A general result

Suppose we have two kinds of particles A, B.

$A + \cdots \to B + \cdots$ with rate $\lambda(A \to B)$

$B + \cdots \to A + \cdots$ with rate $\lambda(B \to A)$.

Suppose further that these are the only processes which change # of A and B.

In equilibrium:

$$n_A^{EQ} \lambda(A \to B) = n_B^{EQ} \lambda(B \to A)$$

$$\Rightarrow \frac{n_A^{EQ}}{n_B^{EQ}} = \frac{\lambda(B \to A)}{\lambda(A \to B)}$$

microscopic results, do not depend on whether or not we have equilibrium.

If we have a system with

$$\frac{n_A}{n_B} < \frac{n_A^{EQ}}{n_B^{EQ}} = \frac{\lambda(B \to A)}{\lambda(A \to B)}$$

$$\Rightarrow n_A \lambda(A \to B) < n_B \lambda(B \to A)$$

$\Rightarrow$ Net conversion of $B \to A$ till we reach equilibrium.

$\to$ takes time $\sim 1/\lambda(B \to A)$.

Examine expected equilibrium abundance at $T = 10^9 K$.

$$Y_d \sim Y_n Y_p 10^{-4}$$

$$Y_{H^3} \sim Y_n^2 Y_p 10^{12}$$

$$Y_{He^3} \sim Y_n Y_p^2 10^{9}$$

$$Y_{He^4} \sim Y_n^2 Y_p^2 10^{97}$$

For $Y_n, Y_p \sim 1$ we expect the reaction rates to be such that we build up H^3, He^3, He^4 from n, p.

Problem: This has to proceed via d and Y_d is still small at $10^9 K$

The reaction rate
$\lambda(p + n \rightarrow d + \gamma)$ is large enough to build d to its equilibrium value at $T \gtrsim 10^9 K$

Even though Y_d is small,
$$d + d \rightarrow He^3 + n$$
$$H^3 + p$$
is large enough to convert 1d to He^3 or H^3

within a Hubble time.

(Meaning of $\frac{\Lambda}{H} \sim 1$ at $T = 10^9 K$)

This increases sharply as $T \downarrow$

$\Rightarrow$ ~~whatever~~ d's produced from $n + p \to d + \gamma$ start getting converted to He^3, H^3 & then to He^4.

As this reduces the density of d's. since more n & p get converted to d.

$$\lambda(n \to d)\, n_n > \lambda(d \to n)\, n_d.$$

This process continues till we run out of n.

Once we run out of n, no new d's are produced.

$$d + d \to H^3 + p \qquad d + H^3 \to He^4 + n$$
$$ He^3 + n, \qquad d + He^3 \to He^4 + p$$

continues.

When does this stop?

When these reaction rates fall below the Hubble expansion rate.

$$d + d \rightleftarrows \begin{matrix} He^3 + n \\ H^3 + p \end{matrix}$$

Total rate ~~per~~ per $d = \sigma v n_d \equiv \Lambda$

Hubble expansion rate H.

$\frac{\Lambda}{H} \lesssim 1$ stops the process.

→ same eq. as before.

Earlier we used it to find T at which nucleosynthesis begins.

Now we shall use it to determine n_d at which nucleosynthesis stops.

$$\sigma = \frac{1}{6} \times (1.8 + 1.6) \times 10^{-17}\ \text{cm}^3/\text{sec}$$

$$\Rightarrow \sigma v n_d = 3.4 \times 10^{-17}\ \text{cm}^3/\text{sec} \times n_d$$

Compare with H.

$$H = \sqrt{\frac{8\pi G}{3}\rho} = \sqrt{\frac{8\pi G}{3}\frac{1}{2}a_B T^4 \left(2 + 6 \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3}\right)}$$

$$a_B = \pi^2/15$$

$$\overset{\text{Ex.}}{=} .28 \left(\frac{T}{10^{10}\,K}\right)^2 \text{sec}^{-1} = .28 \times 10^{-2}\ \text{sec}^{-1}$$

for $T \sim 10^9$ K

Thus nucleosynthesis stops at

$$n_d = .28 \times 10^{-2} \times \frac{1}{3.4 \times 10^{-17}} / cm^3$$

Compare with total no. of nucleons at that time n_N.

$$H_0 = \sqrt{\frac{8\pi G}{3}\rho_c}$$

$$n_N = \frac{\rho_{B0}}{m_p} \times \left(\frac{a_0}{a}\right)^3$$

$$= \frac{1}{m_p} \Omega_B \times \underbrace{\frac{3 H_0^2}{8\pi G}}_{= \rho_c} \times \left(\frac{T}{T_0}\right)^3$$

$$H_0 = h \times 100 \, \underbrace{km}_{= 10^7\,cm} / sec / \underbrace{Mpc}_{= 3 \times 10^{18} \times 10^6\,cm}$$

$$= h \, \frac{1}{3} \times 10^{-17} / sec$$

$$n_N = \Omega_B h^2 \times \frac{1}{9} \times 10^{-34} \, \frac{3}{8\pi G m_p} \times \left(\frac{10^9}{2.73}\right)^3 sec^{-2}$$

$$\frac{n_d}{n_N} = (\Omega_B h^2)^{-1} \times 9 \times 10^{34} \, \frac{8\pi G m_p}{3} \left(\frac{2.73}{10^9}\right)^3$$

$$\times \frac{.28}{3.4} \times 10^{15} \, sec^2/cm^3$$

Either use the values of G & m_p in usual units.

Or set $G = 1$ & use natural units.

$m_p = 1\,GeV = 1/(1.22 \times 10^{19})$, $sec = \frac{10^{44}}{5.39}$, $cm = \frac{10^{33}}{1.62}$

Result: $\frac{n_d}{n_N} = 1.47 \times 10^{-7} (\Omega_b h^2)^{-1}$

~ agrees ~~close~~ with actual numerical results for order of magnitude.

Observations from quasi stellar objects

$n_d / n_H \sim (2.78 \pm .4) \times 10^{-5}$

$\Rightarrow \Omega_B h^2 = .0214 \pm .0029$

Note: Deuterium abundance is sensitive to Ω_b.

→ agrees with other measurement of Ω_b e.g. CMBR.

(Direct observation gives low value)

He^4 measurement from HII regions (of ionized Hydrogen) with low abundance of elements other than H, He

⇒ indicates that this region has not been polluted by emission from stars.

Expect similar abundance of H^3 and He^3 since X-sections are similar.

H^3 decays to He^3 via β-decay.

$\Rightarrow$ ~~Primordial~~ He^3 abundance = (sum of primordial H^3 and He^3 abundance)

$\rightarrow$ can be calculated numerically.

Detailed analysis:

$n_{He^3}/n_H \sim (1.1 \pm 0.2) \times 10^{-5}$ for $\Omega_b h^2 = .0214$

Heavier elements.

Some Li^7 produced by

$$H^3 + He^4 \rightarrow Li^7 + \gamma$$

$$He^3 + He^4 \rightarrow Be^7 + \gamma$$

$$\hookrightarrow Li^7 + \nu_e$$

electron capture

$Li^7/H \sim 10^{-10}$

Li^6 is rarer $\sim 10^{-13}$

Note: Since there are plenty of p and He^4 their reaction rate could be substantial.

$p + He^4 \rightarrow (\cancel{Z}, A=5) + \gamma, e^{\pm}$

$He^4 + He^4 \rightarrow (\cancel{Z=4}, A=8) + \gamma, e^{\pm}$

would have been possible

But there are no stable nuclei with these properties.

Strong interactions processes are prevented by binding energy.

$He^4 + He^4 \rightarrow Be^7 + n$

$\rightarrow Li^7 + p$

Binding energy:

Left: 2×28.3

Right: 39 for Li^7

The agreement of nucleosynthesis with expt ~~⇒~~ constraints on any change in the state of the universe at 10^9 K from what we have used.

e.g. extra matter ⇒ ~~extra~~ large H

⇒ faster expansion rate

⇒ 10^9 K is reached at less time

⇒ Larger X_n ⇒ Larger He^4/H ratio.

Using this we can put bound on # of neutrinos ≤ 4.

μ_{ν_i} would also increase ρ & H.
$\Rightarrow$ puts bound on μ_{ν_i}.

However there is one effect which can occur in opposite direction.

High T equilibrium situation:

$$\frac{n_n}{n_p} = e^{-\frac{(m_n - m_p)}{T} + \frac{\mu_n - \mu_p}{T}}$$

$$\mu_p = \mu_n + \mu_{\nu_e} - \mu_e$$

$$\Rightarrow \frac{n_n}{n_p} = e^{-\frac{Q}{T}} e^{\frac{\mu_e - \mu_{\nu_e}}{T}}$$

μ_e/T is small.
μ_{ν_e}/T positive. $\Rightarrow$ reduced $\frac{n_n}{n_p}$.

Thus it may be possible to allow larger μ_{ν_e}/T and more matter so that X_n remains unchanged.

$\longrightarrow$ requires coincidence.
Possible but unlikely.

We now move forward in time.

Some important events

① Photons go out of equilibrium

② Transition from radiation to matter dominated universe.

③ Recombination $\rightarrow$ e^- get bound to nuclei and photons stop scattering
$\rightarrow$ origin of CMBR

Photon electron scattering

$$\sigma = .66525 \times 10^{-24}\ cm^2$$

Rate of scattering

$$\lambda = n_e \sigma c$$

Electrons are massive.
In each scattering photon momentum changes by $\sim T$.
$\Rightarrow$ electron momentum also changes by $\sim T$

$$\Delta E_e \sim \frac{T^2}{m_e} \Rightarrow \Delta E_\gamma \sim \frac{T^2}{m_e}$$

Thus to exchange energy of order T we need at least m_e/T scattering

Need $n_e \sigma c \frac{T}{m_e} > H$

$$n_e = n_p = .88 \times \frac{\Omega_B \rho_c}{m_p} \times \left(\frac{T}{T_0}\right)^3$$

$$= .88 \times \Omega_B . \frac{3 H_0^2}{8\pi G m_p} \left(\frac{T}{T_0}\right)^3 \qquad T_0 = 2.73\,K$$

$H_0 = 100 \times h$ Km/sec/Mpc

$$l.h.s. = 9 \times 10^{-29} \left(\frac{T}{T_0}\right)^4 \Omega_B h^2 / sec.$$

$$H = \sqrt{\frac{8\pi G}{3} \frac{1}{2} a_B T^4 \left(2 + 6 \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3}\right)}$$

$$= 2.1 \times 10^{-20} \left(\frac{T}{T_0}\right)^2 / sec.$$

$$\Rightarrow T_{freeze} = 1.5 \times 10^4 K \left(\Omega_B h^2\right)^{-1/2}$$

$$\Omega_B h^2 = .02 \Rightarrow T_{freeze} \sim 10^5 K.$$

Actually nothing special happens at T_{freeze}.

For $T > T_{freeze}$ the system is in equilibrium.

~~Constrained~~ Independent variables

a, T, n_e, n_p, n_{He}

Independent ~~...~~ equation:

Friedman eq: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{tot}$

$$\frac{d}{dt}(n_e a^3) = 0, \quad \frac{d}{dt}(n_p a^3) = 0$$

$$\frac{d}{dt}(n_{He} a^3) = 0$$

$$\frac{d}{dt}(s_{tot} a^3) = 0$$

$s_\gamma + \underbrace{s_e + s_p + s_{He}}$

negligible due to check small # of them.

$\frac{4}{3} a_B T^3$

$$\Rightarrow \frac{d}{dt}(T^3 a^3) = 0 \Rightarrow T \sim \frac{1}{a}$$

After γ freeze out:

$T \propto 1/a$ still (redshift).

$n_e, n_H, n_p \propto \frac{1}{a^3}$

What about ~~the~~ temperature of e, H, p system?

Rate of ~~...~~ collision per electron

$= n_\gamma \sigma c \to$ factor of 10^9 larger due to $n_\gamma / n_e \sim 10^9$.

Thus the rate of energy transfer to/from electrons is still large enough to keep them in thermal equilibrium at γ temperature T.

Same is true for p and He (extra suppression by $\frac{m_e}{m_p, m_{He}} \sim 10^{-3} - 10^{-4}$ but still large enough).

Thus even after freeze out the system evolves as if it is in equilibrium.

② Matter radiation equality revisited.

$$\frac{1}{2} a_B T^4 \left(2 + 6 \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3}\right)$$

$$= \Omega_m \cdot \frac{3 H_0^2}{8\pi G} \times \left(\frac{T}{T_0}\right)^3 \rightarrow \text{from } \left(\frac{a_0}{a}\right)^3$$

$$\Rightarrow T = 6.56 \times 10^4 \, K \, (\underbrace{\Omega_M h^2}_{.15})$$

$$\simeq 10^4 \, K$$

$$10^4 \, K \cong 10^4 \times 8.62 \times 10^{-5} \, eV = .86 \, eV$$

Recombination

~~The~~ Electron binding energy to He is larger. (Larger Z)

⇒ First electron gets bound to He. ~~[illegible]~~ ⇒ Has no drastic effect on the rest of the system.

There are still plenty of free electrons left to scatter γ even though energy exchange is small.

Important event: e^- binding to p. γ's stop since after that photons stop scattering altogether.

Consider a temperature where He has already formed atoms.

⇒ We have e^-, γ, p ~~[illegible]~~ ~~[illegible]~~ and (n l) states of H in thermal equilibrium.

$$e^- + p \rightleftarrows (n\,l).$$

Conserved charges: $n_e + \sum_{n,l} n_{(nl)}$

$$n_p + \sum_{n,l} n_{(n,l)}$$

$$\tilde{\mu}_1 \left(n_e + \sum_{(n,l)} n_{(n,l)}\right) + \tilde{\mu}_2 \left(n_p + \sum_{(n,l)} n_{(n,l)}\right)$$

$$\Rightarrow \quad \mu_e = \tilde{\mu}_1, \qquad \mu_p = \tilde{\mu}_2$$

$$\mu_{n,l} = \tilde{\mu}_1 + \tilde{\mu}_2 = \mu_e + \mu_p$$

Use general formula ~~for low density~~

$$n_\alpha = \frac{g_\alpha}{8\pi^3} \int_0^\infty 4\pi k^2\, dk \, \frac{1}{e^{\frac{m_\alpha + k^2/2m_\alpha}{T} - \frac{\mu_\alpha}{T}} \pm 1}$$

Ignore at low density.

Otherwise $n_\alpha \sim n_\gamma$

$\alpha = e,\ p,\ (n,l).$

$$n_\alpha = \frac{g_\alpha}{8\pi^3} \int_0^\infty 4\pi k^2\, dk \; e^{-\frac{(m_\alpha - \mu_\alpha)}{T} - \frac{k^2}{2m_\alpha T}}$$

$$= g_\alpha \, e^{-\frac{m_\alpha - \mu_\alpha}{T}} \left(\frac{m_\alpha T}{2\pi}\right)^{3/2}$$

$$g_e = 2 \qquad g_p = 2.$$

$$g_{(n,l)} = \underbrace{2}_{e\text{-spin}} \times \underbrace{2}_{p\text{-spin}} \times \underbrace{(2l+1)}_{\text{angular momentum}}$$

$$n_{(nl)} = 4(2l+1)\, e^{-\frac{m_{nl}}{T} + \frac{\mu_e + \mu_p}{T}} \left(\frac{m_{nl} T}{2\pi}\right)^{3/2}$$

$$n_e = 2\, e^{-m_e/T + \frac{\mu_e}{T}} \left(\frac{m_e T}{2\pi}\right)^{3/2}$$

$$n_p = 2\, e^{-\frac{m_p}{T} + \frac{\mu_p}{T}} \left(\frac{m_p T}{2\pi}\right)^{3/2}$$

$$\frac{n_{n\ell}}{n_e n_p} = (2\ell+1)\, e^{(m_e+m_p-m_{n\ell})/T} \left(\frac{m_e T}{2\pi}\right)^{-3/2} \times \left(\frac{m_{n\ell}}{m_p}\right)^{3/2}$$

$$\left(\frac{m_{n\ell}}{m_p}\right)^{3/2} \simeq 1$$

$$\simeq (2\ell+1)\, e^{B_{n\ell}/T} \left(\frac{m_e T}{2\pi}\right)^{-3/2}$$

$n_e = n_p$

$$n_{n\ell} = (2\ell+1)\, n_p^2\, e^{B_{n\ell}/T} \left(\frac{m_e T}{2\pi}\right)^{-3/2}$$

Note: $B_{1s} > B_{n\ell}$ by ~ 13.6 eV

$\Rightarrow$ For $T << 13.6$ eV only 1s states are present.

Define $\dfrac{n_p}{n_p + n_{1s}} = X$

$$(1-X) = X^2 (2\ell+1)(n_p + n_{1s})\, e^{B_{1s}/T} \left(\frac{m_e T}{2\pi}\right)^{-3/2}$$

$$(n_p + n_{1s}) = .88 \times \Omega_B \frac{3 H_0^2}{8\pi G m_p} \left(\frac{T}{T_0}\right)^3$$

$$(1-X) = X^2 S$$

$$S = 1.747 \times 10^{-22}\, e^{157894^{K}/T} \left(\frac{T}{1K}\right)^{3/2} \Omega_B h^2$$

$157894\,K = 13.6$ eV

Note small coefficient 1.747×10^{-22}

$\Rightarrow$ For $T \sim B_{1s}$, S is small.

$\Rightarrow X \simeq 1 \Rightarrow n_{1s}$ small

H-atom forms when T is sufficiently below $13.6\ eV = 157894\ K$.

In this case $e^{157894/T}$ is a rapidly varying fn.

$\Rightarrow$ H-atom formation should happen rapidly.

For $\Omega_B h^2 = .02$

T	X
4500	.998
4000	.900
3800	.634
3600	.290
3400	~~.280~~ .094
3200	.024

In actual practice this process is slower because equilibrium is not achieved.

Dominant mechanism:

$e + p \to 1S + \gamma$

This γ is energetic ~ 13.6 eV and cannot thermalize within Hubble time.

Eventually it hits an 1S state and ionizes it.

Even if it loses energy by Hubble expansion, it can raise it to excited state.

$\Rightarrow$ No net gain in 1S state.

Similarly transition from nl to 1S will eject energetic photon which will eventually excite another 1S state.

Dominant process

$2p \to 1S + \gamma$ loses energy so that it cannot excite $1S \to 2p$ any more.

$2s \to 1s + \gamma + \gamma$

slow — needed for parity + angular momentum

* Full set of equations:

$$\frac{d}{dt}\left(n_{n\ell}\, a^3\right) = \sum_\beta R(\beta \to n\ell) - \sum_\beta R(n\ell \to \beta)$$

total rates

Rates can be calculated from microscopic theory → functions of $n_{n\ell}$ (independent variables)

⇒ A set of differential equations

$(n\ell)$: run over ∞ no. of values.

→ ∞ no. of eqs. to be solved.

States with small energy difference can make transition into each other via thermal photons and hence can equilibriate.

For high n states we can assume thermal equilibrium.

Simplified assumption: All states with $n \geq 2$ are in thermal equilibrium with each other.

$$n_{n\ell} = (2\ell+1)\, n_{2s} \exp\left((B_2 - B_{n\ell})/T\right)$$

Model: 3 systems. (Independent parameters n_e, n_{1s}. ($\sum_{n\geq 2} n_{nl} = n - n_e - n_{1s}$) $\Rightarrow$ fixes n_{2s}.

① Free electrons.

② Thermal system of (nl) $n \geq 2$ States

③ 1s states.

The free electrons $\to$ system 2 $\to$ system 3.

Note: System 2 consists of dominantly 2s states.

n_e = ~~#~~ ~~of~~ density of free electrons

$$\frac{d}{dt}\left\{\left(n_e + \sum_{(n,l)} n_{nl} + n_{1s}\right) a^3\right\} = 0.$$

$$\frac{d}{dt}\left\{n_e a^3\right\} = \alpha(T)\, n_e a^3\, n_p - \beta(T)\, n_{2s}\, a^3$$

$n_p = n_e$

after expressing n_{nl} in terms of n_{2s}.

$a^3 \to \frac{1}{n}$

In equilibrium ~~[illegible]~~

$$n_{2s} = n_e^2 \left(\frac{m_e T}{2\pi}\right)^{-3/2} \exp(B_2/T)$$

$$\Rightarrow \quad \beta(T) = \alpha(T) \left(\frac{m_e T}{2\pi}\right)^{3/2} \exp(-B_2/T)$$

$$\frac{d}{dt}\left(\left(\sum_{n\ell} n_{n\ell}\right) a^3\right) = \left\{ \alpha\, n_e a^3 n_p - \beta\, n_{2s} a^3 \right\}$$

$$- \left\{ \left(\Gamma_{2s} + 3 P\, \Gamma_{2p} \right) n_{2s} - \varepsilon\, n_{1s} \right\} a^3$$

Γ_{2s}: transition probability from $2s \to 1s$

P: Probability of a $2p \to 1s$ γ to escape exciting 1s state to 2p state (Lyman α γ).

ε: Probability of exciting 1s to 2s or 2p except the ones by Lyman α γ

We assume this to be zero.

The individual rates are sufficiently high so that $\left(\sum_{n\ell} n_{n\ell}\right)$ has already reached the dynamical equilibrium value.

System ① → System ② → System ③

equal rate

$(n\ell) \to 1s$ transitions cancel the opposite transition.

(High energy photons are produced & needed)

This gives one relation between

n_{1s}, n_{2s}, n_e

Another relation $n_{1s} + n_{2s} + n_e = n$

total electron density.

$\Rightarrow$ one independent variable

(Take to be n_e or $X = n_e/n$)

$\Rightarrow$ a differential equation for X.

(Instead of two differential eq. in two variables n_e, n_{1s} we have one differential eq. in one variable).

$$\frac{\xi}{\Gamma_{2s} + 3P\Gamma_{2p}} = \left(\frac{n_{2s}}{n_{1s}}\right)_{\text{equilibrium}}$$

$$= \exp\left(-(B_1 - B_2)/T\right)$$

Γ_{2s}, Γ_{2p}, α are calculable from microscopic theory (or measured in the laboratory).

P is the only unknown.

Final result for $\frac{dn_e}{dt}$:

$X = n_e/n$

$$\frac{dX}{dt} = \frac{\Gamma_{2s} + 3P\Gamma_{2p}}{\Gamma_{2s} + 3P\Gamma_{2p} + \beta}\, \alpha n \left(-X^2 + S^{-1}(1-X)\right)$$

Calculation of P: Probability of escape of the Lyman α photon.

$$P(t) = \int_{-\infty}^{\infty} d\omega\, \mathcal{P}(\omega) \exp\left[-\int_t^{\infty} dt'\, n_{1s}(t')\, c\, \sigma(\omega a(t)/a(t'))\right]$$

$c = 1$

$\mathcal{P}(\omega)\, d\omega$ = Probability that the Lyman α photon has energy between $(\omega, \omega + d\omega)$

$$\mathcal{P}(\omega) = \frac{\Gamma_{2p}}{2\pi} \frac{1}{(\omega - \omega_\alpha)^2 + \Gamma_{2p}^2/4}$$

$\omega_\alpha = B_{1s} - B_{2p} \rightarrow$ mean energy.

$\sigma(\omega')$ = absorption cross section of a photon of energy ω' by 1s state.

$c\, n_{1s}(t')\, \sigma(\omega')$ = total probability of absorption at time t'.

Survival probability $g(t)$ of an initial photon of energy ω satisfies

$$\frac{dg}{dt'} = -c\, n_{1s}(t')\, \sigma\left(\omega \frac{a(t)}{a(t')}\right) g(t')$$

$\Rightarrow$ Total survival probability at the end

$$g^{(\infty)} = \exp\left(-\int_t^\infty dt'\, c\, n_{1s}(t')\, \sigma\left(\omega \frac{a(t)}{a(t')}\right)\right)$$

$$\sigma(\omega) = \frac{3}{2} \underbrace{\frac{2\pi^2 \Gamma_{2p}}{\omega_\alpha^2}} \phi(\omega)$$

$\rightarrow$ Breit-Wigner formula relating decay to absorption.

Physically we expect most of the absorption to take place around $t' = t$.

$\Rightarrow$ ~~Except for~~ (otherwise energy falls below the threshold)

$\Rightarrow$ pull out $n_{1s}(t')$ as $n_{1s}(t)$

σ is rapidly varying $\Rightarrow$ cannot replace $t' = t$ there.

$$a(t') = a(t)\left(1 + H(t)(t'-t)\right)$$

$$a(t)/a(t') = 1 - H(t)(t'-t)$$

$$\omega' = \omega \frac{a(t)}{a(t')} = \omega\left(1 - H(t)(t'-t)\right)$$

change variable from $t' \to \omega'$

$$\Rightarrow dt' = -\frac{d\omega'}{\omega H(t)}$$

$$\Rightarrow P(t) = \int_{-\infty}^{\infty} d\omega\, P(\omega)\, \exp\left[-\frac{3\pi^2 \Gamma_{2p}}{\omega_\alpha^2\, \omega H(t)} \int_{-\infty}^{\omega} d\omega'\, P(\omega')\right]$$

rapidly varying

slowly varying $\omega \to \omega_\alpha$.

rapidly varying

$$\Rightarrow P(t) = F\left(\frac{3\pi^2 \Gamma_{2p}\, n_{1s}(t)}{\omega_\alpha^3\, H(t)}\right)$$

$$F(x) = \frac{1 - e^{-x}}{x}$$

For $T < 6000K$, x large. ~~for~~ tested by ~~rotation~~ (can only be tested by calculating n_{1s})

$$F(x) \simeq 1/x \Rightarrow P = \frac{\omega_\alpha^3 H(t)}{3\pi^2 \Gamma_{2p}\, n_{1s}(t)}$$

We could also use the full expression.

$$3 P \Gamma_{2p} = \frac{\omega_\alpha^3 H}{\pi^2 n_{1s}(t)}$$

Some numbers:

$\Gamma_{2s} = 8.22458$ /sec.

Γ_{2p} not needed. (Need to check $x > 10$)

$\alpha = 2.84 \times 10^{-11}\, T^{-1/2}$ cm^3/sec.

(Simple model of electron capture rate)

More accurate estimate. (numerical)

$$\alpha = \frac{1.4377 \times 10^{-10}\, T^{-0.6166}}{1 + 5.085 \times 10^{-3}\, T^{0.5300}} \text{ cm}^3/\text{sec.}$$

Substitute & solve for X:

For H we need matter + radiation domination.

$\Omega_B h^2 = .02$

T	X
	.984
4226	.958
4090	.815
3818	.645
3545	.236
3136	.154 (Saha eq: .00491)
3000	

(Agrees with more elaborate calculation)

$e^- p$ recombination

Full analysis → requires setting up coupled differential eq for $n_{n\ell}$.

(Also Helium states)

→ done in Seager, Sasselov, Scott, astro-ph/9909275

(Uses about 300 level system)

Result: At low temperature ionization (# of free e^-) is 10% smaller than the simple minded calculation.

→ We shall continue with the simple minded calculation.

3 systems

① Free electrons

② Electrons in $n\ell$ state ($n \geq 2$) in equilibrium (dominated by $n=2$)

③ Electrons in 1s state

Variables n_e $\quad n_{2s}$ $\quad n_{1s}$

free electron density, determines $n_{n\ell}$ for $n \geq 2$

$$n_e + n_{1s} + \sum_{n,\ell\,(n\geq 2)} n_{n\ell} = n \quad \text{total \# density}$$

$n = C/a^3 = C' T^3 \to$ Known

$\Rightarrow$ Two variables. n_e, n_{1s}

$$\frac{d}{dt}\left(\frac{n_e}{n}\right) = -\frac{\alpha n_e^2}{n} + \beta \frac{n_{2s}}{n}$$

$$\frac{\beta}{\alpha} = \left(\frac{m_e T}{2\pi}\right)^{3/2} \exp\left(-\frac{B_2}{T}\right)$$

$$\frac{d}{dt}\left(\frac{n_{1s}}{n}\right) = \left(\Gamma_{2s} + 3P\Gamma_{2p}\right)\frac{n_{2s}}{n} - \varepsilon \frac{n_{1s}}{n}$$

$$\frac{\varepsilon}{\Gamma_{2s} + 3P\Gamma_{2p}} = \exp\left(-\frac{B_1 - B_2}{T}\right)$$

We could solve this coupled differential eq. but we simplify further by assuming

$$\frac{d}{dt}\left(\frac{n_e}{n}\right) = -\frac{d}{dt}\left(\frac{n_{1s}}{n}\right)$$

$\left(\sum_{n \geq 2, l} n_{nl} / n\right.$ remains almost constant$)$

$$\frac{\alpha n_e^2}{n} - \beta\frac{n_{2s}}{n} = \left(\Gamma_{2s} + 3P\Gamma_{2p}\right)\frac{n_{2s}}{n} - \varepsilon\frac{n_{1s}}{n}$$

$\to$ use it to eliminate n_{2s}.

$$n_e + n_{1s} + \sum_{\substack{n,l \\ n \geq 2}} n_{nl} = n$$

$$\Downarrow \simeq n_{2s} + 3n_{2p} \simeq 4 n_{2s}$$

Ex. $n_{2s} = \dfrac{\alpha n_e^2 + \xi(n - n_e)}{\Gamma_{2s} + 3P\Gamma_{2p} + \beta + 4\xi}$

substitute

$$\Rightarrow \frac{d}{dt}\left(\frac{n_e}{n}\right) = \frac{\Gamma_{2s} + 3P\Gamma_{2p}}{(\Gamma_{2s} + 3P\Gamma_{2p})\left(1 + e^{-\frac{B_1 - B_2}{T}}\right) + \beta}$$

$$\times \left\{ -\left(1 + e^{-\frac{B_1-B_2}{T}}\right)\frac{\alpha n_e^2}{n} + e^{-\frac{B_1-B_2}{T}}\frac{\beta(n - n_e)}{n} \right\}$$

neglect these at low T.

$$\simeq \frac{\Gamma_{2s} + 3P\Gamma_{2p}}{\Gamma_{2s} + 3P\Gamma_{2p} + \beta}\left\{ -\frac{\alpha n_e^2}{n} + e^{-\frac{B_1-B_2}{T}}\frac{\beta(n - n_e)}{n} \right\}$$

$X = n_e/n$.

$$\frac{dX}{dt} = \frac{\Gamma_{2s} + 3P\Gamma_{2p}}{\Gamma_{2s} + 3P\Gamma_{2p} + \beta}\,\alpha n\left\{ -X^2 + S^{-1}(1 - X) \right\}$$

$$S = n\left(\frac{m_e T}{2\pi}\right)^{-3/2}\exp\left(\frac{B_1}{T}\right)$$

(used result for β/α.)

Γ_{2s}, Γ_{2p}: Decay constants from $2s, 2p \to 1s$

α: & Capture ~~...~~ ~~section~~ rate/electron

P: probability that a Lyman α photon escapes absorption by a 1S state.
$\to$ Need to be calculated.

$p(\omega)\, d\omega$: Probability of a Lyman α photon carrying energy between $(\omega, \omega + d\omega)$

$$p(\omega) = \frac{\Gamma_{2p}}{2\pi} \frac{1}{(\omega - \omega_\alpha)^2 + \Gamma_{2p}^2/4}$$

$$\int_0^\infty p(\omega)\, d\omega = 1 \qquad \Gamma_{2p} \ll \omega_\alpha$$

ω_α: $(B_1 - B_2) \to$ central value.

$\sigma(\omega)$: absorption x-section of a photon of energy ω by a 1S state $\to$ 2p state

$$\sigma(\omega) = \frac{3}{2} \frac{2\pi^2 \Gamma_{2p}}{\omega_\alpha^2} \, p(\omega) \to \text{Breit-Wigner formula}$$

Now consider a Lyman α photon of energy ω emitted at time t.
$P(t')$: probability that it escapes capture by 1S Hydrogen up to time t'.

$$\frac{d}{dt'} P(t') = - P(t')\, n_{1s}(t')\, \sigma\left(\omega \frac{a(t)}{a(t')}\right)$$

$$P(\infty) = \exp\left[- \int_t^\infty dt'\, n_{1s}(t')\, \sigma\left(\omega \frac{a(t)}{a(t')}\right)\right]$$

$\Rightarrow$ Total survival probability of a Lyman α photon emitted at time t:

$$P = \int_0^\infty p(\omega)\, d\omega\, \exp\left[- \int_t^\infty dt'\, n_{1s}(t')\, \sigma\left(\omega \frac{a(t)}{a(t')}\right)\right]$$

We now make approximations to simplify this integral.

Most of the absorption will be at $t' \simeq t$.

(otherwise $\omega \frac{a(t)}{a(t')}$ becomes $\ll \omega_\alpha - \Gamma_{2p}$ and σ drops).

$\Rightarrow$ We can replace t' by t in slowly varying functions of t.

$$n_{1s}(t') \to n_{1s}(t)$$

$$\frac{a(t)}{a(t')} = \frac{1}{1 + (t'-t)\, H(t) + \cdots} = 1 - (t'-t)\, H(t) + \cdots$$

ignore

$$\omega' \equiv \omega \cdot \frac{a(t)}{a(t')} = \omega \{ 1 - (t'-t)\, H(t) \}$$

Trade in integration variable t' by ω'.

$$d\omega' = -\,\omega H(t)\, dt'$$

$\omega' = \omega$ at $t' = t$, $\omega' = 0$ at $t' = \infty$.

$$P = \int_0^\infty p(\omega)\, d\omega \exp\left(-\frac{n_{1s}(t)}{H(t)\,\omega} \int_0^\omega \sigma(\omega')\right)$$

$$\sigma(\omega') = \frac{3\pi^2 \Gamma_{2p}}{\omega_\alpha^2}\, p(\omega')$$

$$= \int_{-\infty}^\infty p(\omega)\, d\omega \exp\left[-\frac{3\pi^2 \Gamma_{2p}\, n_{1s}(t)}{H(t)\,\omega\,\omega_\alpha^2} \int_0^\omega p(\omega')\, d\omega'\right]$$

replace by ω_α

$$u = \int_0^\omega p(\omega')\, d\omega'$$

$$du = p(\omega)\, d\omega$$

$u = 0$ at $\omega = 0$, $u = 1$ at $\omega = \infty$

$$\Rightarrow P = \int_0^1 du \exp(-c\,u) \quad \Big/ \quad c = \frac{3\pi^2 \Gamma_{2p}\, n_{1s}(t)}{H(t)\,\omega_\alpha^3}$$

$$= \frac{1}{c}\left(1 - e^{-c}\right)$$

Some numbers

$\Gamma_{2p} = 4.699 \times 10^{8}$ /sec

$\Gamma_{2s} = 8.22458$ /sec.

$\alpha = 2.84 \times 10^{-11}\, T^{-1/2}\, cm^3/s$ (simple minded capture)

More detailed calculation fits with

$$\alpha = \frac{1.4377 \times 10^{-10}\, T^{-.6166}\, cm^3/s}{1 + 5.085 \times 10^{-3}\, T^{0.5300}}$$

Using this we can calculate $X(t)$ by solving the differential equation.

$$\frac{dX}{dT} = \frac{dX}{dt}\left(\frac{dT}{dt}\right)^{-1} = -(TH)^{-1}\frac{dX}{dt}$$

For $\Omega_B h^2 = .02$

z	T	X	X_{saha}
1550	4226	.992	
	3818	.861	
	3409	.526	
	3000	.205 →	.00491
	2591	.0405	
	2183	.00662	
		⋮	

Opacity at temperature T:

$O(T)$ = probability that a photon emitted at time $t(T)$ will undergo at least one scattering before today

~~$\frac{dO(T)}{dt'}$~~

$S(t')$: Probability for no scattering till time t'.

$$\frac{dS(t')}{dt'} = -c\,\sigma_T\, n_e(t')$$

σ_T: Thomson scattering x-section

$$\Rightarrow S(t_0) = \exp\left(-\int_t^{t_0} c\,\sigma_T\, n_e(t')\,dt'\right)$$

$$O(T) = 1 - S(t_0)$$
$$= 1 - \exp\left(-\int_t^{t_0} c\,\sigma_T\, n_e(t')\,dt'\right)$$

$$dt' = \frac{dT'}{T'H'}$$

$$= 1 - \exp\left(-\sigma_T \int_{T_0}^{T} \frac{n_e(T')}{H'T'}\,dT'\right)$$

$1-O(T)$ = probability that the last scattering of a CMB photon was before temperature T.

$1-O(T)$ fraction of the photons at temperature T reaches us ~~today~~, without scattering.

$O'(T)\,dT$: probability that the last scattering was between T, $T+dT$.

→ can be calculated.

$O'(T)$ is peaked at T_L with standard deviation σ

2954 253

for $\Omega_B h^2 = .02$