Cosmology 2

158

We now begin exploring the universe at $T > 100$ GeV

$\rightarrow$ unknown physics.

Known end products.

1. Baryon number $\neq 0$ lepton number ?
2. Dark matter.

If ϕ inflation set all conserved charges to zero, these must have been produced after inflation.

$$n_f = n_{in} \times \left(\frac{a_i}{a}\right)^3$$

exponentially small.

Baryogenesis (Leptogenesis)

Sakharov's 3 conditions for baryogenesis

1. The microscopic theory must have B violating interactions.

---

2. The microscopic theory must have C and CP violation.

If initial state has equal # of particles & anti-particles in equilibrium, it is invariant under C and CP.

Final state is not invariant under C and CP (more particles than anti-particles)

Note: CPT is always ~~broken~~ a symmetry of the microscopic theory but T is broken by the expansion of the universe.

---

③ The universe must be out of equilibrium during baryon # creation.

→ consequence of a general result.

Suppose $\tilde{n}_1, \ldots \tilde{n}_K$ are strictly conserved number densities.

(cannot be changed in the microscopic theory and hence must remain zero at all times).

If ~~then~~ $\tilde{n}_\ell = 0 \ \forall \ell \Rightarrow n_i - \bar{n}_i = 0 \ \forall i$ in equilibrium.

# of particles of type i

$\Rightarrow B = \sum_i b_i (n_i - \bar{n}_i)$ will vanish

b_i: baryon number carried by ith particle.

Proof of $n_i - \bar{n}_i = 0 \ \forall i$:

Independent chemical potentials: $\tilde{\mu}_\ell$

$$\sum_\ell \tilde{\mu}_\ell \tilde{n}_\ell = \sum_{\ell, i} \tilde{\mu}_\ell \sum_i c_i^{(\ell)} (n_i - \bar{n}_i)$$

$$= \sum_i \left(\sum_\ell c_i^{(\ell)} \tilde{\mu}_\ell \right) (n_i - \bar{n}_i)$$

$$\Rightarrow \quad \mu_i = \sum_\ell c_i^{(\ell)} \tilde{\mu}_\ell$$

$$\bar{\mu}_i = - \mu_i$$

$$n_i = \frac{g_i}{8\pi^3} \int 4\pi k^2 \, dk \; \frac{1}{e^{\frac{\sqrt{k^2+m_i^2}-\mu_i}{T}} \pm 1}$$

$$\bar{n}_i = \frac{g_i}{8\pi^3} \int 4\pi k^2 \, dk \; \frac{1}{e^{\frac{\sqrt{k^2+m_i^2}-\mu_i}{T}} \pm 1}$$

$$\tilde{n}_\ell = \sum_\ell c_i^{(\ell)} (n_i - \bar{n}_i)$$

$$= \sum_\ell c_i^{(\ell)} \frac{g_i}{8\pi^3} \int 4\pi k^2 \, dk$$

$$\times \left\{ \frac{1}{e^{\frac{\sqrt{k^2+m_i^2}-\mu_i}{T}} \pm 1} - \frac{1}{e^{\frac{\sqrt{k^2+m_i^2}+\mu_i}{T}} \pm 1} \right\}$$

at $\mu_i = \sum_\ell c_i^{(\ell)} \tilde{\mu}_\ell$.

$\tilde{n}_\ell$ are given.

$\Rightarrow$ K equations in K unknowns

$\tilde{\mu}_1, \ldots \tilde{\mu}_\ell$.

One solution: $\tilde{\mu}_1 = \ldots \tilde{\mu}_k = 0$ if

$\tilde{n}_1 = \ldots \tilde{n}_k = 0$

$\rightarrow$ can be shown to be unique in dilute gas approximation

This shows that even if B, C, CP violating interactions are present we need to be away from equilibrium to generate non-zero baryon number.

Similar results hold for lepton number violation (L).

All 3 conditions can be satisfied.

1. Even SM violate C and CP.
2. ~~Some~~ Grand unified theories which unifies strong, weak, em interactions violate B conservation.
3. Out of equilibrium situation can arise.

e.g. SU(5) GUT contains addition massive vectors X (leptoquark) $X \to q\ell$

X, $\bar{X}$ decay can produce non-zero baryon no. due to different decay rates into B-violating channels. ($T < M_{GUT}$)

If this happens late enough so that B violating interactions have fallen out of equilibrium then we can create B-asymmetry.
(Otherwise inverse process will destroy B-asymmetry)

Electroweak ~~baryon~~ baryon number violation:

Baryon number current is anomalous.

$\partial_\mu J^\mu_B \neq 0$ even though it is classically conserved.

$\Rightarrow$ Can produce baryon asymmetry.

Suppressed at low temperature by a factor of $\exp(-8\pi^2/g_w^2) \sim 10^{-162}$.

$\Rightarrow$ has no direct observable effect today.

(Tunneling through barrier)

However at high temperature this barrier is removed.

We have tunneling between states of different baryon number via ~~…~~ configuration called sphalerons.

$\Rightarrow$ Baryon number violating interactions not suppressed at $T \gtrsim 300$ GeV.

Problem: ~~The~~ ~~…~~ universe is at equilibrium at these temperatures.

$\Rightarrow$ no net B violation.

Additional problem: Any earlier B production will be wiped out by these B-violating interactions.

One possible solution: if the equilibrium condition can be violated at a temperature close to 300 GeV then new B-asymmetry may be created by electroweak physics.

One possible source: The electroweak phase transition ($SU(2) \times U(1) \to U(1)$)

We need this to be first order for having non-equilibrium dynamics.

not favoured by observed Higgs mass ~ 125 GeV

Need $m_H < 72$ GeV for 1st order transition.

$\Rightarrow$ Electroweak baryon asymmetry generation is disfavoured.

+ ~~All~~ Previous asymmetry gets erased.

Possible Solution:

B-L is conserved and non-anomalous in ~~electroweak~~ the Standard model.

If (B-L) asymmetry is produced in the early universe, it may get converted to B asymmetry.

Are there (B-L) violating interaction?

Neutrino mass:

Current best theory → see-saw mechanism.

Net effect at low energy.

$m_{ij} \, \overline{\nu^c_{iL}} \, \nu_{jL}$

c: charge conjugation.

ν_{jL} has $L = 1$

ν^c_{iL} has $L = -1$ $\overline{\nu^c_{iL}}$ has $L = 1$.

⇒ Violated L by 2.

$\Delta B = 0$ $\Delta L = \pm 2$. ⇒ $\Delta (B-L) = \mp 2$

~~and this gave There must be some int~~

The neutrino mass term is not invariant under $SU(2)_L \times U(1)$.

so must come from some gauge invariant term.

$$\frac{1}{M} \left(\overline{\nu^c_L} \; \overline{e^c_L} \right) \begin{pmatrix} \phi^0 \\ \phi^+ \end{pmatrix} \left(\phi^0 \;\; \phi^+ \right) \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

Large mass $\gtrsim 10^{10}$ Gev.

$(\phi^0 \;\; \phi^+)$: Higgs fields in SU(2) doublet.

$\langle \phi^0 \rangle = v$ generates $(v^2/M) \, \overline{\nu^c_L} \, \nu_L$ term.

Since $M \gtrsim 10^{10}$ GeV the effect of this interaction is small at ~ 300 GeV.

$\Rightarrow$ Cannot erase $B-L$.

However existence of this $B-L$ violating interaction shows that high energy we could produce $(B-L) \neq 0$ if other conditions are right.

$\rightarrow$ survives till today.

Q. What will be its effect on B and L?

$$n_i - \bar{n}_i = \frac{g_i}{8\pi^3} \int 4\pi k^2\, dk \cdot \left\{ \frac{1}{e^{\sqrt{k^2+m_i^2}/T - \frac{\mu_i}{T}} \pm 1} - \frac{1}{e^{\sqrt{k^2+m_i^2}/T + \frac{\mu_i}{T}} \pm 1} \right\}$$

assuming all particles relativistic & μ_i small

$$= \frac{g_i}{2\pi^2} \frac{2\mu_i}{T} \underbrace{\int_0^\infty k^2\, dk \frac{e^{k/T}}{\{e^{k/T} \pm 1\}^2}}$$

$T^3 \times \frac{\pi^2}{6}$ for fermion

$T^3 \times \frac{\pi^2}{3}$ for bosons.

$$= \frac{1}{6} T^2 \mu_i \bar{g}_i$$

$\bar{g}_i \rightarrow$ g_i for fermions, $2g_i$ for bosons.

166

$c_i^{(\ell)}$: $\tilde{n}_\ell$ charge carried by i-th particle

$$\tilde{n}_\ell = \sum_i c_i^{(\ell)} (n_i - \bar{n}_i)$$

$$\tilde{\mu}_i = \sum_k c_j^{(k)} \tilde{\mu}_j$$

$$\Rightarrow \tilde{n}_\ell = \sum_i \frac{1}{6} c_i^{(\ell)} \times \frac{1}{6} T^2 \mu_i \bar{g}_i$$

$$= \frac{1}{6} T^2 \sum_i c_i^{(\ell)} \bar{g}_i \sum_k c_j^{(k)} \tilde{\mu}_k$$

$$= \frac{1}{6} T^2 \sum_k \Big(\underbrace{\sum_i c_i^{(\ell)} c_i^{(k)} \bar{g}_i}_{M_{\ell k}} \Big) \tilde{\mu}_k$$

$$= \frac{1}{6} T^2 M_{\ell k} \tilde{\mu}_k$$

+ve definite matrix.

$$\Rightarrow \tilde{\mu}_k = \frac{6}{T^2} \sum_\ell (M^{-1})_{k\ell} \tilde{n}_\ell$$

$$\Rightarrow n_i - \bar{n}_i = \frac{1}{6} T^2 \mu_i \bar{g}_i$$

$$= \frac{1}{6} T^2 \bar{g}_i \sum_k \tilde{c}_i^{(k)} \tilde{\mu}_k$$

$$= \bar{g}_i \sum_k \tilde{c}_i^{(k)} \sum_\ell (M^{-1})_{k\ell} \tilde{n}_\ell$$

$$\Rightarrow B - \bar{B} = \sum_i B_i (n_i - \bar{n}_i)$$

$$= \sum_{i,k,\ell} B_i \bar{g}_i \tilde{c}_i^{(k)} (M^{-1})_{k\ell} \tilde{n}_\ell$$

$$= \sum_{k,\ell} v_k (M^{-1})_{k\ell} \tilde{n}_\ell \qquad v_k = \sum_i B_i \bar{g}_i c_i^{(k)}$$

Standard model results. $Q = I_3 - Y$ ↓ weak hypercharge

Particle	B	L	Y	$\bar{g}$
u_L	1/3	0	−1/6	3 → Colour
d_L	1/3	0	−1/6	3
u_R	1/3	0	−2/3	3
d_R	1/3	0	1/3	3
ν_L	0	1	1/2	1
e_L	0	1	1/2	1
e_R	0	1	1	1
ϕ^+	0	0	−1/2	2
ϕ^0	0	0	−1/2	2

Count only particles (or anti-particles) but not both

Note W, Z, gluons do not carry B, L, Y.

$$M_{B-L,\,B-L} = \sum_i \bar{g}_i (B-L)_i (B-L)_i$$

$$= N_g \left(\tfrac{1}{3} \times \tfrac{1}{\cancel{3}} \times \cancel{3} \times 4 + (-1)\times(-1)\times 1 \times 3 \right)$$

N_g = no. of generations = 3

$$+ N_d \times 0$$

N_d = # of Higgs doublets = 1

$$= \frac{13}{3} N_g$$

$$M_{B-L,\,Y} = \sum_i \bar{g}_i (B-L)_i Y_i$$

$$= N_g \Big(\tfrac{1}{3} \times \left(-\tfrac{1}{\cancel{6}}\right) \times \cancel{3} \times 2 + \tfrac{1}{\cancel{3}} \times \left(-\tfrac{2}{3}\right) \times \cancel{3} \; \cancel{+ \tfrac{1}{3} \times \tfrac{2}{3} \times 3}$$

$$+ \tfrac{1}{3} \times \tfrac{1}{\cancel{3}} \times \cancel{3} + (-1) \times \tfrac{1}{2} \times 1 \times \cancel{2} + (-1) \times 1 \times 1 \Big)$$

$$+ N_d \times 0 = -\frac{8}{3} N_g$$

168

$$M_{yy} = \sum \bar{g}_i \, Y_i \, Y_i$$

$$= N_g \left(\frac{1}{6} \times \frac{1}{\cancel{6}} \times \cancel{3} \times \cancel{2} + \frac{2}{3} \times \frac{2}{\cancel{3}} \times \cancel{3} + \frac{1}{3} \times \frac{1}{\cancel{3}} \times \cancel{3} \right.$$

$$\left. + \frac{1}{2} \times \frac{1}{\cancel{2}} \times 1 \times \cancel{2} + 1 \times 1 \times 1 \right)$$

$$+ N_d \left(\frac{1}{2} \times \frac{1}{2} \times 2 \times 2 \right)$$

$$= \frac{10}{3} N_g + N_d$$

$$M = \begin{pmatrix} \frac{13}{3} N_g & -\frac{8}{3} N_g \\ -\frac{8}{3} N_g & \frac{10}{3} N_g + N_d \end{pmatrix}$$

$$M^{-1} = \frac{1}{D} \begin{pmatrix} \frac{10}{3} N_g + N_d & \frac{8}{3} N_g \\ \frac{8}{3} N_g & \frac{13}{3} N_g \end{pmatrix}$$

$$D = N_g \left\{ \frac{13}{3} \left(\frac{10}{3} N_g + N_d \right) - \frac{64}{9} N_g \right\}$$

$$= N_g \left\{ \frac{22}{3} N_g + \frac{13}{3} N_d \right\}$$

$$\vec{\tilde{n}} = \begin{pmatrix} \tilde{n}_{B-L} \\ n_y = 0 \end{pmatrix}$$

$$M^{-1} \vec{\tilde{n}} = \frac{\tilde{n}_{B-L}}{D} \begin{pmatrix} \frac{10}{3} N_g + N_d \\ \frac{8}{3} N_g \end{pmatrix}$$

(169)

$$\cancel{n_B = \sum b_i \bar{g}_i (B-L)_i}$$

$$\vartheta_k = \sum B_i \, \bar{g}_i \, C_i^{(k)}$$

$$\Rightarrow \vartheta_{B-L} = \sum B_i \, \bar{g}_i \, (B-L)_i$$

$$\vartheta_Y = \sum B_i \, \bar{g}_i \, Y_i$$

$$\vartheta_{B-L} = N_g \left(\frac{1}{3} \times \frac{1}{3} \times 3 \times 4 \right) = \frac{4}{3} N_g$$

$$\vartheta_Y = N_g \left(\frac{1}{3} \times \left(-\frac{1}{\cancel{6}}\right) \times \cancel{3} \times \cancel{2} + \frac{1}{3} \times \left(-\frac{2}{\cancel{3}}\right) \times \cancel{3} + \frac{1}{3} \times \frac{1}{\cancel{3}} \times \cancel{3} \right)$$

$$= -\frac{2}{3} N_g$$

$$\Rightarrow n_B = \sum_{k,l} \vartheta_k \left(M^{-1} \right)_{kl} \tilde{n}_l$$

$$= \begin{pmatrix} \frac{4}{3} N_g & -\frac{2}{3} N_g \end{pmatrix} \begin{pmatrix} \frac{10}{3} N_g + N_d \\ \frac{8}{3} N_g \end{pmatrix} \frac{1}{D} \tilde{n}_{B-L}$$

$$= \frac{\cancel{N_g} \left\{ \frac{4}{\cancel{3}} \left(\frac{10}{3} N_g + N_d \right) - \frac{2}{\cancel{3}} \times \frac{8}{3} N_g \right\}}{\cancel{N_g} \left\{ \frac{22}{\cancel{3}} N_g + \frac{13}{\cancel{3}} N_d \right\}}$$

$$= \frac{8 N_g + 4 N_d}{22 N_g + 13 N_d} \tilde{n}_{B-L}$$

$$n_L = n_B - \tilde{n}_{B-L} = -\frac{16 N_g + 9 N_d}{22 N_g + 13 N_d} \tilde{n}_{B-L}$$

This gives a definite prediction of the ratio of $\frac{n_L}{n_{B-L}}$.

Note: SM also has.

$L_i - L_j \quad (i, j = 1, 2, 3)$

as conserved quantities.

$\Rightarrow \tilde{n}_{L_e - L_\mu}, \tilde{n}_{L_e - L_\tau}$ are two more independent conserved charges.

$\rightarrow$ Do not affect the $\frac{n_B}{n_L}$ calculation but these are not determined by the above analysis.

$\rightarrow$ depends on the detailed microscopic process which generate non-zero lepton number.

Dark matter.

Best candidate: Weakly interacting massive particles (WIMPS) → need to be stable. ⇒ denote by L

Weak interaction $\nRightarrow$ the interaction necessarily induced by $SU(2) \times U(1)$.

strength vs weak.

⇒ Would have decoupled early.

However, there will be equal no. of particles and anti particles (unless ∃ conserved charge)

→ Could annihilate.

→ need some detailed analysis to determine how much remains today. L

A simplified assumption: ~~WIMP~~ L is its own anti-particle.

n : density of ~~WIMP~~ dark matter particle, L at time t.

Annihilation X-section per particle σ

⇒ Total annihilation rate / particle

$n\langle\sigma v\rangle$ → averaged over all velocities

→ typically constant as $v \to 0$ if annihilation possible from s-state ($l=0$).

~~Annihilation effect~~

$$\Rightarrow \frac{d}{dt}(n a^3) = -n a^3 n \langle \sigma v \rangle$$

+ Rate of production from thermal background

independent of n if n is small.

$$= n_{eq}^2 \, a^3 \langle \sigma v \rangle$$

equilibrium value of n.

$$\frac{d}{dt}(n a^3) = -a^3 (n^2 - n_{eq}^2) \langle \sigma v \rangle$$

• Some properties of the eq:

At late time ($T \ll m_L$) n_{eq} is small.

$$\Rightarrow \frac{d}{dt}(n a^3) = -a^3 n^2 \langle \sigma v \rangle$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{n a^3}\right) = \frac{\langle \sigma v \rangle}{a^3}$$

$$\frac{1}{n(t) a(t)^3} = \frac{1}{n(t_1) a(t_1)^3} + \int_{t_1}^{t} \frac{\langle \sigma v \rangle}{a(t')^3} dt'$$

Radiation dominated:

$$a(t') = c \, (t')^{1/2}$$

$\Rightarrow$ r.h.s. as $t \to \infty$

$$\Rightarrow \frac{1}{n(t_1) a(t_1)^3} + \int_{t_1}^{\infty} \frac{\langle \sigma v \rangle}{c^3} t'^{-3/2} dt' = 2 \frac{\langle \sigma v \rangle}{c^3} t_1^{-1/2}$$

Main point: $na^3 \to$ finite value as $t \to \infty$.

$\Rightarrow$ There is some finite # of L particles / comoving volume left over.

More detailed analysis:

(Assume radiation dominated universe)

$$\frac{\dot{T}}{T} = -\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3} a_B T^4 \frac{N}{2}}$$ $\to$ effective no. of dof.

$$\Rightarrow \frac{dT}{T^3} = -dt \sqrt{\frac{4\pi G}{3} a_B N}$$

$$\frac{d}{dt}(na^3) = -a^3 \langle \sigma v \rangle (n^2 - n_{eq}^2)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{n}{T^3}\right) = -\frac{1}{T^3} \langle \sigma v \rangle (n^2 - n_{eq}^2)$$

$$\frac{d}{dT}\left(\frac{n}{T^3}\right) = \sqrt{\frac{3}{4\pi G a_B N}} \langle \sigma v \rangle \left(\left(\frac{n}{T^3}\right)^2 - \left(\frac{n_{eq}}{T^3}\right)^2\right)$$

$$u \equiv \frac{n}{T^3}, \quad x = \frac{T}{m_L}$$

$$\frac{du}{dx} = \underbrace{\sqrt{\frac{45}{4\pi^3 G N}} \langle \sigma v \rangle m_L}_{B} (u^2 - u_{eq}^2)$$

$$n_{eq} = (2s_L + 1) \frac{1}{8\pi^3} \int 4\pi k^2 dk \frac{1}{e^{\frac{\sqrt{k^2 + m_L^2}}{T}} \pm 1}$$

$$u_{eq} = \frac{n_{eq}}{T^3} = (2s_L + 1) \frac{1}{8\pi^3} \int 4\pi y^2 dy \frac{1}{e^{\sqrt{y^2 + x^{-2}}} \pm 1}$$

u_{eq} = a known fn. of x.

$\Rightarrow$ can solve the differential eq. numerically and ~~study~~ find u_{eq} as $x \to \infty$.

$\Rightarrow$ n/T^3 as $T \to 0$.

Mass density = $n m_L$.

Comparing this with known dark matter density we get constraint involving m_L and $\langle\sigma v\rangle$

$$\Omega_L = 11.5\, h^{-2} \left(m_L (\text{GeV})\right)^{-1.85} \left(\mathcal{F}/\sqrt{N}\right)^{-0.95}$$

if we parametrize $\langle\sigma v\rangle$ as

$$\langle\sigma v\rangle = \underbrace{G_{wk}^2\, m_L^2}_{\text{rate if L had weak interaction}}\ \underbrace{\mathcal{F}/2\pi}_{\text{some constant parametrizing } \langle\sigma v\rangle}$$

Equating Ω_L to Ω_M we get

$$m_L \left(\mathcal{F}/\sqrt{N}\right)^{.51} = 3.7 \left(\Omega_M h^2\right)^{-.54} \text{GeV}.$$

Inflation: Solves Horizon problem.

~~Two~~ On the last scattering surface two diametrically opposite points were outside the horizon in standard big bang picture extending all the way to $T = \infty$. ($a = 0$).

Inflation: Rapid exponential expansion which takes two neighbouring points and separates them by large distance.

Thus two far away points on last scattering surface could have been close to each other & within each others horizon before the beginning of inflation.

$\Rightarrow$ solution to the horizon problem.

Inflation also solves the flatness problem.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\Big(\overbrace{\frac{\rho_{r0}\, a_0^4}{a^4} + \frac{\rho_{m0}\, a_0^3}{a^3} + \rho_{\Lambda 0}}^{\rho}\Big) - \frac{k}{a^2}$$

smaller   same order today

$3 \times \frac{7}{8} \times \left(\frac{4}{11}\right)^{4/3}$. $\Omega_r \simeq 5 \times 10^{-5}$ (ignoring neutrinos) $\Omega_m = .27$, $\Omega_\Lambda = .73$

For $a < a_{EQ}$, radiation dominates.

Today:

$$\frac{8\pi G}{3} \frac{\rho_{r_0} a_0^4}{a^4} \Big/ \left|\frac{k}{a^2}\right| \gtrsim 5\times10^{-5}$$

In the early universe:

$$\frac{8\pi G}{3} \frac{\rho_{r_0} a_0^4}{a^4} \Big/ \left|\frac{k}{a^2}\right| \gtrsim \left(\frac{a_0}{a}\right)^2 \times 5\times10^{-5}$$

$$\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\left(\frac{T}{T_0}\right)^2 \times 5\times10^{-5}}$$

$T_0 = 2.73\,K = 2.73\times10^{-5} \times 8.62$ eV.

At $T = 100$ GeV.

$$\left(\frac{T}{T_0}\right)^2 \times 5\times10^{-5} = \left(\frac{100\times10^9}{2.73\times8.62\times10^{-5}}\right)^2 \times 5\times10^{-5}$$

$$= 10^{30} \times 5\times10^{-5} \gg 1.$$

If we believe that standard big bang picture holds up to Planck scale 10^{19} GeV, then

at $T \sim 10^{19}$ GeV

$$\left(\frac{T}{T_0}\right)^2 \times 5\times10^{-5} = \left(\frac{10^{19}\times10^9}{2.73\times8.62\times10^{-5}}\right)^2 \times 5\times10^{-5}$$

$$= 10^{\cancel{40}\,59} \gg 1.$$

Natural expectation: At Planck scale different terms in the eq. should be of the same order unless there is some reason.

$\Rightarrow$ Need fine tuning of 1 part in 10^{59} to keep the k/a^2 term small.

$\Rightarrow$ Flatness problem.

Inflation provides a soln.

T_I : temperature after reheating at the end of inflation.

$$\frac{a_0}{a_I} = \frac{T_I}{T_0}$$

$$\Rightarrow \frac{8\pi G}{3} \frac{\rho_{r0} a_0^4}{a_I^4} \Big/ \frac{k}{|a_I|^2} \quad \text{at } T = T_I$$

$$= \left(\frac{T_I}{T_0}\right)^2 \times 5 \times 10^{-5} \rightarrow \text{large}$$

Suppose a_{in} is the scale factor at the beginning of inflation.

$$\frac{k}{a_I^2} = \frac{k}{a_{in}^2} e^{-2N}$$ $N \rightarrow$ # of e-foldings.

Energy density during inflation $\Lambda_I \sim \frac{\rho_{r0} a_0^4}{a_I^4}$

Suppose at the beginning of inflation the two terms, $\frac{k}{a_{in}^2}$ and $\frac{8\pi G}{3}\rho$ were comparable.

$$\frac{k}{a_{in}^2} \simeq \frac{8\pi G}{3} \frac{\rho_{r_0} a_0^4}{a_I^4}$$

$$\frac{k}{a_I^2} = \frac{8\pi G}{3} \frac{\rho_{r_0} a_0^4}{a_I^4} \times e^{-2N}$$

$\Rightarrow$ explains why.

$$\frac{8\pi G}{3} \rho_I \Big/ \frac{k}{a_I^2} = e^{2N} \gg 1$$

If $T_I \sim 10^{19}$ GeV.

$$\Rightarrow \frac{8\pi G}{3} \rho_I \Big/ \frac{k}{a_I^2} \sim 10^{59}$$

Need $e^{2N} \gg 10^{59}$

$$\Rightarrow N > \frac{1}{2} \times 59 \ln 10 \simeq 68.$$

Recall that for solving the horizon problem also we needed 60 e-foldings.

Scalar field driven inflation:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \qquad p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$$H = \sqrt{\frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)}$$

Slow roll condition: $\frac{\dot{a}}{a} = H$

H^{-1} : Time by which a changes by a finite factor.

H should have small fractional change during this time.

$$\frac{\dot{H}}{H} H^{-1} << 1 \Rightarrow \dot{H} << H^2.$$

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \rightarrow \text{first slow roll parameter}$$

Guarantees $\frac{\dot{a}}{a} \simeq$ constant

$\Rightarrow$ nearly exponential expansion.

$$\eta \equiv -\frac{\ddot{\phi}}{\dot{\phi}} H^{-1} \Rightarrow \text{second slow roll parameter (not needed to be small)}$$

$\Rightarrow$ Fractional change of $\dot{\phi}$ during expansion is small.

Condition on $V(\phi)$:

$$\mathcal{H}^2 = \frac{8\pi G}{3}\left(\tfrac{1}{2}\dot\phi^2 + V(\phi)\right)$$

$$2\mathcal{H}\dot{\mathcal{H}} = \frac{8\pi G}{3}\dot\phi\left(\ddot\phi + V'(\phi)\right)$$

$$\ddot\phi + 3\mathcal{H}\dot\phi + V'(\phi) = 0$$

$$\Rightarrow 2\mathcal{H}\dot{\mathcal{H}} = \frac{8\pi G}{3}\left(-3\mathcal{H}\dot\phi^2\right) = -8\pi G\dot\phi^2$$

$$\epsilon = \frac{\dot{\mathcal{H}}}{\mathcal{H}^2} = -\frac{1}{2}\frac{8\pi G\dot\phi^2}{\frac{8\pi G}{3}\left(\frac{1}{2}\dot\phi^2 + V(\phi)\right)} = -\frac{\frac{3}{2}\dot\phi^2}{\frac{1}{2}\dot\phi^2 + V(\phi)}$$

Thus $|\epsilon| \ll 1 \Rightarrow \dot\phi^2 \ll V(\phi)$.

$$\eta = \frac{\ddot\phi}{\dot\phi\mathcal{H}}$$

If $|\eta| \ll 1$ then $3\mathcal{H}\dot\phi + V'(\phi) \simeq 0$

$$\dot\phi \simeq -\frac{V'(\phi)}{3\mathcal{H}} \Rightarrow |\epsilon| \ll 1 \Rightarrow \left(V'(\phi)\right)^2 \ll 9\mathcal{H}^2 V(\phi)$$

$$\qquad\qquad\qquad\qquad // \; 24\pi G\, V(\phi)^2$$

$$\ddot\phi = -\frac{V''(\phi)}{3\mathcal{H}}\dot\phi + \frac{V'(\phi)}{3\mathcal{H}^2}\dot{\mathcal{H}}$$

$$= \frac{V''(\phi)V'(\phi)}{9\mathcal{H}^2} + \frac{V'(\phi)}{3\mathcal{H}^2}\dot{\mathcal{H}}$$

$$= \cancel{\frac{V'(\phi)}{9\mathcal{H}^2}\left\{V''(\phi) + 3\right.}$$

$$\eta = \frac{\ddot\phi}{\dot\phi\mathcal{H}} = -\frac{V''}{3\mathcal{H}^2} - \frac{\dot{\mathcal{H}}}{\mathcal{H}^2}$$

$= \epsilon$ small

Small $|\eta|$ requires

$$|V''| << 3H^2 = 8\pi G\, V(\phi)$$

Two conditions:

$$(V')^2 << 24\pi G\, V(\phi)^2$$

$$|V''| << 8\pi G\, V(\phi)$$

Another condition: ~~the~~ Quantum gravity corrections should be small.

$$\Rightarrow |V(\phi)| << M_{Pl}^4 \sim (4\pi G)^{-2} \quad (\hbar = c = 1 \text{ unit}).$$

In natural units $|V(\phi)| << 1$.

→ ensures that the curvature of space-time remains low at the Planck scale.

Example: $V(\phi) = g\,\phi^\alpha$

$$V'(\phi) = \alpha\, g\, \phi^{\alpha-1}$$

$$V''(\phi) = \alpha(\alpha-1)\, g\, \phi^{\alpha-2}$$

$$|V'(\phi)|^2 << 24\pi G\, |V(\phi)|^2$$

$$\Rightarrow \alpha^2 g^2 \phi^{2\alpha-2} << 24\pi G.\, g^2 \phi^{2\alpha}$$

$$\Rightarrow \phi^2 >> \alpha^2/(24\pi G)$$

$V''(\phi) >> 4\pi G\, V(\phi)$

$\Rightarrow \alpha(\alpha-1)\, g\, \phi^{\alpha-2} >> 4\pi G\, g\, \phi^{\alpha}$

$\Rightarrow \phi^2 >> \alpha(\alpha-1)/(4\pi G)$

Thus for $\alpha \geq 2$ both conditions are easy to satisfy by taking ϕ to be large.

$V(\phi) << (4\pi G)^{-2}$

$\Rightarrow g\, \phi^{\alpha} << (4\pi G)^{-2}$

$\alpha = 4$: $g\, \phi^{4} << (4\pi G)^{-2}$

$\phi^4 >> (4\pi G)^{-2}$

$\Rightarrow g << 1$

Suppose inflation lasts from $\phi = \phi_1$ to $\phi = \phi_2$.

$\frac{\dot{a}}{a} = H$

$\Rightarrow a(t_2) = a(t_1) \exp\left(\int_{t_1}^{t_2} H\, dt \right)$

$3H \frac{d\phi}{dt} = -V'(\phi)$

$\Rightarrow dt = -\frac{3H\, d\phi}{V'(\phi)}$

$$a(t_2) = a(t_1) \exp\left(\int_{\phi_2}^{\phi_1} \frac{3H^2 \, d\phi}{V'(\phi)} \right)$$

$$= a(t_1) \exp\left(\int_{\phi_2}^{\phi_1} 8\pi G \frac{V(\phi)}{V'(\phi)} \, d\phi \right)$$

If $V = g\phi^\alpha$

$V'(\phi) = g\alpha\phi^{\alpha-1}$

$$\Rightarrow a(t_2) = a(t_1) \exp\left(\int_{\phi_2}^{\phi_1} \frac{8\pi G}{\alpha} \phi \, d\phi \right)$$

$$= a(t_1) \exp\left(\frac{4\pi G}{\alpha} (\phi_1^2 - \phi_2^2) \right)$$

Need $\frac{4\pi G}{\alpha}(\phi_1^2 - \phi_2^2) > 60$.

$$\phi_1^2 > \frac{60\alpha}{4\pi G}$$

For $\alpha = 4$.

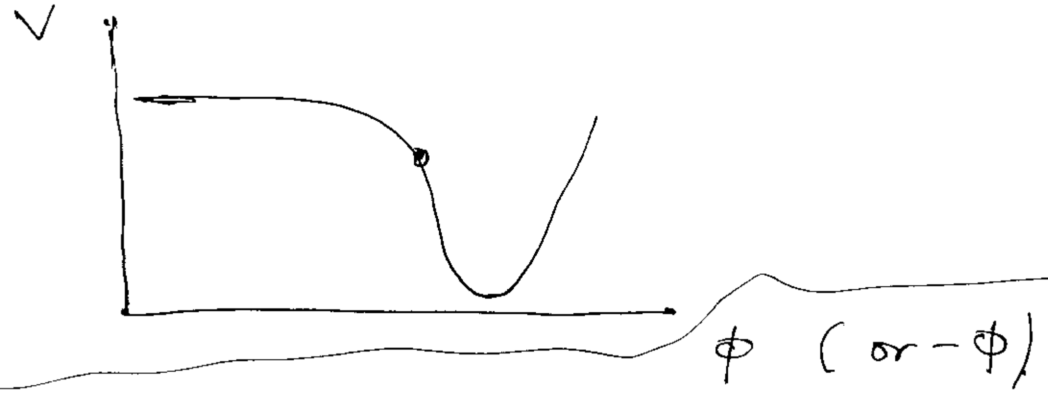
$g\phi^4 << (4\pi G)^{-2}$

$g(240)^2 << 1$.

Ex. Check that inflation is possible even with free massive scalar for sufficiently low mass.

$V(\phi) = \frac{1}{2}m^2\phi^2 \qquad \alpha = 2, \; g = \frac{1}{2}m^2$.

Reheating



When slow roll conditions are violated.

- Simplified assumption: Happens abruptly.

$$V(\phi) = \frac{1}{2} m^2 (\phi - \phi_0)^2$$

- $\phi = \phi_{cl} \Rightarrow$ a coherent state.

$\Rightarrow$ Linear combination of highly excited states of a harmonic oscillator.

- In QFT: $\Rightarrow$ a state with large no. of ϕ particles of mass m.

$\vec{\nabla}\phi \simeq 0$ (due to inflation)

$\Rightarrow$ In Fourier transform space momenta $\simeq 0$.

$\Rightarrow$ A gas of non-relativistic particles of mass m.

$$\rho_\phi(t) \simeq \rho_\phi(t_I) \left(a(t_I)/a(t)\right)^3.$$

In order to make connection with the standard big bang scenario, ϕ must coupled to ordinary fields ~~as~~ & hence a massive ϕ particle will have some decay ~~probabili~~ rate to (nearly) massless particles.

$\downarrow$ gives radiation.

(ρ_M, p_M) : (energy density, pressure) of ordinary matter.

Γ : decay rate of ϕ to ordinary matter

$$\rho_\phi = \rho_\phi(t_I)\left(\frac{a(t_I)}{a(t)}\right)^3 e^{-\Gamma t}$$

$$p_\phi = 0.$$

$$\frac{d}{dt}\left((\rho_M+\rho_\phi)a^3\right) = -3(p_\phi + p_M)\, a^2 \dot{a}$$

Since $\frac{d}{dt}\left(\rho_\phi a^3\right) = -\Gamma \rho_\phi a^3$, $\dot{a} = H a$, $p_\phi = 0$

$$\Rightarrow \quad \frac{d\rho_M}{dt} + 3H(\rho_M + p_M) = \Gamma \rho_\phi.$$

$p_M = \frac{\rho_M}{3}$ (radiation dominated)

$$\frac{d}{dt}\left(\rho_M a^4\right) = \Gamma \rho_\phi a^4$$
$$= \Gamma \rho_\phi(t_I)\, a(t_I)^3\, a(t)\, e^{-\Gamma t}$$

$$\rho_M(t) = (a(t))^{-4}\, \rho_\phi(t_I)\, \Gamma\, a(t_I)^3 \int_{t_I}^{t} dt' \exp\left(-\Gamma(t'-t_I)\right) a(t')$$

Important issue: What is the maximum value of $\rho_M(t)$

$\Rightarrow$ Maximum $T(t)$ via

$$\rho_M = \frac{N}{2} a_B T^4.$$

$\Rightarrow$ tells us what ~~particles~~ processes could have ~~been~~ ~~prod~~ taken place after inflation.

(Baryogenesis, dark matter production etc. must happen after inflation).

Two extreme cases:

① $\Gamma \gg H(t)$.

$$\int_{t_I}^{t} dt' \underbrace{\exp\left(-\Gamma(t'-t_I)\right)}_{-\frac{1}{\Gamma}\frac{d}{dt'}\exp(-\Gamma(t'-t_I))} a(t')$$

Integrate by parts.

$$\frac{1}{\Gamma}\left(a(t_I) - a(t)\, e^{-\Gamma(t-t_I)}\right) + \frac{1}{\Gamma}\int_{t_I}^{t} dt' \exp\left(-\Gamma(t'-t_I)\right) \dot{a}(t') \rightarrow \text{suppressed by } H/\Gamma$$

leading term:

$$a(t)^{-4}\, \rho_{\phi}(t_I)\, a(t_I)^4 \left(1 - \frac{a(t)}{a(t_I)} e^{-\Gamma(t-t_I)}\right)$$

$\Rightarrow$ is maximum at $t = t_I$ & then decays.

$$\rho_M|_{max} = \rho_\phi(t_I).$$

i.e. if the inflaton can transfer its energy quickly to ordinary radiation then the energy density of radiation will rise to $\rho_\phi(t_I)$

② $\Gamma << H(t_I)$

$\Rightarrow$ The rate is very slow.

$$\rho_M(t) = a(t)^{-4}\, \rho_\phi(t_I)\, \Gamma\, a(t_I)^3$$

$$\int_{t_I}^{t} dt' \exp\left(-\Gamma(t'-t_I)\right) a(t')$$

ignore this

$$a(t') \simeq a(t_I)\left(t'/t_I\right)^{2/3}$$

(matter, i.e. inflaton, dominated)

$$a(t_I) \cdot (t_I)^{-2/3} \cdot \frac{3}{5}\left(t^{5/3} - t_I^{5/3}\right).$$

188

$$\rho_M(t) = \rho_\phi(t_I)\, a(t_I)^{-4} \left(\frac{t}{t_I}\right)^{-8/3} \Gamma\, a(t_I)^3$$

$$a(t_I)\,(t_I)^{-2/3}\,\frac{3}{5}\,\left(t^{5/3} - t_I^{5/3}\right)$$

$$= \Gamma\, \rho_\phi(t_I)\, t_I\, \frac{3}{5}\left(\left(\frac{t}{t_I}\right)^{-1} - \left(\frac{t}{t_I}\right)^{-8/3}\right)$$

Maximum at:

$$-\left(\frac{t}{t_I}\right)^{-2} + \frac{8}{3}\left(\frac{t}{t_I}\right)^{-\frac{11}{3}} = 0$$

$$\Rightarrow \left(\frac{t}{t_I}\right)^{\frac{5}{3}} = \frac{8}{3} \qquad \frac{t}{t_I} = \left(\frac{8}{3}\right)^{3/5}.$$

$$\rho_M\Big|_{max} = \frac{3}{5}\,\Gamma\,\rho_\phi(t_I)\, t_I \left(\left(\frac{8}{3}\right)^{-3/5} - \left(\frac{8}{3}\right)^{-8/5}\right)$$

Note: origin of time is counted so that big bang at $t=0$ if there was no inflation.

$$H(t_I) = \frac{2}{3}\, t_I^{-1} \quad \text{(for matter dominated)}$$

$$\rho_M\Big|_{max} = \frac{2}{5}\,\Gamma\,\rho_\phi(t_I)\, H(t_I)^{-1}\left(\left(\frac{8}{3}\right)^{-3/5} - \left(\frac{8}{3}\right)^{-8/5}\right)$$

~~Thus~~ ~~[illegible]~~ Inflation makes the universe homogeneous.

Then why is the universe not completely homogeneous?

Last scattering surface shows inhomogeneities ~ 1 in 10^5.

Where does this come from?

Possible answer: Thermal fluctuations after inflation.

However we know that since inflation ~~[illegible]~~ signal cannot propagate beyond the Horizon size calculated earlier ignoring inflation.

$$\text{Horizon} \sim H^{-1} \sim t$$

$\Rightarrow$ spans about 2° ~~in~~ on the last scattering surface.

Thus ρ, averaged over 2° scale, should have been the same in all directions.

~~$\langle \rho(\vec{x},t)\rho(\vec{y},t)\rangle$ = correlation fn.~~

~~$P(k) = \int e^{i\vec{k}\cdot\vec{x}} \rho(\vec{x},t)$~~

However this is not so.

Fluctuations exist at all scales.
$\Rightarrow$ Could not have been produced by thermal fluctuations.

Answer: These fluctuations are produced by quantum effects during inflation.

~~Given~~ ~~as~~ Furthermore fluctuations at scales $> 2°$ on the last scattering surface is unaffected by the details of the evolution after inflation

$\Rightarrow$ Gives us direct information about microscopic theory behind inflation.

Fluctuations at scale $< 2°$ on the ~~microscopic~~ LSS gives us information about the ~~the~~ evolution after inflation.

$\Rightarrow$ Important to understand how fluctuations are produced during inflation and how they evolve in a given microscopic theory.

Planck units: $\hbar = c = G = 1$.

Reduced Planck units: $\hbar = c = 8\pi G = 1$.

Reduced Planck mass $= \frac{1}{\sqrt{8\pi}} \times$ Planck mass.

Reduced Planck length $= \sqrt{8\pi} \times$ Planck length

Reduced Planck time $= \sqrt{8\pi} \times$ Planck time.

We shall work in these units to get rid of ~~some~~ 8π factors.

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\rho = \frac{1}{3}\rho.$$

$\frac{\dot{a}}{a} = H$. $\Rightarrow$ The universe expands to a finite multiple in time $\sim H^{-1}$.

Comoving distance travelled by light in time H^{-1}:

$$dr = \frac{dt}{a(t)} \Rightarrow \Delta r \sim \frac{1}{a(t)} \Delta t = \frac{1}{H(t)\,a(t)}$$

at time t

$\Rightarrow$ The microscopic dynamics should affect physics at comoving distance $\lesssim (H(t)\,a(t))^{-1}$.

Physics at larger scale should not be affected by microscopic physics at time t.

Some notations:

X_i: some physical quantity.

e.g. scalar field, some component of metric, some component of $T_{\mu\nu}$ etc.

$\tilde{X}_i$: deviation from the average $\bar{X}_i^{(t)}$

The measure fluctuation we need to analyze correlation fn.:

$$\langle \tilde{X}_i(\vec{x},t)\ \tilde{X}_j(\vec{y},t)\rangle$$

average over ensembles / quantum average

If $\tilde{X}_i(\vec{x},t)$ is independent of $\vec{x}$ then $\langle \tilde{X}_i(\vec{x},t)\ \tilde{X}_j(\vec{y},t)\rangle$ will be independent of $\vec{x},\vec{y}$.

$\Rightarrow$ Deformation of the background.

By translation invariance we expect

$$\langle \tilde{X}_i(\vec{x},t)\ \tilde{X}_j(\vec{y},t)\rangle = f_{ij}(\vec{x}-\vec{y},t)$$

It is useful to use Fourier transformed variables:

$$\hat{X}_i(\vec{k},t) = \int d^3x\ e^{-i\vec{k}\cdot\vec{x}}\ \tilde{X}_i(\vec{x},t)$$

$$\tilde{X}_i(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\ e^{i\vec{k}\cdot\vec{x}}\ \hat{X}_i(\vec{k},t).$$

$$\langle \hat{x}_i(\vec{k},t)\,\hat{x}_j(\vec{k}',t)\rangle$$

$$= \int d^3x\, d^3y\, e^{-i(\vec{k}\cdot\vec{x}+\vec{k}'\cdot\vec{y})}\, f_{ij}(\underbrace{\vec{x}-\vec{y}}_{=\vec{z}})$$

$$= \int d^3z\, d^3y\, e^{-i(\vec{k}+\vec{k}')\cdot\vec{y} - i\vec{k}\cdot\vec{z}}\, f_{ij}(\vec{z})$$

$$= (2\pi)^3\,\delta^{(3)}(\vec{k}+\vec{k}')\,\underbrace{\int d^3z\, e^{-i\vec{k}\cdot\vec{z}} f_{ij}(\vec{z})}_{\hat{f}_{ij}(\vec{k})}$$

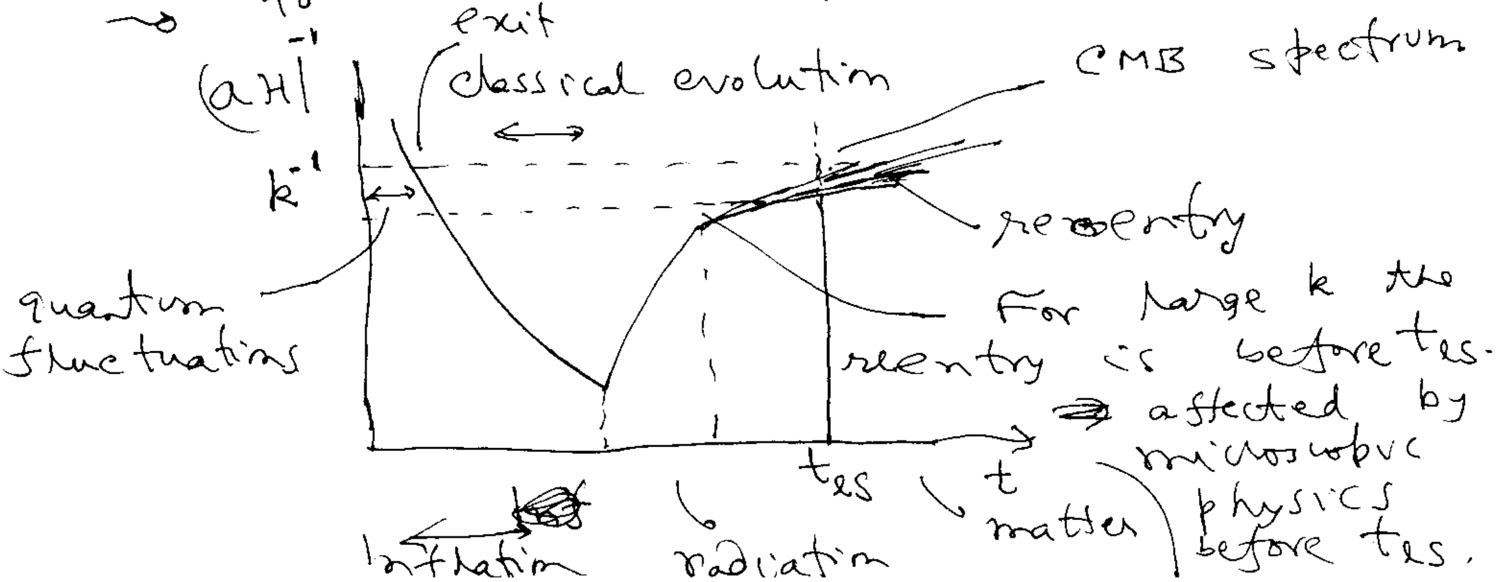
We want to calculate $\hat{f}_{ij}(\vec{k})$ for various quantities $\hat{x}_i$.

Behaviour of $\hat{f}_{ij}(\vec{k})$ at small $|\vec{k}|$

$\Leftrightarrow$ behaviour of $f_{ij}(\vec{x})$ at large $|\vec{x}| \sim \frac{1}{|\vec{k}|}$

$\Rightarrow$ $\hat{f}_{ij}(\vec{k})$ for $|\vec{k}|^{-1} > a^{-1}H^{-1}$ is not affected by microscopic quantum physics.

$\rightarrow$ to be made precise later.



(194)

Inflation : $a = c\, e^{Ht}$ $H = \frac{\dot a}{a}$ constant

$(aH)^{-1} = c^{-1} H^{-1} e^{-Ht}$

Radiation dominated : $a = c_1 t^{1/2}$

$H = \frac{\dot a}{a} = \frac{1}{2t}$

$a^{-1} H^{-1} \sim 2 c_1^{-1} t^{1/2}$

Matter dominated : $a = c_2 t^{2/3}$

$H = \frac{2}{3t}$

$(aH)^{-1} = \frac{3}{2} c_2^{-1} t^{1/3}$

Study the physics before horizon entry $\leftrightarrow$ quantum fluctuations.

Slow roll.

$\Rightarrow$ extreme approximation $V(\phi) = \text{constant } V_0$

$\phi = \phi_0 \rightarrow$ arbitrary constant

$\tilde\phi = \phi - \chi_0$

$$\mathcal{L} = \int \sqrt{\det g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \tilde\phi \, \partial_\nu \tilde\phi - V_0 + \frac{1}{16\pi G} R \right]$$

$\frac{1}{16\pi G} = \frac{1}{2}$ in reduced planck unit.

Background metric:

$$ds^2 = -dt^2 + c^2 e^{2Ht}\, d\vec{x}^2 \qquad H = \sqrt{\frac{V_0}{3}}$$

Define $d\tau = \dfrac{dt}{c e^{Ht}} = c^{-1} e^{-Ht}\, dt$

$$\tau = -c^{-1} H^{-1} e^{-Ht} = -H^{-1} a^{-1}$$

$$-\infty < t < \infty \Rightarrow -\infty < \tau < 0.$$

$$ds^2 = c^2 e^{2Ht} \left(-d\tau^2 + d\vec{x}^2\right)$$

$$= \frac{1}{H^2\tau^2}\left(-d\tau^2 + d\vec{x}^2\right)$$

$$S_{\tilde{\phi}} = \int d^3x\, d\tau\, H^{-4}\tau^{-4}\, H^2\tau^2 \left((\partial_\tau \tilde{\phi})^2 - (\vec{\nabla}\tilde{\phi})^2 \right)$$

$$= \int d^3x\, d\tau\, \frac{1}{2H^2\tau^2}\left((\partial_\tau \tilde{\phi})^2 - (\vec{\nabla}\tilde{\phi})^2 \right)$$

$$\tilde{\phi}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \hat{\phi}(\vec{k},t)$$

$$\Rightarrow S = \int \frac{d^3k}{(2\pi)^3}\, d\tau\, \frac{1}{2H^2\tau^2} \Big(\partial_\tau \hat{\phi}(-\vec{k},\tau)\, \partial_\tau \hat{\phi}(\vec{k},t)$$

$$- \vec{k}^2\, \hat{\phi}(-\vec{k},\tau)\, \hat{\phi}(\vec{k},\tau) \Big)$$

Quantization. $\tilde{\Pi}(\vec{x},\tau) = \dfrac{\partial L}{\partial(\partial_\tau \tilde{\phi}(\vec{x},\tau))} = \dfrac{1}{H^2\tau^2}\tilde{\phi}(\vec{x},\tau)$

$$\Rightarrow \left[\tilde{\phi}(\vec{x},\tau), \tilde{\Pi}(\vec{y},\tau)\right] = i\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$[\tilde{\phi}(\vec{x},\tau),\ \partial_\tau \tilde{\phi}(\vec{y},\tau)] = i\mathcal{H}^2\tau^2\ \delta^{(3)}(\vec{x}-\vec{y})$$

Note: As $t \to \infty$, $\tau \to 0$, the commutator $\to 0$.

The theory becomes classical.

$$[\hat{\phi}(\vec{k},\tau),\ \partial_\tau \hat{\phi}(\vec{k}',\tau)]$$

$$= i\,(2\pi)^3\ \delta^{(3)}(\vec{k}+\vec{k}')\ \mathcal{H}^2\tau^2$$

If we can identify the "vacuum" state then we could ~~identify~~ calculate the average fluctuation

Use more conventional normalization:

$$\tilde{\chi}(\vec{x},\tau) = \frac{1}{\mathcal{H}\tau}\ \tilde{\phi}(\vec{x},\tau)$$

$$\hat{\chi}(\vec{k},\tau) = \frac{1}{\mathcal{H}\tau}\ \hat{\phi}(\vec{k},\tau)$$

$$\partial_\tau \hat{\phi} = \mathcal{H}\tau\ \partial_\tau \hat{\chi} + \mathcal{H}\ \hat{\chi}$$

$$\Rightarrow S = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\, d\tau\ \Big[\big(\partial_\tau \hat{\chi}(-\vec{k},\tau) + \tfrac{1}{\tau}\hat{\chi}(-\vec{k},\tau)\big)\big(\partial_\tau \hat{\chi}(\vec{k},\tau) + \tfrac{1}{\tau}\hat{\chi}(\vec{k},\tau)\big) - \vec{k}^2\ \hat{\chi}(-\vec{k},\tau)\,\hat{\chi}(\vec{k},\tau)\Big]$$

For fixed $\vec{k}$, as $\tau \to -\infty$ it reduces to conventional free scalar action.

Recall the quantization rules of free scalar field theory in flat space

(no $\frac{1}{\tau}$ term) $= \frac{1}{2}\int d^3x\, d\tau\, [(\partial_\tau \tilde{\chi}(\vec{x},\tau))^2 - (\vec{\nabla}\tilde{\chi}(\vec{x},\tau))^2]$

$$S = \frac{1}{2}\int d\tau\, d^3k\, [\partial_\tau \hat{\chi}(-\vec{k},\tau)\, \partial_\tau \hat{\chi}(\vec{k},\tau) - \vec{k}^2 \hat{\chi}(-\vec{k},\tau)\, \hat{\chi}(\vec{k},\tau)]$$

$\hat{\chi}(-\vec{k},\tau)^* = \hat{\chi}(\vec{k},\tau)$

Steps for quantization

① Eqs. of motion: $(\partial_\tau^2 + \omega^2)\,\hat{\chi}(\vec{k},\tau) = 0$

$$\omega = |\vec{k}|$$

② Solution: $v(\vec{k},\tau) = \frac{1}{\sqrt{2\omega}\,(2\pi)^{3/2}} e^{-i\omega\tau}$

$$v(\vec{k},\tau)^* = \frac{1}{\sqrt{2\omega}\,(2\pi)^{3/2}} e^{+i\omega\tau}$$

$v\,\partial_\tau v^* - v^*\partial_\tau v = (2\pi)^3 i$

⇒ Two independent solutions

③ $\hat{\chi}(k,\tau) = a(\vec{k})\, v(\vec{k},\tau) + a(-\vec{k})^*\, v(-\vec{k},\tau)^*$

(satisfies $\hat{\chi}(\vec{k},\tau)^* = \hat{\chi}(-\vec{k},\tau)$)

Choose $v(-\vec{k},\tau) = v(\vec{k},\tau)$.

$$\Rightarrow \hat{\chi}(\vec{k},\tau) = a(\vec{k})\, v(\vec{k},\tau) + a(-\vec{k})^*\, v(\vec{k},\tau)^*$$

④ Promote a, a^* to τ-independent operators.

$$\hat{\chi}(\vec{k},\tau) = a(\vec{k})\, v(\vec{k},\tau) + a(\vec{k})^\dagger\, v(\vec{k},\tau)^*$$

⑤ $[\hat{\chi}(\vec{k},\tau), \hat{\chi}(\vec{k}',\tau)] = 0$

$[\partial_\tau\hat{\chi}(\vec{k},\tau), \partial_\tau\hat{\chi}(\vec{k}',\tau)] = 0$

$[\hat{\chi}(\vec{k},\tau), \partial_\tau\hat{\chi}(\vec{k}',\tau)] = i\,\delta^{(3)}(\vec{k}+\vec{k}')\,(2\pi)^3$

$\rightarrow$ follows from

$[\tilde{\chi}(\vec{x},\tau), \tilde{\chi}(\vec{y},\tau)] = 0$

$[\partial_\tau\tilde{\chi}(\vec{x},\tau), \partial_\tau\tilde{\chi}(\vec{y},\tau)] = 0$

$[\tilde{\chi}(\vec{x},\tau), \partial_\tau\tilde{\chi}(\vec{y},\tau)] = i\,\delta^{(3)}(\vec{x}-\vec{y})$

e.g. $[\hat{\chi}(\vec{k},\tau), \partial_\tau\hat{\chi}(\vec{k}',\tau)]$

$$= \int d^3x\, d^3y\, e^{i(\vec{k}\cdot\vec{x}+\vec{k}'\cdot\vec{y})}\,\underbrace{[\hat{\chi}(\vec{x},\tau), \partial_\tau\hat{\chi}(\vec{y},\tau)]}_{i\,\delta^{(3)}(x-\vec{y})}$$

$$= i\,(2\pi)^3\,\delta^{(3)}(\vec{k}+\vec{k}')$$

⑥ This translates to:

$$[a(\vec{k}), a(\vec{k}')] = 0 = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')]$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \cancel{(2\pi)^3}\,\delta^{(3)}(\vec{k}-\vec{k}')$$

e.g. $[\hat{\chi}(\vec{k},\tau), \partial_\tau\hat{\chi}(\vec{k}',\tau)]$

$$= [a(\vec{k}), a(-\vec{k}')^\dagger]\, v(\vec{k},\tau)\,\partial_\tau v(\vec{k},\tau)^* + [a(\vec{k})^\dagger, a(-\vec{k}')]\, v(\vec{k},\tau)^*\,\partial_\tau v(\vec{k},\tau)$$

$$= \cancel{(2\pi)^3}\ \delta(\vec{k}+\vec{k}')\ \left[v(\vec{k},\tau)\, \partial_\tau v(\vec{k},\tau)^* - \partial_\tau v(\vec{k},\tau)\, v(\vec{k},\tau)^* \right]$$

$$(2\pi)^3 \frac{1}{2\omega} (i\omega + i\omega) = (2\pi)^3 i$$

$$= (2\pi)^3\ i\ \delta^{(3)}(\vec{k}+\vec{k}').$$

→ Correct answer.

Similarly for other commutators.

(7) Define $|0\rangle$: $a(\vec{k})|0\rangle = 0$ $\forall \vec{k}$ → minimizes H

Now return to the more general case

$$S = \int d^3k\, d\tau\ \frac{1}{2}\left[\left(\partial_\tau \hat{\chi}(-\vec{k},\tau) + \frac{1}{\tau}\hat{\chi}(-\vec{k},\tau)\right) \left(\partial_\tau \hat{\chi}(\vec{k},\tau) + \frac{1}{\tau}\hat{\chi}(\vec{k},\tau)\right) - \vec{k}^2\ \hat{\chi}(-\vec{k},\tau)\hat{\chi}(\vec{k},\tau)\right]$$

① Eq. of motion:

$$\left(-\partial_\tau + \frac{1}{\tau}\right)\left(\partial_\tau + \frac{1}{\tau}\right)\hat{\chi}(\vec{k},\tau) - \omega^2\,\hat{\chi}(\vec{k},\tau) = 0$$

② Solution: (check)

$$v(\vec{k},\tau) = \frac{1}{\sqrt{2\omega}}(2\pi)^{3/2} e^{-i\omega\tau}\left(1 + \frac{1}{i\omega\tau}\right)$$

$$v(\vec{k},\tau)^* = \frac{1}{\sqrt{2\omega}}(2\pi)^{3/2} e^{i\omega\tau}\left(1 - \frac{1}{i\omega\tau}\right)$$

→ approaches the earlier soln. for $\tau \to -\infty$, ③ $\hat{\chi}(\vec{k},t) = a(\vec{k})\, \overset{(200)}{\upsilon}(\vec{k},\tau) + a(-\vec{k})^{+}\, \upsilon^{*}(\vec{k},\tau)$

④ Promote a to operators

$\Rightarrow a^{*} \rightarrow a^{+}$

⑤ Need:

$$[\chi(\vec{x},\tau), \chi(\vec{y},\tau)] = 0$$

$$[\chi(\vec{x},\tau), \partial_\tau \chi(\vec{y},\tau) + \tfrac{1}{\tau}\chi(\vec{y},\tau)] = i\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$\Rightarrow [\chi(\vec{x},\tau), \partial_\tau \chi(\vec{y},\tau)] = i\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$[\partial_\tau \chi(\vec{x},\tau) + \tfrac{1}{\tau}\chi(\vec{x},\tau), \partial_\tau \chi(\vec{y},\tau) + \tfrac{1}{\tau}\chi(\vec{y},\tau)] = 0$$

$$\Rightarrow [\partial_\tau \chi(\vec{x},\tau), \partial_\tau \chi(\vec{y},\tau)] = 0$$

$\Rightarrow$ Same as before.

⑥ Furthermore

$$\upsilon(\vec{k},\tau)\, \partial_\tau \upsilon(\vec{k},\tau)^{*} - \partial_\tau \upsilon(\vec{k},\tau)\, \upsilon(\vec{k},\tau)^{*}$$

$$= (2\pi)^3 i \quad (\text{check})$$

$$\Rightarrow [a(\vec{k}), a(\vec{k}')] = 0 = [a^{+}(\vec{k}), a^{+}(\vec{k}')]$$

$$[a(\vec{k}), a(\vec{k}')^{+}] = \delta^{(3)}(\vec{k}-\vec{k}')$$

⑦ Define $|0\rangle$: $a(\vec{k})|0\rangle = 0 \quad \forall \vec{k}$

Given this we can calculate

$$\langle \hat{\chi}(\vec{k},\tau), \hat{\chi}(\vec{k}',\tau)\rangle = \langle 0|\, \hat{\chi}(\vec{k},\tau)\, \hat{\chi}(\vec{k}',\tau)\,|0\rangle$$

$$= \langle 0 | a(\vec{k})\, a(-\vec{k}')^{\dagger} | 0 \rangle\, \upsilon(\vec{k},\tau)\, \underbrace{\upsilon(-\vec{k},\tau)^{*}}_{= \upsilon(\vec{k},\tau)^{*}}$$

$$= \delta^{(3)}(\vec{k}+\vec{k}')\,(2\pi)^3\, \frac{1}{2\omega}\left(1+\frac{1}{\omega^2\tau^2}\right)$$

$$\langle 0 | \hat{\phi}(\vec{k},\tau)\, \hat{\phi}(\vec{k}',\tau)^{*} | 0 \rangle$$

$$= (H^2\tau^2)\, \langle 0 | \hat{\chi}(\vec{k},\tau)\, \hat{\chi}(\vec{k}',\tau) | 0 \rangle$$

$$= (2\pi)^3\, \delta^{(3)}(\vec{k}+\vec{k}')\, \frac{H^2\tau^2}{2\omega}\left(1+\frac{1}{\omega^2\tau^2}\right)$$

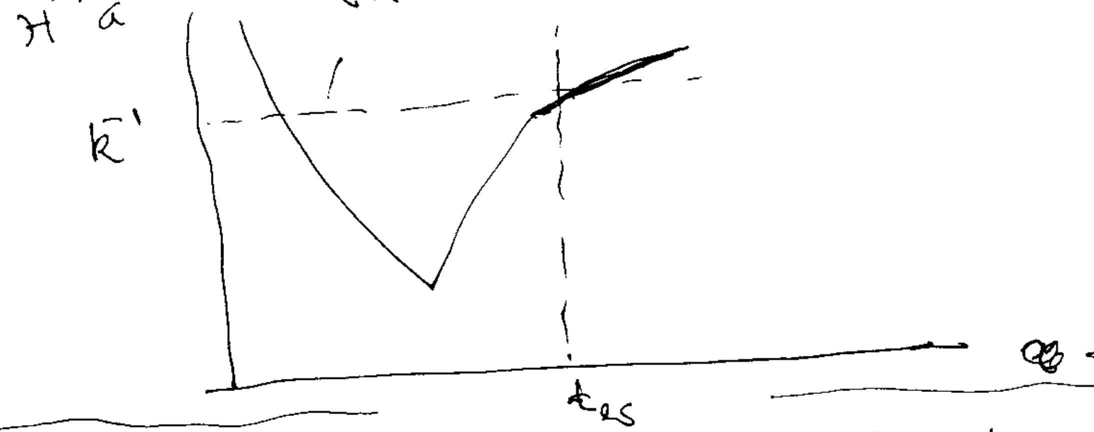
$$\xrightarrow[|\tau| \ll \frac{1}{\omega}]{} (2\pi)^3\, \delta^{(3)}(\vec{k}+\vec{k}')\, \frac{H^2}{2\omega^3}$$

$$\tau = -H^{-1}a^{-1} \qquad \omega = \frac{1}{k}$$

$$|\tau| \ll \frac{1}{k}$$

$$\Rightarrow \quad k^{-1} \gg H^{-1}a^{-1}$$

in this region.



Note: ① The correlation fn. becomes τ independent (not even a phase). before further evolution can be treated classically

⇒ can be related directly to CMB spectrum if $t_{ls} < t_{reentry}$.

⇒ We look at fluctuations over sufficiently large k' i.e. sufficiently large distance scale.

$\gtrsim 1\text{-}2^\circ$ in the sky.

At lower distance scale we have to take into account the effect of microscopic physics between $t_{reentry}$ and t_{ls}.

→ can be done in principle since the physics at this scale is well understood.

A subtle point:

As long as $v(\vec{k},\tau)$ satisfy eqn. of motion and

$$v(\vec{k},\tau)\,\partial_\tau v(\vec{k},\tau)^* - \partial_\tau v(\vec{k},\tau)\,v(\vec{k},\tau)^*$$
$$= (2\pi)^3 i$$

our analysis goes through.

Take

$$u(\vec{k},\tau) = \alpha\, v(\vec{k},\tau) + \beta\, v(\vec{k},\tau)^*$$

→ solves eq. of motion.

Ex. check that

$$u(\vec{k},\tau)\, \partial_\tau u(\vec{k},\tau)^* - \partial_\tau u(\vec{k},\tau)\, u(\vec{k},\tau)^*$$
$$= (2\pi)^3\, i \quad \text{if } \alpha = \cosh\theta,\ \beta = \sinh\theta$$

⇒ We could use $u(\vec{k},\tau)$ also as basis fn.

⇒ Definition of $a, a^\dagger$ would change

⇒ Defn of $|0\rangle$ will change.

(i.e. of original $(a, a^\dagger)$ annihilates $|0\rangle$)

$\langle 0|\, \hat{\phi}(\vec{k},\tau)\, \hat{\phi}(\vec{k}',\tau)\, |0\rangle$ will be different.

For free scalar in flat space the choice of $v(\vec{k},\tau)$ we have made gives minimum energy.

$$H = \int d^3\vec{k}\ |\vec{k}|\ a(\vec{k})^\dagger a(\vec{k})$$

→ takes its minimum value for $a(\vec{k})|0\rangle = 0$.

If we repeat this analysis for de Sitter we find that $\mathcal{H}$ is time dependent.

$\Rightarrow$ We cannot minimize $\mathcal{H}$ at all time.

Our choice: $\mathcal{H}$ is minimized for $\tau \to -\infty$ ($|\omega\tau| \gg 1$) (In this limit we get back free field theory)

$\to$ Bunch - Davies vacuum.

$\to$ Commonly used.

But we should keep in mind that there is in principle an ambiguity in defining vacuum in de Sitter space.

$$\mathcal{H} = \frac{1}{2}\int d^3x\, d\tau \left[\cancel{(\partial_\tau \tilde{\chi} + \tfrac{1}{\tau}\tilde{\chi})^2} \; \tfrac{1}{2}(\partial_\tau \tilde{\chi})^2 - (\tfrac{1}{\tau}\tilde{\chi})^2 + (\vec{\nabla}\tilde{\chi})^2 \right]$$

$$(\partial_\tau \tilde{\chi} + \tfrac{1}{\tau}\tilde{\chi})\, \partial_\tau \tilde{\chi} - \tfrac{1}{2}(\partial_\tau \chi + \tfrac{1}{\tau}\tilde{\chi})^2$$

$$= (\partial_\tau \tilde{\chi} + \tfrac{1}{\tau}\tilde{\chi})(\partial_\tau \tilde{\chi} - \tfrac{1}{2}\partial_\tau \tilde{\chi} - \tfrac{1}{2\tau}\tilde{\chi})$$

$$= \tfrac{1}{2}\left((\partial_\tau \tilde{\chi})^2 - (\tfrac{1}{\tau}\tilde{\chi})^2\right) \qquad \alpha^2 v \partial_\tau v^* - \partial_\tau$$

Metric perturbation

$$ds^2 = -(1+2\Phi)\,dt^2 + 2a(t)\,B_i\,dx^i\,dt + a(t)^2(\delta_{ij} + h_{ij})\,dx^i\,dx^j$$

$$a(t) = c\,e^{Ht}$$

Decompose B_i, h_{ij} as

$$B_i = \partial_i B \pm S_i \qquad \partial_i S_i = 0.$$

$$h_{ij} = -2\Psi\,\delta_{ij} + 2\,\partial_i\partial_j E + (\partial_i F_j + \partial_j F_i) + h^T_{ij}$$

$$\partial_i F_i = 0, \qquad \partial_i h^T_{ij} = h^T_{ii} = 0.$$

Meaning of this decomposition is more transparent in Fourier space:

$$\hat{B}_i = i k_i \hat{B} + \hat{S}_i \qquad k_i \hat{S}_i = 0$$

$\Downarrow$ component $\parallel \vec{k}$. $\quad$ components $\perp \vec{k}$

Similarly: $\hat{h}^T_{ij}$: components $\perp \vec{k}$ and traceless.

Φ, B, Ψ, E : scalar components

S_i, F_i : vector components

h^T_{ij} : tensor component.

Expand Einstein's action $-\frac{1}{16\pi G}\int\sqrt{-\det g}\, R$ to quadratic order in the fluctuations.

→ Cannot contain any X-terms between scalars, vectors, tensors.

(impossible to contract indices maintaining rotational invariance)

e.g. scalar-vector:

$$\underbrace{\phi\, \partial_i S_i}_{=0}$$

vector-tensor $S_i\, h^T_{kl}$

one of the indices of h^T must contract with a derivative.

⇒ Eqs. of motion for scalars, vectors, tensors decouple.

Quantization → can be done separately for scalars, vectors, tensors.

Must take into account "gauge invariance"

⇓

general coordinate transformation

$$t \to t \pm \xi^0(t,\vec{x})$$
$$x^i \to x^i \pm \xi^i(t,\vec{x})$$

Changes original background leading to trivial deformations.

$$\xi^i = \partial_i \xi(t,\vec{x}) + \zeta_i(t,\vec{x})$$

$$\partial_i \zeta_i = 0.$$

$\xi^0, \xi \to$ scalar parameters

$\to$ transform only scalar modes at the linearized level.

$\zeta_i \to$ vector parameter

$\to$ affects only vector modes at the linearized level.

$$h^T_{ij} \to h^T_{ij} \pm \partial_i \zeta_j \pm \partial_j \zeta_i \quad \text{possible?}$$

$$h^T_{ii} = 0 \Rightarrow \partial_i \zeta_i = 0 \to \text{automatic.}$$

$$\partial_i h^T_{ij} = 0 \Rightarrow \vec{\nabla}^2 \zeta_j + \underbrace{\partial_i \partial_j \zeta_i}_{=0} = 0$$

$$\Rightarrow \vec{k}^2 \hat{\zeta}_j = 0.$$

$\to$ not possible for generic $\vec{k}$.

Analyze the scalar, tensor and vector mode fluctuations separately.

Tensor modes: gauge invariant.

$$h_{ij}(\vec{x},\tau) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \hat{h}_{ij}(\vec{k},\tau)$$

$$\hat{h}^T_{ij} : \frac{3\times 4}{2} - 3 - 1 = 2 \text{ components}$$

$$\Downarrow \qquad \hat{h}^T_{ii} = 0$$

$$\partial_i \hat{h}^T_{ij} = 0$$

$$\hat{h}^T_{ij}(\vec{k},\tau) = \sum_s \hat{h}_s(\vec{k},\tau)\, e^{(s)}_{ij}(\vec{k})$$

$$k_i e^{(s)}_{ij} = 0 = e^{(s)}_{ii}$$

$$\sum_{i,j} e^{(s)}_{ij} e^{(s')}_{ij} = \delta_{ss'}$$

Ex. $S_{tensor} = \frac{1}{8} \int \frac{d^3k}{(2\pi)^3} \frac{d\tau}{H^2\tau^2}$

$$\left(\partial_\tau \hat{h}_s(-\vec{k},\tau)\, \partial_\tau \hat{h}_s(\vec{k},\tau) - \vec{k}^2 \hat{h}_s(-\vec{k},\tau)\, \hat{h}_s(\vec{k},\tau)\right)$$

$\frac{1}{2} \hat{h}_s$ has same dynamics as the scalar mode $\hat{\phi}$

$$\Rightarrow \langle \hat{h}_s(\vec{k},\tau)\, \hat{h}_{s'}(\vec{k}',\tau)\rangle$$

$$= 4 \times \delta_{ss'} \times (2\pi)^3 \delta^{(3)}(\vec{k}+\vec{k}') \frac{H^2}{2\omega^3}(1+\omega^2\tau^2)$$

Late time $\tau \to 0$ $|\omega\tau| \ll 1$

$$\langle \hat{h}_s(\vec{k},\tau)\, \hat{h}_{s'}(\vec{k}',\tau)\rangle = 4\, \delta_{ss'} (2\pi)^3 \delta^{(3)}(\vec{k}+\vec{k}')\, \frac{H^2}{2\omega^3}$$

This gives the amplitude for tensor perturbation.

One can show ~~that~~ scalars & vectors have no dynamics.

Eqs. of motion sets the modes to zero up to gauge transformations.

~~Comme~~ The above analysis gives the fluctuations at horizon exit.

$\Rightarrow$ controls the fluctuations at reentry / last scattering via classical evolution.

$\Rightarrow$ Must study these evolution equations.

Vector perturbation:

$$ds^2 = -dt^2 + 2a(t) S_i \, dx^i \, dt + a(t)^2 \left(\delta_{ij} + (\partial_i F_j + \partial_j F_i) \right) dx^i dx^j$$

$\partial_i S_i = 0, \quad \partial_i F_i = 0.$

Vector gauge transformation

$$t \to t, \quad x^i \to x^i - \xi^i(t, \vec{x}), \quad \partial_i \xi^i = 0.$$

$$ds^2 \to -dt^2 - 2a\,S_i(t)\left(dx^i - \partial_j \xi^i dx^j - \partial_t \xi^i dt\right) dt$$

$$+ a(t)^2 \{\delta_{ij} + \partial_i F_j + \partial_j F_i\}(dx^i - \partial_k \xi^i dx^k - \partial_t \xi^i dt)(dx^j - \partial_l \xi^j dx^l - \partial_t \xi^j dt)$$

$$\cong -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$$

$$- 2a\,S_i(t)\, dx^i dt + a(t)^2(\partial_i F_j + \partial_j F_i) dx^i dx^j$$

$$+ a(t)^2 dx^j (-\partial_l \xi^j dx^l - \partial_t \xi^j dt)$$

$$+ a(t)^2 dx^i (-\partial_k \xi^i dx^k - \partial_t \xi^i dt)$$

$$\cong -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$$

$$- 2a\{S_i(t) + a(t)\partial_t \xi^i\} dx^i dt$$

$$+ a(t)^2\{\partial_i F_j + \partial_j F_i - \partial_i \xi_j - \partial_j \xi_i\} dx^i dx^j$$

$$\Rightarrow \quad S_i \to S_i + a(t)\partial_t \xi^i$$

$$F_i \to F_i - \xi_i$$

$\partial_i \xi^i = 0$ consistent with $\partial_i S_i = 0 = \partial_i F_i$

$S_i/a + \dot{F}_i$ is gauge invariant

Einstein equation

$$\Rightarrow \quad \vec{\nabla}^2 (S_i/a + \dot{F}_i) = 0$$

Fourier space $\omega^2(\hat{S}_i/a(t) + \hat{\dot{F}}_i) = 0$

$\Rightarrow \hat{S}_i/a + \hat{\dot{F}}_i = 0 \to$ only gauge invariant variable.

Scalar perturbations.

Show that under $x^0 \to x^0 - \xi^0(\vec{x},t)$, $x^i \to x^i - \partial_i \xi(\vec{x},t)$
$\overset{\alpha}{\shortparallel}$ $= x^i - \partial_i \beta$.

$$\Phi \to \Phi - \dot{\alpha}$$

$$B \to B + a'\alpha - a\dot{\beta}$$

$$E \to E - \beta$$

$$\Psi \to \Psi + \mathcal{H}\alpha$$

$\Rightarrow$ Gauge invariant combinations:

$$\Phi_B = \Phi - \frac{d}{dt}\left(a^2(\dot{E} - B/a)\right)$$

$$\Psi_B = \Psi + a^2\mathcal{H}(\dot{E} - B/a)$$

$\Phi_B = \Phi$
$\Psi_B = \Psi$
in $E = B = 0$ gauge.

Einstein's equation (in Fourier space)

$$k_i\left(\dot{\hat{\Psi}}_B + \mathcal{H}\hat{\Phi}_B\right) = 0$$

$$3\mathcal{H}\left(\dot{\hat{\Psi}}_B + \mathcal{H}\hat{\Phi}_B\right) + \frac{k^2}{a^2}\hat{\Psi}_B = 0$$

$\Rightarrow \hat{\Psi}_B = 0, \quad \hat{\Phi}_B = 0.$ for generic $\vec{k}$.

$\Rightarrow$ all gauge invariant scalar perturbations vanish.

No dynamics in scalar and vector perturbations.

(212)

$$ds^2 = -dt^2 + a(t)^2\, d\vec{x}^2$$

$$\to -\left(dt - \partial_t \xi^0\, dt - \partial_i \xi^0\, dx^i\right)^2$$

$$+ \left(a(t) - \dot{a}(t)\,\xi^0\right)^2 \left(dx^i - \partial_t \xi^i\, dt - \partial_k \xi^i\, dx^k\right)^2$$

$$= -dt^2 + 2\,\partial_t \xi^0\, dt^2 + 2\,\partial_i \xi^0\, dt\, dx^i$$

$$- a(t)^2\, dx^i dx^i - 2\, a\dot{a}\,\xi^0\, dx^i dx^i$$

$$- 2a(t)^2\, dx^i\, \partial_t \xi^i\, dt - a(t)^2\, dx^i dx^j \left(\partial_i \xi_j + \partial_j \xi_i\right)$$

Compare with: $\quad \xi^i = \partial_i \xi + \zeta_i$

$$ds^2 = -(1+2\Phi)\, dt^2 + 2a(t)\, B_i\, dx^i\, dt$$

$$+ a(t)^2 \left(\delta_{ij} + h_{ij}\right) dx^i dx^j$$

$$B_i = \partial_i B - S_i$$

$$h_{ij} = -2\Psi\,\delta_{ij} + 2\,\partial_i \partial_j E + \partial_i F_j + \partial_j F_i$$

$$+ h^T_{ij}$$

$$\delta\Phi = -\partial_t \xi^0$$

$$\delta B = a^{-1}\xi^0 - a\,\partial_t \xi$$

$$\delta\Psi = H\,\xi^0$$

$$\delta S_i = + a\,\partial_t \zeta_i$$

$$\delta F_i = -\zeta_i$$

$$\delta E = -\xi$$

$$\delta h_{ij} = 0$$

$$\delta B_i = a^{-1}\left(\partial_i \xi^0 - a^2\,\partial_t \xi^i\right)$$

$$\delta h_{ij} = -\partial_i \xi_j - \partial_j \xi_i$$

$$\Rightarrow \delta B_i = a^{-1}\left(\partial_i \xi^0 - a^2\,\partial_t \zeta^i - a^2\,\partial_t \partial_i \xi\right)$$

$$\delta h_{ij} = -\left(\partial_i \zeta_j + \partial_j \zeta_i\right) - 2\,\partial_i \partial_j \xi$$

Slow roll inflation

So far we have proceeded assuming that $\dot{\bar{\Phi}} = c$ and $V(\phi) =$ Constant exactly.
⇓
background.

Now we shall consider slow roll approximation in which $\dot{\bar{\Phi}}$ and $V'(\phi)$ are small but non-zero.

So We expect our previous computation to still go through but with some subtleties related to gauge invariance.

Vector and tensor perturbations do not mix with scalar perturbation of ϕ & remain unaffected.

Focus on scalar perturbations:

$$ds^2 = -(1+2\Phi)\,dt^2 + 2a(t)\,\partial_i B\, dx^i\, dt + a(t)^2\left[(1-2\Psi)\,\delta_{ij} + 2\partial_i\partial_j E\right] dx^i dx^j$$

$$\phi(\vec{x},t) = \bar{\Phi}(t) + \tilde{\Phi}(\vec{x},t).$$

Gauge transformation:

$$t \to t - \xi^0, \quad x^i \to x^i - \partial_i \xi.$$

$$\Phi \to \Phi - \partial_t \xi^0$$

$$B \to B + a^{-1} \xi^0 - a \partial_t \xi$$

$$\Psi \to \Psi + H \xi^0$$

$$E \to E - \xi$$

$$\phi \to \bar{\phi}(t - \xi^0) = \bar{\phi}(t) - \xi^0 \partial_t \bar{\phi}$$

$$\Rightarrow \tilde{\phi} \to \tilde{\phi} - \xi^0 \partial_t \bar{\phi}$$

We had earlier introduced gauge invariant observables:

$$\Phi_B = \Phi - \frac{d}{dt}\left(a^2 (\partial_t E - B/a)\right)$$

$$\Psi_B = \Psi + a^2 H (\partial_t E - B/a)$$

We need one more

$$\zeta = -\left(\Psi + \frac{H}{\dot{\phi}} \tilde{\phi}\right)$$

$$\mathcal{R} = \Psi + \frac{H}{\dot{\phi}} \tilde{\phi}$$

Note $\mathcal{R} = -\zeta$.

(later we'll see that in the more general case $\mathcal{R}$ and ζ could be different).

We have ~~seen~~ that

$\Psi_B \simeq 0 \quad \Phi_B \simeq 0$

$\Rightarrow$ In the $E=0$, $B=0$ gauge

$\Psi \simeq 0, \quad \Phi \simeq 0.$

~~In the same gau~~

$\Rightarrow ds^2 = -dt^2 + a(t)^2 \, dx^i dx^j$

In this background we have calculated

$$\langle \hat{\varphi}(\vec{k},\tau) \, \hat{\varphi}(\vec{k},\tau) \rangle$$
$$= (2\pi)^3 \, \delta^{(3)}(\vec{k}+\vec{k}') \, \frac{H^2}{2\omega^3} \quad \text{for } \tau \to 0$$

Now in the $E=B=0$ gauge

$\Psi_B = \Psi, \quad \Phi_B = \Phi$

$$R = -\zeta = \frac{H}{\dot{\phi}} \, \hat{\varphi}$$

gauge independent

$$\Rightarrow \langle \hat{R}(\vec{k},\tau) \, \hat{R}(\vec{k}',\tau) \rangle$$
$$= (2\pi)^3 \, \delta^{(3)}(\vec{k}+\vec{k}') \, \frac{H^2}{\dot{\phi}^2} \, \frac{H^2}{2\omega^3}$$

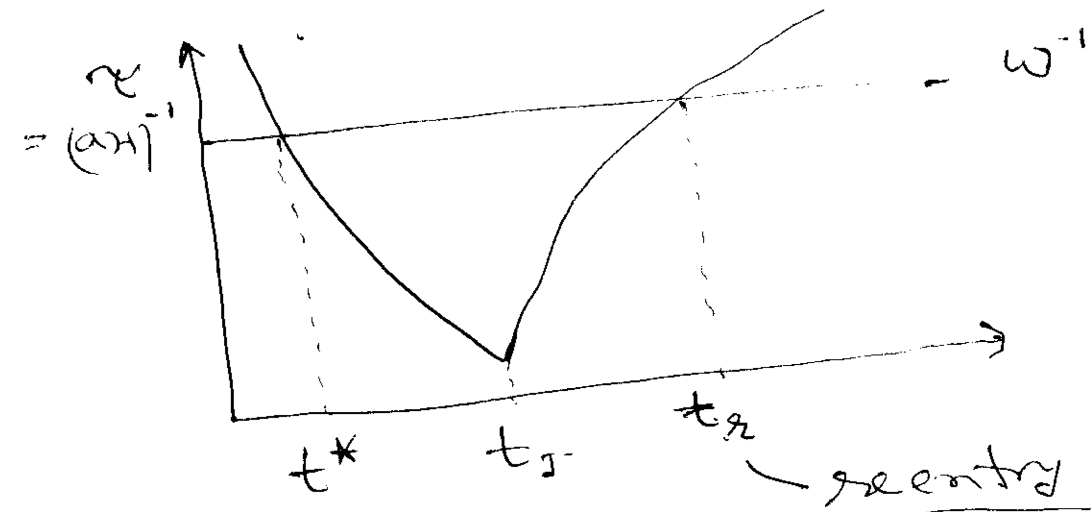
for $\tau \to 0$.

$$\langle \hat{h}^{(s)}(\vec{k},\tau) \, \hat{h}^{(s')}(\vec{k}',\tau) \rangle$$
$$= (2\pi)^3 \, \delta^{(3)}(\vec{k}+\vec{k}') \, \delta_{ss'} \, \frac{2H^2}{\omega^3}$$

Note: $\mathcal{H}$ and $\dot{\bar{\Phi}}$ changes slowly.

$\epsilon = -\frac{\dot{\mathcal{H}}}{\mathcal{H}^2}$, $\eta = -\frac{\ddot{\bar{\Phi}}}{\dot{\bar{\Phi}} H}$ are small.

Q. At which time should we compute the r.h.s.?



At $t^* \to$ horizon x-ing.

We'll show later that after horizon x-ing $\mathcal{R}$ and $h^{(s)}$ becomes constant

---

(Like $\hat{\Phi}$ and $h^{(s)}$ for de Sitter space - constant H)

Fluctuations in $\hat{\mathcal{R}}$ and $\hat{h}^{(s)}$ are directly observed in the CMB if $t_R < t_{l.s.}$

$$\sum_{s=1}^{2} \langle h^{(s)}(\vec{k},\tau)\, h^{(s)}(\vec{k}',\tau) \rangle$$

$$= (2\pi)^3\, \delta^{(3)}(\vec{k}+\vec{k}') \left(\frac{4H^2}{\omega^3}\right) \equiv P_T$$

tensor power spectrum

$$\langle \mathcal{R}(\vec{k},\tau)\, \mathcal{R}(\vec{k}',\tau) \rangle$$

$$= (2\pi)^3\, \delta^{(3)}(\vec{k}+\vec{k}') \left(\frac{H^4}{2\dot{\Phi}^2 \omega^3}\right) \equiv P_R$$

scalar power spectrum

$$\Delta_s^2 = \frac{\omega^3}{2\pi^2} P_R = \frac{H^4}{4\pi^2 \dot{\Phi}^2}$$

$$\Delta_T^2 = \frac{\omega^3}{2\pi^2} P_T = 2H^2/\pi^2$$

---

tensor to scalar ratio

$$r = \frac{\Delta_T^2}{\Delta_s^2} = \frac{8\dot{\Phi}^2}{H^2}$$ → claimed to be observed by BICEP2 to be .2

Δ_s observed in CMB by COBE, WMAP, PLANCK, ...

$$\Delta_s \simeq 10^{-5}$$

Implications

① $\frac{2\mathcal{H}^2}{\pi^2} = \Delta_T^2 \simeq .2 \times \Delta_S^2 \simeq .2 \times 10^{-10}$

$\mathcal{H}^2 = \frac{V}{3}$ in $8\pi G = 1$ unit.

$\Rightarrow \frac{2V}{3\pi^2} \simeq .2 \times 10^{-10}$

$V = 3\pi^2 \times 10^{-11}$

M_I = scale of inflation

$V = M_I^4$ (from dimensional analysis in $\hbar = c = 1$ unit)

energy density.

$M_I^4 = 3\pi^2 \times 10^{-11}$

$M_I \simeq (3\pi^2 \times 10^{-11})^{1/4} \times M_{Pl,\,reduced}$

$M_{Pl,\,reduced} = \frac{1.22 \times 10^{19}\ \text{GeV}}{\sqrt{8\pi}}$

$\simeq 10^{16}$ GeV

Fixes the scale of inflation.

② $r = \frac{8\dot{\Phi}^2}{\mathcal{H}^2}$

$\Rightarrow \dot{\Phi} = \sqrt{\frac{r}{8}}\,\mathcal{H}$

If inflation begins at t_1 and ends at t_2 then:

$$\Phi(t_2) - \Phi(t_1) = \int_{t_1}^{t_2} \dot{\Phi}(t)\, dt$$

$$= \int_{t_1}^{t_2} \sqrt{\frac{r}{8}}\, H(t)\, dt$$

$$\simeq \sqrt{\frac{r}{8}}\, N$$

$\hookrightarrow$ number of e-foldings.

$$\Delta\phi = \sqrt{\frac{r}{8}} \times N.$$

Now $N > 60$.

$$\Rightarrow \Delta\phi > \sqrt{\frac{r}{8}} \times 60.$$

If we want $\Delta\phi < 1$ ← in Planck units

then $\frac{r}{8} < \frac{1}{3600}$

$$\Rightarrow r < \frac{1}{500} = .002$$

(Lyth bound)

$$r = .2 \Rightarrow \Delta\phi > \sqrt{\frac{.2}{8}} \times 60 \simeq 10.$$

$\Rightarrow$ The inflaton travels at least 10 Planck units.

– In general it is difficult to find $V(\phi)$ that remains constant over such large scales.

Spectral index:

$$n_s - 1 = \frac{d \ln \Delta_s^2}{d(\ln \omega)}$$

$$n_T = \frac{d \ln \Delta_T^2}{d(\ln \omega)}$$

$$\ln \Delta_s^2 = \ln \frac{H^4}{\dot{\Phi}^2}\Big|_* - \ln(4\pi^2)$$

$$\ln \Delta_T^2 = \ln H^2|_* + \ln(2/\pi^2)$$

ω dependence comes through H_*, $\dot{\Phi}_*$.

$\omega + \Delta\omega \Rightarrow$ crossing at $t + \Delta t$.

$$\omega = aH$$

$$d\omega = \dot{a} H\, dt + a \underset{\text{small}}{\dot{H}}\, dt$$

$$= H^2 a\, dt$$

$$\Rightarrow \omega \frac{d}{d\omega} \ln \Delta_s^2 = \frac{aH}{H^2 a} \frac{d}{dt} \ln \frac{H^4}{\dot{\Phi}^2}\Big|_*$$

$$= \frac{1}{H}\left\{ 4\frac{\dot{H}}{H} - 2\frac{\ddot{\Phi}}{\dot{\Phi}} \right\}_* = -4\epsilon^* + 2\eta^*$$

$$n_s - 1 = -4\epsilon^* + 2\eta^*$$

$$n_T = \omega \frac{d}{d\omega} \ln \Delta_T^2 = \frac{1}{H}\left(2\frac{\dot{H}}{H}\right)\Big|_* = -2\epsilon^*$$

$$n_T = -2\epsilon^*$$

Physical interpretation of $\mathcal{R} = \Psi + \frac{H}{\dot{\bar\phi}} \tilde\phi$

Recall gauge trs:

$$\tilde\phi \to \tilde\phi - \xi^0 \partial_t \bar\phi \,, \quad \Psi \to \Psi + H\xi^0$$

In the ~~usual~~ $B = E = 0$ gauge $\tilde\phi$ represents scalar fluctuations.

$\Rightarrow$ In different points of space $\tilde\phi$ takes different values.

+ve ~~Larger~~ $\tilde\phi \Rightarrow$ inflation ends early

-ve $\tilde\phi \Rightarrow$ Inflation ends late.

ρ begins to decrease early.

$\Rightarrow$ smaller ρ at the end.

ρ begins to decrease late.

$\Rightarrow$ larger ρ at the end.

$\Rightarrow \tilde\phi$ fluctuation gets converted to density fluctuation.

However by choosing $\xi^0 = + \tilde\phi / \dot{\bar\phi}$ we can ensure that $\tilde\phi = 0$.

At ~~the~~ a given time $\tilde\phi$ has the same value everywhere.

In this gauge $\mathcal{R} = \Psi$

$-\vec{\nabla}^2 \psi = \vec{k}^2 \psi$ has the interpretation of a Ricci scalar of the 3-manifold on constant t surface

$\Downarrow$

Constant Φ surface

In this description the fluctuation is completely in the metric degree of freedom.

$\rightsquigarrow$ affects CMB by providing different gravitational potential at different points & hence different apparent temperature due to gravitational red shift.