Conclusion: $\frac{n_p}{n_r}$ and $\frac{\rho_p}{\rho_r}$ is negligible at $T \sim$ MeV.

Similar argument for all matter (neutron, dark matter etc.)

$\Rightarrow$ Can be ignored.

Thus we include

γ, $e^{\pm}$, $\nu_e, \bar{\nu}_e$, $\nu_\mu, \bar{\nu}_\mu$, $\nu_\tau, \bar{\nu}_\tau$

$kT \gg m_i \Rightarrow$ neglect m_i.

What about μ_i's?

$$\frac{n_{e^+} - n_{e^-}}{n_\gamma} \sim \frac{n_p - n_{\bar{p}}}{n_\gamma} \sim 10^{-9}$$

$$n_{e^\mp} = \frac{1}{2\pi^2} \times 2 \int 4\pi k^2 dk \frac{1}{e^{\beta(k \mp \mu_e)} + 1}$$

$n_{e^-} - n_{e^+}$ small $\Rightarrow \mu_e$ small.

$\Rightarrow$ Set $\mu_e = 0$.

~~For relativistic Bose & f~~

What about the neutrinos?

~~The~~ We do not know if the present universe has large $(n_{\nu_i} - n_{\bar{\nu}_i})$.

$\Rightarrow$ cannot say much about μ_{ν_i} $i = e, \mu, \tau$.

Proceed by assuming $\mu_{\nu_i} = 0$.

Later try to relax this assumption.

In this case ρ_i, p_i can be calculated analytically

Ex. General results.

$k = u/\beta$ and int. by parts

$\rightarrow \int_0^\infty u^3 \, du \, \frac{1}{e^u \mp 1}$

$$\rho_i = \frac{g_i}{2} a_B T^4 \times \begin{matrix} 1 \rightarrow \text{boson} \\ 7/8 \rightarrow \text{fermion} \end{matrix}$$

$$p_i = \frac{g_i}{6} a_B T^4 \times \begin{matrix} 1 \rightarrow \text{boson} \\ 7/8 \rightarrow \text{fermion} \end{matrix}$$

$$a_B = \frac{\pi^2}{15}$$

$$s_i = \frac{p_i + \rho_i}{T} = \frac{g_i}{2} \times \frac{4}{3} a_B T^3 \times \begin{matrix} 1 - \text{boson} \\ 7/8 \rightarrow \text{fermion} \end{matrix}$$

$$n_i = c \frac{g_i}{2} a_B T^3 \times \begin{matrix} 1 \rightarrow \text{boson} \\ 3/4 \rightarrow \text{fermion} \end{matrix}$$

numerical constant.

$$s_{tot} = \frac{2}{3} a_B T^3 \left(\underset{\gamma}{2} + \frac{7}{8}\left(\underset{e^-}{2} + \underset{e^+}{2} + \underset{\nu}{1} + \underset{\bar{\nu}}{1} \times 3 \right) \right)$$

e.g. τ

$$= \frac{2}{3} a_B T^3 \left(2 + \frac{35}{4} \right)$$

$= 43/4 = N$ (effective no. of degrees of freedom.

$$\rho_{tot} = \frac{1}{2} a_B T^4 N$$

$$\frac{d}{dt}(s a^3) = \text{constant} \Rightarrow T a = \text{constant}.$$

$$T \propto 1/a.$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_{tot}$$

$$\parallel$$

$$\left(\frac{\dot{T}}{T}\right)^2 = \frac{8\pi G}{3} \frac{1}{2} a_B T^4 N$$

$$\frac{dT}{T^3} = -\sqrt{\frac{4\pi G}{3} a_B N}\, dt$$

$$\frac{1}{2} \frac{1}{T^2} = \sqrt{\frac{4\pi G}{3} a_B N}\,(t + c)$$

$$T = \left(\frac{16\pi G}{3} a_B N\right)^{-1/2} (t + c)^{-1/2}$$

could put $c = 0$

(count time from when T was high.)

~~Did~~ Was the system in equilibrium?

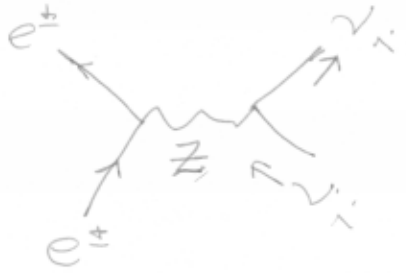
Interactions:

$e^{\pm}\gamma \rightleftarrows e^{\pm}\gamma \qquad e^+e^- \rightleftarrows \gamma\gamma$ → electromagnetic

$e^+e^- \rightleftarrows e^+e^-$ etc.

$e^{\pm}\nu_i \rightleftarrows e^{\pm}\nu_i$

$\bar{\nu}_i \qquad \bar{\nu}_i$



$e^+e^- \rightleftarrows \nu_i\bar{\nu}_i$ etc.

→ specific to ν_e.

need M^2

$\sigma_{wk} = |G_{wk}|^2 \times T^2$ | $\sigma_{ee} \sim T^{-2}$ × e.m. coupling.

$\Downarrow$

dim. L^2 $\sim M^{-2}$

$\left(\frac{g_w^2}{m_w^2}\right)^2 \sim M^{-4}$

ν-e scattering

σ_{wk} falls as T falls

Reaction rate / neutrino

$= n\,\sigma_{wk} \sim T^3 \times |G_{wk}|^2 T^2$

density

Hubble constant $\frac{\dot{a}}{a} \sim \sqrt{\frac{8\pi G}{3}}\, T^2$

$$\frac{\text{reaction rate}}{\text{expansion rate}} = \sqrt{\frac{3}{8\pi G}}\, (G_{wk})^2\, T^3$$

$$\simeq \sqrt{\frac{3}{8\pi G}}\, \underbrace{(G_{wk})^2}_{1.16\times10^{-5}\,\mathrm{GeV}^{-2}}\, (1\,\mathrm{MeV})^3 \left(\frac{T}{\mathrm{MeV}}\right)^3$$

$$\underset{\text{Planck Scale}}{\sim} \left(\frac{1.16\times10^{-5}}{1.22\times10^{19}}\right)^{-2} \times \left(\frac{10^{-3}}{1.22\times10^{19}}\right)^3 \times \left(\frac{T}{\mathrm{MeV}}\right)^3$$

$$\sim 10^{10-9-19} \left(\frac{T}{\mathrm{MeV}}\right)^3 \sim \left(\frac{T}{\mathrm{MeV}}\right)^3$$

$\Rightarrow$ For $T < \mathrm{MeV}$ the neutrino reaction rate falls sharply.

After this the neutrino temp. is decoupled from $\gamma - e^+ - e^-$ temp.

$T_\nu \sim \frac{1}{a}$ $\quad T_{\gamma e^+ e^-} \sim \frac{1}{a}$

equal due to initial condition.

~~$T \ll \mathrm{MeV}$~~

For $T \ll$ MeV. e^+e^- pairs have disappeared

γe^+e^- still in equilibrium.

$(\rho_{\gamma e^+e^-} a^3)$ remains constant.

For $T = T_1 \gg$ MeV $< \frac{11}{2}$

$$\rho_{\gamma e^+e^-} = \frac{2}{3} a_B T_1^3 \left(\underset{\gamma}{2} + \left(\underset{e^+}{2} + \underset{e^-}{2} \right) \frac{7}{8} \right)$$

For $T = T_2 \ll$ MeV

$$\rho_{\gamma e^+e^-} = \frac{2}{3} a_B T_1^3 \times 2.$$

$\Rightarrow \rho a^3 =$ constant

$$\Rightarrow T_1^3 \times \frac{11}{2} \times a_1^3 = T_2^3 \times 2 \times a_2^3$$

$$\Rightarrow T_2 = \left(\frac{11}{4}\right)^{1/3} T_1 \frac{a_1}{a_2}$$

$$(T_\nu)_2 = (T_\nu)_1 \times \frac{a_1}{a_2}$$

$$\Rightarrow T_{\nu_2} = T_2 \left(\frac{4}{11}\right)^{1/3}$$

For $T \ll$ MeV $\quad T_\nu = T_\gamma \left(\frac{4}{11}\right)^{1/3}$

Since then $T_\gamma, T_\nu \propto \frac{1}{a} \Rightarrow T_\nu / T_\gamma$ remains constant, (still ν becomes non-relativistic)

$T \sim 10$ MeV

$1\,K = 8.6\times10^{-5}$ eV. $\Rightarrow T \sim \dfrac{10\times10^{6}}{8.6\times10^{-5}}\,K \sim 10^{11}\,K$

Relavant particles.

$e^+, e^-, \gamma, \nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$

p, n number small. $\rightarrow$ can be ignored for overall evolution.

Conserved ~~charges~~ densities.

$n_{e^+} - n_{e^-}$, $n_{\nu_e} - n_{\bar{\nu}_e}$, $n_{\nu_\mu} - n_{\bar{\nu}_\mu}$, $n_{\nu_\tau} - n_{\bar{\nu}_\tau}$

check conservation for possible reaction:

Electromagnetic.

$e^+e^- \rightarrow \gamma\gamma\gamma\cdots\gamma$, $e^+e^- \rightarrow e^+e^-\gamma\gamma\cdots\gamma$

$e^\pm\gamma \rightarrow e^\pm\gamma$ etc.

Weak.

$e^-\nu_e \rightarrow e^-\nu_e$

or

Neutrino oscillations irrelevant.

$e^-\nu_\mu \rightarrow e^-\nu_\mu$

$e^+e^- \to \nu_\mu \bar{\nu}_\mu$

$$\tilde{\mu}_1 (n_{e^-} - n_{e^+}) + \tilde{\mu}_2 (n_{\nu_e} - n_{\bar{\nu}_e}) + \tilde{\mu}_3 (n_{\nu_\mu} - n_{\bar{\nu}_\mu}) + \tilde{\mu}_4 (n_{\nu_\tau} - n_{\bar{\nu}_\tau})$$

$\Rightarrow \quad \mu_{e^-} = \tilde{\mu}_1$

$\mu_{e^+} = -\tilde{\mu}_1 = -\mu_{e^-}$

$\mu_{\nu_e} = \tilde{\mu}_2, \quad \mu_{\bar{\nu}_e} = -\tilde{\mu}_2 = -\mu_{\nu_e}$ etc.

Take $\mu_{e^-}, \mu_{\nu_e}, \mu_{\nu_\mu}, \mu_{\nu_\tau}$ as independent variables.

Extra information:

$(n_{e^-} - n_{e^+})/n_\gamma$ is very small.

$$n_{e^-} = \frac{2}{(2\pi)^3} \int 4\pi k^2 \, dk \, \frac{1}{e^{\beta(\sqrt{k^2+m_e^2} - \mu_e)} + 1}$$

$$n_{e^+} = \frac{2}{(2\pi)^3} \int 4\pi k^2 \, dk \, \frac{1}{e^{\beta(\sqrt{k^2+m_e^2} + \mu_e)} - 1}$$

$n_{e^-} - n_{e^+}$ small $\Rightarrow \beta \cdot \mu_e$ small.

Set $n_{e^-} - n_{e^+} = 0$ to very good approximation.

$$\frac{d}{dt}\left((n_{e^-} - n_{e^+})\, a^3\right) = 0.$$

$\Rightarrow$ $n_{e^-} - n_{e^+} = 0$ $\Rightarrow$ $n_{e^-} - n_{e^+} = 0$ at all t..
at $t = t_1$

Additional assumption:

$n_{\nu_e} - n_{\bar{\nu}_e} = 0$, $n_{\nu_\mu} - n_{\bar{\nu}_\mu} = 0$, $n_{\nu_\tau} - n_{\bar{\nu}_\tau} = 0$.

$\Rightarrow$ they also remain zero.

$\Rightarrow$ $\mu_{\nu_e} = 0$, $\mu_{\nu_\mu} = 0$, $\mu_{\nu_\tau} = 0$

at all times.

In this case the charge conservation laws are automatically satisfied.

Relevant equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

$$\frac{d}{dt}\left(s a^3\right) = 0..$$

For $T \sim 10$ MeV we can ignore $e^\pm$ mass.

$\Rightarrow$ All particles effectively massless.

$$\frac{1}{V}\ln Q_i = \mp \frac{g_i}{(2\pi)^3}\int_0^\infty 4\pi k^2\,dk\,\ln(1\mp e^{-\beta k})$$

$$\underset{\beta k = u}{=} \mp \frac{g_i T^3}{2\pi^2}\int_0^\infty u^2\,du\,\ln(1\mp e^{-u})$$

$$= \frac{g_i T^3}{2\pi^2}\cdot\frac{1}{3}\times\frac{\pi^4}{15}\times \begin{cases} 1 \to \text{for boson} \\ \frac{7}{8} \to \text{for fermion} \end{cases}$$

$$p_i = \frac{T}{V}\ln Q = \frac{g_i}{6}T^4 a_B \times \begin{cases} 1 & \text{for boson} \\ \frac{7}{8} & \text{for fermion} \end{cases}$$

$$a_B = \frac{\pi^2}{15}$$

Similarly

$$\rho_i = \frac{g_i}{2} a_B T^4 \times \begin{cases} 1 & \text{for boson} \\ \frac{7}{8} & \text{for fermion} \end{cases}$$

$$n_i = \frac{g_i}{2}\cancel{a_B}\, T^3 \times \frac{2\zeta(3)}{\pi^2}\begin{cases} 1 & \text{for boson} \\ \frac{3}{4} & \text{for fermion} \end{cases}$$

$$s_i = \frac{p_i+\rho_i}{T} = \frac{g_i}{2} a_B T^3 \times \frac{4}{3}\times\begin{cases} 1 & \text{for boson} \\ \frac{3}{4} & \text{for fermion} \end{cases}$$

$$\rho = \frac{1}{2}a_B T^4\left\{\underset{1}{2} + \frac{7}{8}(2+2+2+2+2)\right\}$$

$$= \frac{1}{2}a_B T^4 \underset{\simeq N}{\frac{43}{4}} = \frac{1}{2}N a_B T^4$$

effective # of DoF.

$$s = \frac{4}{3}a_B T^3 \times \frac{1}{2}\times N = \frac{43}{4}$$

$T \propto 1/a$ (radiation domination)

$\frac{d}{dt}(s_i a^3) = 0 \;\; \forall i$ separately.

Note: $n_\gamma, n_{e^\pm}, n_{\nu_i}, n_{\bar{\nu}_i}$ for $i = e, \mu, \tau$ are of the same order.

Now we need to test if the condition for thermal equilibrium is satisfied.

Expansion rate $\cdot\ H = \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3} \times \frac{1}{2} N a_B T^4}$

Reaction rate of a particle:

$\nu_i, \bar{\nu}_i$ (weak interaction)

$$\sigma_{WK} \propto (G_{WK})^2$$

$$\Downarrow$$

$$\frac{g_W^2}{m_W^2} \qquad \underset{W}{\overset{g_W \quad g_W}{\sim\sim\sim}}$$

σ_{WK}: dimension of area $\sim M^{-2}$ | $\hbar = c = 1$ unit

G_{WK}: dimensional of M^{-2} (not $G_N = 1$)

Need something with unit of M^2 to multiply $(G_{WK})^2$

$$\sigma_{WK} \propto (G_{WK})^2 T^2$$

$$\sigma_{el} \propto (\alpha_{em})^2 T^{-2}$$

Reaction rate / Expansion rate

$$= \sigma_{wk}\, n / H$$

$$= (G_{wk})^2\, T^2 \times T^3 \times \sqrt{\frac{3}{4\pi G N a_B T^4}}$$

$$\simeq (G_{wk})^2\, T^3$$

$\hbar = c = G = 1$
unit

$$G_{wk} = 1.16\, 10^{-5}\ \mathrm{GeV}^{-2}$$

$$m_{pl} = 1.22 \times 10^{19}\ \mathrm{GeV}.$$

$$\Rightarrow G_{wk} = \frac{1.16 \times 10^{-5}}{(1.22 \times 10^{19})^2}$$

$$T = 10\ \mathrm{MeV} = \frac{10 \times 10^{-3}}{1.22 \times 10^{19}}$$

$$T^3 = \left(\frac{1}{1.22 \times 10^{21}}\right)^3$$

$\Rightarrow$ Reaction rate / Expansion rate

$$= \left(10^{-5} \times 10^{38}\right)^2 \times \frac{1}{(10^{21})^3} \left(\frac{T}{10\ \mathrm{MeV}}\right)^3$$

$$= 10^{3} \times \left(\frac{T}{10\ \mathrm{MeV}}\right)^3$$

$\rightarrow$ large at $T \sim 10$ MeV.

~ 1 at $T \simeq 1$ MeV.

At $T \sim 1\,\text{MeV}$ the neutrinos fall out of equilibrium.

$$\frac{\sigma_{el}\, n}{H} \sim (\alpha_{em})^2\, T^{-2}\, T^3\, \frac{1}{T^2} \quad (\text{in } \hbar = c = G = 1)$$

$$\sim (\alpha_{em})^2\, T^{-1}$$

$$\sim \left(\frac{1}{137}\right)^2 \times \left(\frac{T}{10\,\text{MeV}}\right)^{-1} \times \left(10\,\text{MeV}\right)^{-1}$$

$$\Downarrow$$

$$\left(\frac{10 \times 10^{-3}}{1.22 \times 10^{19}}\right)^{-1}$$

$$\sim \left(\frac{1}{137}\right)^2 \left(\frac{T}{10\,\text{MeV}}\right)^{-1} \times 10^{21} \gg 1$$

$\sim e^+e^-\gamma$ remains in equilibrium.

For $T < 1\,\text{MeV}$, neutrinos begin free expansion.

$$T_\nu \propto \frac{1}{a} \quad (\text{Redshift of wavelength})$$

$$s_\nu = \frac{4}{3} a_B T_\nu^3\, \frac{1}{2} \underbrace{(1+1+1+1+1+1)}_{\nu_e\ \bar{\nu}_e\ \nu_\mu\ \bar{\nu}_\mu\ \nu_\tau\ \bar{\nu}_\tau}$$

$\frac{d}{dt}(s_\nu a^3) = 0$ for different reason.

$T_1 \gg 1$ MeV.

$(T_\nu)_1 = T_1$ scale factor a_1.

$\rho_{e^+e^-\gamma} = \frac{4}{3} a_B T_1^3 \frac{1}{2} \times (2 + \frac{7}{8} \times 4)$

$T_2 \ll .5$ MeV.

mass of $e^\pm$.

$\Rightarrow \frac{n_{e^+}}{n_\gamma}, \frac{n_{e^-}}{n_\gamma}$ small.

so is $\frac{\rho_{e^\pm}}{\rho_\gamma}, \frac{\rho_{e^\pm}}{\rho_\gamma}$ etc.

$e^{-m/T}$ factor suppresses all terms.

$\Rightarrow \rho_{e^+e^-\gamma} = \rho_\gamma = \frac{4}{3} a_B T_2^3 \frac{1}{2} \times 2.$

$= \frac{4}{3} a_B T_2^3$

$(\rho_{e^+e^-\gamma}|_{T_1} \times a_1^3) = (\rho_{e^+e^-\gamma}|_{T_2} \times a_2^3$

$\Rightarrow \frac{4}{3} a_B T_1^3 \cdot \frac{1}{2} \times \frac{11}{2} \cdot a_1^3$

$= \frac{4}{3} a_B T_2^3 \; a_2^3$

$\Rightarrow T_1 = T_2 \frac{a_2}{a_1} \times \left(\frac{4}{11}\right)^{1/3}$

$(T_\nu)_2 = (T_\nu)_1 \cdot \frac{a_1}{a_2} = T_1 \frac{a_1}{a_2} = T_2 \left(\frac{4}{11}\right)^{1/3}.$

From this point onwards both ~~photons~~ T_γ and T_ν falls as $1/a$.

$$\Rightarrow \frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} \rightarrow \text{fixed.}$$

(till ν's become non-relativistic)

Our next goal will be to make a detailed study of the evolution from $T >> 1$ MeV to $T << .5$ MeV.

$T >> 1$ MeV.

$$\rho = \frac{1}{2} a_B T^4 N \qquad N = \frac{43}{4}$$

$$s = \frac{1}{2} a_B T^3 \cdot \frac{4}{3} N$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho$$

$$\frac{d}{dt}(s a^3) = 0 \Rightarrow Ta = \text{constant.}$$

$$\Rightarrow \left(\frac{\dot{T}}{T}\right)^2 = \frac{8\pi G}{3} \cdot \frac{1}{2} a_B T^4 N$$

$$\Rightarrow \frac{\dot{T}}{T^3} = -\sqrt{\frac{4\pi G a_B N}{3}} \qquad \dot{T} < 0$$

$$\Rightarrow \frac{1}{2T^2} = \sqrt{\frac{4\pi G}{3} a_B N}\,(t + c) \qquad c = 0$$

$$t = \sqrt{\frac{3}{16\pi G a_B N}}\,\frac{1}{T^2}$$

Natural units:

$$t = \sqrt{\frac{3}{16\pi \frac{43}{4}} \frac{15}{\pi^2} \frac{1}{T^2}}$$

$$= \sqrt{\frac{45}{172\pi^3}} \left(\frac{1\,\text{MeV}}{T}\right)^2 \frac{1}{(1\,\text{MeV})^2}$$

$$= \sqrt{\frac{45}{172\pi^3}} \left(\frac{1\,\text{MeV}}{T}\right)^2 \times \left(1.22\times10^{19}\times10^{3}\right)^2$$

$$= \sqrt{\frac{45}{172\pi^3}} \times (1.22)^2 \times 10^{44} \times \left(\frac{1\,\text{MeV}}{T}\right)^2$$

$$\times\, 5.39 \times 10^{-44} \text{ sec.}$$

$$= .736\left(\frac{\text{MeV}}{T}\right)^2 = .736 \times \left(\frac{\text{MeV}}{10^{10}\,\text{K}}\right)^2 \left(\frac{10^{10}\,\text{K}}{T}\right)^2$$

$$= .736 \times \left(\frac{\text{MeV}\,10^{6}}{8.62\times10^{-5}\times10^{10}}\right)^2 \left(\frac{10^{10}\,\text{K}}{T}\right)^2$$

$$= .99\,s \left(\frac{10^{10}\,\text{K}}{T}\right)^2$$

$\rightarrow$ Valid for $T \gg \text{MeV}$ or $T \gg 10^{10}\,\text{K}$.

Now consider the time evolution during the epoch $T \sim \text{MeV}$.

$$\frac{d}{dt}\left(s_{e^+e^-\gamma}\, a^3\right) = 0.$$

$$\Rightarrow \quad s_{e^+e^-\gamma} = a_B T^4 S(x) \qquad x = \frac{m_e}{T}.$$

$$S(0) = 1 + \frac{7}{8}\times 2 = \frac{11}{4}, \qquad S(\infty) = 1$$

$$\rho_\gamma = \frac{1}{2\pi^3} \times 2 \int_0^\infty 4\pi k^2\, dk \; \frac{k}{e^{k/T}-1} = a_B T^4$$

$$\rho_{e^-} = \frac{1}{8\pi^3} \times 2 \times \int_0^\infty 4\pi k^2\, dk \; \frac{\sqrt{k^2+m^2}}{e^{\sqrt{k^2+m^2}/T}+1}$$

$$= \frac{1}{8\pi^3} T^4 \times 2 \times \int_0^\infty 4\pi u^2\, du \; \frac{\sqrt{x^2+u^2}}{e^{\sqrt{x^2+u^2}}+1}$$

$k = Tu$, $x = \frac{m}{T}$

$$\rho_{e^+} = \rho_{e^-}$$

$$p_\gamma = \frac{1}{3}\rho_\gamma = \frac{1}{3} a_B T^4$$

$$p_{e^-} = \frac{T}{8\pi^3} \times 2 \times \int 4\pi k^2\, dk \; \ln\left(1+e^{\frac{\sqrt{k^2+m^2}}{T}}\right)$$

$$= \frac{T^4}{\pi^2} \times \int_0^\infty u^2\, du \; \ln\left(1+e^{\sqrt{x^2+u^2}}\right)$$

$$p_{e^+} = p_{e^-}$$

$$\Rightarrow \quad s_{e^+e^-\gamma} = \frac{p+\rho}{T} = \frac{4}{3} a_B T^3 s(x)$$

$$s(x) = \left\{ 1 + \frac{3}{4a_B} \times \frac{2}{\pi^2} \int_0^\infty u^2\, du \; \frac{\sqrt{x^2+u^2}}{e^{\sqrt{x^2+u^2}}+1} \right.$$

$$\left. + \frac{3}{4a_B} \times \frac{2}{\pi^2} \int_0^\infty u^2\, du \; \ln\left(1+e^{\sqrt{x^2+u^2}}\right) \right\}$$

$$= 1 + \frac{45}{2\pi^4} \int_0^\infty u^2\, du \left\{ \frac{\sqrt{x^2+u^2}}{e^{\sqrt{x^2+u^2}}+1} + \ln\left(1+e^{\sqrt{x^2+u^2}}\right) \right\}$$

$$s_{e^+e^-\gamma} = a_B T^3 s(x) \qquad s(0) = \frac{11}{4}, \quad s(\infty) = 1$$

$$s_{e^+e^-\gamma} \times a^3 = \text{constant}$$

$$\Rightarrow \quad T^3 a^3 s\left(\frac{m_e}{T}\right) = \text{constant}$$

$\Rightarrow T_\nu = \text{constant}$

$\Rightarrow T_\nu = c\, T\, s\left(\frac{m_e}{T}\right)^{1/3}$

$c = \left(\frac{4}{11}\right)^{1/3} \rightarrow$ from $T = \infty$

$$3\frac{\dot{T}}{T} + 3\frac{\dot{a}}{a} - \frac{m_e}{T^2}\dot{T}\,\frac{s'(x)}{s(x)} = 0$$

$$\Rightarrow \quad \frac{\dot{a}}{a} = -\frac{\dot{T}}{T}\left(1 - \frac{x}{3}\frac{s'(x)}{s(x)}\right)$$

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\rho_{tot}}$$

$$\rho_{tot} = \rho_\gamma + \rho_{e^+e^-} + \rho_\nu$$

$$= a_B T^4 \left\{ 1 + \frac{1}{\pi^2 a_B} \times 2 \int_0^\infty u^2\, du\, \frac{\sqrt{x^2+u^2}}{e^{\sqrt{x^2+u^2}}+1} + 3\left(\frac{T_\nu}{T}\right)^4 \times \frac{7}{8} \right\} = a_B T^4\, \Sigma(x)$$

$$T_\nu \propto a^{-1} \propto T\, s\left(\frac{m_e}{T}\right)^{1/3}$$

As $T \to \infty$, $T_\nu \to T$, $s\left(\frac{m_e}{T}\right)^{1/3} \to \frac{11}{4}$

$$\Rightarrow \boxed{T_\nu = T\left(\frac{4}{11}\right)^{1/3} s\left(\frac{m_e}{T}\right)^{1/3}}$$

$$\Sigma(x) = 1 + \frac{30}{\pi^4} \times \int_0^\infty u^2\, du\, \frac{\sqrt{x^2+u^2}}{e^{\sqrt{x^2+u^2}}+1} + \frac{21}{8}\left(\frac{4}{11}\right)^{4/3} s(x)^{4/3}$$

$$\frac{\dot{x}}{x} = -\frac{\dot{T}}{T} \times \left(1 - \frac{x}{3}\frac{\rho'(x)}{\rho(x)}\right) = \sqrt{\frac{8\pi G}{3} a_B}\, T^2 \cdot \xi(x)^{1/2}$$

$\Rightarrow$ t + constant

$$= \int_T^{\infty} \frac{dT}{T}\left(1 - \frac{x}{3}\frac{\rho'(x)}{\rho(x)}\right) \sqrt{\frac{3}{8\pi G a_B}} \times T^{-2}\, \xi(x)^{-1/2}$$

$$x = \frac{m_e}{T} \qquad \frac{dT}{T} = -\frac{dx}{x}$$

t + constant

$$= \sqrt{\frac{3}{8\pi G a_B m_e^4}} \int_0^{x} \frac{d\tilde{x}}{\tilde{x}}\left(1 - \frac{\tilde{x}}{3}\frac{\rho'(\tilde{x})}{\rho(\tilde{x})}\right) \tilde{x}^2\, \xi(\tilde{x})^{-1/2}$$

$t = 0$ at $x = 0$ $(T = \infty)$

$$t + C = \frac{1}{\sqrt{24\pi G a_B m_e^4}} \int_0^{x} \tilde{x}\, d\tilde{x}\, \xi(x)^{-1/2}\left(3 - \tilde{x}\frac{\rho'(\tilde{x})}{\rho(\tilde{x})}\right)$$

known fns

$$\parallel$$

$t_e = 4.3694$ sec. $\rightarrow$ Can be numerically integrated.

This can be evaluated to get t as fn. of $x = m_e/T$ and hence as fn. of T

$T(K)$	T/T_ν	$t(sec)$
$8.6\,MeV \equiv 10^{11}$	1.000	0
3×10^{10}	1.001	.101
10^{10}	1.008	.998
3×10^{9}	1.08	12.66
$= 10^{9}$	1.345	168
$.086\,MeV$ 3×10^{8}	1.401	1980
10^{8}	1.401	1.78×10^{4}

Free Neutron life-time ~ 886 sec.

$M_n - M_p \simeq 1.293\,MeV.$

~> Comparable to the energy scale and time scale of the e^+e^- ~~decoupling~~ annihilation era.

Effect of extra decoupled d.o.f.

ρ gets increased.

$\Rightarrow$ expansion is faster.

e.g. for 2 extra $\nu\bar{\nu}$ after e^+e^- annihilation

$(T \ll .5\,MeV,\ t > 100\,s)$

$$\rho = \frac{1}{2} a_B T^4 \times \left(2 + \frac{7}{8}\times\left(6 + 2l \times \left(\frac{4}{11}\right)^{4/3}\right)\right)$$

$$\cancel{= \frac{1}{2} a_B T^4 \left(\frac{29}{4} + g\right) = \frac{29}{8} a_B T^4 \left(1 + \frac{4g}{29}\right)}$$

$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\rho}$ increases by a factor of $\sqrt{\frac{2+\frac{21}{4}\left(\frac{4}{11}\right)^{4/3}\left(1+\frac{l}{3}\right)}{2+\frac{21}{4}\left(\frac{4}{11}\right)^{4/3}}}$

$\Rightarrow$ t decreases by a factor of

$$\sqrt{\left\{1+\frac{21}{8}\left(\frac{4}{11}\right)^{4/3}\right\} \Big/ \left\{1+\frac{21}{8}\left(\frac{4}{11}\right)^{4/3}\left(1+\frac{l}{3}\right)\right\}}$$

Same T is reached at shorter time.

For $l=1$ this is about .94

Effect of neutrino chemical potential:

$$n_\nu - n_{\bar{\nu}} = \frac{1}{8\pi^3}\int_0^\infty 4\pi k^2\,dk\left\{\frac{1}{e^{k/T-\frac{\mu}{T}}+1} - \frac{1}{e^{\frac{k}{T}+\frac{\mu}{T}}+1}\right\}$$

$$\underset{k=Tu}{=} \frac{1}{2\pi^2}T^3\int_0^\infty u^2\,du\left\{\frac{1}{e^{u-\frac{\mu}{T}}+1} - \frac{1}{e^{u+\frac{\mu}{T}}+1}\right\}$$

$$= T^3 f\left(\frac{\mu}{T}\right)$$

$$\rho_\nu + \rho_{\bar{\nu}} = \frac{1}{8\pi^3}\int_0^\infty 4\pi k^2\,dk\left\{\frac{k}{e^{\frac{k}{T}-\frac{\mu}{T}}+1} + \frac{k}{e^{\frac{k}{T}+\frac{\mu}{T}}+1}\right\}$$

$$= T^4 g\left(\frac{\mu}{T}\right)$$

$$s_\nu + s_{\bar{\nu}} = \frac{4}{3}T^3 g\left(\frac{\mu}{T}\right)$$

$$s_{e^+e^-\gamma} = \frac{4}{3}a_B T^3 \times \frac{1}{2}\left\{2+\frac{7}{8}\times(2+2)\right\}$$

The equations determining T and μ_i evolution in equilibrium:

$$\frac{d}{dt}\left\{(n_\nu - n_{\bar\nu})a^3\right\} = 0$$

$$\Rightarrow \frac{d}{dt}\left\{T^3 a^3 \sum_{i=e,\mu,\tau} f\left(\frac{\mu_{\nu_i}}{T}\right)\right\} = 0$$

$$\frac{d}{dt}\left\{\left(\rho_{e^+e^-\gamma} + \sum_{i=e,\mu,\tau} \rho_{\nu_i}\right)a^3\right\} = 0$$

$$\Rightarrow \frac{d}{dt}\left\{T^3 a^3 \left(\frac{4}{3}a_B \times \frac{11}{4} + \sum_i g\left(\frac{\mu_{\nu_i}}{T}\right)\right)\right\} = 0$$

4-equations for four variables Ta, μ_{ν_i} $(i = e, \mu, \tau)$.

$\rightarrow$ Can be solved by taking

$$Ta = \text{constant}, \quad \frac{\mu_{\nu_i}}{T} = \text{constant}$$

4 unknown constants.

correct number.

Thus during equilibrium μ_{ν_i}/T remains constant and $T \propto 1/a$.

After decoupling

$$\rho_{\nu_i} = \frac{1}{2\pi^2}\int_0^\infty k^2\,dk\,\frac{1}{e^{\frac{k-\mu_{\nu_i}}{T}}+1}$$

Expansion: $k \to \lambda k$. $\equiv$ $T \to \lambda T$, $\frac{\mu_{\nu_i}}{T}$ constant.

Thus both in equilibrium and out of equilibrium, $T_\nu \propto 1/a$ and μ_{ν_i}/T remains constant.

What is the effect of $\mu_\nu/T \neq 0$ on the Friedman equation?

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

$$\rho = \rho_{e^+e^-\gamma} + \sum_i \rho_{\nu_i}$$

$$\rho_\nu + \rho_{\bar{\nu}} = \frac{1}{2\pi^2}\int_0^\infty k^3\,dk\left(\frac{1}{e^{\frac{k}{T}-\frac{\mu}{T}}+1} + \frac{1}{e^{\frac{k}{T}+\frac{\mu}{T}}+1}\right)$$

$$= \frac{1}{2\pi^2}\int_0^\infty k^3\,dk\,\frac{2}{e^{\frac{k}{T}}+1}$$

$$+\frac{1}{2\pi^2}\int_0^\infty k^3\,dk\left\{\frac{1}{e^{\frac{k}{T}-\frac{\mu}{T}}+1} + \frac{1}{e^{\frac{k}{T}+\frac{\mu}{T}}+1} - \frac{2}{e^{\frac{k}{T}}+1}\right\}$$

strictly positive.

Thus $\mu_\nu \neq 0$ increases ρ_ν.

$\rightarrow$ makes the expansion rate larger.

Large positive μ/T:

$$\frac{1}{e^{k/T - \frac{\mu}{T}} + 1} \simeq 1 \quad \text{for } \frac{k}{\mu} < 1$$

$$\simeq 0 \quad \text{for } \frac{k}{\mu} > 1$$

$$\frac{1}{2\pi^2} \int_0^\infty k^3 \, dk \, \frac{1}{e^{\frac{k}{T} - \frac{\mu}{T}} + 1}$$

$$\sim \frac{1}{2\pi^2} \int_0^{\mu} k^3 \, dk \sim \frac{1}{2\pi^2} \left(\mu\right)^4 \times \frac{1}{3}$$

$\therefore \Rightarrow \rho$ is enhanced by a factor of order $(\mu/T)^4$

Direct experimental bounds on μ_{ν_e} are very weak.

For $\mu_e \sim$ eV it affects β-decay results

$^3H \to {}^3He + e^- + \bar{\nu}_e \sim$ can be measured if it has energy $\sim$ eV. (from $^3H - {}^3He$ e^- energy difference)

only observe this.

$\mu_e \sim$ eV $\Rightarrow$ there are neutrinos with energy $\sim$ eV in background.

$\nu_e + {}^3H \to {}^3He + e^- \rightsquigarrow$ will enhance the rate (visible near the tail)

If $\mu_e \sim$ eV then there are $\bar{\nu}_e$ up to energy eV in background $\Rightarrow$ suppress β-decay due to Pauli exclusion

Between $T = 10^{10}$ K ($\sim .86$ MeV) and $T = 10^9$ K ($.086$ MeV) several important changes happen.

① e^+e^- goes non-relativistic.

② Neutrinos go out of equilibrium.

$T_\nu = T$ for $T \gg 10^{10}$ K ~~to~~ to $T_\nu = \left(\frac{4}{11}\right)^{1/3} T$ for $T \ll 10^9$ K

$\frac{1}{1.4}$

Time scale: Takes about 168 sec to reach 10^9 K from very high T

For $T \ll 10^9$ K ~~After~~ ~~this~~ $t \propto T^{-2}$

$$\frac{t}{168 \text{ sec}} = \left(\frac{T}{10^9 \text{ K}}\right)^{-2} \rightarrow \text{radiation dominated}$$

Q. What happens to ~~&~~ protons and neutrons during this time?

Expectation $\frac{n_p}{n_\gamma}, \frac{n_n}{n_\gamma}$ small

$\Rightarrow$ cannot affect the state of $e^+e^-\gamma\nu$ system.

However $e^+e^-\gamma\nu$ system can affect the state of p, n.

~~We can regard~~ For $T \gg 10^{10}$ K when $T_\nu = T$, we can regard p, n to be described by equilibrium stat mech. system in contact with the thermal bath of $e^+ e^- \gamma \nu$ system.

Interactions:

Electromagnetic. $p\gamma \rightleftarrows p\gamma$ etc. → very high rate.

Weak: $n + \nu_e \rightleftarrows p + e^-$, $n + e^+ \rightleftarrows p + \bar{\nu}_e$,

$n \rightarrow p + e^- + \bar{\nu}_e$

Define $Q = m_n - m_p = 1.293$ MeV

As long as the rates for these reactions are larger than H, the p, n system will be in thermal and chemical equilibrium with e-γ-ν system.

(To be tested later).

Conserved charges:

Electric charge: $n_p - n_{\bar{p}} - (n_{e^-} - n_{e^+})$

Electron lepton number: $n_{e^-} - n_{e^+} + n_{\nu_e} - n_{\bar{\nu}_e}$

Baryon number: $n_p - n_{\bar{p}} + n_n - n_{\bar{n}}$

$$\tilde{\mu}_1 (n_p - n_{\bar{p}} - n_{e^-} + n_{e^+}) + \tilde{\mu}_2 (n_{e^-} - n_{e^+} + n_{\nu_e} - n_{\bar{\nu}_e})$$
$$+ \tilde{\mu}_3 (n_p - n_{\bar{p}} + n_n - n_{\bar{n}})$$

Note: $n_{\nu_\mu} - n_{\bar{\nu}_\mu}$ and $n_{\nu_\tau} - n_{\bar{\nu}_\tau}$ are separately conserved (ignoring neutrino oscillation)

$$\mu_e = -\tilde{\mu}_1 + \tilde{\mu}_2 \Rightarrow \tilde{\mu}_1 = -\mu_e + \mu_\nu$$
$$\mu_\nu = \tilde{\mu}_2 \qquad \tilde{\mu}_2 = \mu_\nu$$
$$\mu_B = \tilde{\mu}_3 \qquad \tilde{\mu}_3 = \mu_B$$

$$\Rightarrow \mu_e (n_{e^-} - n_{e^+}) + \mu_\nu (n_{\nu_e} - n_{\bar{\nu}_e})$$
$$+ (\mu_B + \mu_\nu - \mu_e)(n_p - n_{\bar{p}}) + \mu_B (n_n - n_{\bar{n}})$$

$$\Rightarrow \mu_p = \mu_B + \mu_\nu - \mu_e$$
$$\mu_n = \mu_B$$

$$n_p = \frac{1}{8\pi^3} \times 2 \times \int_0^\infty 4\pi k^2 \, dk \, \frac{1}{e^{\frac{\sqrt{m_p^2 + k^2} - \mu_p}{T}} + 1}$$

$$\simeq \frac{1}{2\pi^2} \times 2 \times \int_0^\infty e^{-\frac{\sqrt{m_p^2 + k^2}}{T} + \frac{\mu_p}{T}} k^2 \, dk$$

$$\frac{\sqrt{m_p^2 + k^2}}{T} = \frac{m_p}{T} + \frac{k^2}{2 m_p T}$$

$$= \frac{1}{2\pi^2} \times 2 \times e^{-\frac{m_p}{T} + \frac{\mu_p}{T}}$$

$$\times \int_0^\infty k^2\, dk\, e^{-\frac{k^2}{2mT}}$$

$$= 2\, e^{-\frac{m_p - \mu_p}{T}} \left(\frac{m_p T}{2\pi}\right)^{3/2}$$

Similarly

$$n_n = 2\, e^{-\frac{m_n - \mu_n}{T}} \left(\frac{m_n T}{2\pi}\right)^{3/2}$$

$\frac{n_p + n_n}{n_\gamma} \sim 10^{-9}$ fixes μ_B.

(If $\mu_e = \mu_\nu = 0$, then μ_B is the only independent parameter).

Ex. ~~check that~~ ~~write~~ Compute μ_B and check that $n_{\bar{p}}$ and $n_{\bar{n}}$ computed from these formula are ~~ent~~ negligible compared to n_p and n_n.

$$X_n \equiv \frac{n_n}{n_p + n_n} = \text{fraction of neutrons}$$

$$X_n \simeq e^{-\frac{m_n - m_p}{T} + \frac{\mu_e - \mu_\nu}{T}} \Big/ \left(1 + e^{-\frac{m_n - m_p}{T} + \frac{\mu_e - \mu_\nu}{T}}\right)$$

For large T, $e^{\pm}$ are relativistic

$\Rightarrow n_{e^+} + n_{e^-} \gg n_{e^+} - n_{e^-}$

$\Rightarrow e^{\mu_e/T}$ is small.

About $e^{\mu_\nu/T}$ we cannot be sure but we proceed as before by assuming μ_ν/T to be small.

$$\Rightarrow X_n \simeq \frac{e^{-Q/T}}{1+e^{-Q/T}} \qquad Q \equiv m_n - m_p$$

Condition for equilibrium: Calculate the rates of $n \rightleftarrows p$.

$n \to p$ can occur via

$n \to p + e^- + \bar{\nu}_e$

$n + e^+ \to p + \bar{\nu}_e$

$n + \nu_e \to p + e^-$

$p \to n$ via reverse processes.

Example: Take $n + e^+ \to p + \bar{\nu}_e$

Rate per neutron (assumed at rest)

$$\sigma v_{e^+} n_{e^+}$$

$$\sigma = 2\pi^2 A \, E_\nu^2 / v_{e^+}$$

(Counts both directions)

$$A = \frac{G_{WK}^2 (1+3g_A^2) \cos^2\theta_c}{2\pi^3}$$

$G_{WK} = 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}$, $g_A = 1.257$, $\cos\theta_c = .975$

n_{e^+} between p_e and $p_e + dp_e$

$$\frac{1}{8\pi^3} \times \cancel{2} \times 4\pi p_e^2 \, dp_e \, \frac{1}{e^{\frac{E_{e^+}}{T}} + 1}$$

$$E_e = \sqrt{p_e^2 + m_e^2} \qquad v_e = \frac{p_e}{\sqrt{p_e^2 + m_e^2}}$$

Correction: We need the final state neutrino energy levels to be unfilled.

$$1 - \frac{1}{1 + e^{E_\nu / T_\nu}} = \frac{1}{1 + e^{-E_{\bar\nu}/T_\nu}}$$

$$\lambda(n + e^+ \to p + \bar\nu)$$

$$= A \int p_e^2 \, dp_e \, E_\nu^2 \, \frac{1}{1 + e^{E_{e^+}/T}} \, \frac{1}{1 + e^{-E_{\bar\nu}/T_\nu}}$$

$$E_{\bar\nu} - E_{e^+} = Q$$

$$\overset{\parallel\!\!\parallel}{-q} \qquad \overset{\parallel}{\sqrt{p_e^2 + m_e^2}}$$

$$E_{e^+} = -q - Q$$

$$E_{\bar\nu} = -q$$

$$-q = Q + E_{e^+} > Q + m_e$$

$$-\infty < q < -Q - m_e$$

$$p_e^2 + m_e^2 = (q + Q)^2$$

$$p_e \, dp_e = (q + Q) \, dq$$

$$p_e = (q + Q) \left\{ 1 - \frac{m_e^2}{(q+Q)^2} \right\}^{1/2}$$

$\lambda(n + e^+ \to p + \bar{\nu})$

$$= A \int_{-\infty}^{-Q-m_e} dq \left(1 - \frac{m_e^2}{(q+Q)^2}\right)^{1/2} \frac{q^2 (Q+q)^2}{\left(1 + e^{-\frac{q+Q}{T}}\right)\left(1 + e^{\frac{q}{T_\nu}}\right)}$$

Total rate

$\lambda(n \to p)$

$$= A \int dq \left(1 - \frac{m_e^2}{(q+Q)^2}\right)^{1/2} \frac{q^2 (Q+q)^2}{\left(1 + e^{-\frac{q+Q}{T}}\right)\left(1 + e^{\frac{q}{T_\nu}}\right)}$$

$-\infty \le q \le -Q - m_e$,	$-Q + m_e \le q \le \cancel{-m_e}\ 0$
$0\ \cancel{m_e} \le q \le \infty$	$\downarrow$
$\searrow$	$n \to p + e^- + \bar{\nu}_e$
$n + \nu_e \to p + e^-$	

Total rate

$\lambda(p \to n)$

$$= A \int dq \left(1 - \frac{m_e^2}{(Q+q)^2}\right)^{1/2} \frac{(Q+q)^2\, q^2\, dq}{\left(1 + e^{-\frac{q}{T_\nu}}\right)\left(1 + e^{\frac{Q+q}{T}}\right)}$$

$A = G_{WK}^2 \left(1 + 3 g_A^2\right) \cos^2\theta_c / 2\pi^3$ $\quad$ ($\hbar = c = 1$ units)

$G_{WK} = 1.166 \times 10^{-5}\ (\mathrm{GeV})^{-2}$, $g_A = 1.257$, $\cos\theta_c = 0.975$

$$\frac{dX_n}{dt} = -\lambda(n\to p)\, X_n + \lambda(p\to n)\,(1-X_n)$$

For ~~large~~ $T = T_\nu$,

$$\lambda(p\to n) = e^{-\frac{Q}{T}}\,\lambda(n\to p)$$

$\frac{dX_n}{dt} \propto \frac{dT}{dt}$ → small compared to λ.

$$\frac{dX_n}{dt} = 0 \quad \text{for} \quad \frac{X_n}{1-X_n} = e^{-Q/T}$$

⇒ equilibrium value.

Estimate of $\lambda(n\to p)$ & $\lambda(p\to n)$.

For $T \cong T_\nu \sim$ MeV, we can add $Q+q$ by q to replace $e^{\pm Q/T}$ by 1 in ~~$(1+e^{-\frac{q+Q}{T}})(1+e^{\frac{Q}{T}})$~~

Get an order of magnitude estimate

$$\Rightarrow \lambda = A\int dq \frac{q^4}{(1+e^{q/T})(1+e^{-q/T})}$$

$$= \frac{7}{15}\pi^4 A T^5$$

$$= \cdot 400\ \text{sec}^{-1}\left(\frac{T}{10^{10}\,K}\right)^5$$

Note: Same T^5 as in $e^+ + \nu_e \to e^+ + \nu_e$ etc. (weak interaction rate)

$$\mathcal{H} = \sqrt{\frac{8\pi G}{3}\frac{1}{2}a_B T^4 \times \frac{43}{4}} = N \qquad a_B = \frac{\pi^2}{15}$$

$$\frac{\lambda}{\mathcal{H}} = .8 \times \left(\frac{T}{10^{10}K}\right)^3$$

Thus $\frac{\lambda}{\mathcal{H}}$ falls below 1 at $T \lesssim 10^{10}K$
$= .86$ MeV

~ same temperature at which ~~fixed T stat differing~~ neutrinos fall out of equilibrium.

Below this temperature we cannot use equilibrium statistical mechanics.

Nevertheless we can solve

$$\frac{dX_n}{dt} = -\lambda(n\to p)\, X_n + \lambda(p\to n)\,(1-X_n)$$

numerically to find $X_n(t)$.

Input data: $T(t)$, $T_\nu(t)$ from earlier calculation.

Small $T \Rightarrow$ Small T_ν

$\lambda(p\to n)$ contains $\frac{1}{e^{-Q/T_\nu}+1}$ $\quad \frac{1}{e^{+\frac{Q+Q}{T}}+1}$

$\Downarrow$ large for $Q < 0$ $\quad \Downarrow$ large for $Q > -Q$

$\Rightarrow \lambda(p \to n) \simeq 0$

$\lambda(n \to p)$: contains

$$\frac{1}{\left(e^{-\frac{Q+q}{T}}+1\right)\left(e^{\frac{q}{T_\nu}}+1\right)}$$

large for $q>0$

$\Downarrow$

large for $q+Q<0$

$\Rightarrow -Q < q < 0$ contributes

In this range $\dfrac{1}{e^{-\frac{Q+q}{T}}+1} \cdot \dfrac{1}{e^{\frac{q}{T_\nu}}+1} \simeq 1$

except at the edges.

$$\Rightarrow \lambda(n \to p) \simeq A \int_{-Q}^{0} dq \left(1 - \frac{m_e^2}{(Q+q)^2}\right)^{1/2} q^2 (Q+q)^2$$

$= \tau_n^{-1}$ neutrons decay rate

$$\Rightarrow \frac{dX_n}{dt} = -\frac{1}{\tau_n} X_n$$

$$\Rightarrow X_n = X_n^0 \exp(-t/\tau_n)$$

$\tau_n = 885.7$ sec. $\quad X_n^0 = .1609$

$\Downarrow$

numerical evolution

For $t \gg \tau_n$, X_n decreases rapidly.

$Q = 1.293$ MeV

T(K)	t(sec)	$X_n \rightarrow \frac{1}{e^{Q/T}+1}$
86 MeV $= 10^{12}$	.0001	.4962
10^{11}	.0099	.4626
3×10^{10}	.1106	.3798
10^{10}	1.008	.2386
3×10^{9}	12.67	.1654
10^{9}	168.1	.1333
3×10^{8}	1980	.0172
10^{8}	17780	3.07×10^{-10}

The neutron population gets highly depleted for $t \gg 800$ s.

→ Consequence of $\tau_n \simeq 885.7$ sec.

The process stops by formation of heavier nuclei.

Neutron is stable inside heavier nuclei e.g. deuteron $d \equiv H^2$ or helium He^4.

The process of nuclei formation

$p + p \rightleftarrows d + \gamma$ (electromagnetic)

$$\left.\begin{array}{l} d + d \rightleftarrows He^3 + n \\ d + d \rightleftarrows H^3 + p \\ d + H^3 \rightleftarrows He^4 + n \\ d + He^3 \rightleftarrows He^4 + p \end{array}\right\} \text{strong}$$

There are also slower electromagnetic process.

Suppose the strong interaction 'switches on' at time t_1. ~~and~~

→ The system equilibriates very fast

→ Can ignore $\frac{dX_n}{dt}$, and take $X_n(t_1)$ as the given ratio between $(n_n)_{tot}$ and $(n_p + n_n)_{tot}$

$n_N = (n_n)_{tot} + (n_p)_{tot}$ ⇒ total no. of $n + p$.

from Ω_B & H_0 — given — given

$n_n(t_1) = n_N X_n(t_1)$ $\quad n_p(t_1) = n_N(1 - X_n(t_1))$

After equilibrium let:

n_n = # of free neutrons

n_p = # of free protons

n_i = # of nuclei of type i

Z_i = # of protons in ith nucleus

$A_i - Z_i$ = # of neutrons in ith nucleus

(ignore anti-particles)

Conserved charges:

$$(n_p)_{tot} = n_p + \sum_i Z_i n_i = n_N (1 - X_n(t_0))$$

$$(n_n)_{tot} = n_n + \sum_i (A_i - Z_i) n_i = n_N X_n(t_0)$$

$$\tilde{\mu}_1 (n_n)_{tot} + \tilde{\mu}_2 (n_p)_{tot}$$

$$= \tilde{\mu}_1 \left(n_n + \sum_i (A_i - Z_i) n_i \right) + \tilde{\mu}_2 \left(n_p + \sum_i Z_i n_i \right)$$

$$\mu_p = \tilde{\mu}_2 \qquad \mu_n = \tilde{\mu}_1$$

$$\mu_i = (A_i - Z_i) \tilde{\mu}_1 + Z_i \tilde{\mu}_2 = Z_i \mu_p + (A_i - Z_i) \mu_n$$

$$n_i = \frac{1}{8\pi^3} g_i \int_0^\infty 4\pi k^2 dk \frac{1}{e^{\frac{\sqrt{M_i^2+k^2}-\mu_i}{T}}+1}$$

g_i: # of spin states

$$\simeq \frac{1}{8\pi^3} g_i \int_0^\infty 4\pi k^2 dk \, e^{-\frac{M_i-\mu_i}{T} - \frac{k^2}{2M_iT}}$$

$$= g_i e^{-\frac{M_i-\mu_i}{T}} \left(\frac{M_iT}{2\pi}\right)^{3/2}$$

$$n_p = 2 e^{-\frac{m_p-\mu_p}{T}} \left(\frac{m_pT}{2\pi}\right)^{3/2}$$

$$n_n = 2 e^{-\frac{m_n-\mu_n}{T}} \left(\frac{m_nT}{2\pi}\right)^{3/2}$$

Take $m_n \simeq m_p$

$$M_i = Z_i m_p + (A_i - Z_i) m_n$$

except in the exponents

$$\frac{n_i}{n_p^{Z_i} n_n^{A_i-Z_i}} = g_i 2^{-A_i} e^{-\frac{1}{T}(M_i-(A_i-Z_i)m_n - Z_i m_p)} \times \left(\frac{T}{2\pi}\right)^{\frac{3}{2}(1-Z_i-(A_i-Z_i))} \frac{M_i^{3/2}}{(m_p)^{3A_i/2}}$$

$A_i m_p$

$$Y_i = \frac{n_i}{n_N} \qquad Y_n = \frac{n_n}{n_N} \qquad Y_p = \frac{n_p}{n_N}$$

$$\Rightarrow \frac{Y_i}{Y_n^{A_i - Z_i}\, Y_p^{Z_i}} = \frac{g_i}{2} (A_i)^{3/2} e^{B_i/T} \epsilon^{A_i - 1}$$

$$B_i = Z_i m_p + (A_i - Z_i) m_n - M_i$$

= binding energy

$$\epsilon = \frac{1}{2} n_N \left(\frac{T m_p}{2\pi} \right)^{-3/2}$$

Y_n, Y_p are unknowns determined from:

$$Y_n + \sum_i (A_i - Z_i) Y_i = X_n(t_1) \quad \text{— known}$$

$$Y_p + \sum_i Z_i Y_i = 1 - X_n(t_1)$$

$\Rightarrow$ Two eqs. for two unknowns.

$$n_N = \left(\frac{a_0}{a}\right)^3 \times \frac{\Omega_B \rho_c}{m_p}$$

$$\rho_c = \cancel{\frac{8\pi G}{3}} \; \frac{3H_0^2}{8\pi G}$$

$$\Omega_B \rho_c = \Omega_B \frac{3}{8\pi G} H_0^2$$

$$= \Omega_B \frac{3}{8\pi G} \left(100 \text{ km/s/Mpc} \times h\right)^2$$

$$= \Omega_B h^2 \frac{3}{8\pi G} \left(100 \text{ km/s}/3.086 \times 10^{24} \text{ cm}\right)^2$$

$$\Omega_B \simeq .04, \quad h \simeq .7 \Rightarrow \Omega_B h^2 \simeq .02$$

$$\frac{a_0}{a} = \left(\frac{T}{T_0}\right)$$

$$n_N = \left(\frac{T}{T_0}\right)^3 \frac{3}{8\pi G m_p} \left(\frac{1}{3.086 \times 10^{19} \text{ sec}}\right)^2 \Omega_B h^2$$

$$\epsilon = 1.46 \times 10^{-12} \left(\frac{T}{10^{10} K}\right)^{3/2} \Omega_B h^2$$

using $T_0 = 2.7 K$, $m_p = 1$ GeV $\rightarrow$ convert to Planck scale

Convert 1 sec to Planck scale

set $G = 1$

A general note

For detailed analysis of nucleosynthesis one can use evolution equations

$$\frac{dn_\alpha}{dt} = (\text{no of events with } \alpha \text{ in final state}) - (\text{no of events with } \alpha \text{ in initial state})$$

per unit volume per unit time

$\alpha = n, p, i$

$$\Rightarrow \frac{dn_\alpha}{dt} = F(n_n, n_p, \{n_i\}, T)$$

Known

$\Rightarrow$ can be solved to find $n_\alpha(t)$.

Our goal here is to show that equilibrium statistical mechanics can give a fairly accurate estimate

$$Y_i = \frac{g_i}{2} Y_n^{A_i - Z_i} Y_p^{Z_i} (A_i)^{3/2} e^{B_N/T} \epsilon^{A_i - 1}$$

$$Y_n + \sum_i (A_i - Z_i) Y_i = X_n(t_1)$$

$$Y_p + \sum_i Z_i Y_i = 1 - X_n(t_1)$$

$\Rightarrow$ Used to determine Y_n and Y_p.

$B_d = 2.225$ MeV

$B_{H^3} = 8.482$ MeV

$B_{He^3} = 7.718$ MeV

$B_{He^4} = 28.301$ MeV

Suppose $T = 10^9$ K $= .086$ MeV.

$\exp(B_{He}/T) \sim 10^{142}$

$\exp(B_{He^3}/T) \sim 10^{39}$

$\exp(B_{H^3}/T) \sim 10^{42}$

$\exp(B_d/T) \sim 10^{11}$

$\epsilon \sim \cancel{7.8} \times 10^{-15}$ $\Omega_B = .04$ $h \simeq .7$

$$Y_{He} \sim Y_n^2 Y_p^2 \, 10^{142-45} \sim Y_n^2 Y_p^2 \, 10^{97}$$

$$Y_{He^3} \sim Y_n Y_p^2 \, 10^{39-30} \sim Y_n Y_p^2 \, 10^{9}$$

$$Y_{H^3} \sim Y_n^2 Y_p \, 10^{42-30} \sim Y_n^2 Y_p \, 10^{12}$$

$$Y_d \sim Y_n Y_p \, 10^{11-15} \sim Y_n Y_p \, 10^{-4}$$

Y_n, Y_p to be obtained by solving

$$Y_n + Y_d + Y_{He^3} + 2Y_{H^3} + 2Y_{He^4} = X_n$$
$$Y_p + Y_d + 2Y_{He^3} + Y_{H^3} + 2Y_{He^4} = 1 - X_n$$
at t_1

At $T \sim 10^9$, $X_n \sim .133$

$$Y_{He^4} \sim Y_n^2 Y_p^2 \times 10^{97}$$

$\Rightarrow$ either Y_n or Y_p or both must be small.

Since $X_n < 1 - X_n$ we cannot find a solution by taking Y_p small.

$\Rightarrow$ Y_n is small.

$$Y_{He^4} \simeq \frac{1}{2} X_n, \quad Y_p \simeq 1 - X_n - X_n = 1 - 2X_n$$

$$2 \times Y_n^2 \underbrace{Y_p^2}_{\sim 1} \cdot 10^{97} \sim .13$$

$$Y_n^2 \sim 10^{-97} \qquad Y_n \sim 10^{-48}$$

With this

$$Y_{He^3} \sim Y_n Y_p^2 \, 10^9 \sim 10^{-48+9} \sim 10^{-39}$$
$$Y_{H^3} \sim Y_n^2 Y_p \, 10^{12} \sim 10^{12-96} \sim 10^{-84}$$
$$Y_d \sim Y_n Y_p \, 10^{-4} \sim 10^{-4-48} \sim 10^{-52}$$

Lesson: Once the system equilibriates the final state consists of only ~~neutrons~~ He^4 and protons.

$$Y_{He^4} = .133/2$$

$$Y_p = 1 - 2 \times .133$$

By weight:

$$\frac{M_{He^4}}{M_p} = \frac{4 \times .133/2}{1 - 2 \times .133} = \frac{.27}{.73}$$

$\Rightarrow$ 27% He^4, 73% H by mass.

For any other T we can carry out a similar analysis.

Q. How to determine T?

We have to compute reaction rates and compare with H.

$p + n \rightleftharpoons d + \gamma$ → needed first.

We need to check first if this is fast enough.

For this assume that we only have p, n, d and carry out the analysis.

$\lambda_d \equiv$ Rate of deuteron production per neutron

$$= \underbrace{\sigma_d v}_{4.55\times10^{-12}\ cm^3/sec} n_p \longrightarrow Y_p n_N$$

$$\simeq 2.52 \times 10^4\ sec^{-1} \left(\frac{T}{10^{10}K}\right)^3 Y_p \Omega_B h^2$$

$$\mathcal{H} = \sqrt{\frac{8\pi G}{3}\rho} \rightarrow \frac{1}{2} a_B T^4 \left(2 + \frac{7}{8}\times 6\right)$$

$$= .28 \times \left(\frac{T}{10^{10}K}\right)^2 sec^{-1} \qquad \pi^2/15$$

$$\frac{\lambda_d}{\mathcal{H}} \simeq 9\times10^4 \left(\frac{T}{10^{10}K}\right) \underbrace{Y_p}_{\sim 1} \underbrace{\Omega_B h^2}_{.04\times(.7)^2 \sim .02}$$

$\rightarrow$ significantly larger than 1 for $T > 10^9$

$\Rightarrow$ neutrons are efficiently converted to deuterons.

$\Rightarrow$ Y_d is given by its equilibrium value in the absence of other reactions.

$$Y_d \simeq 3\sqrt{3}\, Y_p Y_n \epsilon \exp(B_d/T)$$

Around $T \sim 10^9 K$

$$Y_d \sim Y_p Y_n \, 10^{-15} \times 10^{11} \sim 10^{-4} Y_p Y_n$$

$\Rightarrow Y_d << Y_n$

$\Rightarrow Y_n \simeq X_n, \quad Y_p \simeq X_p.$

Now examine rate for further reactions.

$$d + d \rightarrow H^3 + p \Rightarrow \sigma = \frac{1.8 \times 10^{-17}}{v} cm^3/sec$$

$$d + d \rightarrow He^3 + n \quad \sigma = \frac{1.6 \times 10^{-17}}{v} cm^3/sec.$$

$\hat{\Lambda}$ = Total rate per deuteron

$$= \sigma v n_d \simeq Y_d n_N$$

$$= 1.9 \times 10^7 \times \left(\frac{T}{10^{10} K}\right)^3 \Omega_B h^2 Y_d$$

$$\mathcal{H} = .28 \left(\frac{T}{10^{10} K}\right)^2 sec^{-1}$$

$$\frac{\hat{\Lambda}}{\mathcal{H}} = \frac{1.9 \times 10^7}{.28} \left(\frac{T}{10^{10} K}\right) \Omega_B h^2 \, 3\sqrt{3}\, Y_p Y_n \epsilon \exp(B_d/T)$$

At $T \simeq 10^9 K$

$$\frac{\hat{\Lambda}}{\mathcal{H}} \sim 10^8 \times 10^{-1} (.02) \times 5 \times .13 \times .87 \times 10^{-15} \times 10^{11}$$

$$\sim 10^{19-1-1-1-15} \sim 10$$

The rate for

$$H^3 + d \rightleftarrows He^4 + n$$

$$He^3 + d \rightleftarrows He^4 + p$$

are similar for H^3 & He^3 since ~~density~~ cross sections are of same order and n_d is the same.

⇒ Around $T \sim 10^9$ K the system equilibriates and the ~~system~~ settles down to dominantly He^4 and p.

Q. Could it be $\frac{1}{2} \times 10^9$ K or 2×10^9 K

→ will make a difference since $X_n(t)$ at those times will be different.

T dependence of $\frac{\Lambda}{H}$

$$T \times \underbrace{T^{+3/2}}_{\text{from } \epsilon} \times \exp(B_d/T) \sim \underbrace{T^{+5/2}}_{\text{mild.}} \exp(B_d/T)$$

$\Delta T = 10\% \Rightarrow 25\%$ change.

$B_d = 2.225$ MeV.

For $T = 10^9$ K $= .086$ MeV

$B_d/T \simeq 25.87$

10% change in T or $1/T$

$\Rightarrow \Delta\left(\frac{B_d}{T}\right) \simeq 2.587$

$e^{\Delta(B_d/T)} = e^{2.587} = 13.28$

$\Rightarrow$ 10% drop in T increases $\frac{\Lambda}{H}$ by a factor of 13.

10% rise in T decreases $\frac{\Lambda}{H}$ by a factor of 13.

$\Rightarrow$ T is within about 10% of 10^9 K.

At $T = 10^9$ K, $t \sim 168$ sec.

10% change in T $\Rightarrow$ 20% change in t.

$t \sim T^{-2}$

$\Rightarrow$ 168 sec $\pm$ 33.6 sec.

$X_n \to X_n \exp(\pm 33.6/885)$

$\sim X_n (1.04)^{\pm 1}$

$X_n|_{10^9 \times 1.1 K} = -1385$

$X_n|_{10^9 K \times .9} = -1268$

X_n differs from its value at 10^9 K by 4%

$M_{He}/M_{tot} = .27(1 \pm .04)$

Correction

$$\frac{\Lambda}{H} = \frac{1.9 \times 10^7}{.28} \left(\frac{T}{10^{10}K}\right) \Omega_B h^2 \, 3\sqrt{3} \, Y_p Y_n \, \epsilon \, \exp(B_d/T)$$

$d+d \to H^3 + p$
$He^3 + n$

At the beginning $Y_p \simeq X_p$ $Y_n \simeq X_n$

$\to$ mild T dependence.

$$\epsilon \propto \frac{1}{2} n_N \left(\frac{m_p T}{2\pi}\right)^{-3/2} \propto T^{3/2}$$

$$\propto \left(\frac{a_0}{a}\right)^3 \propto \left(\frac{T}{T_0}\right)^3$$

$$\frac{\Lambda}{H} \propto T^{5/2} \exp(B_d/T)$$

10% change in T $\Rightarrow$ ~30% change in $T^{5/2}$

13 times in $\exp(B/T)$

$\Rightarrow$ $\frac{\Lambda}{H}$ changes by a factor of 10.

This fixes $T \simeq 10^9$ K within 10%

as the beginning of nucleosynthesis

Even though we used results of equilibrium statistical mechanics to get the results it is useful to understand the non-equilibrium process behind it.