$M_{pl} = 1.22 \times 10^{19}$ GeV $\qquad \boxed{G=1 \quad \hbar = c = 1}$

$L_{pl} = 1.62 \times 10^{-33}$ cm

$T_{pl} = 5.39 \times 10^{-44}$ sec.

$k_B = 8.62 \times 10^{-5}$ eV/K

$8\pi G = 1$ (reduced Planck scale)

$M_{pl} \simeq 2.43 \times 10^{18}$ GeV etc.

Cosmology: Study of universe (past, present & future) as a whole

complicated object containing many stars, galaxies, ----

⇒ Need some simplifying assumption

→ may be relaxed later.

Simplifying assumptions: Homogeneity and isotropy.

Homogeneity: At a fixed instant of time, different parts of the universe look the same.

(Not true at small scales where there are stars, galaxies etc., but appears to be true when we consider average properties over large distance scales)

Isotropy: The universe looks the same in all directions, at a fixed time.

Note: The above statements require some prefered choice of time coordinate t.

Later we shall explore the origin of these properties but for now we proceed with these assumptions.

Metric: $c=1$

$$ds^2 = -f_1(\vec{x},t)\,dt^2 + f_{2i}(\vec{x},t)\,dx^i dt$$
$$+ f_{3ij}(\vec{x},t)\,dx^i dx^j$$

$t$: prefered time coordinate

$\vec{x} = (x^1, x^2, x^3)$: space coordinates.

We can remove $f_{2i}$ by coordinate trs.

$$x^i = \phi^i(\vec{x}', t')$$
$$t = t'$$

$$ds^2 = -f_1\,dt'^2 + f_{2i}\,\partial_j'\phi^i\,dx'^j\,dt'$$
$$+ f_{2i}\,\partial_{t'}\phi^i\,dt'^2 + f_{3ij}\,\partial_k'\phi^i\,\partial_l'\phi^j\,dx'^k\,dx'^l$$
$$+ 2 f_{3ij}\,\partial_{t'}\phi^i\,\partial_k'\phi^j\,dt'\,dx'^k$$
$$+ f_{3ij}\,\partial_{t'}\phi^i\,\partial_{t'}\phi^j\,(dt')^2$$

$$= -(f_1 - f_{2i}\,\partial_{t'}\phi^i + f_{3ij}\,\partial_{t'}\phi^i\,\partial_{t'}\phi^j)\,dt'^2$$
$$+ (f_{2i}\,\partial_{j'}\phi^i + 2 f_{3ij}\,\partial_{t'}\phi^i\,\partial_k'\phi^j)\,dt'\,dx'^k$$
$$+ f_{3ij}\,\partial_k'\phi^i\,\partial_l'\phi^j\,dx'^k\,dx'^l$$

$\phi^i$: 3 functions of $\vec{x}, t$

Adjust them to set

$$f_{2i}\,\partial_j'\phi^i + 2 f_{3ij}\,\partial_{t'}\phi^i\,\partial_k'\phi^j = 0$$

If the initial $f_{2i}$ is small (assuming close to flat metric) then this can be done iteratively.

$$f_1 \simeq 1 \quad , \quad f_{3ij} \simeq \delta_{ij}$$

Take $\phi^i(\vec{x}',t') = x'^i + \underbrace{\psi^i(\vec{x}',t')}_{\text{small.}}$

$$2 f_{3ij}\, \partial_{t'} \psi^i (\delta_{jk} + \partial_k' \psi^i) + f_{2i}(\delta_{ij} + \partial_j' \psi^i) = 0$$

$$2 f_{3ik}\, \partial_{t'} \psi^i = -2 f_{3ij}\, \partial_{t'} \psi^i \partial_k' \psi^i - \underbrace{f_{2kj}}_{\text{Leading term.}} - f_{2i}\, \partial_j' \psi^i$$

$$\partial_t' \psi^i = -\frac{1}{2} (f_3^{-1})_{ik} f_{2k} \rightarrow \text{leading soln.}$$

(Substitute & iterate to find $\psi^i$

$$\psi^i = \int_{t_0'}^{t'} dt'' \left(-\frac{1}{2} \left(f_3(\vec{x}', t'')\right)^{-1}\right)_{ik} f_{2k}(\vec{x}', t'')$$

→ leading order solution.

Consider the metric:

$$ds^2 = -f_1(\vec{x},t)\, dt^2 + f_{3ij}(\vec{x},t)\, dx^i dx^j$$

Now impose the condition for homogeneity and isotropy.

Homogeneity: Naive guess: $f_1, f_3$ should be independent of $\vec{x}$.

→ too restrictive.

General condition: Given any two points $(\vec{x}_{(1)}, t)$ and $(\vec{x}_{(2)}, t)$, there is a diffeomorphism:

$$t = t', \quad x^i = \Phi^i(\vec{x}', t')$$

such that ① in the $\vec{x}', t'$ coordinates the metric retains the same form:

$$ds^2 = -f_1(\vec{x}',t')\, dt'^2 + f_{3ij}(\vec{x},t')\, dx'^i dx'^j$$

② $\Phi^i(\vec{x}^*_{(1)}, t^*) = x^i_{(2)}$

It is enough to check these conditions for $\vec{x}_{(1)}, \vec{x}_{(1)}$ infinitesimally close.

$\Rightarrow \Phi^i$ is infinitesimal diffeomorphism

Take $\Phi^i(\vec{x}', t') = x'^i + \epsilon \xi^i(\vec{x}', t')$

$t = t'$

$x^i = x'^i + \epsilon\, \xi^i(\vec{x}', t')$

~~$ds^2$~~ $dt = dt'$

$dx^i = dx'^i + \epsilon\, \partial'_j \xi^i\, dx'^j + \epsilon\, \partial'_t \xi^i\, dt'$

$$ds^2 = -\overset{f_1(\vec{x}',t')}{\wedge} dt'^2 + f_{3ij}^{(\vec{x}',t')} ~~\epsilon~~ dx'^i\, dx'^j - \epsilon \xi^i\, \partial'_i f_1 (dt')^2$$

$$+ \epsilon\, f_{3ij}\, dx'^i\, \partial'_k \xi^j\, dx'^k$$

$$+ \epsilon\, f_{3ij}\, \partial'_k \xi^i\, dx'^k\, dx'^j$$

$$+ \epsilon\, f_{3ij}\, \partial'_t \xi^i\, dt'\, dx'^j + \epsilon\, f'_{3ij}\, dx'^i\, dt'\, \partial_{t'} \xi^j$$

$$= ~~dt'^2 + f_{3ij}~~ + \epsilon\, \xi^k\, \partial'_k f_{3ij}^{(\vec{x}',t')}\, dx'^i\, dx'^j$$

$$\underset{\text{want}}{=} -f_1 dt'^2 + f_{3ij}(\vec{x}', t')\, dx'^i\, dx'^j$$

$$\Rightarrow \Big( f_{3ij}\, \partial'_k \xi^j + f_{3ji}\, \partial'_k \xi^j + \xi^l\, \partial'_l f_{3ik} \Big| + (k \leftrightarrow i) = 0.$$

$$f_{3ij}\, \partial_{t'} \xi^i + f_{3ji}\, \partial_{t'} \xi^i = 0$$

$$f_{3ij}\, \partial_{t'} \xi^i = 0 \Rightarrow \partial_{t'} \xi^i = 0.$$

$\xi^i \partial'_i f_1 = 0 \Rightarrow \partial'_i f_1 = 0$ since we need 3 indep $\xi^i$'s.

$\Rightarrow \xi^i(\vec{x}',t')$ is independent of $t'$ and should satisfy:

$$2 f_{3ij}(\vec{x}',t')\,\partial'_k \xi^j + f_{3\,\cancel{\;}\,ki}(\vec{x}',t')\,\partial_j \overset{f_{3ki}}{\cancel{\;}}$$

$$+ (k \Leftrightarrow i) = 0 \Rightarrow \xi^j \text{ is Killing vector of } f_{3ij}.$$

~~⇒~~ ① There must exist 3 linearly independent $\xi^i$ satisfying this condition so that from $\vec{x}$ we can move to any neighbouring point.

$\Rightarrow$ The spatial metric $f_{3ij}(\vec{x},t)$ at any fixed $t$ is homogeneous.

Isotropy: Consider a point $(\vec{x}_{(0)},t)$ and two points $(\vec{x}_{(1)},t)$ and $(\vec{x}_{(2)},t)$ at fixed geodesic distance from $(\vec{x}_{(0)},t)$.

There must exist a diffeomorphism $t'=t$, $\vec{x}'^i = \psi^i(\vec{x},t)$ such that

① Metric in the $(\vec{x}',t')$ coordinate system retains the same form, and

② $\psi^i(\vec{x}_{(1)},t) = x^i_{(2)}$, $\psi^i(\vec{x}_{(0)},t) = \vec{x}_{(0)}$.

Consider infinitesimal diffeomorphism:

$$x'^i = x^i + \epsilon \eta^i(\vec{x},t)$$

---

Condition ① ⇒ identical condition on $\eta^i(\vec{x},t)$.

$$2 f_{3ij}(\vec{x},t)\,\partial_k \eta^j + f_{3ki}(\vec{x},t)\,\partial_j f_{3ki} + (k \leftrightarrow i) = 0.$$

---

Condition ② ⇒ Given $(\vec{x}_{(0)}, t)$ & two nearby points $(\vec{x}_{(0)} + \epsilon \vec{x}_{(1)}, t)$, $(\vec{x}_{(0)} + \epsilon \vec{x}_{(2)})$ satisfying

$$f_{3ij}\, x^i_{(1)} x^j_{(1)} = f_{3ij}\, x^i_{(2)} x^j_{(2)}$$

there must exist $\eta^i$ satisfying condition 1 and satisfying:

$$x^i_{(2)} = x^i_{(1)} + \eta^i.$$

---

Summary: For every $t$, the spatial metric $f_{3ij}(\vec{x},t)$ is homogeneous and isotropic.

---

$$ds^2 = -f_1(t)\,dt^2 + f_{3ij}(\vec{x},t)\,dx^i dx^j$$

↳ set $f_1(t) = 1$ by $t \to x(t)$.

Classify homogeneous and isotropic metrics in 3-space dimensions.

---

Examples:

① $ds_3^2 = dx^2 + dy^2 + dz^2$

→ Invariant under $(x, y, z) \to (x+a, y+b, z+c)$

⇒ homogeneous.

---

Invariant under $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to R \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $R^T R = I$

→ isotropic around $\vec{0}$.

---

Note: Homogeneity + isotropy around a single point ⇒ isotropy around any point.

---

space-time metric:

$$ds^2 = - dt^2 + a(t)^2 \left( dx^2 + dy^2 + dz^2 \right)$$

---

② Consider the euclidean 3-sphere $S^3$:

$x^2 + y^2 + z^2 + \omega^2 = a^2$ ⇒ 3-d space.

$ds^2 = dx^2 + dy^2 + dz^2 + d\omega^2$

---

⇒ Invariant under:

$$\begin{pmatrix} x \\ y \\ z \\ \omega \end{pmatrix} \to S \begin{pmatrix} x \\ y \\ z \\ \omega \end{pmatrix} \qquad \underset{4 \times 4}{S^T S = I}$$

Given any two points $(x, y, z, w)$ and $(x', y', z', w')$ we can find ~~[illegible]~~ an $S$ such that

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = S \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

⇒ Homogeneous.

Consider the point $\begin{pmatrix} 0 \\ 0 \\ 0 \\ a \end{pmatrix}$.

Take 2 points: $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}$. equidistant from $\begin{pmatrix} 0 \\ 0 \\ 0 \\ a \end{pmatrix}$

$x_i, y_i, z_i$ small.

$$w_1 = w_2 \Leftrightarrow x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2$$

⇒ the points are equidistant from $\begin{pmatrix} 0 \\ 0 \\ 0 \\ a \end{pmatrix}$.

∃ a diffeomorphism:

$$S = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \qquad R^T R = I, \ 3 \times 3.$$

⇒ isotropic.

Such that ~~that~~ ~~[illegible]~~ takes $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 = w_1 \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}$

Explicit metric:

$$w = \rho \cos\psi$$

$$z = \rho \sin\psi \cos\theta$$

$$y = \rho \sin\psi \sin\theta \sin\phi$$

$$x = \rho \sin\psi \sin\theta \cos\phi$$

$$dx^2 + dy^2 + dz^2 + dw^2$$

$$= d\rho^2 + \rho^2 \{ d\psi^2 + \sin^2\psi \, (d\theta^2 + \sin^2\theta \, d\phi^2) \}$$

$$x^2 + y^2 + z^2 + w^2 = a^2 \Rightarrow \rho = a.$$

$$\Rightarrow d\rho = 0.$$

$$ds_3^2 = a^2 \{ d\psi^2 + \sin^2\psi \, (d\theta^2 + \sin^2\theta \, d\phi^2) \}$$

$$\boxed{r = \sin\psi}$$

$$\Rightarrow ds_3^2 = a^2 \left\{ \frac{dr^2}{1-r^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right\}$$

$$\Rightarrow ds^2 = - \cancel{f(t)} \, dt^2 + a(t)^2 \left\{ \frac{dr^2}{1-r^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right\}$$

Example 3: Hyperboloid:

$w^2 - x^2 - y^2 - z^2 = a^2 \qquad w > 0$

metric = $dx^2 + dy^2 + dz^2 - dw^2$

Eqs. & metric invariant under:

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \to \Lambda \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

$\Lambda$: SO(3,1) Lorentz trs. matrix.

① Given two points: $(w, x, y, z)$ and $(w', x', y', z')$ satisfying the constraint $\exists\ \Lambda$ that takes $(w, x, y, z) \to (w', x', y', z')$.

⇒ Homogeneous.

② Consider the point:

$$\begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ \& two nearly points: } \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} w_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

with $w_1 = w_2$

$\Rightarrow x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2$

⇒ equidistant from $\begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

②

$$\Rightarrow \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \qquad \underset{3\times 3}{R^{T}R = I}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} w_2 = w_1 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\Rightarrow R\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = R\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$\Rightarrow$ isotropic around $\begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Parametrization:

$w = a\cosh\psi$

$z = a\sinh\psi\cos\theta$

$x = a\sinh\psi\sin\theta\cos\phi$

$y = a\sinh\psi\sin\theta\sin\phi$

$\Rightarrow$ ① satisfies the constraint

② $ds_3^2 = a^2\{d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2$

$\boxed{\sinh\psi = r}$

$$\Rightarrow ds_3^2 = a^2\{\frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi$$

$$\Rightarrow ds^2 = -dt^2 + a(t)^2\{\frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi$$

Exhausts all homogeneous and isotropic metric in $D=3$.

$$\Rightarrow \quad ds^2 = -dt^2 + a(t)^2 \left\{ \frac{dr^2}{1-Kr^2} + \underset{\substack{\downarrow \\ \text{FRW}}}{r^2(d\theta^2 + \sin^2\theta\, d\phi^2)} \right\}$$

$K = 1 \to$ sphere (3d)

$= 0 \to$ flat

$= -1 \to$ hyperboloid (3d)

} in 3-d. metric

Present universe has low curvature ⇒ metric almost flat.

Can we reconcile this with ~~previous~~ the above form?

$t_0$ : Present time.

$a_0 = a(t_0)$

$$ds^2 = -dt^2 + \frac{a(t)^2}{a_0^2} \left\{ a_0^2 \frac{dr^2}{1-Kr^2} + a_0^2 r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \right\}$$

$a_0 r \equiv \rho$

$$ds^2 = -dt^2 + \left(\frac{a(t)}{a_0}\right)^2 \left\{ \frac{d\rho^2}{1-K\rho^2/a_0^2} + \rho^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \right\}$$

→ nearly flat metric if

① $a(t)$ is slowly varying ⇒ $\frac{\dot{a}(t_0)}{a_0}$ small

② $a_0$ is large → not needed for $K=0$.

Physical significance of $a(t)$:

Consider light travelling from at point A at $t_1$ to point B at $t_0$ $(t_0 > t_1)$.

Homogeneity: Can take B at $(r=0,\ \theta=0\ \phi=0)$

Isotropy: Take A to be at $(r = r_1,\ \theta=0,\ \phi=0)$.

Ex: Find path of the light ray solving the geodesic eq:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$\tau$: affine parameter.

& show that the path lies along $(\theta=0,\ \phi=0)$.

$$ds^2 = 0 \Rightarrow -dt^2 + a(t)^2\frac{dr^2}{1-kr^2} = 0$$

$$\Rightarrow \int_0^{r_1}\frac{dr}{\sqrt{1-kr^2}} = \underset{\substack{\text{relevant}\\ \text{since } t_1<t_0}}{\pm}\int_{t_0}^{t_1}\frac{dt}{a(t)} = \int_{t_1}^{t_0}\frac{dt}{a(t)}$$

Suppose the light has frequence $\nu_0$ when being emitted.

~~The next peak~~ If at $t_1$ the wave has a peak then the next peak will be at $t_1 + \delta t_1$ $= 1/\nu$.

Suppose it reaches B at $t_0 + \delta t_0$

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}}$$

Difference:

$$\frac{\delta t_0}{a(t_0)} - \frac{\delta t_1}{a(t_1)} = 0.$$

$\Rightarrow$ Observed frequency $\tilde{\nu} = 1/\delta t_0$

$$\nu_0 = 1/\delta t_1 = \frac{a(t_0)}{a(t_1)} \, 1/\delta t_0 = \frac{a(t_0)}{a(t_1)} \tilde{\nu}.$$

Observation $\tilde{\nu} < \nu$ (spectrum from distant stars is red shifted)

$\Rightarrow a(t_1) < a(t_0)$

$\Rightarrow$ Universe is expanding on average.

Assumption: observer & the stars are at rest in $(t, r, \theta, \phi)$ coordinates.
→ will be justified later.

Some definitions.

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \cdots$$

present

$H_0$ : measures current rate of expansion

→ Hubble parameter.

$q_0$ : measures how the expansion is slowing down ~~the rate of deceleration~~

→ deceleration parameter.

Current observation ⇒ $q_0 < 0$

⇒ The universe is accelerating.

For distant objects whose light emitted at time $t_1$ is reaching us ~~today~~,

$$z = \frac{a(t_0)}{a(t_1)} - 1 \rightarrow \text{redshift parameter.}$$

⇒ a measure of the distance of the star.

Small $z$ : closer objects

Large $z$ : Further objects.

If $a(t)$ is known then the precise relation between distance & $z$ is given by

$$d_0 \equiv a_0 \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}} = a_0 \int_{t_1}^{t_0} \frac{dt}{a(t)}$$

Knowing $z$ as fn. of $d_0$ ⇒ info about $a(t)/a_0$

$$ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right) \quad " \tilde{g}_{ij} dx^i dx^j$$

Q. How can ~~be~~ we theoretically determine $a(t)$?

→ by solving Einstein's equation.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

Ex. Show that

$$R_{00} = \frac{3\ddot{a}}{a}, \quad R_{i0} = 0, \quad R_{ij} \qquad x^0 = t$$

$$R_{ij} = -(a\ddot{a} + 2\dot{a}^2 + 2k) \tilde{g}_{ij}$$

$$= -\left( \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2} \right) g_{ij}, \quad g_{ij} = a^2 \tilde{g}_{ij}$$

$$R = g^{00} R_{00} + g^{ij} R_{ij}$$

$$= -\frac{3\ddot{a}}{a} - \left( \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2} \right) \underbrace{g^{ij} g_{ij}}_{=3}$$

$$= -\frac{6\ddot{a}}{a} - \frac{2\dot{a}^2}{a^2} - \frac{2k}{a^2}$$

For consistency $T_{\mu\nu}$ must have the form

$$T_{00} = \rho(t) = -\rho(t) g_{00}, \qquad T_{i0} = 0$$

$$T_{ij} = p(t) g_{ij}$$

⇒ Also follows from requiring that $T_{\mu\nu}$ is consistent with homogeneity & isotropy.

Interpretation:

$T^{00} = T_{00} = \rho$ : energy density

$T^{i0}$ = momentum density

$\overset{''}{0}$

$\Rightarrow$ On the average the momentum of matter in the $(t, r, \theta, \phi)$ coordinate system should vanish.

$\Rightarrow$ Matter is a rest on the average in this coordinate system.

$T_{ij} = p\, g_{ij}$

Go to a coordinate system in which $g_{ij} = \delta_{ij}$ at some time $t_1$.

$$ds^2 = -dt^2 + \underbrace{\frac{a(t)^2}{a(t_1)^2}}_{\substack{1 \text{ at} \\ t = t_1}} \left\{ \underbrace{\frac{d\rho^2}{1 - k\frac{\rho^2}{a(t_1)^2}} + \rho^2 d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2}_{} \right\}$$

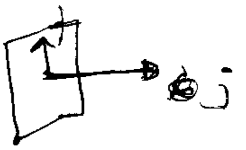
Can always change coordinates to make $d\hat{x}^i d\hat{x}^j$

$\hat{T}_{ij} = p\, \delta_{ij}$

same quantity.

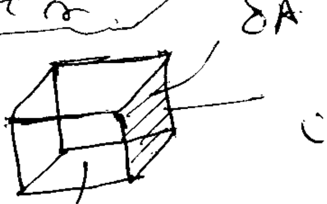
$\hat{T}_{ij} = \hat{T}^{ij}$ : Flux of i-th ^component of momentum $P_i$ along j-direction.

$\hat{T}_{ij} = 0$ for $i \neq j$ On the average



$P_i$ flows along jth direction = 0.

$T_{jj} = p$ ⇒ matter carrying +ve momentum along j will flow along j-th direction & matter carrying -ve momentum along j will flow along -j direction.



Total momentum incident on this per unit time = $\frac{p}{2} \cdot \delta A$

momentum is incident from one side only.

This gets elastically reflected

⇒ Total momentum transfer per unit time = $p\,\delta A$ ⇒ Force

⇒ Force/area = p = pressure

p is called the pressure even if the source of $T_{ij}$ is not particles or radiation

Now substitute into Einstein's eq.

$$R_{00} - \frac{1}{2} R g_{00} = - 8\pi G T_{00} \qquad x^0 \equiv t$$

$$\Rightarrow \quad \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \quad \rightarrow \text{Friedman eq.}$$

$$R_{ij} - \frac{1}{2} R g_{ij} = - 8\pi G T_{ij}$$

$$\Rightarrow \quad \frac{2\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = -8\pi G p$$

$$\frac{d}{dt}(\rho a^3) = \frac{3}{8\pi G}\frac{d}{dt}\left(\dot a^2 a + k a\right)$$

$$= \frac{3}{8\pi G}\left(2\ddot a \dot a a + \dot a^3 + k\dot a\right)$$

$$= \frac{3}{8\pi G}\dot a a^2 \left(\underbrace{\frac{2\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{k}{a^2}}_{-8\pi G p}\right)$$

$$= -3p\dot a a^2$$

$$\Rightarrow \quad \frac{d}{dt}(\rho a^3) = -3p\dot a a^2$$

Ex. check that this is equivalent to

$$D_\mu T^{\mu\nu} = 0 \qquad \text{(Local energy-momentum conservation)}$$

Follows from the identity

$$D_\mu\left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}\right) = 0$$

We have ① 3 unknowns $a(t), \rho(t), p(t)$

② Two differential equations (Friedman eq. & energy-momentum conservation).

How do we determine the evolution? The third eq. is provided by the equation of state.

~ a relation between $\rho$ and $p$.

Examples.

① Radiation: $p = \frac{\rho}{3}$. (Cosmic background radiation).

② Non-relativistic matter:

$$\rho \simeq m n, \quad p = n k_B T$$

mass of individual particle

# density of the particles

For protons : $m \sim 1$ GeV $\simeq 10^9$ eV.

$k_B \simeq 8.62 \times 10^{-5}$ eV/K.

If $T = 10^4$ K then $k_B T \sim 1$ eV $<< m$.

In practice most of the matter is very cold

$\Rightarrow$ $p \ll \rho$. We take $p=0$ as eq. of state.

③ Cosmological constant: (dark energy)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \Lambda g_{\mu\nu}\right)$$

a constant (cosmological constant)

eqs. are general coordinate trs. invariant.

We can rewrite this as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G\left(T_{\mu\nu} + T^{\Lambda}_{\mu\nu}\right)$$

$$T^{\Lambda}_{\mu\nu} = -\Lambda g_{\mu\nu}$$

$$T_{00} = -\Lambda g_{00} = \Lambda \rightarrow \rho$$

$$T_{ij} = -\Lambda g_{ij} \Rightarrow p = -\Lambda = -\rho$$

(Note: negative pressure)

Thus the equation of state is

$$p = -\rho.$$

If only one component is present the eq. of state ~~gives~~ relates $p$ & $\rho$ $\rightarrow$ third equation $\rightarrow$ can be used to solve the eqs. for $a(t)$ and $\rho(t)$.

What if all 3 components are present?

$$T_{\mu\nu} = T^{\Lambda}_{\mu\nu} + T^{m}_{\mu\nu} + T^{r}_{\mu\nu}$$

Make simplifying assumption: The different components do not interact except through gravity. (Will see how to relax this assumption later).

~~Note~~ $\Rightarrow$ $D_\mu T^{\Lambda\,\mu\nu} = 0$ $\quad D_\mu T^{m\,\mu\nu} = 0$

$D_\mu T^{r\,\mu\nu} = 0$. $\searrow$ always true due to $D_\mu g^{\mu\nu} = 0$.

$$\Rightarrow \frac{d}{dt}(\rho_m a^3) = -3 p_m \dot a a^2 = 0$$

$$\Rightarrow \rho_m = \frac{\rho_{m0} a_0^3}{a^3}.$$

Over a time scale $1/H_0$, a photon has negligible probability of scatteri

$$\frac{d}{dt}(\rho_r a^3) = -3 p_r \dot a a^2 = -\rho_r \dot a a^2$$

$$\underset{\text{Ex.}}{\Rightarrow} \frac{d}{dt}(\rho_r a^4) = 0 \Rightarrow \rho_r = \frac{\rho_{r0} a_0^4}{a^4}$$

$$\frac{d}{dt}(\rho_\Lambda a^3) = -3 p_\Lambda \dot a a^2 = -3 \rho_\Lambda \dot a a^2$$

$$\underset{\text{Ex}}{\Rightarrow} \frac{d}{dt}(\rho_\Lambda) = 0 \Rightarrow \rho_\Lambda = \text{constant} = \rho_{\Lambda 0}$$

$\gamma$-H collisions in period $H_0^{-1}$ ~~inelastic~~ : $\underbrace{n}_{\#\text{ of H/cc}} \times \underbrace{\sigma}_{\text{X-section}} \times H_0^{-1}$ ~~inelastic~~

Note: For large $a$ $\rho_\Lambda$ dominates, for small $a$, $\rho_r$ dominates, for intermediate $a$ the situation depends on the detailed values of $\rho_{\Lambda 0}$, $\rho_{m0}$, $\rho_{r0}$.

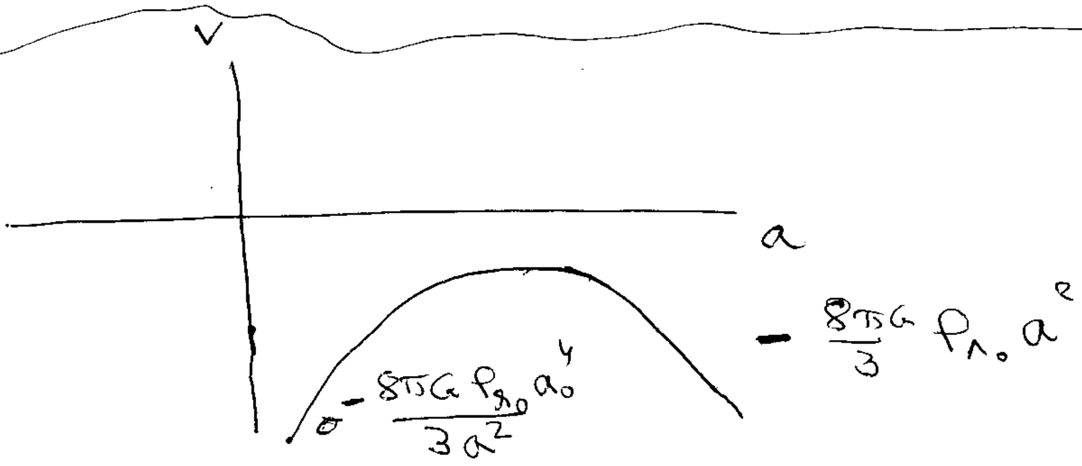
Friedman equation:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left(\rho_r + \rho_m + \rho_\Lambda\right)$$

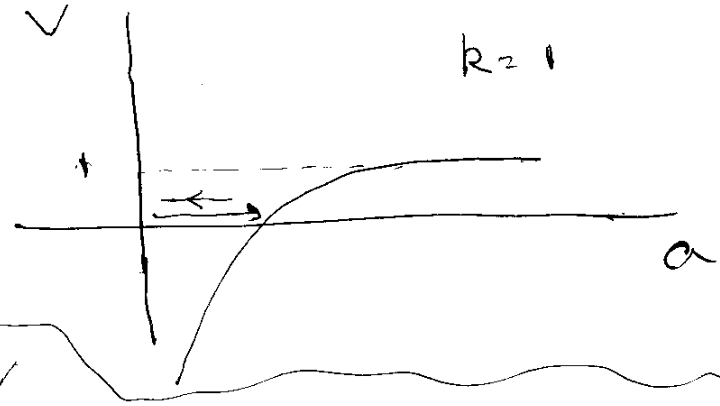
$$= \frac{8\pi G}{3}\left(\frac{\rho_{r0}a_0^4}{a^4} + \frac{\rho_{m0}a_0^3}{a^3} + \rho_{\Lambda 0}\right)$$

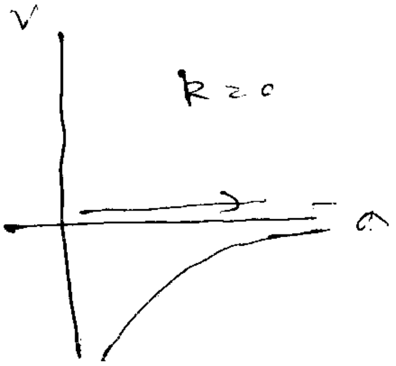
$$\Rightarrow \frac{1}{2}\dot{a}^2 + \frac{1}{2}k \underbrace{- \frac{\cancel{8}^{4}\pi G}{3}\left(\frac{\rho_{r0}a_0^4}{a^2} + \frac{\rho_{m0}a_0^3}{a} + \rho_{\Lambda 0}a^2\right)}_{+V(a)} = 0$$

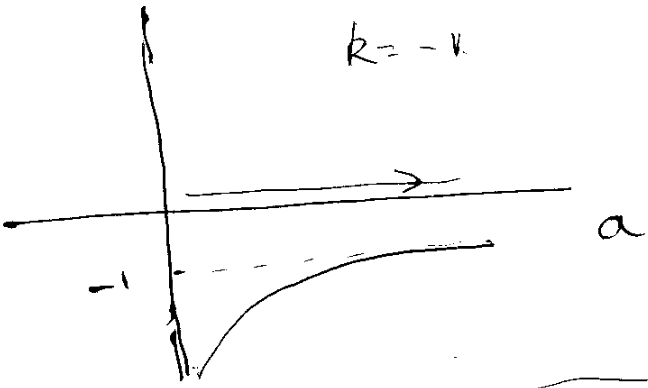
→ Motion of a particle of mass 1 with potential $V(a)$ and total energy 0.



20 years ago it was thought that $\rho_\Lambda = 0$.







k = 1 ⇒ Expansion is reversed.

k = 0, -1 : Expansion continues for ever.

→ not relevant ~~today~~.

With $\rho_{\Lambda 0}$ the expansion continues for ever.

Late time behaviour (cosmological constant dominated)

$$\dot{a}^2 = \frac{8\pi G}{3} \rho_{\Lambda 0} a^2 \Rightarrow a = \exp\left(\sqrt{\frac{8\pi G}{3}\rho_{\Lambda 0}}\, t\right)$$

Early time behaviour (radiation dominated)

$$\dot{a}^2 = \frac{8\pi G}{3} \frac{\rho_{r0} a_0^4}{a^2} \Rightarrow a\dot{a} = \sqrt{\frac{8\pi G}{3}\rho_{r0} a_0^4}$$

$$\frac{1}{2} a^2 = \sqrt{\frac{8\pi G \rho_{r0} a_0^4}{3}}\,(t+c)$$

$$a = 2^{1/2}\left(\frac{8\pi G \rho_{r0} a_0^4}{3}\right)^{1/4}(t+c)^{1/2}$$

---

Note: • At $t=-c$ we have a singularity (take $c=0$).

→ Known as big bang singularity.

---

Intermediate - time scale: The universe is matter dominated (to be seen later).

$$\cancel{\frac{1}{2}}\,\dot{a}^2 = \frac{8\pi G}{3}\rho_{m0}\frac{a_0^3}{a}$$

---

$$\Rightarrow \quad a^{1/2}\dot{a} = \left(\frac{8\pi G}{3}\rho_{m0} a_0^3\right)^{1/2}$$

---

$$\frac{2}{3} a^{3/2} = \left(\frac{8\pi G}{3}\rho_{m0} a_0^3\right)^{1/2}(t+c_1)$$

---

$$a = \left(\frac{3}{2}\right)^{2/3}\left(\frac{8\pi G}{3}\rho_{m0} a_0^3\right)^{1/3}(t+c_1)^{2/3}.$$

## Current values of parameters

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left(\rho_r + \rho_m + \rho_\Lambda\right)$$

At $t = t_0$

$$H_0^2 + \frac{k}{a_0^2} = \frac{8\pi G}{3}\left(\rho_{r0} + \rho_{m0} + \rho_{\Lambda 0}\right)$$

$$\Rightarrow \frac{k}{a_0^2} = \frac{8\pi G}{3}\left\{\rho_{r0} + \rho_{m0} + \rho_{\Lambda 0} - \frac{3H_0^2}{8\pi G}\right\}$$

Define $\rho_c = \frac{3H_0^2}{8\pi G}$ → critical density.

Current value :

$H_0 = 60 - 70$ Km/sec/megaparsec

parsec $= 3.086 \times 10^{18}$ cm.

Megaparsec $= 10^6$ parsec.

Note: $H_0$ has dimension of 1/time.

$1/H_0$ : time scale over which $a(t)$ changes appreciably.

$\rho_c = 8 \times 10^{-30}$ gm/cc for $H_0 = 65$ Km/s/Mpc

If $\rho_{r0} + \rho_{m0} + \rho_{\Lambda 0} > \rho_c \Rightarrow k = 1$

$< \rho_c \Rightarrow k = -1$

$\cong \rho_c \Rightarrow k = 0$ or $a_0$ lar

$a_0$ can be calculated from $\rho_{r0} + \rho_{m0} + \rho_{\Lambda 0}$ & $H_0$

Define $\Omega_m = \frac{\rho_m}{\rho_c}$, $\Omega_r = \frac{\rho_r}{\rho_c}$, $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}$

$$\Rightarrow \frac{k}{a_0^2} = \frac{8\pi G}{3}\rho_c\left(\Omega_m + \Omega_r + \Omega_\Lambda - 1\right)$$

Determination of $\Omega_m, \Omega_r, \Omega_\Lambda$:

$\rho\cancel{\Omega}_r = K T^4$ → black body radiation law.

$K \rightarrow \frac{8\pi^5 k_B^4}{15 h^3 c^3}$

$T = 2.7\,K$

⇒ K: Known constant (Ignoring neutrinos)

$\Rightarrow \Omega_r \simeq 5\times10^{-5}$ for $H = 65\,\text{km/s/Mpc}$ $\quad 3\times\frac{7}{8}\times\left(\frac{4}{11}\right)^{4/3}$

$\Omega_m$: Naive estimate from observed matter (stars, galaxies, clusters etc.)

$\Rightarrow (\Omega_m)_{\text{observed}} = .04$

But this is not all.

Observe motion of stars at the edge of galaxies.

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

M: Total mass inside radius r.

r: distance of the star from galaxy center

v = velocity of the star.

$$v = \sqrt{\frac{GM}{r}}$$

Observed :

$\Rightarrow M \sim r$. The mass enclosed inside a volume of radius $r$ grows as $r$ even when there seem to be very little visible matter at that radius.

$\Rightarrow$ M is larger than observed mass of the galaxy.

$\Rightarrow \exists$ dark matter (not optically visible) $\sim$ most likely new kind of elementary particles.

$(\Omega_{m0}) \simeq .3$

Measurement of $\Omega_{m0}$ : indirect

$$\frac{a(t)}{a_0} = 1 + H_0 (t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2$$

$$\Rightarrow \frac{\ddot{a}}{a_0} = -q_0 H_0^2$$

$$\dot a^2 = -k + \frac{8\pi G}{3}\rho a^2 = -k + \frac{8\pi G}{3}\left(\rho_{m0}\frac{a_0^3}{a} + \rho_{r0}\frac{a_0^4}{a^2} + \rho_{\Lambda_0} a^2\right)$$

~~$2\dot a\ddot a = \frac{8\pi G}{3}$~~

$$2\dot a\ddot a = \frac{8\pi G}{3}\left(-\rho_{m0}\frac{a_0^3}{a^2}\dot a - 2\rho_{r0}\frac{a_0^4}{a^3}\dot a + 2\rho_{\Lambda_0}\dot a a\right)$$

$$\Rightarrow \frac{\ddot a}{a} = \frac{4\pi G}{3}\left(-\rho_{m0}\frac{a_0^3}{a^3} - 2\rho_{r0}\frac{a_0^4}{a^4} + 2\rho_{\Lambda_0}\right)$$

$$\left.\frac{\ddot a}{a}\right|_{t=t_0} = \frac{4\pi G}{3}\left(2\rho_{\Lambda_0} - \rho_{m0} - 2\rho_{r0}\right)$$

$$\|$$

$$-q_0 H_0^2$$

$$\Rightarrow 2\rho_{\Lambda_0} - \rho_{m0} - 2\rho_{r0} = -\frac{3}{4\pi G}q_0 H_0^2$$

Knowing $q_0$ and $H_0$ we know

$2\rho_{\Lambda_0} - \rho_{m0} - 2\rho_{r0}$.

$\Rightarrow \rho_{\Lambda_0}$ since $\rho_{m0}$ & $\rho_{r0}$ are known.

$$\Rightarrow \Omega_{\Lambda_0} = \frac{\rho_{\Lambda_0}}{\rho_c} \simeq .7$$

$$\frac{k}{a_0^2} = \frac{8\pi G}{3}\rho_c\left(\Omega_m + \Omega_\Lambda + \Omega_r - 1\right)$$

$= 0$ within experimental erro

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Either k=0 or $a_0$ is very large.

In either case the present universe can be studied by setting k=0 to a good approximation since $k/a^2$ is the relevant term.

$$\frac{\dot{a}^2}{a^2} + \underset{\substack{\uparrow \\ \text{negligible} \\ \text{today.}}}{\frac{k}{a^2}} - \frac{8\pi G}{3}\left(\rho_{r0}\frac{a_0^4}{a^4} + \rho_{m0}\frac{a_0^3}{a^3} + \rho_{\Lambda 0}\right) = 0$$

$\rho_{m0} = .3\rho_c$, $\rho_{\Lambda 0} = .7\rho_c$, $\rho_{r0} = 5\times10^{-5}\rho_c$

$\rho_\Lambda >> \rho_{m}, \rho_r$ for $\frac{a^3}{a_0^3} >> \frac{.3}{.7} = \frac{3}{7}$

i.e. $a >> \left(\frac{3}{7}\right)^{1/3} a_0 \Big\}$ Λ-dominated

$\rho_m >> \rho_\Lambda$ for $a << \left(\frac{3}{7}\right)^{1/3} a_0$

$\rho_m >> \rho_r$ for $a >> \frac{5\times10^{-5}}{.3} a_0$

so matter dominated.

$\rho_r >> \rho_m, \rho_\Lambda$ for $\frac{a}{a_0} << \frac{5\times10^{-5}}{.3} \sim 10^{-4}$

At redshift $z \sim 10^4$ radiation & matter energy densities are comparable

Temperature: $\rho =$ ~~[illegible]~~ $K T^4$

$$\| \\ \rho_{R0} \frac{a_0^4}{a^4}$$

$$\Rightarrow \left(\frac{T}{T_0}\right)^4 = \left(\frac{a_0}{a}\right)^4$$

$$\Rightarrow T = T_0 \frac{a_0}{a} = T_0 \times 6 \times 10^3$$

At matter radiation equality.

$$T_{eq} = 2.7 \times \frac{-3}{5 \times 10^{-5}} = 1.6 \times 10^4 \text{ K}$$

$$k_B = 8.62 \times 10^{-5} \text{ ev/K}$$

$$\Rightarrow k_B T_{eq} = 8.62 \times 10^{-5} \times 1.6 \times 10^4 \sim 1.4 \text{ ev}$$

Go further back to the past.

T increases.

At $k_B T \sim 13.6$ eV something interesting happens.

Photon is energetic enough to knock an electron out of the Hydrogen atom.

Happens at $T = 13.6/(8.62 \times 10^{-5})$ K

$= 1.6 \times 10^5$ K

$$\frac{a_0}{a} = \frac{T}{T_0} = \frac{1.6\times10^5}{2.7} = 6\times10^4$$

---

$\mathcal{H}_0 = 100h$ Km/s/Mpc $\simeq 70$ Km/s/Mpc

$\Omega_m \simeq .27$, $\Omega_\Lambda \simeq .73$, $\Omega_B \simeq .04$

$$\Omega_r = 2.47\times10^{-5}\, h^{-2} \simeq 5\times10^{-5}$$

Review

$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left(\rho_{r0}\frac{a_0^4}{a^4} + \rho_{m0}\frac{a_0^3}{a^3} + \rho_{\Lambda 0}\right)$$

$$= H_0^2\left(\Omega_r\frac{a_0^4}{a^4} + \Omega_m\frac{a_0^3}{a^3} + \Omega_\Lambda\right)$$

---

$\Omega_r = \frac{\rho_{r0}}{\rho_c}$ , $\rho_c = \frac{3H_0^2}{8\pi G}$

$\Omega_m = \rho_m/\rho_c$ , $\Omega_\Lambda = \rho_\Lambda/\rho_c$

---

Current best values.

$H_0 = 71$ Km/s/Mpc,

$\Omega_m = .27$ , $\Omega_\Lambda = .73$ , $\Omega_B = .02 = \frac{\rho_B}{\rho_c}$

$\Omega_r \simeq 5\times10^{-5}$ (only photons)

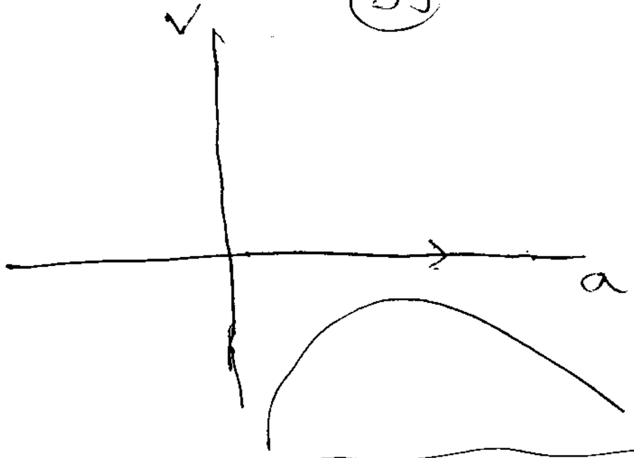
$k \simeq 0$

Mpc = $10^6$ pc = $10^6 \times 3.086\times10^{18}$ cm.

$H_0^{-1} = (10^6\times3.086\times10^{18}/71\times10^5)$ s

$\simeq 4.3\times10^{17}$ s $\simeq 1.38\times10^{10}$ years

$$\frac{1}{2}\dot a^2 - \underbrace{\frac{H_0^2}{2}\left(\Omega_r a_0^4/a^2 + \Omega_m a_0^3/a + \Omega_\Lambda a^2\right)}_{V(a)}$$



⇒ The universe expands forever.

$\lambda \equiv a/a_0$ present value

$$\left(\frac{\dot{\lambda}}{\lambda}\right)^2 = H_0^2\left(\Omega_r \lambda^{-4} + \Omega_m \lambda^{-3} + \Omega_\Lambda\right)$$

$$\int_{t_1}^{t} dt = H_0^{-1}\int_{\lambda_1}^{\lambda} \frac{d\lambda'}{\lambda'\sqrt{\Omega_r \lambda'^{-4} + \Omega_m \lambda'^{-3} + \Omega_\Lambda}}$$

$$= H_0^{-1}\int_{\lambda_1}^{\lambda} \frac{\lambda' \, d\lambda'}{\sqrt{\Omega_r + \Omega_m \lambda' + \Omega_\Lambda \lambda'^4}}$$

~~Earliest epoch: radiation dominated~~

3 periods.

$$\lambda \gg \underset{\sim 1}{\underset{\downarrow}{\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3}}}, \underset{\text{small}}{\underset{\downarrow}{\left(\frac{\Omega_r}{\Omega_\Lambda}\right)^{1/4}}}$$ : Cosmological constant dominated.

$$\underset{\sim 1}{\left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3}} \gg \lambda \gg \frac{\Omega_r}{\Omega_m}$$ : matter dominated

$$\sim 5\times10^{-5}/.3 = 1.6\times10^{-4}$$

$\lambda << \frac{\Omega_r}{\Omega_m}$ ⇒ radiation dominated.

$\lambda_{eq} = \frac{\Omega_r}{\Omega_m} \simeq 1'6 \times 10^{-4}$ ⇒ matter-radiation equality.

→ underestimate by a factor ~1

Neutrinoes also add to $\Omega_{r,0}$ (will be seen later).

Study of $\lambda(t)$.

$$\int_{t_1}^{t} dt' = H_0^{-1} \int_{\lambda_1}^{\lambda} \frac{\lambda' d\lambda'}{\sqrt{\Omega_{r} \lambda'^{-4} + \Omega_m \lambda'^{-3} + \Omega_\Lambda}}$$

Choose the origin of time.

$\lambda = 0$ at $t = 0$

$$\int_0^{t} dt' = H_0^{-1} \int_0^{\lambda} \frac{\lambda' d\lambda'}{\sqrt{\Omega_r + \Omega_m \lambda' + \Omega_\Lambda \lambda'^4}}$$

✓ no divergence from $\lambda' = 0$ end.