Classical evolution.

$$ds^2 = -(1+2\Phi)\,dt^2 + 2aB_i\,dx^i dt + a(t)^2\left[\delta_{ij} + h_{ij}\right] dx^i dx^j$$

General energy-momentum tensor (not necessarily the one due to scalar fields).

$$T_\mu{}^\nu = \underbrace{\bar{T}_\mu{}^\nu}_{\text{background}} + \underbrace{\tilde{T}_\mu{}^\nu}_{\text{fluctuations.}}$$

$$\bar{T}_0{}^0 = -\bar{\rho}$$

$$\bar{T}_i{}^j = \bar{p}\,\delta_i{}^j \qquad \bar{T}_i{}^0 = \bar{T}_0{}^i = 0.$$

$$\tilde{T}_0{}^0 = \tilde{\rho}$$

$$\tilde{T}_i{}^0 = \tilde{q}_i = \partial_i \tilde{q} + Q_i \qquad \partial_i Q_i = 0.$$

Ex. $\tilde{T}_0{}^i = -a^{-2}\tilde{q}_i + (\bar{\rho} + \bar{p})\,B_i/a$

$$\tilde{T}_j{}^i = \delta^i_j\,\tilde{p} + \underbrace{\Sigma_i{}^j}_{\text{anisotropic stress tensor.}}$$

$$\Sigma_i{}^i = 0$$

Can be decomposed into transverse traceless tensor, vector, scalar.

Gauge transformation:

$$x^i \to y^i = x^i - \xi^i$$

$$\xi^i = \partial_i \xi + \zeta^i$$

$$x^0 \to y^0 = x^0 - \xi^0$$

$$\cancel{T_{\mu\nu} dx^\mu dx^\nu = (T_{\mu\nu} + \delta T_{\mu\nu}) dy^\mu dy^\nu}$$

~~To linear order~~

$$(T_{\mu\nu} + \delta T_{\mu\nu}) dx^\mu dx^\nu$$

$$= T_{\mu\nu}(y)\, dy^\mu dy^\nu$$

$$\bar{T}^{\mu}{}_{\nu} + \delta T_{\mu}{}^{\nu} = \bar{T}^{\mu}{}_{\rho}{}^{\sigma} \frac{\partial y^\rho}{\partial x^\mu} \frac{\partial x^\nu}{\partial y^\sigma}$$

Ex. check that to linear order:

$$\tilde{q}_i = \partial_i \tilde{q} + \theta_i$$

$$\tilde{\rho} \to \tilde{\rho} - \dot{\bar{\rho}}\, \xi^0$$

$$\tilde{p} \to \tilde{p} - \dot{\bar{p}}\, \xi^0$$

$$\tilde{q} \to \tilde{q} + (\bar{\rho} + \bar{p})\, \xi^0$$

$$\Sigma_i{}^j \to \Sigma_i{}^j$$

$$\theta_i \to \theta_i$$

gauge invariant

Recall:

$$\Phi \to \Phi - \partial_t \xi^0$$

$$B \to B + a^{-1} \xi^0 - a \partial_t \xi$$

$$E \to E - \xi$$

$$\Psi \to \Psi + \mathcal{H} \xi^0$$

Earlier we had introduced gauge invariant combinations Ψ_B, Φ_B.

~~Three~~ Four more gauge invariant scalars:

$$-\zeta = \Psi + \frac{\mathcal{H}}{\dot{\bar{\rho}}}\tilde{\rho}$$

$$\mathcal{R} = \Psi - \frac{\mathcal{H}}{\bar{\rho}+\bar{p}}\tilde{q}$$

$$\Sigma \quad p_{en} = \tilde{p} - \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}}\tilde{\rho}$$

Check for single scalar field in slow roll

$-\zeta = \mathcal{R}$ = earlier expression

$$\Sigma_{ij} = \Sigma^{T}_{ij} + (\partial_i \Sigma_j + \partial_j \Sigma_i) + \left(\partial_i \partial_j \Sigma - \frac{1}{3}\delta_{ij}\vec{\nabla}^2\Sigma\right)$$

$$\partial_i \Sigma_i = 0$$

Einstein eq. + $D_\mu T^{\mu\nu} = 0$ $\Rightarrow$ scalar sector

$$3\mathcal{H}(\dot{\Psi}_B + \mathcal{H}\Phi_B) + \frac{k^2}{a^2}\Psi_B = 4\pi G\frac{\dot{\bar{\rho}}}{\mathcal{H}}(\zeta + \Psi_B)$$

$$\dot{\Psi}_B + \mathcal{H}\Phi_B = 4\pi G\frac{\bar{\rho}+\bar{p}}{\mathcal{H}}(\mathcal{R} - \Psi_B)$$

$$(\Psi_B - \Phi_B) = 8\pi G a^2 \delta\Sigma$$

$$\dot{\zeta} = -\mathcal{H}\frac{p_{en}}{\bar{\rho}+\bar{p}} - \Pi$$

$$\frac{\Pi}{\mathcal{H}} \equiv -\frac{k^2}{3a^2\mathcal{H}^2}\left[\zeta - \Psi_B\left(1 - \frac{2\bar{\rho}}{9(\bar{\rho}+\bar{p})}\frac{k^2}{a^2\mathcal{H}^2}\right)\right]$$

Initial condition after horizon exit.

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \left(\eta^{\rho\sigma} \partial_\rho \phi \, \partial_\sigma \phi + V(\phi) \right)$$

$$\Rightarrow \quad T^0_i = -\partial_i \phi \, \partial_0 \phi = -\dot{\bar{\phi}} \, \partial_i \tilde{\phi}$$

$$\equiv \partial_i \tilde{q}$$

$$\Rightarrow \tilde{q} = -\dot{\bar{\phi}} \, \tilde{\phi}$$

$$R = \psi - \frac{H}{\bar{\rho} + \bar{p}} \tilde{q} = \psi + \frac{H}{\dot{\bar{\phi}}^2} \dot{\bar{\phi}} \tilde{\phi}$$

$$= \psi + \frac{H}{\dot{\bar{\phi}}} \tilde{\phi} \rightarrow \text{old definition}$$

R at horizon exist is a constant given by our $\langle \hat{R} \hat{R} \rangle$ calculation.

$\psi_B, \Phi_B = 0$ at horizon exit.

What about S?

$$-S = \psi + \frac{H}{\dot{\bar{\rho}}} \tilde{\rho}$$

$$\bar{\rho} = \frac{1}{2} (\partial_0 \phi)^2 + V(\phi)$$

$$\tilde{\rho} = \rho - \bar{\rho} = \partial_0 \bar{\phi} \, \partial_0 \tilde{\phi} + V'(\phi) \tilde{\phi}$$

$$= \dot{\bar{\phi}} \dot{\tilde{\phi}} - 3H \dot{\bar{\phi}} \tilde{\phi}$$

$$\dot{\bar{\rho}} = \ddot{\phi} \dot{\phi} + V'(\phi) \dot{\phi} = -3H \dot{\bar{\phi}}^2$$

$$-\zeta = \psi + \frac{H}{\dot{\bar{\rho}}}\,\tilde{\rho}$$

$$= \psi + \frac{H}{(-3H\dot{\bar{\phi}}^2)}\left(\dot{\bar{\phi}}\,\dot{\tilde{\phi}} - 3H\,\dot{\bar{\phi}}\,\tilde{\phi}\right)$$

$$= \psi + \frac{H}{\dot{\bar{\phi}}}\,\tilde{\phi} - \frac{1}{3\dot{\bar{\phi}}}\,\dot{\tilde{\phi}}$$

$$= R - \frac{1}{3\dot{\bar{\phi}}}\,\dot{\tilde{\phi}} \qquad , \quad \tau \propto e^{-Ht}$$

$$\dot{\tilde{\phi}} = \partial_t \tilde{\phi} = \frac{d\tau}{dt}\,\partial_\tau \tilde{\phi}$$

$$\sim H\tau\,\omega\,\tilde{\phi} \qquad \to i\omega\tilde{\phi}$$

$$\Rightarrow -\zeta = R - \frac{1}{3\dot{\bar{\phi}}}\,H\tau\omega\tilde{\phi}$$

$$= R - \frac{H}{\dot{\bar{\phi}}}\,\frac{1}{3}\,\underbrace{\omega\tau}_{\ll 1}\,\tilde{\phi}$$

$\ll 1$ after horizon exit.

$\Rightarrow$ On superhorizon scale $-\zeta \simeq R$.

With these boundary condition we can evolve.

$\rightarrow$ Need analog of eq. of state

① $\Sigma^{0j}_{i} = 0 \Rightarrow \Sigma^{ij}_{i} = \Sigma_i = \Sigma = 0$

(Perfect fluid form)

(2) $p_{en} = \tilde{p} - \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}} \tilde{\rho} = 0$

true if fluctuations follow same eq. of state as background.

$$\bar{p} = f(\bar{\rho}) \qquad \dot{\bar{p}} = f'(\bar{\rho})\,\dot{\bar{\rho}}$$

$$\bar{p} + \tilde{p} = f(\bar{\rho} + \tilde{\rho}) \Rightarrow \tilde{p} = f'(\bar{\rho})\,\tilde{\rho}$$

$$\Rightarrow \tilde{p} - \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}}\tilde{\rho} = \tilde{p} - f'(\bar{\rho})\,\tilde{\rho} = 0$$

$\Rightarrow$ 4 eqs. in 4 unknowns

Ψ_B, Φ_B, ζ, R.

We proceed by assuming that Ψ_B, Φ_B, ζ, R remains ~~constant~~ finite during evolution.

$\rightarrow$ to be verified afterwards.

$$\dot{\zeta} = -\Pi \simeq 0 \quad \text{for } k^2 \ll a^2 H^2$$

$\Rightarrow \zeta$ is constant.

$\Psi_B = \Phi_B$ for $\delta\Sigma = 0$.

① $3\mathcal{H}(\dot{\Psi}_B + \mathcal{H}\Phi_B) + \frac{k^2}{a^2}\Psi_B = 4\pi G \frac{\dot{\bar{\rho}}}{\mathcal{H}}(\zeta + \Psi_B)$

② $\dot{\Psi}_B + \mathcal{H}\Phi_B = 4\pi G \frac{\bar{\rho} + \bar{p}}{\mathcal{H}}(R - \Psi_B)$

$$\frac{d}{dt}(\bar{\rho} a^3) + 3\bar{p} a^2 \dot{a} = 0$$

$$\Rightarrow \dot{\bar{\rho}} + 3\mathcal{H}\bar{\rho} + 3\mathcal{H}\bar{p} = 0$$

$$\dot{\bar{\rho}} = -3\mathcal{H}(\bar{\rho} + \bar{p})$$

① $\Rightarrow$ $\dot{\Psi}_B + \mathcal{H}\Phi_B + \frac{k^2}{3\mathcal{H}a^2}\Psi_B = 4\pi G\left(-\frac{\bar{\rho}+\bar{p}}{\mathcal{H}}\right)(\zeta + \Psi_B)$

①'

①'-2 $\Rightarrow$ $\frac{k^2}{3\mathcal{H}a^2}\Psi_B = 4\pi G \frac{\bar{\rho}+\bar{p}}{\mathcal{H}}(-\zeta - R)$

$$\Rightarrow \zeta + R = -\frac{\mathcal{H}}{\bar{\rho}+\bar{p}} \frac{1}{4\pi G} \frac{k^2}{3\mathcal{H}a^2}\Psi_B$$

$$\mathcal{H}^2 = \frac{8\pi G}{3}\bar{\rho} \Big| = -\frac{2\bar{\rho}}{3(\bar{\rho}+\bar{p})} \underbrace{\frac{k^2}{3\mathcal{H}^2 a^2}}_{\text{small}} \Psi_B$$

$\Rightarrow \zeta = -R$ on super horizon scale.

Thus we have learned

$\zeta = -R =$ constant $\rightarrow$ determined from $\langle RR \rangle$

$\Psi_B = \Phi_B$

At $t = 0$, $\Psi_B = \Phi_B = 0$

Now we have to find time evolution of $\psi_B = \Phi_B$.

$$\dot{\psi}_B + \mathcal{H}\,\underset{\psi_B}{\underset{\parallel}{\Phi_B}} = 4\pi G\,\frac{\bar{\rho}+\bar{p}}{\mathcal{H}}\,(R-\psi_B)$$

$$\dot{\psi}_B + \Big(\mathcal{H} + \underset{\frac{1}{2}}{\underset{\parallel}{4\pi G}}\,\frac{\bar{\rho}+\bar{p}}{\mathcal{H}}\Big) = \underset{\frac{1}{2}}{\underset{\parallel}{4\pi G}}\,\frac{\bar{\rho}+\bar{p}}{\mathcal{H}}\,R$$

$$\dot{\psi}_B + \mathcal{H}^{-1}\Big(\underset{=\frac{\bar{\rho}}{3}}{\mathcal{H}^2} + \tfrac{1}{2}(\bar{\rho}+\bar{p})\Big) = \frac{1}{2}\,\frac{\bar{\rho}+\bar{p}}{\mathcal{H}}\,R$$

$$\dot{\psi}_B + \mathcal{H}^{-1}\,\frac{1}{6}\,(5\bar{\rho}+3\bar{p}) = \frac{1}{2}\,\frac{\bar{\rho}+\bar{p}}{\mathcal{H}}\,R$$

$$\bar{p} = \omega\bar{\rho} \quad (\text{say})$$

$$\dot{\psi}_B + \mathcal{H}^{-1}\,\frac{1}{6}\,(5+3\omega)\bar{\rho} = \frac{1}{2}(1+\omega)\,\frac{\bar{\rho}}{\mathcal{H}}\,R$$

$$\frac{d}{dt}(\bar{\rho}\,a^3) + 3a^2\dot{a}\,\underset{=\omega\bar{\rho}}{p} = 0$$

$$\Rightarrow \quad \bar{\rho} = \rho_0\,a^{-3(1+\omega)}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\,\bar{\rho} = \frac{1}{3}\,\rho_0\,a^{-3(1+\omega)}$$

$$\Rightarrow \quad a = c\,t^{\frac{2}{3(1+\omega)}} \qquad \mathcal{H} = \frac{\dot{a}}{a} = \frac{2}{3(1+\omega)\,t}$$

$$\bar{\rho} = 3\mathcal{H}^2 = \frac{4}{3(1+\omega)^2\,t^2}$$

$$\Rightarrow \quad \dot{\psi}_B + \underset{A}{\underset{\parallel}{\frac{5+3\omega}{3(1+\omega)}}}\,\psi_B = \frac{1}{t}\,R$$

$$\psi_B = \frac{R}{A} + \underline{c\,t^{-A}} \quad \text{integration constant}$$

Thus for large t in any given epoch

$$\Psi_B \approx \frac{R}{A}.$$

For matter dominated epoch (during $t_{R.S.}$) we have $\omega = 0$.

$$\Rightarrow \Psi_B = \frac{3}{5} R \Rightarrow \Phi_B = \frac{3}{5} R.$$

Vector perturbation

Metric degrees of freedom:

$$F_i, S_i$$

$$B_i = \partial_i B - S_i$$

$$h_{ij} = h^T_{ij} + (\partial_i F_j + \partial_j F_i) + 2\partial_i \partial_j E$$

Gauge invariant combination:

$$\tilde{F}_i + S_i/a$$

For $T_{\mu\nu}$ we get

$$Q_i, \Sigma_i \to \text{both gauge invariant}$$

Einsteins eq. + $D_\mu T^{\mu\nu} = 0$.

$$\Rightarrow \dot{Q}_i + 3HQ_i = k^2 \Sigma_i$$

$$k^2(\dot{F}_i + S_i/a) = 16\pi G Q_i$$

If $\Sigma_i = 0$

$\Rightarrow \dot{Q}_i + 3\mathcal{H} Q_i = 0$

$$Q_i = \exp\left(-3\int_{t_0}^{t} \mathcal{H}(t')\,dt'\right) Q_i(t_0)$$

$\Rightarrow Q_i$ decays exponentially.

$\Rightarrow (\dot{F}_i + S_i/a)$ also decays exponentially

$\Rightarrow$ Vector perturbations vanish ~~at~~ ~~[illegible]~~ soon after t_*.
$\hookrightarrow$ exit

Metric perturbation

Metric $h^T_{ij} \rightarrow h^{(s)}$ $s = 1, 2$.

$T_\mu{}^\nu \rightarrow \Sigma^T_{ij}$ — assumed to be 0

Einsteins eq:

$$\ddot{h}^{(s)} + 3\mathcal{H}\dot{h}^{(s)} + \frac{k^2}{a^2} h = 0$$

For k^2/a^2 small

$$\ddot{h}^{(s)} + 3\mathcal{H}\dot{h}^{(s)} = 0$$

$$\Rightarrow \dot{h}^{(s)} = \exp\left(-3\int_{t_0}^{t} \mathcal{H}(t')\,dt'\right) \rightarrow 0$$

for $t \gg t_*$

$h^{(s)} \rightarrow$ constant for $t \gg t_*$ conformal
$\rightarrow$ equal to what we have from quantum effects

Scalar perturbation

$$ds^2 = -(1+2\Phi)\,dt^2 + 2a\,\partial_i B\,dx^i\,dt$$
$$+ a(t)^2\left[(1-2\Psi)\,\delta_{ij} + 2\,\partial_i\partial_j E\right]dx^i\,dx^j$$

$T_0^{\ 0} = -(\bar{\rho} + \underbrace{\tilde{\rho}}_{})$ note correct sign.

$T_i^{\ 0} = -\partial_i \tilde{q}$

$T_i^{\ j} = (\bar{p} + \tilde{p})\,\delta_i^{\ j} + \left(\partial_i\partial_j \Sigma - \frac{1}{3}\delta_{ij}\vec{\nabla}^2\Sigma\right)$

$\bar{p}$, $\bar{\rho}$, $a(t)$: background values.

Gauge invariant variables

$\Phi_B = \Phi - \frac{d}{dt}\left(a^2(\dot{E} - B/a)\right)$

$\Psi_B = \Psi + a^2 H(\dot{E} - B/a)$ or $\dot{\Psi}/a$

$S = -\Psi - \frac{H}{\dot{\bar{\rho}}}\tilde{\rho}$

$R = \Psi - \frac{H}{\bar{\rho} + \bar{p}}\tilde{q}$

$p_{eff} = \tilde{p} - \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}}\tilde{\rho}$

Σ

Einstein's eq. + $D_\mu T^{\mu\nu} = 0$ in scalar sector

① $3\mathcal{H}(\dot{\Psi}_B + \mathcal{H}\Phi_B) + \frac{k^2}{a^2}\Psi_B = 4\pi G \frac{\bar{\rho}}{\mathcal{H}}(\delta + \Psi_B)$.

② $\dot{\Psi}_B + \mathcal{H}\Phi_B = 4\pi G \frac{\bar{\rho}+\bar{p}}{\mathcal{H}}(R - \Psi_B)$

③ $\Psi_B - \Phi_B = 8\pi G a^2 \Sigma$

④ $\dot{\delta} = -\mathcal{H}\frac{\delta p_{en}}{\bar{\rho}+\bar{p}} - \Pi$

$$\Pi/\mathcal{H} = -\frac{k^2}{3a^2\mathcal{H}^2}\left[\delta - \Psi_B\left(1 - \frac{2\bar{\rho}}{\mathcal{H}(\bar{\rho}+\bar{p})}\frac{k^2}{a^2\mathcal{H}^2}\right)\right]$$

We assume

$\delta p_{en} = 0, \quad \Sigma = 0.$

$\Rightarrow \Psi_B = \Phi_B$

At superhorizon scale $\frac{k}{a\mathcal{H}} \ll 1$

$\dot{\delta} = 0.$

Eqs. ① + ② & $\dot{\bar{\rho}} = -3(\bar{\rho}+\bar{p})\mathcal{H}$

$\Rightarrow \quad \delta = R$

$\Psi_B = \Phi_B$ is obtained by solving second equation:

$$\dot{\Psi}_B + \mathcal{H}\Psi_B + 4\pi G\frac{\bar{\rho}+\bar{p}}{\mathcal{H}}\Psi_B = 4\pi G\frac{\bar{\rho}+\bar{p}}{\mathcal{H}}\underbrace{R}_{\text{constant}}$$

For $\bar{p} = \omega\bar{\rho}$

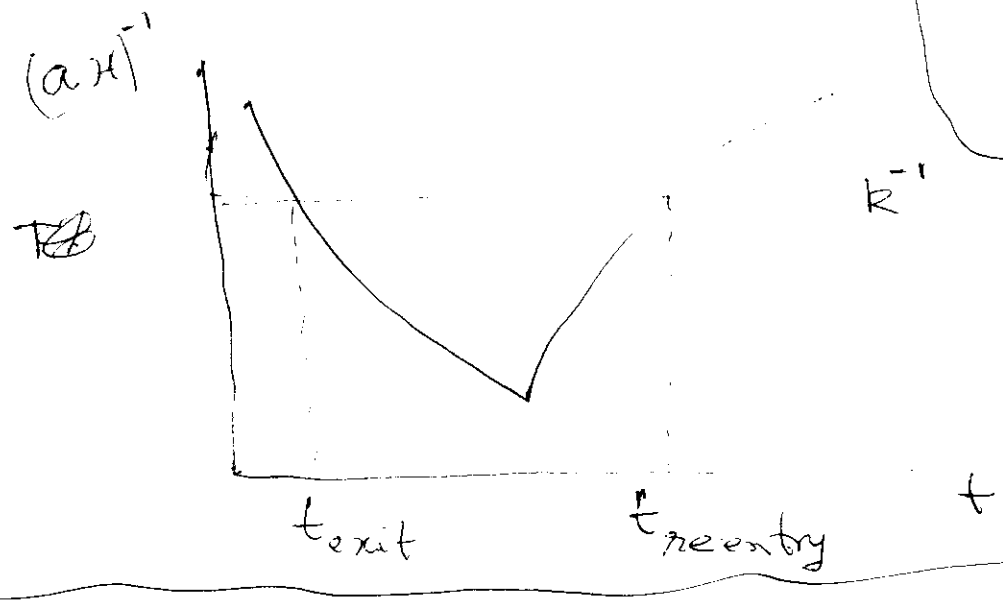
$\Psi_B = \frac{R}{A} + \underbrace{c}_{\text{integration const.}} t^{-A}$ $\quad A = \frac{5+3\omega}{3(1+\omega)}$

Everything is determined at the superhorizon scale in terms of $R(\vec{k})$.

$R(\vec{k})$ is determined from inflation:

$$\langle R(\vec{k})\, R(\vec{k}') \rangle = (2\pi)^3\, \delta^{(3)}(\vec{k}+\vec{k}')\, P_R$$

$$P_R = \frac{\mathcal{H}''}{2\,\dot{\phi}^2\,\omega^3}\Big|_{t_{exit}}, \quad \omega = |\vec{k}|$$



These provide us with the boundary condition at $t_{reentry}$

→ can be evolved further using known physics.

All fluctuations (metric, matter density, CMB etc.) → related to P_R and

P_T → tensor power spectrum

$$= \frac{4\mathcal{H}^2}{\omega^3}\Big|_{t_{exit}}$$

We'll now give intuitive derivation of two of the 4 eqs.

→ intuitive understanding of fluctuations.

For small k, fluctuations are slowly varying function of $\vec{x}$.

→ ignore spatial derivatives.

Newtonian gauge: $B = E = 0$.

$$ds^2 = -(1+2\Phi_{(t)})\,dt^2 + (1-2\Psi_{(t)})\,a(t)^2\,d\vec{x}^2$$

$\Phi_B = \Phi$, $\Psi_B = \Psi$ averages over distance $\sim k^{-1}$

Define $d\hat{t} = (1+\Phi(t))\,dt$

$$ds^2 = -d\hat{t}^2 + \underbrace{a(t)^2\,(1-2\Psi(t))}_{\hat{a}(t)^2}\,d\vec{x}^2$$

Use Friedman equation:

$$\frac{1}{\hat{a}^2}\left(\frac{d}{d\hat{t}}\hat{a}(t)\right)^2 = (\bar{\rho} + \tilde{\rho}) \times \frac{8\pi G}{3}$$

$\Downarrow$

$$(1+2\Psi)\,a^{-2}\left\{(1+\Phi)^{-1}\frac{d}{dt}\left((1-\Psi)\,a(t)\right)\right\}^2$$

$$= (1+2\Psi-2\Phi)\,a^{-2}\left\{(1-\Psi)\,\dot{a} - \dot{\Psi}\,a\right\}^2$$

$$= (1+2\Psi-2\Phi)\,a^{-2}\,\dot{a}^2\left\{1-\Psi-\frac{\dot{\Psi}}{H}\right\}^2$$

$$= \mathcal{H}^2 \left(1 - 2\Phi - \frac{2\dot{\Psi}}{\mathcal{H}}\right)$$

$$\Rightarrow \quad \mathcal{H}^2 \left(1 - 2\Phi - \frac{2\dot{\Psi}}{\mathcal{H}}\right) = \frac{8\pi G}{3} (\bar{\rho} + \tilde{\rho})$$

$$\Rightarrow \quad \dot{\Psi} + \mathcal{H}\Phi = -\frac{4\pi G}{3\mathcal{H}} \tilde{\rho}$$

Compare with eq. (1) for small k:

$$3\mathcal{H}(\dot{\Psi}_B + \mathcal{H}\Phi_B) \simeq 4\pi G \frac{\dot{\bar{\rho}}}{\mathcal{H}} (\delta + \underbrace{\Psi_B}_{\Psi})$$

$$= 4\pi G \frac{\dot{\bar{\rho}}}{\mathcal{H}} \left(-\frac{\mathcal{H}}{\dot{\bar{\rho}}} \tilde{\rho}\right)$$

$$= -4\pi G \tilde{\rho}$$

$\rightarrow$ same equation.

Next consider energy-momentum conservation

$$\frac{d\bar{\rho}}{dt} = -3(\bar{\rho} + \bar{p})\mathcal{H}$$

$$\Rightarrow \quad \frac{d}{d\hat{t}} (\bar{\rho} + \tilde{\rho}) = -3(\bar{\rho} + \bar{p} + \tilde{\rho} + \tilde{p})\hat{\mathcal{H}}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$(1 - \Phi)\frac{d}{dt} \qquad\qquad \frac{1}{\hat{a}}\frac{d\hat{a}}{d\hat{t}}$$

$$\Downarrow$$

$$\mathcal{H}\left(1 - \Phi - \frac{\Psi}{\mathcal{H}}\right)$$

Use $\tilde{p} = \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}} \tilde{\rho}$ $\qquad \dot{\bar{\rho}} = -3\mathcal{H}(\bar{\rho} + \bar{p})$.

$$\frac{d}{dt}\tilde{\rho} = \Phi\dot{\bar{\rho}} + 3(\bar{\rho}+\bar{p})(\psi + H\Phi) - 3H(\tilde{\rho}+\tilde{p})$$

$$= 3(\bar{\rho}+\bar{p})\dot{\psi} - 3H(\tilde{\rho}+\tilde{p})$$

$$\frac{d}{dt}\left(\psi + \frac{H}{\dot{\bar{\rho}}}\tilde{\rho}\right)$$

$$= \frac{d\psi}{dt} + \frac{H}{\dot{\bar{\rho}}}\frac{d\tilde{\rho}}{dt} + \tilde{\rho}\frac{d}{dt}\left(\frac{H}{\dot{\bar{\rho}}}\right) \qquad = -\frac{1}{3(\bar{\rho}+\bar{p})}$$

$$= \frac{d\psi}{dt} - \frac{1}{3(\bar{\rho}+\bar{p})}\frac{d\tilde{\rho}}{dt} + \tilde{\rho}\frac{1}{3(\bar{\rho}+\bar{p})^2}(\dot{\bar{\rho}}+\dot{\bar{p}})$$

$$= \frac{d\psi}{dt} - \frac{1}{3(\bar{\rho}+\bar{p})}\frac{d\tilde{\rho}}{dt} + \frac{\dot{\bar{\rho}}}{3(\bar{\rho}+\bar{p})^2}(\tilde{\rho}+\tilde{p})$$

$$= \frac{d\psi}{dt} - \frac{3}{3(\bar{\rho}+\bar{p})}\frac{d\tilde{\rho}}{dt} - \frac{H}{3(\bar{\rho}+\bar{p})}(\tilde{\rho}+\tilde{p})$$

$$= 0$$

$\Rightarrow$ The $\frac{dS}{dt} = 0$ equation

Thus eq. (1) & (4) are simply the original eq. for homogeneous & isotropic universe, averaged over superhorizon scales.

Fluctuations $\Rightarrow$ different choice of cosmic time in different regions.

Other two eqs. to flow of matter due to the fluctuation.

In Einstein's eq., $D_\mu T^{\mu\nu} = 0$ eq. these come multiplied by k_λ

$\rightarrow$ vanish as $k_\lambda \rightarrow 0$ & hence not derivable from the original FRW equations.

Nevertheless there is no new integration constant needed.

Solutions are completely controlled by the integration constants in the other two equations.

From the analysis of the superhorizon evolution it is clear that for each $\vec{k}$ the fluctuations are controlled by $R(\vec{k})$.

(Two more from tensors).

After reentry the superhorizon approximation breaks down but they continue to evolve according to the full set of linear equations, as long as they stay small.

If they grow to large values we can still evolve them using eqs. of motion but now we need the full non-linear equations.

so doable in principle since the physics is known.

We focus on the linear regime.

Let χ be some scalar fluctuation that we want to compute.

e.g. $\chi = \frac{\delta T}{T}$.

Convention: use same symbol χ for position ~~rep~~ space representation, momentum space representation etc.

(Argument k or x distinguish the two).

$$\chi(\vec{k}) = f(\vec{k})\, \mathcal{R}(\vec{k})$$

some known function (computable from microscopic physics)

Rotational invariance of the background $\Rightarrow$ $f(\vec{k}) = f(\underset{k}{\underset{|||}{|\vec{k}|}})$

$$\chi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \chi(\vec{k})$$

(r, θ, ϕ). $r=0$: Our position in comoving coordinates.

$$x(r,\theta,\phi) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} f(k) R(\vec{k})$$

$$= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i)^l \underbrace{Y_{lm}(\hat{x})}_{(\theta,\phi)} \underbrace{Y^*_{lm}(\hat{k})}_{(\theta_{\vec{k}},\phi_{\vec{k}})} j_l(kr)$$

$$\equiv \sum X_{lm}(r) Y_{lm}(\theta,\phi)$$

$\Delta_{xl}(k)$

$$X_{lm}(r) = \int \frac{d^3k}{(2\pi)^3} R(\vec{k}) f(k) j_l(kr) \, 4\pi (-i)^l Y^*_{lm}(\hat{k})$$

r: Fixed by observation.

e.g. for CMB fluctuations

$$r = \int_{t_{rec}}^{t_0} \frac{dt}{a(t)}$$

$(2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')$

$|P_R(k)|^2$

$$\langle X_{lm}(r) X_{l'm'}(r)^* \rangle$$

$R(-\vec{k}')$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \langle R(\vec{k}) R(\vec{k}')^* \rangle f(k) j_l(kr)$$

$$f(k')^* j_l(k'r) \, 16\pi^2 (-i)^l (i)^{l'}$$

$$Y^*_{lm}(\hat{k}) Y^*_{l'm'}(\hat{k}') \delta^{(3)}(\vec{k}+\vec{k}')$$

$$= \frac{16\pi^2}{(2\pi)^3} \int d^3k \; \mathcal{P}_R(k) \; |f(k)|^2 \, j_\ell(kr)^2$$

$$Y_{\ell m}(\hat{k})^* \; Y_{\ell' m'}(\hat{k})^*$$

$$= \frac{2}{\pi} \int dk \; k^2 \; |f(k)|^2 \; j_\ell(kr)^2$$

$$\mathcal{P}_R(k) \; \delta_{\ell\ell'} \; \delta_{mm'}$$

$$= \delta_{\ell\ell'} \; \delta_{mm'} \; C_\ell^{XX} \qquad \Delta_{X\ell}(k)^2$$

$$C_\ell^{XX} = \frac{2}{\pi} \int dk \; k^2 \; |f(k)|^2 \; (j_\ell(kr))^2$$

$$\mathcal{P}_R(k) \simeq \frac{2\pi^2}{k^3} \underbrace{\Delta_s^2(k)}_{\text{slowly varying function}}$$

$$= 4\pi \int \frac{dk}{k} \; |f(k)|^2 \; j_\ell(kr)^2 \; \Delta_s^2(k)$$

peaked around $kr \sim \ell$

$k \sim \ell/r$

$$\sim 4\pi \; \left|f\left(\frac{\ell}{r}\right)\right|^2 \Delta_s^2(\ell/r) \int \frac{dk}{k} \; j_\ell(kr)^2$$

$$\Downarrow$$

$$\frac{1}{2\ell(\ell+1)}$$

Note: Large/small k $\leftrightarrow$ Large/small ℓ.

$k \simeq aH$ for $\Delta\theta = 2^\circ$

last scattering

$k^{-1} > r\,\Delta\theta \Rightarrow k^{-1} > (aH)^{-1}$ at t_{LS}

$\Downarrow$

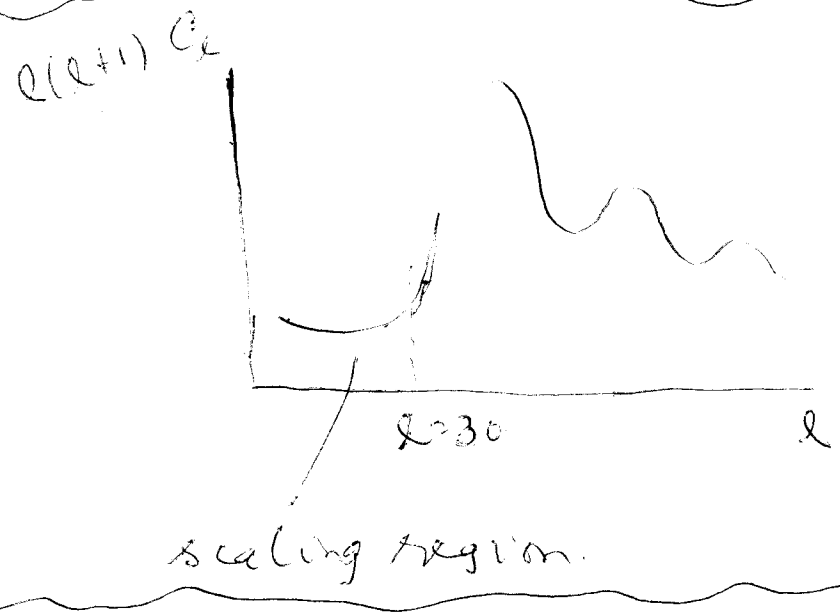
$\frac{r}{\ell} > r\,\Delta\theta \Rightarrow \ell < \frac{1}{\Delta\theta} = \frac{180^\circ}{2^\circ}\frac{1}{\pi}$

In this case: ~ 30

$f\left(\frac{\ell}{r}\right) \sim f(0)$

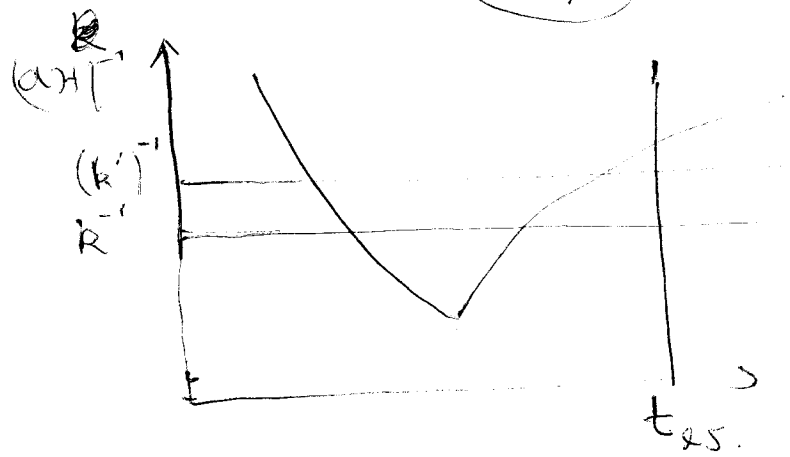
$\Rightarrow \ell(\ell+1)\, C_\ell^{XX} \sim 2\pi\, |f(0)|^2\, \Delta_s^2\left(\frac{\ell}{r}\right)$

slowly varying fn of ℓ/r



Q. What is the origin of the oscillation?

For $\ell \gtrsim 30$ we are in a regime where the perturbation enters the horizon before last scattering.



We need to study evolution after reentry

Eqs. of motion:

③ $\Psi_B = \Phi_B$

① $3H(\dot{\Psi}_B + H\Phi_B) + \frac{k^2}{a^2}\Psi_B = 4\pi G \frac{\bar{\rho}}{H}(\zeta + \Psi_B)$

④ $\dot{\zeta} = -\Pi$

$$\frac{\Pi}{H} = -\frac{k^2}{3a^2H^2}\left[\zeta - \Psi_B\left(1 - \frac{2\bar{\rho}}{9(\bar{\rho}+\bar{p})}\frac{k^2}{a^2H^2}\right)\right]$$

(The other equation gives R in terms of ζ, Ψ_B via algebric eq.)

Use ①, ③ to express ζ, Ψ_B in terms of Φ_B, $\dot{\Phi}_B$

④ $\Rightarrow$ second order differential eq. Φ_B

$$\ddot{\Phi}_B + A\dot{\Phi}_B + B\Phi_B = 0$$

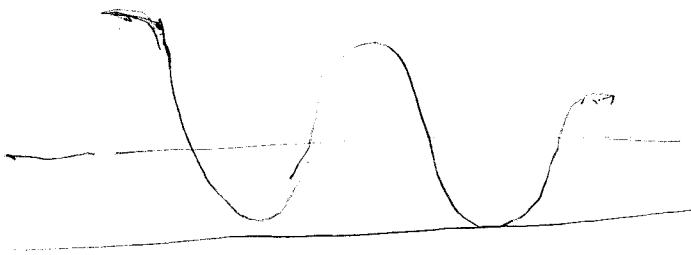
computable.

Initial condition: provided by superhorizon solution.

e.g. if the exit occurs deep in the matter dominated region then

$\dot{\Phi}_B = 0$, $\Phi_B = \frac{3}{5} R$ at exit.

After exit it evolves as (damped) harmonic oscillator.



Now compare what happens for k with what happens for k':

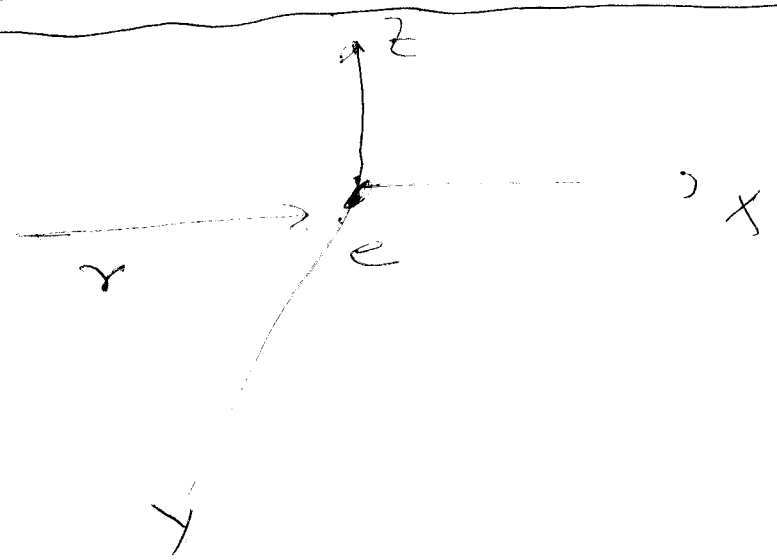
If $k > k'$ then the system evolves for a longer time after exit.

$\Rightarrow$ whether at t_{es} we have a crest or trough or something in between depends on k.

$\Rightarrow$ oscillation in $\Phi|_{t_{es}}$.

Polarization

→ arises from Thompson scattering (scattering of photons from an electron at rest).



γ can be polarized along Y or Z.

Suppose after scattering γ moves in some fixed direction

Polarization of the final γ determined by polarization of initial γ in plane ⊥ the final γ momentum.

e.g. if the final γ goes along Z then it is polarized along Y

Thus even if the initial γ was moving along a fixed direction unpolarized, the final γ becomes polarized.

However this effect is lost if the initial γ were isotropic.

Metric fluctuations $\Rightarrow$ Incident γ non-isotropic.

$\Rightarrow$ Final γ carries net polarization.

Quantitative measure:

From a given direction there are many photons.

(many e.m. wave packets)

$$I_{ij} = \text{Constant} \langle E_i E_j \rangle$$

~~Intensity~~ electric field.

$\Rightarrow$ Intensity tensor.

(2×2 matrix after we choose a basis $\hat{e}_1, \hat{e}_2 \perp$ γ-momentum and decompose

$$\vec{E} = E_1 \hat{e}_1 + E_2 \hat{e}_2 .)$$

$$Q = \frac{1}{4}(I_{11} - I_{22})$$

$$U = \frac{1}{2} I_{12}$$

$$T = \frac{1}{4}(I_{11} + I_{22}) \rightarrow \text{fixes normalization of } I$$

T : invariant under rotation of $(\hat{e}_1, \hat{e}_2) \rightarrow$ behaves as a scalar.

$$(Q \pm iU) \rightarrow e^{\mp 2i\theta}(Q \pm iU)$$

under $\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}$

$$T = \sum_{lm} T_{lm} Y_{lm}(\theta, \phi)$$

$$Q \pm iU = \sum_{lm} a_{\pm 2, lm} \; {}_{\pm 2}Y_{lm}(\theta, \phi)$$

tensor spherical harmonics.

transforms in $(l\,m)$ representation under rotation about the origin.

$$a_{E,lm} = -\frac{1}{2}(a_{2,lm} + a_{-2,lm})$$

$$a_{B,lm} = -\frac{1}{2i}(a_{2,lm} - a_{-2,lm})$$

A similar decomposition can be carried out for $h_{ij}(\vec{k})$.

E and B modes.

Result: Symmetry analysis

$\Rightarrow$ E-modes mix with scalars but B-modes do not.

$\Rightarrow$ (E mode of polarization, T) = linear combination of scalar fluctuation & E mode of tensor fluctuation.

B-mode of polarization

$\times$ B-mode of h fluctuations

$\Downarrow$

direct signal of tensor perturbation.

General expression:

$$\chi(\vec{k}) = f(k)\, R(\vec{k})$$

quantity of interest — computable in principle — created by inflation

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Goal of today's lecture: Compute $f(k)$ in a ~~simple~~ the case of CMB temperature fluctuations at superhorizon scale.

When the photons were in thermal equilibrium & the universe was radiation dominated:

$$\rho = c\, T^4.$$

$\Rightarrow$ expect $\frac{\delta\rho}{\rho} = 4\, \frac{\delta T}{T}$.

Work in $B = E = 0$ gauge.

$$ds^2 = -(1+2\Phi)\, dt^2 + (1-2\Psi)\, d\vec{x}^2$$

for scalar fluctuation.

Recall: defn. of gauge invariant variables:

$$\Psi_B = \Psi + a^2 \mathcal{H} (\dot{E} - B/a)$$

$$\zeta = -\Psi - \frac{\mathcal{H}}{\bar{\rho}} \tilde{\rho} \qquad \Phi_B = \Phi - \frac{d}{dt}\left(a^2 (E - \frac{B}{a})\right)$$

In $E = B = 0$ gauge $\Psi_B = \Psi$

$$\Rightarrow \tilde{\rho} = \frac{\dot{\bar{\rho}}}{\mathcal{H}} (\zeta + \Psi_B)$$

$$\dot{\bar{\rho}} = -3\mathcal{H}(\bar{\rho} + \bar{p}).$$

$$\zeta = -R, \quad \underset{\Phi_B}{\underset{\parallel}{\Psi_B}} = \frac{3(1+w)}{5+3w} R \Big|_{w=\frac{1}{3}} = \frac{4}{7} R = \frac{2}{3} R$$

$$\Rightarrow R = \frac{3}{2}\Psi \qquad \zeta = -\frac{3}{2}\Psi$$

$$\frac{\dot{\bar{\rho}}}{\bar{\rho}} = -\frac{3(\bar{\rho}+\bar{p})}{\bar{\rho}} \left(-\frac{3}{2}\Psi + \Psi\right)$$

$$= 3 \times \frac{4}{3} \times \left(-\frac{1}{2}\right) \Psi = -2\Psi$$

$$\frac{\delta T}{T} = \frac{1}{4} \frac{\delta \rho}{\bar{\rho}} = -\frac{1}{2}\Psi = \Psi_R = \frac{2}{3} R$$

during radiation dominated era when the photons are in equilibrium.

Q. How does this evolve when photons fall out of equilibrium & when universe becomes matter dominated?

Need to use Boltzmann transport equation.

Suppose in some locally inertial frame $f(\vec{x}, \vec{p}, t)$ ~~dx~~ $d^3x\, d^3p$ denotes the # of photons in phase space volume $d^3x\, d^3p$.

Then f remains constant along the photon trajectory.

$$\Rightarrow \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p^i}\frac{dp^i}{dt} = 0$$

provided we ignore collisions of the photon with electrons.

(good approximation at superhorizon scale where microscopic dynamics cannot affect the fluctuations)

For complete calculation we need to take into account collision effect

Note: We can use this eq. even if $\vec{x}, \vec{p}$ are not canonically conjugate variables as long as f is ~~measured~~ defined in locally inertial frame.

Take a photon trajectory labelled by the affine parameter λ.

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0.$$

$$ds^2 = 0.$$

$$p^\mu = \frac{dx^\mu}{d\lambda}$$

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda} \Big/ \frac{dt}{d\lambda} = \frac{p^i}{p^0}$$

Along the trajectory:

$$ds^2 = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

$$\Rightarrow -(1+2\Phi)(p^0)^2 + a^2(1-2\Psi)\underbrace{\vec{p}^{\,2}}_{p^i p^i} = 0.$$

$$\Rightarrow \quad p^0 = (1-\Phi)\,\mathfrak{p}$$

$$\mathfrak{p}^2 \equiv a^2(1-2\Psi)\,\vec{p}^{\,2}$$

Introduce $\hat{p}^{i}$ via

$$p^i = c\,\hat{p}^i \qquad \hat{p}^i \hat{p}^i = 1.$$

$\rightarrow$ determines c.

$$p^2 = a^2(1-2\psi)\, p^i p^i = a^2(1-2\psi)\, c^2$$

$$c = \frac{p}{a}(1+\psi) \qquad p^i = \frac{p}{a}(1+\psi)\,\hat{p}^i$$

Regard f as $f(\vec{x}, p, \hat{p}, t)$

$$\frac{dx^i}{dt} = \frac{p^i}{p^0} = \frac{p}{a}(1+\psi)\,\underbrace{\hat{p}^i}_{p(1-\phi)}$$

$$= \frac{1}{a}(1+\psi+\phi)\,\hat{p}^i$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}(1+\psi+\phi)\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt}$$

We want to keep terms up to first order in perturbation.

$\frac{\partial f}{\partial x^i}$ is already first order since f is uniform in zeroth order

$$\Rightarrow \cancel{\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i}}$$

$\frac{\partial f}{\partial \hat{p}^i}$ is first order since leading order distribution fn. is uniform.

$\frac{d\hat{p}^i}{dt}$ is also first order since in FRW background photons travel in straight line in comoving coordinates.

$$\Rightarrow \quad \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{dp}{dt}$$

to first order. ~~ignore~~

We now need to calculate $\frac{dp}{dt}$

$$p^0 = p(1-\phi)$$

$$p = p^0(1+\phi)$$

$$\frac{dp}{dt} = \frac{dp^0}{dt}(1+\phi) + p^0\dot{\phi}$$

$$\frac{dp^0}{dt} = \frac{dp^0}{d\lambda}\Big/\frac{dt}{d\lambda} = \frac{1}{p^0}\frac{dp^0}{d\lambda}$$

$$\frac{dp^0}{d\lambda} = -\Gamma^0_{\alpha\beta}\underbrace{\frac{dx^\alpha}{d\lambda}}_{p^\alpha}\underbrace{\frac{dx^\beta}{d\lambda}}_{p^\beta}$$

can be calculated

Final result: $\cancel{\frac{1}{p^0}\frac{dp^0}{dt}} = \frac{1}{p}\frac{dp}{dt} = -\mathcal{H} + \dot{\psi} - \frac{\hat{p}^i}{a}\frac{\partial\phi}{\partial x^i}$

Some details:

$$p^0 = p(1-\phi)$$
$$p^i = \frac{p}{a}\hat{p}^i(1+\psi)$$

$$\frac{dp^0}{d\lambda} = -\Gamma^0_{\alpha\beta}\, p^\alpha p^\beta$$

$$= (1-2\phi)\, p\left(-\dot{\phi}\, p - 2\frac{\partial\phi}{\partial x^i}\, p\frac{\hat{p}^i}{a} + p\frac{\partial\psi}{\partial t} - pH\right)$$

Using $p^\alpha = p(1-\phi)$.

$$\frac{dt}{d\lambda} = p(1-\phi)$$

$$\cancel{\frac{d}{dt}\left(p(1-\phi)\right) = \frac{1}{p}(1-\phi)\left(-p\dot{\phi} - 2p\frac{\hat{p}^i}{a}\frac{\partial\phi}{\partial x^i} + p\frac{\partial\psi}{\partial t} - pH\right)}$$

$$\frac{dp^0}{dt} = \frac{dp^0}{d\lambda} \Big/ p^0 = \frac{1}{p(1-\phi)}\frac{dp^0}{d\lambda}$$

$$\frac{d}{dt}\left(p(1-\phi)\right) = +(1-\phi)\left(-\dot{\phi}\, p - 2\frac{\partial\phi}{\partial x^i}\, p\frac{\hat{p}^i}{a} + p\frac{\partial\psi}{\partial t} - pH\right)$$

$$\frac{dp}{dt} = (1-\phi) - p\left(\frac{\partial\phi}{\partial t} + \frac{\partial\psi}{\partial x^i}\frac{p}{a}\hat{p}^i\right)$$

$$\Rightarrow \frac{dp}{dt} = p\dot{\phi} + p\frac{\hat{p}^i}{a}\frac{\partial\phi}{\partial x^i} - \dot{\phi}p - 2p\frac{\partial\phi}{\partial x^i}\frac{\hat{p}^i}{a} + p\dot{\psi} - pH$$

$$\frac{1}{p}\frac{dp}{dt} = \left(\dot{\psi} - H - \frac{p}{a}\hat{p}^i\frac{\partial\phi}{\partial x^i}\right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a} \cancel{(1+\psi+\phi)} \frac{\partial f}{\partial x^i}$$

$$+ p \frac{\partial f}{\partial p} \left(\dot{\psi} - H - \frac{p}{a} \hat{p}^i \frac{\partial \phi}{\partial x^i} \right)$$

~~In momentum~~ space:

Parametrize f as:

$$f = \left[\exp\left(\frac{p}{T(t)\,(1+\Theta(\vec{x}, p, \hat{p}, t))} \right) - 1 \right]^{-1}$$

$T(t)$: Background temperature in FRW universe, measured by some inertial observer.

Take as the observer sitting at a fixed comoving coordinate $\vec{x}$.

$$p \frac{\partial f}{\partial p} = -p \frac{\partial f}{\partial T}$$

Now we focus on superhorizon scale

~~An equation for Φ of use~~

Now in absence of collision

$$\frac{df}{dt} = 0.$$

In momentum space

$\frac{\partial}{\partial x^i} \to i k^i \to$ vanishes for superhorizon fluctuation

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial p}(\dot{\psi} - H) = 0$$

$$p\frac{\partial f}{\partial p} = -T\frac{\partial f}{\partial T}$$

$$\Rightarrow \frac{1}{(1+\Theta)}\frac{\partial f}{\partial T}\frac{d}{dt}\left(T(1+\Theta)\right)$$

$$- T\frac{\partial f}{\partial T}(\dot{\psi} - H) = 0$$

$$\Rightarrow \frac{\partial f}{\partial T}\dot{T} + T\frac{\partial f}{\partial T}\dot{\Theta}$$

$$- T\frac{\partial f}{\partial T}\dot{\psi} + T\frac{\partial f}{\partial T}H = 0$$

Now $T \propto \frac{1}{a}$

$$\Rightarrow \frac{\dot{T}}{T} = -\frac{\dot{a}}{a} = -H$$

$$\Rightarrow T\frac{\partial f}{\partial T}(\dot{\Theta} - \dot{\psi}) = 0$$

$$\Rightarrow \dot{\Theta} - \dot{\psi} = 0$$

$$\Theta = +\psi + C$$

$\hookrightarrow$ constant.

Initial condition in radiation dominated era.

$$\Theta = -\frac{1}{2}\psi_r$$

$\hookrightarrow$ radiation dominated result $-\frac{2}{3}$

$$\Rightarrow \quad C + \psi_r = -\frac{1}{2}\psi_r$$

$$C = -\frac{3}{2}\psi_r$$

$$\Rightarrow \quad \Theta = \psi - \frac{3}{2}\psi_r$$

Now during radiation dominated era $\psi = \psi_r$.

$$\Theta = -\frac{1}{2}\psi_r$$

Conclusion: In the radiation dominated era, Θ continues to be equal to $-\frac{1}{2}\psi_r$ even after the photons fall out of thermal equilibrium.

What happens during matter dominated era?

ψ changes from $\psi_r = \frac{2}{3}R$ to

$$\psi_m = \frac{3}{5}R$$

$$\psi_r = \frac{2}{3} \times \frac{5}{3}\psi_m =$$

$$\Theta = \psi_m - \frac{3}{2}\psi_r = \psi_m - \frac{3}{2} \times \frac{2}{3} \times \frac{5}{3}\psi_m$$

$$= -\frac{2}{3}\psi_m$$

Extra effect: climbing the gravitational potential.

---

suppose after t_{es} ϕ and ψ remains constant all the way to t_0.

Follow light ray from $r = r_{es}$ to $r = 0$ → today

t_{es}

$$-\left(1+2\phi(\vec{r})\right)dt^2 + a(t)^2\left(1+2\psi(\vec{r})\right)d\vec{r}^2 = 0$$

$$\int_{t_{es}}^{t_0} \frac{dt}{a(t)} = \int_C \frac{1+2\psi(\vec{r})}{1+2\phi(\vec{r})}\,|d\vec{r}|$$

→ begins at $\vec{r} = \vec{r}_{es}$ and ends at $\vec{r} = 0$

Two successive emission by a source at t and $t + \Delta t$.

$$\int_{t_{es}+\Delta t_1}^{t_0+\Delta t_0} \frac{dt}{a(t)} = \int_C \frac{1+2\psi(\vec{r})}{1+2\phi(\vec{r})}\,|d\vec{r}|$$

$$\Rightarrow \frac{\Delta t_0}{a(t_0)} = \frac{\Delta t_1}{a(t_1)}$$

$$\Rightarrow \quad \Delta t_0 = \frac{a(t_0)}{a(t_1)} \Delta t_1$$

Thus relationship between Δt_0 and Δt_1 does not get affected by fluctuation.

~~However~~

Q. How is Δt_1 related to the intrinsic frequency of photon.

In the frame fixed at $\vec{x}$.

$$d\tau^2 = -(1+2\phi(\vec{x}_1))\, dt^2$$

$$\Rightarrow \quad \Delta\tau = \Delta t\,(1+\phi(\vec{x}_1))$$

$\Rightarrow$ ν measured in t

$= (1+\phi) \times \nu$ measured in ~~inertial~~ inertial frame

$\Rightarrow$ ~~temperature~~ seen

ν measured by observer

$$= \frac{a(t_1)}{a(t_0)} \times (1+\phi(\vec{x}_1)) \times \nu \text{ measured by inertial observer}$$

gets added to Θ

$\frac{1}{5}$

$R(\vec{x}_1) = \Theta|_{seen} = \Theta|_{intrinsic} + \phi(\vec{x}_1) = -\frac{2}{3}\phi(\vec{x}_1) + \phi(\vec{x}_1) = \frac{1}{3}\phi(\vec{x}_1)$

Fluctuations of last scattering surface:

Due to different photon temperature & electron density in different regions, the last scattering surface may fluctuate.

→ Complicated story.

δt_{ls} fluctuation in the last scattering surface.

We can study the effect of this independently of all other fluctuations & add its effect.

(First order perturbation).

$$\delta t_{ls} \Rightarrow \delta a_{ls} = a'(t)\, \delta t_{ls}$$

$$\delta T_{ls} = T'(t)\, \delta t_{ls}$$

background values.

$$a(t)\, T(t) = \text{const.}$$

$$\Rightarrow \frac{\delta a_{ls}}{a_{ls}} = -\frac{\delta T_{ls}}{T_{ls}}$$

Now consider a photon received by us from the last scattering surface.

$$T_{obs} = \underbrace{T_{ls}\, a_{ls}}_{a_0 = 1} \rightsquigarrow \text{red shift}$$

$\Rightarrow \delta T_{obs} = 0$

$\Rightarrow$ Fluctuation of the last scattering surface has no observable effect to first order.

Other effects: We have assumed ϕ & ψ to be constant (t independent)

Not true due to two effects:

① Transition from matter to Λ-dominated era changes ϕ.

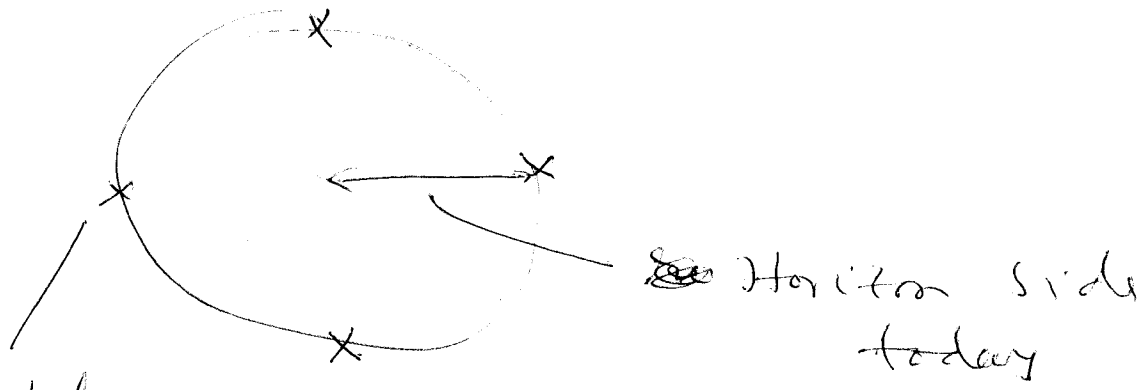
$\rightsquigarrow$ about 10% effect.

② Fluctuations may enter the horizon in between last scattering & today.

$\Rightarrow \phi$ is no longer constant

$\Rightarrow$ photon path more complicated.

We can avoid this problem by looking at modes which are still outside the horizon



dipole.

(hard to measure due to our own peculiar velocity w.r.t. CMB)

Quadrupole: Still outside the horizon.

$\Rightarrow \quad \Theta = \frac{1}{3}\Phi_m$ holds for quadrupole fluctuations.

Matter Density fluctuation

$$\frac{\tilde{\rho}}{\bar{\rho}} = -\frac{5+\psi}{H}\frac{\dot{\bar{\rho}}}{\bar{\rho}}$$

Matter dominated era

$$\zeta = -R = -\frac{5}{3}\psi, \quad \dot{\bar{\rho}} = -3\bar{\rho}H$$

$$\Rightarrow \frac{\tilde{\rho}}{\bar{\rho}} = \frac{2}{3}\psi(-3) = -2\psi_m \rightarrow 6\times\frac{\tilde{T}}{T}$$