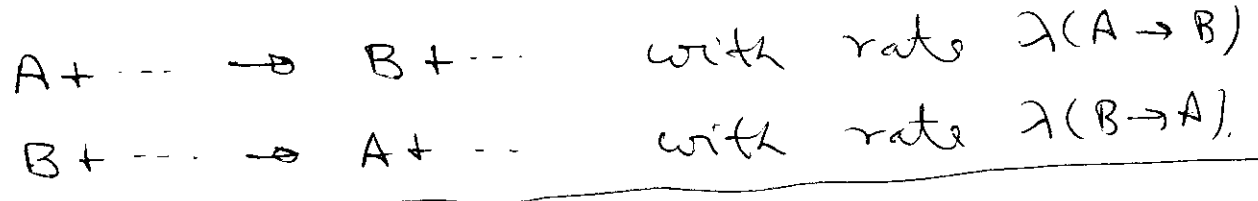


A general result

Suppose we have two kinds of particles A, B.



Suppose further that these are the only processes which change # of A and B.

In equilibrium:

$$n_A^{EQ} \lambda(A \rightarrow B) = n_B^{EQ} \lambda(B \rightarrow A)$$

$$\Rightarrow \frac{n_A^{EQ}}{n_B^{EQ}} = \frac{\lambda(B \rightarrow A)}{\lambda(A \rightarrow B)}$$

microscopic results, do not depend on whether or not we have equilibrium.

If we have a system with

$$\frac{n_A}{n_B} < \frac{n_A^{EQ}}{n_B^{EQ}} = \frac{\lambda(B \rightarrow A)}{\lambda(A \rightarrow B)}$$

$$\Rightarrow n_A \lambda(A \rightarrow B) < n_B \lambda(B \rightarrow A)$$

\Rightarrow Net conversion of B \rightarrow A till we reach equilibrium.

\rightarrow takes time $\sim 1/\lambda(B \rightarrow A)$.

Examine expected equilibrium abundance at $T = 10^9$ K.

$$Y_d \sim Y_n Y_p 10^{-4}$$

$$Y_{H^3} \sim Y_n^2 Y_p 10^{12}$$

$$Y_{He^3} \sim Y_n Y_p^2 10^9$$

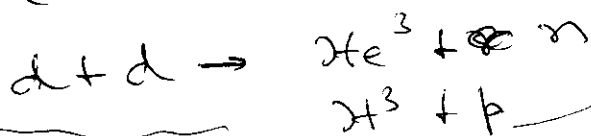
$$Y_{He^4} \sim Y_n^2 Y_p^2 10^{97}$$

For $Y_n, Y_p \sim 1$ we expect He reaction rates to be such that we build up H^3, He^3, He^4 from n, p .

Problem: This has to proceed via d and Y_d is still small at 10^9 K.

The ~~reaction~~ reaction rate $\lambda(p+n \rightarrow d+\gamma)$ is large enough to build d to its equilibrium value at $T \sim 10^9$ K.

Even though Y_d is small,



is large enough to convert $1d$ to He^3 or H^3 .

withing a Hubble time.

(Meaning of $\frac{\Lambda}{H} \sim 1$ at $T = 10^9 K$)

This increases sharply as $T \downarrow$

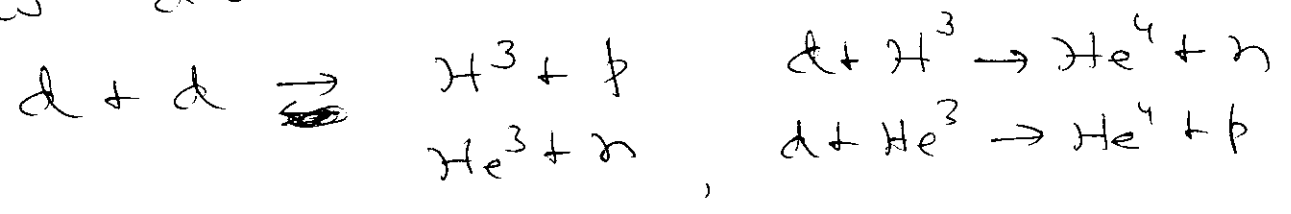
\Rightarrow ~~total~~ d 's produced from $n + p \rightarrow d + \bar{\nu}$ start getting converted to He^3 , H^3 & then to He^4 .

As this reduces the density of d 's, more n & p get converted to d . Since

$$\lambda(n \rightarrow d) n_n \gg \lambda(d \rightarrow n) n_d.$$

This process continues till we run out of n .

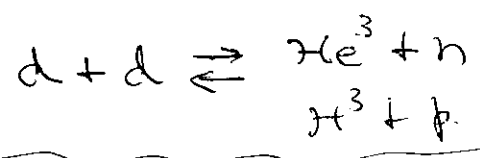
Once we run out of n , no new d 's are produced.



Continues.

When does this stop?

When these reaction rates fall below the Hubble expansion rate.



Total rate ~~per d~~ per d = $\sigma v n_d \equiv \Lambda$

- ~~H~~ Hubble expansion rate H .

$\frac{\Lambda}{H} \lesssim 1$ stops the process.

→ Same eq. as before.

Earlier we used it to find T at which nucleosynthesis begins.

Now we shall use it to determine n_d at which nucleosynthesis stops.

$$\sigma = \frac{1}{6} \times (1.8 + 1.6) \times 10^{-17} \text{ cm}^3/\text{sec}$$

$$\Rightarrow \sigma v n_d = 3.4 \times 10^{-17} \text{ cm}^3/\text{sec} \times n_d$$

Compare with H .

$$H = \sqrt{\frac{8\pi G \rho}{3}} = \sqrt{\frac{8\pi G}{3} \frac{1}{2} a_B T^4 \left(2 + \frac{6}{8} \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3}\right)}$$

$\parallel \frac{\pi^2}{15}$

$$\approx 0.28 \left(\frac{T}{10^{10} \text{ K}}\right)^2 \text{ sec}^{-1} = 0.28 \times 10^{-2} \text{ sec}^{-1}$$

for $T \sim 10^9 \text{ K}$

(127)

Thus ~~the~~ nucleosynthesis stops at

$$n_d = -28 \times 10^{-2} \times \frac{1}{3.4 \times 10^{-17}} / \text{cm}^3$$

Compare with total no. of nucleons at that time n_N . $H_0 = \sqrt{\frac{8\pi G}{3} \rho_c}$

$$\begin{aligned} n_N &= \frac{\rho_{B0}}{m_p} \times \left(\frac{a_0}{a}\right)^3 \\ &= \frac{1}{m_p} \Omega_B \times \sqrt{\frac{3 H_0^2}{8\pi G}} \times \left(\frac{T}{T_0}\right)^3 \end{aligned}$$

$$H_0 = h \times 100 \text{ km/sec/Mpc}$$

$$\text{'' } 10^9 \text{ cm}$$

$$\text{'' } 3 \times 10^{18} \times 10^6 \text{ cm}$$

$$= h \frac{1}{3} \times 10^{-19} / \text{sec}$$

$$n_N = \Omega_B h^2 \times \frac{1}{9} \times 10^{-34} \times \frac{3}{8\pi G m_p} \times \left(\frac{10^9}{2.73}\right)^3 \text{ sec}^{-2}$$

$$\frac{n_d}{n_N} = (\Omega_B h^2)^{-1} \times 9 \times 10^{34} \times \frac{8\pi G m_p}{3} \left(\frac{2.73}{10^9}\right)^3$$

$$\times \frac{.28}{3.4} \times 10^{15} \text{ sec}^2 / \text{cm}^3$$

Either use the values of G & m_p in usual units.

Or set G=1 & use natural units.

$$m_p = 1 \text{ GeV} = (1.22 \times 10^{19}), \text{ sec} = \frac{10^{49}}{5.39}, \text{ cm} = \frac{10^{33}}{1.02}$$

Result: $\frac{n_d}{n_H} = 1.47 \times 10^{-5} (\Omega_b h^2)^{-1}$

no ~~discrepancy~~ agrees with actual numerical results for order of magnitude.

Observations from quasi stellar objects

$$n_d/n_H \sim (2.78 \pm 0.4) \times 10^{-5}$$

$$\Rightarrow \Omega_B h^2 = 0.0214 \pm 0.0029$$

Note: Deuterium abundance ~~is~~ is sensitive to Ω_b .

→ agrees with other measurement of Ω_b e.g. CMBR.

(Direct observation gives low value)

He⁴ measurement from "region (of ionized hydrogen) with low abundance of elements other than H, He

⇒ indicates that this region has not been polluted by emission from stars.

Expect similar abundances of H^3 and He^3 since X-sections are similar.

H^3 decays to He^3 via β -decay.

\Rightarrow ~~Primordial~~ He^3 abundance = (sum of primordial H^3 and He^3 abundance)

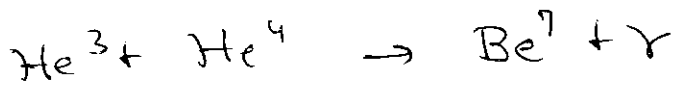
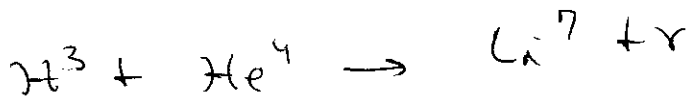
\rightarrow can be calculated numerically.

Detailed analysis:

$$n_{He^3}/n_H \sim (1.1 \pm 0.2) \times 10^{-5} \text{ for } \Omega_b h^2 = 0.0219$$

Heavier elements.

Some Li^7 produced by



$\hookrightarrow Li^7 + \nu_e$
electron capture

$$Li^7/H \sim 10^{-10}$$

Li^6 is rarer. $\sim 10^{-13}$

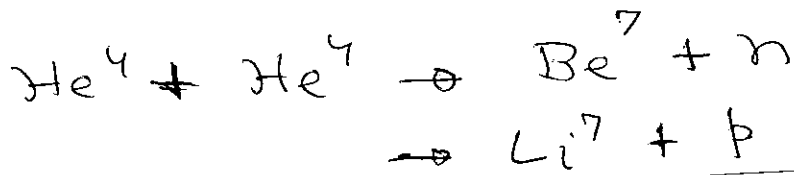
Note: ~~are~~ Sure there are plenty of H^3 and He^4 their reaction rate could be substantial.



would have been possible.

But there are no stable nuclei with these properties.

Strong interactions & processes are prevented by binding energy.



Binding energy:

Left: 2×28.3

Right: 39 for Li^7

The agreement of nucleosynthesis with CMB constraints on any change in the state of the universe at 10^9 K from what we have used.

e.g. extra matter \Rightarrow ~~extra~~ larger H .

\Rightarrow faster expansion rate.

\Rightarrow 10^9 K is reached at less time
 \Rightarrow larger $X_n \Rightarrow$ larger He^4/H ratio.

Using this we can put bound on
of neutrinos ≤ 4 .

μ_{ν_i} would also increase if μ_e & μ_p .
 \Rightarrow puts bound on μ_{ν_i} .

However there is one effect which
can occur in opposite direction.

High T equilibrium situation:

$$\frac{n_n}{n_p} = e^{-\left(\frac{\mu_n - \mu_p}{T}\right) + \frac{\mu_n - \mu_p}{T}}$$

$$\mu_p = \mu_n + \mu_{\nu_e} - \mu_e$$

$$\Rightarrow \frac{n_n}{n_p} = e^{-\frac{\mu_e}{T}} e^{\frac{\mu_n - \mu_{\nu_e}}{T}}$$

μ_e/T is small.

μ_{ν_e}/T positive. \Rightarrow reduced $\frac{n_n}{n_p}$.

Thus it may be possible to
allow larger μ_{ν_e}/T and more matter
so that X_n remains unchanged.

\rightarrow requires coincidence.

Possible but unlikely.

We now move forward in time.

Some important events

- ① Photons go out of equilibrium
- ② Transition from radiation to matter dominated universe.
- ③ Recombination $\rightarrow e^-$ get bound to nuclei and photons stop scattering \rightarrow origin of CMBR.

Photon electron scattering

$$\sigma = 66525 \times 10^{-24} \text{ cm}^2$$

Rate of scattering

$$\lambda = n_e \sigma c$$

Electrons are massive.

In each scattering photon momentum changes by $\sim kT$.

\Rightarrow electron momentum also changes by

$$\sim kT$$

$$\Delta E_e \sim \frac{T^2}{m_e} \Rightarrow \Delta E_r \sim \frac{T^2}{m_e}$$

Thus to exchange energy of order T we need at least $\frac{m_e T}{kT}$ scattering of ~~the~~

Need $n_e \sigma_e \frac{T}{m_e} > H$

$$n_e = n_p = -88 \times \frac{\Omega_B \rho_c}{m_p} \times \left(\frac{T}{T_0}\right)^3$$

$$= -88 \times \Omega_B \cdot \frac{3 H_0^2}{8 \pi G m_p} \left(\frac{T}{T_0}\right)^3 \quad T_0 = 2.73 \text{ K}$$

$$H_0 = 100 h \text{ Km} / \text{sec} / 8 \text{ Mpc}$$

$$\text{L.H.S.} = 9 \times 10^{-29} \left(\frac{T}{T_0}\right)^4 \Omega_B h^2 / \text{sec.}$$

$$H = \sqrt{\frac{8 \pi G}{3} \frac{1}{2} a_B T^4 \left(2 + 6 \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3}\right)}$$

$$= 2.1 \times 10^{-20} \left(\frac{T}{T_0}\right)^2 / \text{sec.}$$

$$\Rightarrow T_{\text{freeze}} = 1.5 \times 10^4 \text{ K} (\Omega_B h^2)^{-1/2}$$

$$\Omega_B h^2 = .02 \Rightarrow T_{\text{freeze}} \sim 10^5 \text{ K.}$$

Actually nothing special happens at T_{freeze} .

For $T > T_{\text{freeze}}$ the system is in equilibrium.

~~connected~~ Independent variables.

- a, T, n_e, n_p, n_{He}

Independent ~~var~~ equation:

Friedman eq: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_{tot}$

$$\frac{d}{dt} (n_e a^3) = 0, \quad \frac{d}{dt} (n_p a^3) = 0$$

$$\frac{d}{dt} (n_{He} a^3) = 0.$$

$$\frac{d}{dt} (\rho_{tot} a^3) = 0$$

↓

$$\rho_r + \rho_e + \rho_p + \rho_{He}$$

negligible due to check
small # of them.

$$\frac{4}{3} a_B T^3$$

$$\Rightarrow \frac{d}{dt} (T^3 a^3) = 0 \Rightarrow T \sim \frac{1}{a}$$

After ~~the~~ freeze out:

$$T \propto 1/a \quad \text{still (redshift)}$$

$$n_e, n_H, n_p \propto \frac{1}{a^3}$$

What about the temperature of

e, H, p system?

Rate of ~~collision~~ collision per electron

$\propto n_p n_e \rightarrow$ factor of 10^9 larger due to $n_p/n_e \sim 10^9$.

Thus the rate of energy transfer to/ from electrons is still large enough to keep them in thermal equilibrium at γ temperature T .

Same is true for p and He (extra suppression by $\frac{m_e}{m_p, m_{He}} \sim 10^{-3} - 10^{-4}$ but still large enough).

Thus even after freeze out the system evolves as if it is in equilibrium.

② Matter radiation equality revisited.

$$\frac{1}{2} a_B T^4 \left(2 + 6 \times \frac{7}{8} \times \left(\frac{4}{11} \right)^{4/3} \right)$$

$$= \Omega_m \cdot \frac{3 H_0^2}{8 \pi G} \times \left(\frac{T}{T_0} \right)^3 \rightarrow \text{from } \left(\frac{a_0}{a} \right)^3$$

$$\Rightarrow T = 656 \times 10^4 \text{ K } (\Omega_m h^2)^{1/3}$$

$$\approx 10^4 \text{ K}$$

$$10^4 \text{ K} \approx 10^4 \times 8.62 \times 10^{-5} \text{ eV} = 0.86 \text{ eV}$$

~~the~~

Recombination

The electron binding energy to He is larger. (Larger Ξ)

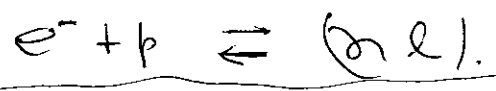
\Rightarrow First electron gets bound to He.
~~which has a focus~~ \rightarrow Has no. drastic effect on the rest of the system.

There are still plenty of free electrons left to scatter γ even though energy exchange is small.

Important event: e^- binding to p since after that photons stop scattering altogether.

Consider a temperature where He has already formed atoms.

\Rightarrow We have e^- , γ , p ~~in~~ and (n,l) states of H in thermal equilibrium.



Conserved charges:

$$n_e + \sum_{n,l} n_{(n,l)}$$
$$n_p + \sum_{n,l} n_{(n,l)}$$

$$\tilde{\mu}_1 (n_e + \sum_{(n,l)} n_{(n,l)}) + \tilde{\mu}_2 (n_p + \sum_{(n,l)} n_{(n,l)})$$

$$\Rightarrow \mu_e = \tilde{\mu}_1, \quad \mu_p = \tilde{\mu}_2$$

$$\mu_{n,l} = \tilde{\mu}_1 + \tilde{\mu}_2 = \mu_e + \mu_p$$

Use general formula ~~for low density~~

$$n_\alpha = \frac{g_\alpha}{8\pi^3} \int_0^\infty 4\pi k^2 dk \frac{1}{e^{\frac{m_\alpha + k^2/2m_\alpha - \mu_\alpha}{T}} \pm 1}$$

Otherwise $n_\alpha \sim n_r$

Ignore at low density.

$\alpha = e, p, (n,l)$.

$$n_\alpha = \frac{g_\alpha}{8\pi^3} \int_0^\infty 4\pi k^2 dk e^{-\frac{(m_\alpha - \mu_\alpha)}{T} - \frac{k^2}{2m_\alpha T}}$$

$$= g_\alpha e^{-\frac{m_\alpha - \mu_\alpha}{T}} \left(\frac{2m_\alpha T}{2\pi} \right)^{3/2}$$

$$g_e = 2 \quad g_p = 2$$

$$g_{(n,l)} = \underbrace{2}_{e\text{-spin}} \times \underbrace{2}_{p\text{-spin}} \times \underbrace{(2l+1)}_{\text{angular momentum}}$$

$$n_{(n,l)} = 4(2l+1) e^{-\frac{m_{n,l} - \mu_{n,l}}{T}} \left(\frac{2m_{n,l} T}{2\pi} \right)^{3/2}$$

$$n_e = 2 e^{-\frac{m_e - \mu_e}{T}} \left(\frac{m_e T}{2\pi} \right)^{3/2}$$

$$n_p = 2 e^{-\frac{m_p - \mu_p}{T}} \left(\frac{m_p T}{2\pi} \right)^{3/2}$$

$$\frac{n_{nl}}{n_e n_p} = (2l+1) e^{(m_e + m_p - m_{nl})/T} \left(\frac{m_e T}{2\pi} \right)^{-3/2}$$

$$\approx (2l+1) e^{B_{nl}/T} \left(\frac{m_e T}{2\pi} \right)^{-3/2} \times \left(\frac{m_{nl}}{m_p} \right)^{3/2} \approx 1$$

$n_e = n_p$

$$n_{nl} = (2l+1) n_p^2 e^{B_{nl}/T} \left(\frac{m_e T}{2\pi} \right)^{-3/2}$$

Note: $B_{nl} > B_{nl}$ by $\sim 13.6 \text{ eV}$

\Rightarrow For $T < 13.6 \text{ eV}$ only 1 s

states are present.

Define $\frac{n_{nl}}{n_p + n_{nl}} = X$

$$(1-X) = X^2 (2l+1) (n_p + n_{nl}) e^{B_{nl}/T} \left(\frac{m_e T}{2\pi} \right)^{-3/2}$$

$$\approx 88 \times \Omega_B \frac{3 H_0^2}{8\pi G m_p} \left(\frac{T}{T_0} \right)^3$$

$$(1-X) = X^2 S$$

$$S = 1.747 \times 10^{-22} e^{157894/T} \left(\frac{T}{1\text{K}} \right)^{3/2} \Omega_B h^2$$

13.6 eV

Note small coefficient 1.749×10^{-21}

\Rightarrow For $T \sim B_{15}$, S is small.

$\Rightarrow X \approx 1 \Rightarrow n_{15}$ small

H-atom forms when T is sufficiently below $13.6 \text{ eV} = 157894 \text{ K}$.

In this case $e^{157894/T}$ is a rapidly varying fr.

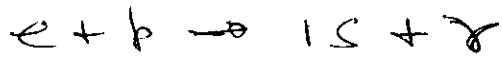
\Rightarrow ~~the~~ H-~~is~~ atom formation should happen rapidly.

For $\Omega_B h^2 = -0.2$

T	X
4500	.998
4000	.900
3800	.634
3600	.296
3400	.296 -0.094
3200	.024

In actual practice this process is slower because equilibrium is not achieved.

Dominant mechanism:



This γ is energetic ~ 13.6 eV and cannot thermalize within Hubble time.

Eventually it hits an 1s state and ionizes it.

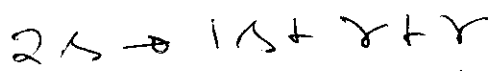
Even if it loses energy by Hubble expansion, it can raise it to excited state.

\Rightarrow No net gain in 1s state.

Similarly transition from $n \geq 2$ to 1s will eject energetic photon which will eventually excite another 1s state.

~~On~~ Dominant process

$2p \rightarrow 1s + \gamma$ loses energy so that it cannot excite $1s \rightarrow 2p$ any more.



slow / needed for parity & angular momentum

(141)

* Full set of equations:

$$\frac{d(n_{nl}^{a^3})}{dt} = \sum_{\beta} R(\beta \rightarrow nl) - \sum_{\beta} R(nl \rightarrow \beta)$$

↓
total rates

Rates can be calculated from microscopic theory \rightarrow functions of n_{nl} (independent variables)

\Rightarrow A set of differential equations

(nl) : run over ∞ no. of values.

$\rightarrow \infty$ no. of eqs. to be solved.

States with small energy difference can make transition into each other via thermal photons and hence can equilibrate.

For high n states we can assume thermal equilibrium.

Simplified assumption: All states with $n \geq 2$ are in thermal equilibrium with each other.

$$n_{nl} = (2l+1) n_{2s} \exp((B_2 - B_{nl})/T)$$

independent parameters
 n_e, n_{2s}
 $(\sum n_{nl} = n - n_e - n_{1s})$
 $n_{2s} \Rightarrow$ fixes n_{2s} .

Model: 3 systems.

- ① Free electrons.
- ② Thermal system of (nl) $n \geq 2$ states
- ③ $1s$ states.

The free electrons \rightarrow system 2 \rightarrow system 3.

Note: System 2 consists of dominantly $2s$ states.

$n_e =$ density of free electrons

$$\frac{d}{dt} \{ (n_e + \sum_{(nl)} n_{nl} + n_{1s}) a^3 \} = 0.$$

$$\frac{d}{dt} \{ n_e a^3 \} = \alpha(T) n_e a^3 n_p = n_e$$

$$- \beta(T) n_{2s} a^3$$

after expressing n_{nl} in terms of n_{2s} .

$$a^3 \rightarrow \frac{1}{n}$$

In equilibrium ~~condition~~

$$n_{2s} = n_e^2 \left(\frac{m_e T}{2\pi} \right)^{-3/2} \exp(B_2/T)$$

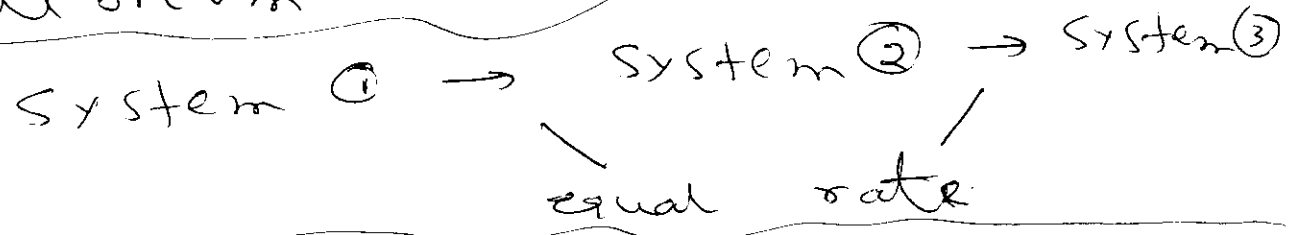
$$\Rightarrow \beta(T) = \alpha(T) \left(\frac{m_e T}{2\pi} \right)^{3/2} \exp(-B_2/T)$$

$$\frac{d}{dt} \left(\sum_{ne} n_{ne} a^3 \right) = \left\{ \alpha n_e a^3 n_p - \beta n_{2s} a^3 \right\}$$

$-\left\{ (\Gamma_{2s} + 3P \Gamma_{2p}) n_{2s} - \epsilon n_{1s} \right\} a^3$
 transition probability from $2s \rightarrow 1s$
 Probability of exciting $1s$ state to $2p$ state (Lyman α γ).
 Probability of exciting $1s$ to $2s$ or $2p$ except the ones by γ .
 a $2p \rightarrow 1s$ γ to escape

We assume this to be zero.

The individual rates are sufficiently high so that $\left(\sum_{ne} n_{ne} \right)$ has already reached the dynamical equilibrium value.



$n_{e1} \rightarrow 1s$ transitions equal the opposite transition.

(High energy photo ions are produced & needed)

This gives one relation between

n_{1s}, n_{2s}, n_e

Another relation $n_{1s} + n_{2s} + n_e = n$
total electron density,

⇒ one independent variable.

(Take to be n_e or $X = n_e/n$)

⇒ a differential equation for X.

(Instead of two differential eq. in two variables n_e, n_{1s} we have one differential eq. in one variable)

$$\frac{\xi}{n_{2s} + 3P n_{2p}} = \left(\frac{n_{2s}}{n_{1s}} \right)_{\text{equilibrium}} = \exp(- (B_1 - B_2) / T)$$

n_{2s}, n_{2p}, α are calculable from microscopic theory (or measured in the laboratory).

P is the only unknown.

Final result for $\frac{dn_e}{dt}$:

$$X \equiv n_e/n$$

$$\frac{dX}{dt} = \frac{\Gamma_{2s} + 3P\Gamma_{2p}}{\Gamma_{2s} + 3P\Gamma_{2p} + \beta} \alpha n (-X^2 + S^{-1}(1-X))$$

Calculation of P : Probability of escape of the Lyman α photon.

$$P(t) = \int_{-\infty}^{\infty} d\omega P(\omega) \exp\left[-\int_t^{\infty} dt' n_{1s}(t')\right]$$

$$= e^{-\sigma(\omega) n_{1s}(t)/a(t)}$$

$P(\omega) d\omega$ = Probability that the Lyman α photon has energy between $(\omega, \omega+d\omega)$

$$P(\omega) = \frac{\Gamma_{2p}}{2\pi} \frac{1}{(\omega - \omega_\alpha)^2 + \Gamma_{2p}^2/4}$$

$$\omega_\alpha = B_{1s} - B_{2p} \rightarrow \text{mean energy.}$$

~~$n_{1s}(t')$~~ $\sigma(\omega')$ = absorption cross section of a photon of energy ω' by 1s state.

$e n_{1s}(t') \sigma(\omega')$ = total probability of absorption at time t' .

(146) of an initial photon of survival probability $g(t)$ satisfies energy ω

$$\frac{dg}{dt'} = -c n_{is}(t') \sigma\left(\omega \frac{a(t)}{a(t')}\right) g(t')$$

\Rightarrow Total survival probability at the end

$$g^{(\infty)} = \exp\left(-\int_t^\infty dt' c n_{is}(t') \sigma\left(\omega \frac{a(t)}{a(t')}\right)\right)$$

$$\sigma(\omega) = \frac{3}{2} \frac{2\pi^2 P_{2p}}{\omega_\alpha^2} F(\omega)$$

\rightarrow Breit-Wigner formula relating decay to absorption.

Physically we expect most of the absorption to take place around $t'=t$.

\Rightarrow ~~except for~~ (otherwise energy falls below the threshold)

\Rightarrow pull out $n_{is}(t')$ as $n_{is}(t)$

σ is rapidly varying \Rightarrow cannot replace $t'=t$ there.

$$a(t') = a(t) (1 + \mathcal{H}(t)(t'-t))$$

$$a(t)/a(t') = 1 - \mathcal{H}(t)(t'-t)$$

$$\omega' = \omega \frac{a(t)}{a(t')} = \omega (1 - H(t)(t' - t))$$

change variable from $t' \rightarrow \omega'$

$$\Rightarrow dt' = \frac{d\omega'}{\omega H(t)}$$

$$\Rightarrow P(t) = \int_{-\infty}^{\infty} d\omega P(\omega) \exp\left[-\frac{3\pi^2 P_{2p}}{\omega_x^2 \omega H(t)}\right]$$

rapidly varying
slowly varying $\omega \rightarrow \omega_x$
rapidly varying

$\int_{-\infty}^{\infty} d\omega' P(\omega')$

$$\Rightarrow P(t) = F\left(\frac{3\pi^2 P_{2p} n_{is}(t)}{\omega_x^3 H(t)}\right)$$

$$F(x) = \frac{1 - e^{-x}}{x}$$

For $\pi < 6000k$, x large. ~~for~~
~~rotation~~ (can only be tested by
 calculating n_{is})

$$F(x) \approx 1/x \Rightarrow P = \frac{\omega_x^3 H(t)}{3\pi^2 P_{2p} n_{is}(t)}$$

~~We~~ We could also use the full expression.

$$3 P P_{2p} = \frac{\omega_x^3 H}{\pi^2 n_{is}(t)}$$

Some numbers:

$$\Gamma_{25} = 8.22458 / \text{sec.}$$

Γ_{2p} not needed. (Need to check $\times 10$)

$$\alpha = 2.84 \times 10^{-11} T^{-1/2} \text{ cm}^3 / \text{sec.}$$

(Simple model of electron capture rate)

More accurate estimate: (numerical)

$$\alpha = \frac{1.4377 \times 10^{-10} T^{-0.6166} \text{ cm}^3 / \text{sec.}}{1 + 5.085 \times 10^{-3} T^{0.5300}}$$

Substitute & solve for X:
For H we need matter + radiation domination.
 $\Omega_B h^2 = .02$

T	X
4226	.984
4090	.958
3818	.815
3545	.645
3136	.236
3000	.154

(also eq: .00491)

(Agrees with more elaborate calculation)

$e^- +$ recombination

Full analysis \rightarrow requires setting up coupled differential eq for n_{ne} .

(Also Helium states)

no done in Seager, Sasselov, Scott,
astro-ph/9909275

(Uses about 300 level system)

Result: At low temperature reionization
(# of free e^-) is 10% smaller than
the simple minded calculation.

\rightarrow We shall continue with the simple
minded calculation.

3 systems

- ① Free electrons
- ② e^- electrons in nl state ($n \geq 2$) in equilibrium
(dominated by $n=2$)
- ③ e^- electrons in $1s$ state

Variables

n_e n_{2s} n_{1s}
 ↓ ↓
 free electron determines
 density, n_{ne} for $n \geq 2$

$$n_e + n_{1s} + \sum_{n \geq 2} n_{nl} = n_{\text{total}} \rightarrow \text{total \# density}$$

$$n = c/a^3 = c' T^3 \rightarrow \text{known}$$

\Rightarrow Two variables. n_e, n_{15}

$$\frac{d}{dt} \left(\frac{n_e}{n} \right) = - \frac{\alpha n_e^2}{n} + \beta \frac{n_{25}}{n}$$

$$\frac{\beta}{\alpha} = \left(\frac{m_e T}{2\pi} \right)^{3/2} \exp \left(- \frac{B_2}{T} \right)$$

$$\frac{d}{dt} \left(\frac{n_{15}}{n} \right) = \left(\Gamma_{25} + 3P P_{2p} \right) \frac{n_{25}}{n} - \epsilon \frac{n_{15}}{n}$$

$$\frac{\epsilon}{\Gamma_{25} + 3P P_{2p}} = \exp \left(- \frac{B_1 - B_2}{T} \right)$$

We could solve this coupled differential eq. but we simplify further by ~~assuming~~ ^{assuming}

$$\frac{d}{dt} \left(\frac{n_e}{n} \right) = - \frac{d}{dt} \left(\frac{n_{15}}{n} \right)$$

$\left(\sum_{n_{22}, n_{12}} n_{nl} / n \right)$ remains almost constant/

$$\frac{\alpha n_e^2}{n} - \beta \frac{n_{25}}{n} = \left(\Gamma_{25} + 3P P_{2p} \right) \frac{n_{25}}{n} - \epsilon \frac{n_{15}}{n}$$

\rightarrow use it to eliminate n_{25} .

$$n_e + n_{15} + \sum_{\substack{n_{12} \\ n_{22}}} n_{nl} = n$$

$$\Rightarrow n_e + n_{15} + 3n_{2p} \approx 4n_{25}$$

$$\text{Ex. } n_{2s} = \frac{\alpha n_e^2 + \beta (n - n_e)}{\Gamma_{2s} + 3P\Gamma_{2p} + \beta + 4\epsilon}$$

substitute

$$\Rightarrow \frac{d}{dt} \left(\frac{n_e}{n} \right) = \frac{\Gamma_{2s} + 3P\Gamma_{2p}}{(\Gamma_{2s} + 3P\Gamma_{2p})(1 + e^{-\frac{B_1 - B_2}{T}}) + \beta} \times \left\{ - (1 + e^{-\frac{B_1 - B_2}{T}}) \frac{\alpha n_e^2}{n} + e^{-\frac{B_1 - B_2}{T}} \frac{\beta (n - n_e)}{n} \right\}$$

neglect these at low n .

$$\approx \frac{\Gamma_{2s} + 3P\Gamma_{2p}}{\Gamma_{2s} + 3P\Gamma_{2p} + \beta} \left\{ - \frac{\alpha n_e^2}{n} + e^{-\frac{B_1 - B_2}{T}} \frac{\beta (n - n_e)}{n} \right\}$$

$$x = n_e/n$$

$$\frac{dx}{dt} = \frac{\Gamma_{2s} + 3P\Gamma_{2p}}{\Gamma_{2s} + 3P\Gamma_{2p} + \beta} \alpha n \left\{ -x^2 + S^{-1} (1-x) \right\}$$

$$S = n \left(\frac{m_e T}{2\pi} \right)^{-3/2} \exp\left(\frac{B_1}{T} \right)$$

(used result for β/α .)

$\{ \Gamma_{2s}, \Gamma_{2p} \}$: Decay constants from $2s, 2p \rightarrow 1s$
 α : capture ~~rate~~ ~~rate~~ ~~rate~~ rate/electron

P : probability that a Lyman α photon escapes absorption by a 1S state.
 \rightarrow Need to be calculated.

$P(\omega) d\omega$: Probability of ~~capture~~ a Lyman α photon carrying energy between $(\omega, \omega + d\omega)$

$$P(\omega) = \frac{\Gamma_{2p}}{2\pi} \frac{1}{(\omega - \omega_x)^2 + \Gamma_{2p}^2/4}$$

$$\int_0^\infty P(\omega) d\omega = 1 \quad \Gamma_{2p} \ll \omega_x$$

$\omega_x: (B_1 - B_2) \rightarrow$ central value.

$\sigma(\omega)$: absorption x-section of a Lyman α photon of energy ω by a 1S state \rightarrow 2p state.

$$\sigma(\omega) = \frac{3}{2} \frac{2\pi^2 \Gamma_{2p}}{\omega_x^2} P(\omega) \rightarrow \text{Breit-Wigner formula}$$

Now consider a Lyman α photon of energy ω emitted at time t .

$P(t')$: probability that it escapes capture by 1S Hydrogen up to time t' .

$$\frac{d}{dt'} P(t') = - P(t') n_{15}(t') \sigma \left(\omega \frac{a(t)}{a(t')} \right)$$

$$P(\infty) = \exp \left[- \int_t^\infty dt' n_{15}(t') \sigma \left(\omega \frac{a(t)}{a(t')} \right) \right]$$

∴ Total survival probability of a Lyman α photon emitted at time t :

$$P = \int_0^\infty p(\omega) d\omega \exp \left[- \int_t^\infty dt' n_{15}(t') \sigma \left(\omega \frac{a(t)}{a(t')} \right) \right]$$

We now make approximations to simplify this integral.

Most of the absorption will be at $t' \approx t$.

(otherwise $\omega \frac{a(t)}{a(t')}$ becomes $\ll \omega_\alpha - \Gamma_{2p}$ and σ drops).

∴ We can ~~take~~ replace t' by t in slowly varying functions of t .

$$n_{15}(t') \rightarrow n_{15}(t)$$

$$\frac{a(t)}{a(t')} = \frac{1}{1 + (t'-t) \mathcal{H}(t) + \dots} = 1 - (t'-t) \mathcal{H}(t) + \dots$$

ignore

$$\omega' \equiv \omega \cdot \frac{a(t)}{a(t')} = \omega \left\{ 1 - (t'-t) \mathcal{H}(t) \right\}$$

Trade in integration variable t' by ω' .

$$d\omega' = -\omega H(t) dt'$$

$$\omega' = \omega \text{ at } t' = t, \quad \omega' = 0 \text{ at } t' = \infty.$$

$$P = \int_0^{\omega} p(\omega) d\omega \exp\left(-\frac{n_{15}(t)}{H(t)\omega} \int_0^{\omega} \sigma(\omega') d\omega'\right)$$

$$\frac{3\pi^2 \Gamma_{2p}}{\omega_\alpha^2} p(\omega')$$

$$= \int_{-\infty}^{\omega} p(\omega) d\omega \exp\left[-\frac{3\pi^2 \Gamma_{2p} n_{15}(t)}{H(t)\omega\omega_\alpha^2} \int_0^{\omega} p(\omega') d\omega'\right]$$

replace by ω_α

$$u = \int_0^{\omega} p(\omega') d\omega'$$

$$du = p(\omega) d\omega$$

$$u = 0 \text{ at } \omega = 0, \quad u = 1 \text{ at } \omega = \infty.$$

$$\Rightarrow P = \int_0^1 du \exp(-cu) \quad / \quad c = \frac{3\pi^2 \Gamma_{2p} n_{15}(t)}{H(t)\omega_\alpha^3}$$

$$= \frac{1}{c} (1 - e^{-c})$$

Some numbers

$$\Gamma_{24} = 4.699 \times 10^8 / \text{sec}$$

$$\Gamma_{25} = 8.22458 / \text{sec.}$$

$$\alpha = 2.84 \times 10^{-11} T^{-1/2} \text{ cm}^3 / \Delta \quad (\text{simple rounded capture})$$

More detailed calculation fits with

$$\alpha = \frac{1.4377 \times 10^{-10} T^{-0.6166} \text{ cm}^3 / \Delta}{1 + 5.085 \times 10^{-3} T^{0.5300}}$$

Using this we can calculate $X(t)$ by solving the differential equation.

$$\text{For } \Omega_B h^2 = .02$$

$$\frac{dX}{dt} = \frac{dX}{dT} \left(\frac{dT}{dt} \right)^{-1} = -\frac{1}{2} \left(\frac{T}{t} \right)^{-3/2} \frac{dX}{dT}$$

\approx	T	X	X_{saha}
1550	4226	.992	
	3818	.861	
	3409	.526	
	3000	.205	→ .00491
	2591	.0405	
	2183	.00662	
		;	

opacity at temperature T : \mathcal{O}

$\mathcal{O}(T) =$ probability that a photon emitted at time $t(T)$ will undergo at least one scattering before today

~~$\frac{d\mathcal{O}(T)}{dT}$~~
 ~~$\frac{d\mathcal{O}(t')}{dt'}$~~

$\mathcal{O}(t')$: Probability for no scattering
 time t_0 time t'

$$\frac{d\mathcal{O}(t')}{dt'} = - c \sigma_T n_e(t')$$

↓
Thomson scattering
x-section

$$\Rightarrow \mathcal{O}(t_0) = \exp\left(-\int_t^{t_0} c \sigma_T n_e(t') dt'\right)$$

$$\mathcal{O}(T) = 1 - \mathcal{O}(t_0) = 1 - \exp\left(-\int_t^{t_0} c \sigma_T n_e(t') dt'\right)$$

$$= 1 - \exp\left(-\int_{T_0}^T \frac{c \sigma_T n_e(T')}{H' T'} dT'\right)$$

$$dt' = \frac{dT'}{H' T'}$$

$1-O(T)$ = probability that the last scattering of a CMB photon was before temperature T .

$1-O(T)$ fraction of all photons at temperature T reaches us today, without scattering.

$G'(T) dT$: probability that the last scattering was between $T, T+dT$.

→ can be calculated.

$G'(T)$ is peaked at T_L with standard deviation σ

2954

253

for $\Omega_B h^2 = 0.02$