor.

By convention, coordinates always carry upper indices

// Consider a q-index object $B^{i_1 \ldots i_q}$ such that:

$$B'^{i_1 \ldots i_q}(x') = \frac{\partial x'^{i_1}}{\partial x^{j_1}} \cdots\cdots \frac{\partial x'^{i_q}}{\partial x^{j_q}}\,B^{j_1 \ldots j_q}(x)$$

In this case $B^{i_1 \ldots i_q}(x)$ is called a contravariant tensor of rank q.

$\underline{\text{Example}}$:-

Define $g^{ij}(x)$: ij component of matrix inverse of g

$\therefore$ $g^{ij}(x)\,g_{jk}(x) = \delta^i_k$

$g'_{ij}(x')$ is known in terms of $g_{ij}(x)$.

(This relation can be written as a matrix prdt.)

Ex. Show that $g^{ij}(x)$ is a rank 2 contravariant tensor

Mixed tensors :-

Rank (p, q) mixed tensor

$$C^{i_1 \dots i_p}{}_{j_1 \dots j_q}(x)$$

We have $C'^{i_1 \dots i_p}{}_{j_1 \dots j_q}(x')$

$$= \frac{\partial x'^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial x'^{i_p}}{\partial x^{k_p}} \frac{\partial x^{l_1}}{\partial x'^{j_1}} \cdots \frac{\partial x^{l_q}}{\partial x'^{j_q}} C^{k_1 \dots k_p}{}_{l_1 \dots l_q}(x)$$

Let us take $A^{i_1 \dots i_{p_1}}{}_{j_1 \dots j_{q_1}}$, $B^{i_1 \dots i_{p_2}}{}_{j_1 \dots j_{q_2}}$

Define :-

$$C^{i_1 \dots i_{p_1+p_2}}{}_{j_1 \dots j_{q_1+q_2}} = A^{i_1 \dots i_{p_1}}{}_{j_1 \dots j_{q_1}}(x) \, B^{i_{p_1+1} \dots i_{p_1+p_2}}{}_{j_{q_1+1} \dots j_{q_1+q_2}}(x)$$

C is a rank (p_1+p_2, q_1+q_2) tensor.

Take a rank (p, q) tensor

$$A^{i_1 \dots i_p}{}_{j_1 \dots j_q}$$

Define :-

$$D^{i_2 \dots i_p}{}_{j_2 \dots j_p}(x) = A^{i_1 i_2 \dots i_p}{}_{i_1 j_2 \dots j_q}(x)$$

(i_1 is a dummy index & is summed over)

It is a tensor of rank $(p-1, q-1)$

Ex: Noting $\underbrace{\frac{\partial x'^{i_1}}{\partial x^{k_1}} \frac{\partial x^{l_1}}{\partial x'^{i_1}}}_{\frac{\partial x^{l_1}}{\partial x^{k_1}} = \delta^{l_1}{}_{k_1}} \cdots A^{k_1 \cdots}{}_{l_1 \cdots}$

prove that D is a tensor

$g_{i_1 i_2} g_{i_3 i_4} \longrightarrow$ rank $(0, 4)$

$g^{i_1 i_2} g^{i_3 i_4} \longrightarrow$ rank $(4, 0)$

$g^{i_1 i_2} g_{j_1 j_2} \longrightarrow$ rank $(2, 2)$

Scalars are rank $(0,0)$ tensors & do not transform & can be used to charactise a manifold
$\longrightarrow$ they are very useful & we construct scalars

---

$g^{i_1 i_2} g_{i_1 i_2}$ is a scalar
but isn't very useful bcos
$g^{i_1 i_2} g_{i_1 i_2} = g^{i_1 i_2} g_{i_2 i_1} = \delta^{i_1}{}_{i_1} = N$
$\rightarrow$ (gives the dim. of space, but doesn't give the geometry
Without using derivatives, all scalars constructed in this manner give info about dim. only)
Not useful to distinguish between geometries.

Need to use derivatives of g_{ij} to find scalars with useful information.
But we saw that
$\partial_{i_1} g_{i_2 i_3}$ is not a tensor.

General result :-

It is impossible to construct a non-trivial tensor out of g_{ij} & its derivatives.

Define :- $\longrightarrow$ Christoffel symbol, Connection coefficients

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk} \right)$$

$\rightarrow$ (not a tensor bcos we can't construct a non-trivial tensor using first deriv. of g_{ij})

Ex. Show that

$$\Gamma'^l_{mn}(x') = \frac{\partial x'^l}{\partial x^i} \frac{\partial x^j}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \Gamma^i_{jk}(x) + \frac{\partial x'^l}{\partial x^k} \frac{\partial^2 x^k}{\partial x'^m \partial x'^n}$$

Define :-

$$R^i{}_{jkl} = \partial_l \Gamma^i_{jk} - \partial_k \Gamma^i_{jl} + \Gamma^m_{jk} \Gamma^i_{lm} - \Gamma^m_{jl} \Gamma^i_{km}$$

Riemann tensor

Ex. Show that $R^i{}_{jkl}$ is a rank $(1,3)$ tensor

Note :- $R_{ijkl} \equiv g_{im} R^m{}_{jkl}$

R_{ijkl} has the property that it is antisymmetric in (ij) & antisymmetric in (kl)

Also, $R_{ijkl} = R_{klij}$

Ex.

Scalars

① $R^i{}_{ikl} = g^{ij} R_{ijkl} = 0$

② $R^i{}_{jil} \equiv R_{jl} \rightarrow$ Ricci tensor

Ex. $R_{jl} = R_{lj}$

③ $g^{jl} R_{jl} \equiv R$ (curvature scalar)

$$R'(x') = R(x)$$

We hope that R will give info about geom.

($\partial_i g_{jk}$ is zero in Cartesian coord. system for Euclidean space & so $R = 0$ for Euclidean space
→ Note that $\partial_k g_{ij}$ is not zero for any choice of coord. But R is a scalar & can be calculated in any coord. system)

Ex: Show that for the 2-dimensional sphere of radius a,

$$R = (\text{const})\, a^{-2}$$

(this also shows that we can't go from radius a_1 to radius a_2 sphere by a coord. trs. bcos a is ~~a scalar~~ involved in the expression of R – changing 'a' means going to a diff. geom.)

Proof of Impossibility of constructing a non-trivial tensor out of first derivatives of the metric:–

Proof:– Assume the contrary.

- There is a tensor $T^{i_1 \dots i_p}{}_{j_1 \dots j_q}(x)$ constructed from $\partial_k g_{ij}$

(we may take powers of the first deriv., contract with g_{kl}, ... etc. → allowed)

- Take an arbitrary point $x_{(0)}$ (on the manifold).
- We'll show that we can choose a coordinate system such that

$$\partial_k g_{ij}(x) = 0 \quad \text{at} \quad x = x_{(0)}$$

Then $T^{i_1 \dots i_p}{}_{j_1 \dots j_q}(x) = 0$ at $x = x_{(0)}$

(except the part of T with contains g_{ij}'s, which of course we have gives info about dim. on contraction)

18/7/07 Riemann tensor can be contracted in diff ways to give diff. scalars :-

e.g. → $R_{ijkl} R_{i'j'k'l'} g^{ii'} g^{jj'} g^{kk'} g^{ll'}$

All these scalars contain diff. info about geom.
How these change (as a fn. of x) on the manifold gives us extra info.
How many indep. scalars can we construct?
→ It. corr. to the no. of indep. comp of the Riemann tensor.

$$\partial_k A^{i_1 \dots i_p}{}_{j_1 \dots j_q}$$

$$\frac{\partial}{\partial x^k}$$

In the primed coordinate system, we have $\partial'_k A'^{i_1 \dots i_p}{}_{j_1 \dots j_q}$

$$= \frac{\partial}{\partial x'^k} \left\{ \frac{\partial x'^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial x'^{i_p}}{\partial x^{m_p}} \frac{\partial x^{n_1}}{\partial x'^{j_1}} \cdots \frac{\partial x^{n_q}}{\partial x'^{j_q}} A^{m_1 \dots m_p}{}_{n_1 \dots n_q}(x) \right\}$$

$$\frac{\partial x^n}{\partial x'^k} \frac{\partial}{\partial x^n}$$

(Bcos of the $p+q$ additional terms the above quantity is a tensor — need to add a non-tensor to construct a tensor in this case)

Scalars are unique quantities whose ordinary derivatives trs. as tensors

Define :-

$D_k A^{i_1 \dots i_p}{}_{j_1 \dots j_q}$ ← Covariant derivative

$$= \partial_k A^{i_1 \dots i_p}{}_{j_1 \dots j_q} + \Gamma^{i_1}_{k l_1} A^{l_1 i_2 \dots i_p}$$

$$+ \Gamma^{i_2}_{k l_2} A^{i_1 l_2 i_3 \dots i_p}{}_{j_1 \dots j_q} + \dots + \Gamma^{i_p}_{k l_p} A^{i_1 \dots i_{p-1} l_p}{}_{j_1 \dots j_q}$$

$$- \Gamma^{m_1}_{k j_1} A^{i_1 \dots i_p}{}_{m_1 j_2 \dots j_q} - \Gamma^{m_2}_{k j_2} A^{i_1 \dots i_p}{}_{j_1 m_2 j_3 \dots j_q} - \dots - \Gamma^{m_q}_{k j_q} A^{i_1 \dots i_p}{}_{j_1 \dots j_{q-1} m_q}$$

$g(x_{(0)})$ is a sym. matrix & hence can be diagonalised by an orthogonal matrix.

$$\therefore \quad g(x_{(0)}) = U^T \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} U$$

for some orthogonal matrix U $(UU^T = \mathbb{1})$

$$\text{Then} \quad g'(x'_{(0)}) = \underbrace{S^T U^T}_{(US)^T} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} US$$

(We want to make sure that g' is an identity matrix. We can of course take $US = \mathbb{1}$ & then g' will be a diagonal matrix - but we want to do better)

Suppose all λ_i's are +ve.

$$\text{Define:-} \quad M = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_N} \end{pmatrix} = M^T$$

$$g'(x'_{(0)}) = S^T U^T M^T M U S = (MUS)^T MUS$$

[S is in our hand as it appeared in coord. trs. - only thing is S should be real & it should have non-zero det. for the trs. to be one-to-one]

Choose S such that $MUS = \underline{\mathbb{1}}$

$$\Rightarrow S = U^{-1} M^{-1}$$

Note:- $U^T = U^{-1}$ & for M^{-1} should be well defined $\lambda_i \neq 0$

$$\therefore \quad \Rightarrow g'(x_{(0)}) = \underline{\mathbb{1}}$$

... quantity transforms as a tensor of rank $(p, q+1)$

Ex. Show that the above statement holds

Now there is an apparent puzzle :-

$D_k g_{ij}$ transforms as a tensor, which is constructed purely out of the first deriv. of g_{ij}, which are trivial.

Ex. Check explicitly that $D_k g_{ij} = 0$.

Q) How close to δ_{ij} can we bring $g_{ij}(x)$ in the neighbourhood of $x_{(0)}$?

→ Consider a coordinate trs.

$$x^i = x^i_{(0)} + S_{ij}\left(x'^j - x'^j_{(0)}\right) + \mathcal{O}\left((x' - x'_{(0)})^2\right)$$

$$\frac{\partial x^i}{\partial x'^j} = S^i{}_j \quad \text{at } x' = x'_{(0)}$$

Taylor series expn.

$$x^i = x^i_{(0)} \quad \text{at } x' = x'_{(0)}$$

∴ $x'_{(0)}$ is the image of the pt. $x_{(0)}$

Now,
$$g'_{ij}(x'_{(0)}) = \left.\frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j} g_{mn}(x)\right|_{\substack{x = x_{(0)} \\ x' = x'_{(0)}}}$$

$$= S^m{}_i \, S^n{}_j \, g_{mn}(x_{(0)})$$

Take $S^i{}_j$ as a matrix $(N \times N)$

i → row index

j → column index

Then $g'(x'_{(0)}) = S^T g(x_{(0)}) S$

Suppose $\lambda_1 < 0$ & $\lambda_2, \lambda_3, \ldots, \lambda_N > 0$

(Since U is real, S will be complex & it won't be a sensible coord. trs. — Note that here M has an imaginary e. value)

$$M = \begin{pmatrix} \sqrt{-\lambda_1} & & & 0 \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\lambda_N} \end{pmatrix}$$

M has been defined in such a way that it is real

$$g'(p'_{(0)}) = \underbrace{S^T U^T M^T}_{\Downarrow \; \mathbb{1}} \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \underbrace{M U S}_{\substack{\Downarrow \\ \text{Choose this} = \mathbb{1}}}$$

$\to$ this has to be taken as M has been defined as a real matrix

$$= \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

This is the best we can do
— the signature of the metric can't be changed by a coord. trs. — the no. of $-$ve e. values of the metric is called the signature of the metric

---

In STR, we often change t to it bcos it is easy to keep track of the i's — but in GTR this shouldn't be done as in that case we must be cautious enough to see which are allowed trs. — Coord. trs. mixes all components

---

We can set first deriv. of metric to zero by adjusting the linear & quadratic terms — Riemann Normal

Ex. Show that

$$\frac{d}{du}\left(g_{ij}(x(u))\, n^i(u)\, n^j(u)\right) = 0$$

as a consequence of the // transport eqn.

[for deriv. of n^i, use the // transport eqn.]

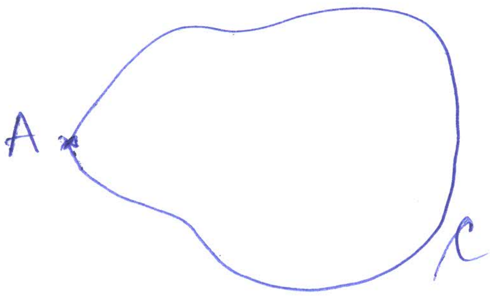
[Initially, in the flat coord. sys., we defined $m^i = n^i + O(\delta x_{(0)})^2$ & the metric is const. & that shows their lengths are the same]

→ length of the vector is preserved &

$M^i{}_j$ must be an $N \times N$ orthogonal matrix if we choose $g_{ij} = \delta_{ij}$ at A & B. otherwise it is a similarity trs. of an orthogonal matrix

as in that case $(\text{length})^2 \neq \sum_i m_i^2$. The $(\text{length})^2$ is $g_{ij} n^i n^j$

Consider closed loops :-



[In general, there is no reason why the final vector will be the same as the initial vector]

Parallel transport along C

$\Rightarrow$ $M^i{}_j(A, A; C)$ → orthogonal matrix

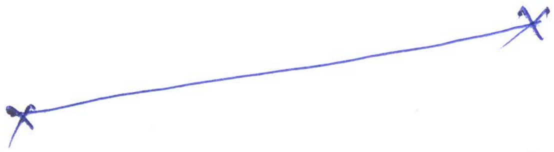
called

holonomy around C

Note :- $M^i{}_j$ is a orthogonal matrix only if the initial g_{ij} is δ_{ij} - otherwise it is related to the orthogonal matrix by a similarity trs.

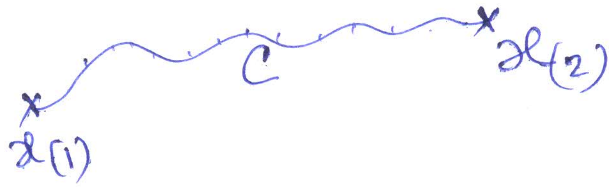
coord. are the closest to the Euclidean metric
- Coord. trs. can be used to trs. to Riemann Normal coord.

Geodesic



→ curve of shortest length connecting two points.

[Right now we assume the signature is +ve
- Defn of length becomes diff. for a -ve signature; we may have -ve $(\text{length})^2$ in some part & +ve in some other part of the path]



Parameter u

$0 \le u \le L$ → some fixed no.

C is specified by

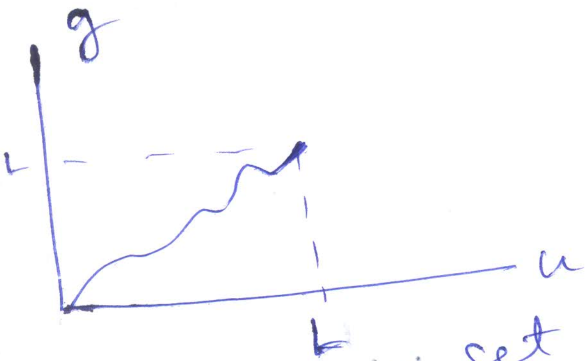
$X^i = f^i(u)$ (such that) $f^i(0) = x^i_{(1)}$

$f^i(L) = x^i_{(2)}$

Specify these N fns to define the curve

But this is somewhat redundant - but this is an equiv. way which treats all X^i's on an equal footing
→ the redundancy is related to the choice of u

Take an arbitrary monotone increasing function $g(u)$ such that $g(0) = 0$ & $g(L) = L$



g^{-1} is also monotone & we can use it as well

Write a new set of eqns :-

$$X^i = f^i(g(u)) \equiv F^i(u)$$

Claim :- this new set of fns describes exactly the same curve

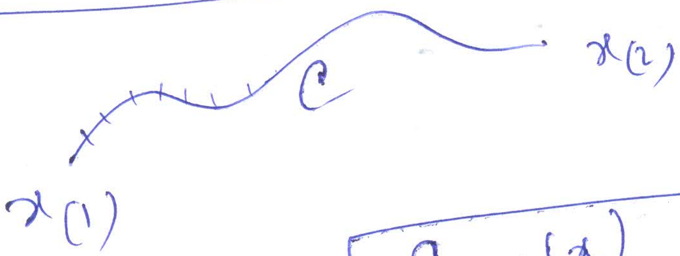
Define :- $v = g(u)$

Then $X^i = f^i(v)$ with $0 \le v \le L$

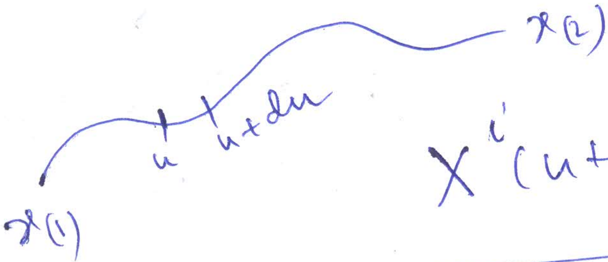
[v is a dummy variable bcos we are supposed to find v in terms of x^1 from the first eqn & then get all the other x^i's]

(Of course $u=1$ (say) corr. to some diff. value of v — what value of u we assign to a particular pt. is upto us — the curve is of course the same)

$$X^i = f^i(u)$$



$$ds = \sqrt{g_{ij}(x)\, dx^i\, dx^j}$$



$$X^i(u+du) - X^i(u) = \frac{dx^i}{du}\, du$$

small segment of coordinat

$$\therefore ds = \sqrt{g_{ij}(x(u))\, \frac{dx^i}{du} \frac{dx^j}{du} (du)^2}$$

$$= du \sqrt{g_{ij}(x(u))\, \frac{dx^i}{du} \frac{dx^j}{du}}$$

Total length of the curve is $\mathcal{L} = \int_0^L du \sqrt{g_{ij}(x(u)) \frac{dx^i}{du} \frac{dx^j}{du}}$

Under small variation, keeping the end points fixed, we have

$$\delta\mathcal{L} = \int_0^L du \, \frac{1}{2} \left\{ g_{mn}(x(u)) \frac{dx^m}{du} \frac{dx^n}{du} \right\}^{-1/2} \delta\left\{ g_{ij}(x(u)) \frac{dx^i}{du} \frac{dx^j}{du} \right\}$$

(The variation is in x as u is changed to $u+du$) →

$$x(u) \longrightarrow x(u) + \delta x(u)$$

Now, $\delta\left\{ g_{ij}(x(u)) \frac{dx^i}{du} \frac{dx^j}{du} \right\}$

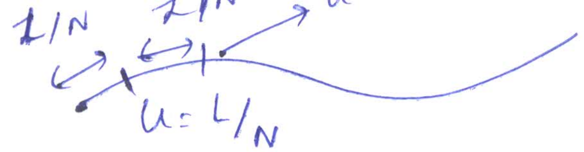
$$= \frac{\partial g_{ij}(x)}{\partial x^k} \delta x^k(u) \frac{dx^i}{du} \frac{dx^j}{du} + 2 g_{ij}(x(u)) \frac{d}{du}(\delta x^i) \frac{dx^j}{du}$$

came from $g_{ij} = g_{ji}$

$$= \frac{\partial g_{ij}(x)}{\partial x^k} \delta x^k \frac{dx^i}{du} \frac{dx^j}{du} + 2 g_{kj}(x(u)) \frac{d(\delta x^k)}{du} \times \frac{dx^j}{du}$$

need to integrate this by parts

(But taking deriv. of $\left(\sqrt{\cdots\cdots}\right)^{-1}$ isn't easy)



$\mathcal{L} \Rightarrow$ total length

$\mathcal{L}/N \rightarrow$ divide into N parts

[Choose u such that for $u = \mathcal{L}/N$, we will cover a distance L/N on the curve, ... etc.]

Equal length segment along $\mathcal{C}$

= Equal interval in u

$$\Rightarrow \sqrt{g_{mn}(x(u)) \frac{dx^m}{du}\frac{dx^n}{du}} = \text{constant along } \mathcal{C}$$

length per unit segment (curve divided into small parts)

$\rightarrow$ doesn't dep. on u

Now,
$$\delta\mathcal{L} = \int_0^L du \, \frac{1}{2}\left\{g_{mn}(x(u)) \frac{dx^m}{du}\frac{dx^n}{du}\right\}^{-\frac{1}{2}} \times \left[\delta x^k \frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{du}\frac{dx^j}{du} - 2\,\delta x^k \frac{d}{du}\left(g_{kj}(x(u))\frac{dx^j}{du}\right)\right]$$

[after int. the 2nd term by parts]

$$\Rightarrow \frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{du}\frac{dx^j}{du} - \frac{\partial g_{kj}}{\partial x^i}\frac{dx^i}{du}\frac{dx^j}{du} - 2\,g_{kj}\frac{d^2x^j}{du^2} = 0 \quad \text{from } \delta\mathcal{L} = 0$$

$$\Rightarrow g_{kj}\frac{d^2x^j}{du^2} + \frac{\partial g_{kj}}{\partial x^i}\frac{dx^i}{du}\frac{dx^j}{du} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{du}\frac{dx^j}{du} = 0$$

$\Downarrow$ multiply by g^{lk}

ex.
$$\frac{d^2x^l}{du^2} + \Gamma^l_{ij}\frac{dx^i}{du}\frac{dx^j}{du} = 0 \rightarrow \text{geodesic eqn.}$$

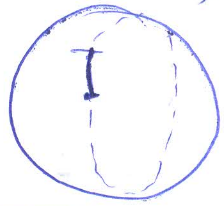
$\longrightarrow$ valid for a parametrization in which

$$\frac{d}{du}\left\{g_{mn}(x(u))\frac{dx^m}{du}\frac{dx^n}{du}\right\} = 0$$

(The whole thing is self-consistent becos) →

Ex. Show that $\frac{d}{du}\left\{g_{mn}\frac{dx^m}{du}\frac{dx^n}{du}\right\} = 0$ follows from the geodesic eqn.

Geodesic → it is an extremum (can be max. or min.)

e.g. →  2 possible paths

---

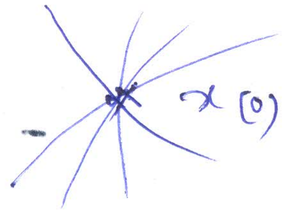
(The geodesic eqn doesn't involve an sq. root & can be applied to a metric of any signature & is a real eqn., so we take this as the → defn. of geodesic in a space of arbitrary signature

(We can't use the notion of shortest dist. for Minkowskian space – but we can of course use the above defn.)

10/7/07

Tangent space

Consider $x_{(0)}$: a point in the manifold



(Consider all geodesics passing through $x_{(0)}$)

satisfy

$$\frac{d^2x^i}{du^2} + \Gamma^i_{jk}\frac{dx^j}{du}\frac{dx^k}{du} = 0$$

where $i = 1, 2, \ldots, N$

(N second-order diff. eqns)

so

Generic soln :- Charaterized by $2N$ boundary conditions.

$$
2N \text{ b.c.}
\begin{cases}
x^i(u)\Big|_{u=0} = x^i_{(0)} & \text{where } i = 1, \dots, N \\[2ex]
\dfrac{dx^i(u)}{du}\Big|_{u=0} = n^i \longrightarrow & \text{other } N \text{ boundary conditions}
\end{cases}
$$

(We can think) →

$\{n^i\}$: N - dimensional vector

Given a vector $\vec{n}$, there is a <u>unique</u> <u>geodesic</u> passing through $x_{(0)}$.

[vector at a given pt. — just an N-comp. object]

Ask the reverse ques. — Given a geodesic, is $\{n^i\}$ fixed? → fixed upto a scaling

$$
\left.
\begin{array}{l}
x^i = f^i(u) \\
x^i = f^i(\lambda u)
\end{array}
\right\} \text{describe the same eqn.}
$$

[Note :- $\dfrac{d}{du}\left(- - - \right)$ is still zero under $u \longrightarrow \lambda u$]

Note :-

$n^i \to \lambda n^i$

$\Downarrow$

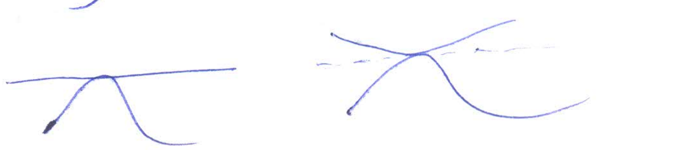
appears as a redundancy in the diff. eqn.

$u \longrightarrow \lambda u$ describes the same set of curves with a diff. parametrisation

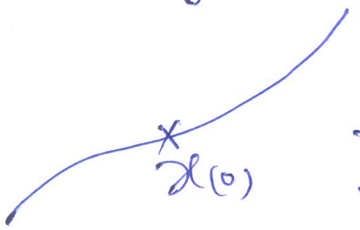
[We could have used unit vectors to avoid the ~~ambiguity~~ redundancy due to scaling, but vector space has nicer prop. than the space of unit vectors]

[Physically, in N dim., after fixing the pt. through which a st. line passes, we need $(N-1)$ more parameters to specify the st. line uniquely]

[Tangents, by defn., are geodesics, which make zero slope with an arbitrary curve's tangent at that pt.]



(The vector space of $\{n^i\}$'s is the tangent space.)



take a geodesic passing through $x_{(0)}$ characterized by the vector $\vec{n}$.

$$\left.\frac{dx^i}{du}\right|_{x=x_{(0)}} = n^i$$

Now change coordinates to x'.

(The same geodesic, in the primed coord. sys. is)

$$n'^i = \left.\frac{dx'^i}{du}\right|_{x=x_{(0)}} = \frac{\partial x'^i}{\partial x^j} \left.\frac{dx^j}{du}\right|_{x=x_{(0)}}$$

$$= \frac{\partial x'^i}{\partial x^j} n^j$$

[formally it seems as if it transforms as a contravariant vector]

[But it is altogether a diff. object — n^i is not a field in any sense, not sth. def. in terms of metric — it is defined at a pt. — diff. from any tensor field defined earlier]

[Given a metric, a tensor field is fixed — but given a metric, we get diff $\{n^i\}$'s for diff. geodesics]

$\{n^i\} \rightarrow \times\ x_{(1)}$ $\qquad$ $\times\ \{m^i\}\ x_{(2)}$

characterizes a geodesic passing through $x_{(1)}$

Is there a notion of $\vec{n}$ being parallel to $\vec{m}$?

Try $n^i = \lambda m^i$ for some λ

is this the ~~?~~ good notion for parallelism?

(this is infact a coord.-dep. criterion & so isn't a good criterion)

coordinate trs. to x'

$$n'^i = \left.\frac{\partial x'^i}{\partial x^j}\right|_{x=x_{(1)}} n^j$$

$$\& \quad m'^i = \left.\frac{\partial x'^i}{\partial x^j}\right|_{x=x_{(2)}} m^j$$

As matrices these two are completely diff. — no relation to each other

$\Rightarrow$ n' & m' are not parallel in general.

Why don't we have this problem in Euclidean space?

[There also, in sph. polar coord., we have the same problem. We define the notion of parallelism in a preffered coord. sys., one in which metric is const.

In curved ~~spa~~ space, metric can't be const. everywhere in any coord. sys. — But at each pt., we can use a preferred coord. sys. where metric is const. (it's a local notion)]

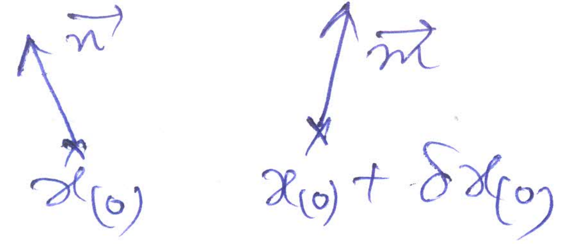
[We can't have first deriv. of metric zero everywhere bcos in that case 2nd " is also zero $\rightarrow$ not possible

So, parallelism can be compared betn. pts. close to the pt. where this local preferred coord. is chosen]

We try to use a preferred coordinate system in which

$$\partial_k g_{ij}(x)\Big|_{x=x_{(0)}} = 0$$

Use this notion to define the notion of parallelness for tangent vectors at points close to $x_{(0)}$.



$\partial_k \, g_{ij}(x) = 0$ at $x = x_{(0)}$

$\Gamma^{i}_{jk}(x_{(0)}) = 0$

$\vec{n}$ & $\vec{m}$ will be called parallel if $\vec{n} = \lambda \vec{m}$ for some λ.

$\vec{m}$ is the parallel transport of $\vec{n}$ if $\vec{m} = \vec{n} + O(\delta x_{(0)}^2)$

[It's not exact bcos it depends on how small we have chosen $\delta x_{(0)}$ to be]

Go to a new coordinate system

$$m'^{i} = \left.\frac{\partial x'^{i}}{\partial x^{j}}\right|_{x_{(0)}} m^{j}$$

$$n'^{i} = \left.\frac{\partial x'^{i}}{\partial x^{j}}\right|_{x_{(0)} + \delta x_{(0)}} n^{j}$$

$$= \left\{ \left.\frac{\partial x'^{i}}{\partial x^{j}}\right|_{x_{(0)}} + \frac{\partial^2 x'^{i}}{\partial x^{j} \partial x^{k}} \delta x^{k}_{(0)} \right\} \left\{ m^{j} + O(\delta x_{(0)}^{2}) \right\}$$

Taylor series expn.

$$= \underbrace{\left.\frac{\partial x'^{i}}{\partial x^{j}}\right|_{x_{(0)}} m^{j}}_{m'^{i}} + \frac{\partial^2 x'^{i}}{\partial x^{j} \partial x^{k}} \delta x^{k}_{(0)} m^{j} + O(\delta x^{2}_{(0)})$$

$$\Rightarrow n'^{i} - m'^{i} = \frac{\partial^2 x'^{i}}{\partial x^{j} \partial x^{k}} m^{j} \delta x^{k}_{(0)}$$

[so, though in a preffered coord. sys. the diff. is zero, it is not so in a general coord. sys.
$\longrightarrow$ Samething is true for Euclidean case also]

Ex. Prove that

$$\Gamma^{i}_{jk}(x) = \frac{\partial x^i}{\partial x'^m}\frac{\partial x'^n}{\partial x^j}\frac{\partial x'^l}{\partial x^k}\Gamma'^m_{ln}(x') + \frac{\partial x^i}{\partial x'^m}\frac{\partial^2 x'^m}{\partial x^j \partial x^k}$$

[In the above relation, set $x = x_{(0)}$]

set $x = x_{(0)} \Rightarrow x' = x'_{(0)}$

$x_{(0)}$ & $x'_{(0)}$ are images of each other

Then, $\Gamma^{i}_{jk}(x_{(0)}) = 0$

$$\Rightarrow \frac{\partial^2 x'^m}{\partial x^j \partial x^k}\bigg|_{x_{(0)}} \frac{\partial x^i}{\partial x'^m}\bigg|_{x_{(0)}} = -\frac{\partial x^i}{\partial x'^m}\bigg|_{x_{(0)}}\frac{\partial x'^n}{\partial x^j}\bigg|_{x_{(0)}}\frac{\partial x'^l}{\partial x^k}\bigg|_{x_{(0)}} \times \Gamma'^m_{ln}(x'_{(0)})$$

(Mult. both sides by $\frac{\partial x'^s}{\partial x^i}\Big|_{x_{(0)}}$) $\Rightarrow$

$$\frac{\partial^2 x'^m}{\partial x^j \partial x^k}\bigg|_{x_{(0)}} \delta_m^s = -\delta_m^s \frac{\partial x'^n}{\partial x^j}\bigg|_{x_{(0)}}\frac{\partial x'^l}{\partial x^k}\bigg|_{x_{(0)}} \times \Gamma'^m_{ln}(x'_{(0)})$$

$$\Rightarrow \frac{\partial^2 x'^s}{\partial x^j \partial x^k}\bigg|_{x_{(0)}} = -\frac{\partial x'^n}{\partial x^j}\bigg|_{x_{(0)}}\frac{\partial x'^l}{\partial x^k}\bigg|_{x_{(0)}}\Gamma'^s_{ln}(x'_{(0)})$$

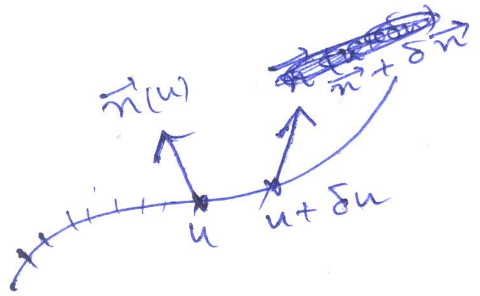
(We are trying to write $n'^i - m'^i$ in terms of some intrinsic quantity of that coord. sys. without referring back to the unprimed coord. sys.)

$$\therefore n'^i - m'^i = -\frac{\partial x'^n}{\partial x^j}\bigg|_{x_{(0)}}\frac{\partial x'^l}{\partial x^k}\bigg|_{x_{(0)}}\Gamma'^i_{ln}(x'_{(0)})\, m^j \times \delta x^k_{(0)}$$

$$= -\Gamma'^i_{ln}(x'_{(0)})\, m'^n\, \delta x'^l_{(0)}$$

[The above is the condn. for parallel transport for any coord. sys. — of course in the preferred coord. sys, $\Gamma^i_{jk} = 0$]

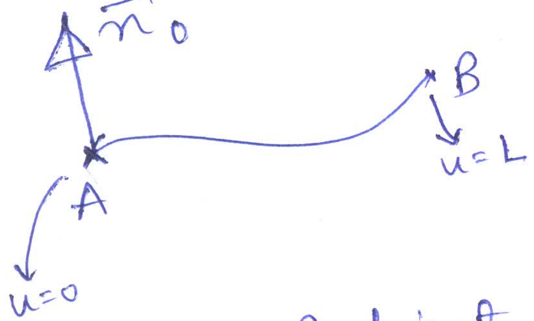
Consider an arbitrary path $x^i = f^i(u)$
[not necessarily geodesic; u not necessarily chosen to satisfy $\frac{d}{du}\left(g_{ij} \frac{dx^i}{du} \frac{dx^j}{du}\right) = 0$ condition]



$\vec{n}$ & $\vec{n} + \delta\vec{n}$ are parallel if

$$\delta n^i = -\Gamma^i_{lk} n^k \delta x^l$$

$$\frac{dn^i}{du} = -\Gamma^i_{lk} n^k \frac{dx^l}{du}$$



Solve the eqn:—

$$\frac{dn^i}{du} = -\Gamma^i_{lk} n^k \frac{dx^l}{du}$$

subject to the boundary condition:—

$n^i(0) = n^i_0$

$n^i(L) = m^i_0$ (say)

$\vec{m}_0$ is the parallel transport of $\vec{n}_0$ from A to B along the curve C.

Prop.:—

① This eqn. is linear in $\vec{n}$

Suppose $\vec{m}_0$ is the $\parallel$ transport of $\vec{n}_0$

& $\vec{\tilde{m}}_0$ is the parallel transport of $\vec{\tilde{n}}_0$

What is the // transport of $\alpha \vec{n}_0 + \beta \vec{\tilde{n}}_0$?

$\xrightarrow{\text{Ans.}}$ $\alpha \vec{m}_0 + \beta \vec{\tilde{m}}_0$ (bcos of the linearity of the diff. eqn.)

If for $\vec{m}_0$, the soln is $\vec{n}(u)$

& for $\vec{\tilde{m}}_0$, the soln is $\vec{\tilde{n}}(u)$, then

$\alpha \vec{n}(u) + \beta \vec{\tilde{n}}(u)$ is the soln. that takes (interpolates) $\alpha \vec{n}_0 + \beta \vec{\tilde{n}}_0$ to $\alpha m_0 + \beta \vec{\tilde{m}}_0$.

Using this, we can write:

$$m_0{}^i = \underbrace{M^i{}_j}\ (\underbrace{A, B}_{\text{Points } A \,\&\, B};\ \underbrace{C}_{\text{curve } C})\ n_0{}^j$$

Suppose $\vec{n}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$; then $\vec{m}_0 = \begin{pmatrix} M^1{}_1 \\ M^2{}_1 \\ \vdots \\ M^N{}_1 \end{pmatrix}$

[// trans. a linear map from original vector space to the final vector space & linear maps can always be rep. by matrics]

Ex. Show that the eqn

$$\frac{dn^i}{du} + \Gamma^i{}_{jk} n^j \frac{dx^k}{du} = 0 \text{ is}$$

invariant under a general coordinate transformation.