15/11/06

$$S_{total} = S + S_{g.f.} + S_{ghost}$$

is invariant under a BRST symmetry

Ex. If we take the most general ~~action~~ Lagrangian density with dimension ≤ 4 terms & require that it is invariant under BRST transformation, we arrive at the action given.

Example → Take the gauge-fixing term as

$$S_{g.f.} = -\int d^4x \, \frac{1}{2\alpha} (\partial_\mu A^{\mu a})^2$$ → Lorentz invariant

Consider a general total Lagrangian density containing all possible Lorentz invariant terms with arbitrary coefficients.

Then require $\delta S_{tot} = 0$ under BRST trs.

Take cubic & quartic coupling constants for A^μ to be diff.

$\xrightarrow[\text{us}]{\text{this gives}}$ relations among the coefficients appearing in S.

set coeff. of each operator equal to zero

Ex. Show that these relations lead to

$$S_{tot} = S + S_{g.f.} + S_{ghost}$$

$$\mathcal{L} = -\tfrac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}\right)$$

$$- \tfrac{g}{2}\left(\partial_\mu A_\nu^c - \partial_\nu A_\mu^c\right) f^{abc} A^{\mu a} A^{\nu b}$$

$$- \tfrac{1}{4} g^2 f^{abc} f^{a'b'c'} A_\mu^a A_\nu^b A^{\mu a'} A^{\nu b'}$$

$$- \frac{1}{2\alpha}\, \partial^\mu A_\mu^c\, \partial^\nu A_\nu^c$$

$$+ b_a(x)\, \partial^\mu \partial_\mu c_a(x)$$

$$+ g f^{bca}\, \partial^\mu b_a(x)\, A_\mu^c(x)\, c_b(x)$$

$$+ \bar{\Psi}_k \left(i\gamma^\mu \partial_\mu - m\right) \Psi_k$$

$$+ g\, \bar{\Psi}_k \gamma^\mu \left(R(T^a)\right)_{kl} \Psi_l A_\mu^a$$

We haven't added scalars

We have just added fermions in a particular repn.

One could add scalars & fermions in diff. repns & couplings betn. scalars & fermions

(We will not directly use BRST sym. here to prove renormalizability)

$$A_\nu^a = \tilde{Z}_A^{1/2} A^a_{\nu R}, \quad b_a = \tilde{Z}_{ghost}^{1/2} b_{aR},$$

$$c_a = \tilde{Z}_{ghost}^{1/2} c_{aR}, \quad \Psi = \tilde{Z}_\Psi^{1/2} \Psi_R,$$

$$g = Z_g\, g_R\, \mu^{\epsilon/2}, \quad \alpha = Z_\alpha \alpha_R, \quad m = Z_m m_R$$

We have manifestly made consts. for b_a & c_a equal bco they appear together

Ex. Check ~~that~~ mass dimensions of g, α, m in $4-\epsilon$ dim.

(g_R, α_R are dimensionless & m_R has dim. of mass)

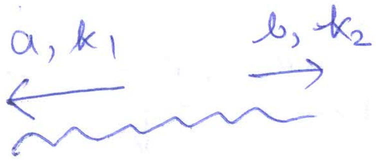
$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{ct}$$

$\mathcal{L}_{ct} \longrightarrow$ counterterm

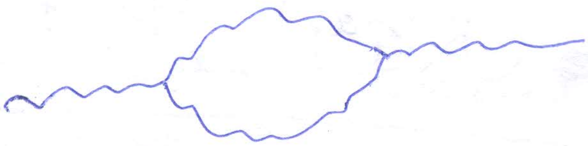
One loop renormalization of one particle irreducible connected feynman diagrams

One particle reducible graphs are built from 1PI graphs — so will be finite when the 1PI's are finite

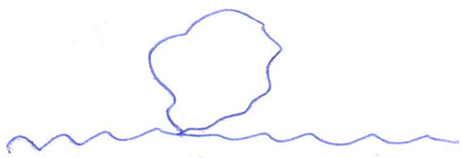
$$\left\langle \tilde{A}^a_{\mu R}(k_1)\, \tilde{A}^b_{\nu R}(k_2) \right\rangle$$



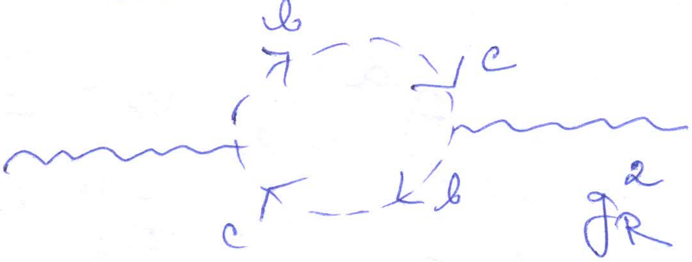
(tree-level)



g_R^2

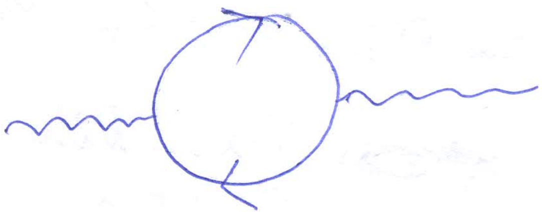


g_R^2



g_R^2

Explicit one-loop graphs from $\mathcal{L}_R$



first term & the gauge-fixing term gives the first counterterm

Counterterms :-

combinatorial factor

$$\text{One term} + \; -i/4\, (\tilde{Z}_A - 1)\, (k_1^2\eta_{\mu\nu} - k_{1\mu}k_{1\nu}) \times 2\; \delta_{ab} \times \overset{\uparrow}{2}$$

$$\times \left[(2\pi)^{4-\epsilon}\, \delta^{(4-\epsilon)}(k_1+k_2)\, \frac{i}{-k_1^2+i\epsilon}\, \frac{i}{-k_2^2+i\epsilon} \right]$$

Fair

$$-i/\mu\,(\tilde{Z}_A - 1)\,(k_1^2 - k_{1\mu} k_{1\nu}) \times 2\,\delta_{ab}$$

$$\times \left[(2\pi)^{4-\epsilon}\, \delta^{(4-\epsilon)}(k_1+k_2)\, \frac{i}{-k_1^2+i\epsilon}\, \frac{i}{-k_2^2+i\epsilon} \right]$$

$\times 2$

(combinatorial factor)

$$- \frac{i}{2\alpha_R}\left(\tilde{Z}_A Z_\alpha^{-1} - 1\right) k_{1\mu} k_{1\nu}\, \delta_{ab}$$

$$\times 2 \times \left[(2\pi)^{4-\epsilon}\, \delta^{(4-\epsilon)}(k_1+k_2) \right]$$

We require that the coefficient of $k_1^2 \eta_{\mu\nu}$ is finite & that the coefficient of $k_{1\mu} k_{1\nu}$ is finite

$\Longrightarrow$ determines $\tilde{Z}_A$ & $\tilde{Z}_A Z_\alpha^{-1}$

We get $\tilde{Z}_A = 1 - \dfrac{g_R^2}{8\pi^2 \epsilon}\left(\frac{4}{3} T_R - \frac{1}{2} C_G\left(\frac{13}{3} - \alpha_R\right)\right)$

gets contribution from

bcos T_R is present

$\tilde{Z}_A Z_\alpha^{-1} = 1$

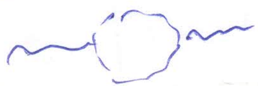
T_R, C_G are defined through

$$\mathrm{Tr}\left(R(T^a) R(T^b)\right) = T_R\, \delta_{ab}, \qquad f^{acd} f^{bcd} = C_G\, \delta^{ab}$$

(these consts of proportionality depends on which repn we are choosing)

define T_R & C_G

$C_G(\ldots)$ gets contribution from



this part won't change if you add more fermion repns or scalars

(The other part will get more contribution if you add more fermion reps or scalars)

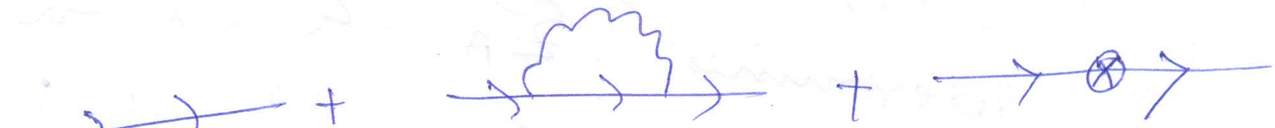
$$\langle \tilde{b}_{aR}(k_1)\, \tilde{c}^{\,b}_R(-k_2) \rangle$$

(tree-level)

+ 

$$\tilde{Z}_{ghost} = 1 + \frac{g_R^2}{32\pi^2\epsilon} C_G (3 - \alpha_R)$$

$$\langle \tilde{\Psi}_\alpha(k_1)\, \tilde{\bar{\Psi}}_\beta(-k_2) \rangle$$



(will det. Z_ψ & Z_m — 2 diff. tensor structures will det. both of them)

$$\tilde{Z}_\psi = 1 - \frac{g_R^2}{8\pi^2\epsilon} C_R \alpha_R$$

$$\tilde{Z}_\psi Z_m = 1 - \frac{3 g_R^2}{8\pi^2\epsilon} C_R$$

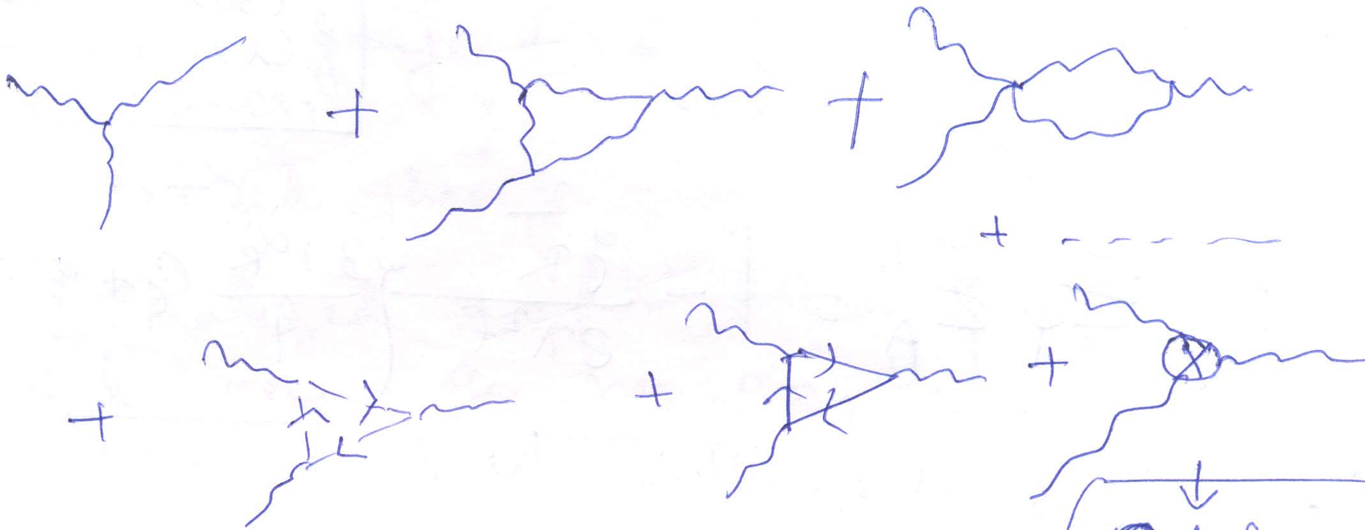
where $\sum_a R(T^a) R(T^a) = C_R \mathbb{1}$

(prop. const.)

defines C_R

One loop renormalization of one particle irreducible connected feynman diagrams (continued) $\rightarrow$ 3-pt. fns.

$$\langle \tilde{A}_{\mu a R}(k_1)\, \tilde{A}_{\nu b R}(k_2)\, \tilde{A}_{\rho c R}(k_3) \rangle$$



got from 2nd term of $\mathcal{L}$

$$Z_g \tilde{Z}_A^{3/2} = 1 - \frac{g_R^2}{8\pi^2 \epsilon}\left\{ C_G\left(-\frac{17}{12} + \frac{3\alpha_R}{4}\right) + \frac{4}{3} T_R \right\}$$

3 from fields
3 from coupling constants
$\Downarrow$
6 renorm. consts. to be determined

(We have got 6 relations $\longrightarrow$ det. 6 renorm. constants)

4-pt. fns :-

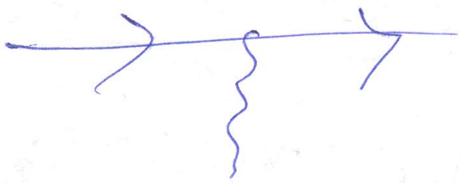
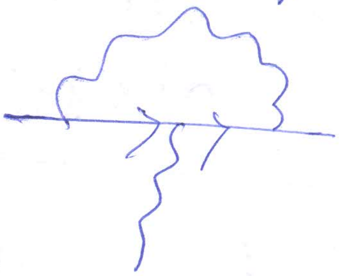
$$\langle \tilde{A}_{\mu a R}(k_1)\, \tilde{A}_{\nu b R}(k_2)\, \tilde{A}_{\rho c R}(k_3)\, \tilde{A}_{\tau d R}(k_4) \rangle$$

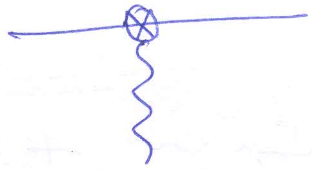
$$\Rightarrow Z_g^2 \tilde{Z}_A^2 = 1 - \frac{g_R^2}{8\pi^2\epsilon}\left[\left(-\frac{2}{3} + \alpha_R\right) C_R + \frac{4}{3} T_R\right]$$

$Z_g^2 \tilde{Z}_A^2$ will be det. see the 3rd term in $\mathcal{L}$

Ex: Check that this (is satisfied) holds automatically.

$$\left\langle \bar\Psi^{k}_{\alpha R}(k_1)\,\Psi^{l}_{\beta R}(k_2)\,A_{\mu a R}(k_3)\right\rangle$$

 +  + 

$+ \;----\; +$ 

Look at the last term in L

$$Z_g\,\tilde Z_\psi\,\tilde Z_A^{1/2} = 1 - \frac{g_R^2}{8\pi^2\epsilon}\left\{\frac{3+\alpha_R}{4}C_G + \alpha_R C_R\right\}$$

Ex: Check that this holds automatically

$$\left\langle \tilde{\bar c}_{aR}(k_1)\,\tilde c_{bR}(k_2)\,\tilde A^{c}_{\mu R}(k_3)\right\rangle$$

Look at the 6th term in L

$$Z_g\,\tilde Z_{ghost}\,\tilde Z_A^{1/2} = 1 - \frac{g_R^2}{16\pi^2\epsilon}C_G\,\alpha_R$$

Ex: Check that this holds automatically.

At one loop, without invoking BRST sym., we see that the th. is renormalisable by explicit calculation.

16/11/06

Renormalization group

We'll consider the gauge theory example.

generally valid for any field th.

Gauge group G with coupling constant g.

Fermion of mass m in representation R.

Original gauge fixed action has parameters g, m, α in the regularized theory.

these are unrenormalised parameters & are dimensionful

Renormalized amplitudes depend on g_R, m_R, α_R, μ

[$\because$ original th. depends on 3 parameters, there must be some redundancy in using the 4 parameters g_R, m_R, α_R, μ]

$$g = Z_g g_R \mu^{\epsilon/2} = f_g(g_R, \alpha_R, m_R, \mu)$$

$$\alpha = Z_\alpha \alpha_R = f_\alpha(g_R, \alpha_R, m_R, \mu)$$

$$m = Z_m m_R = f_m(g_R, \alpha_R, m_R, \mu)$$

We also have ϵ-dependence but we'll keep it fixed.

these fns become ∞ in the $\epsilon \to 0$ limit

If we change μ, g_R, α_R, m_R in such a way that g, α and m remain fixed, then the physical theory is unchanged.

Q) If we change μ, how should we change g_R, m_R and α_R such that g, m and α remain fixed?
(there will be one-parameter family of solns for 3 eqns & 4 unknowns)

→ we require :—

$$\frac{\partial f_g}{\partial g_R}\frac{dg_R}{d\mu} + \frac{\partial f_g}{\partial \alpha_R}\frac{d\alpha_R}{d\mu} + \frac{\partial f_g}{\partial m_R}\frac{dm_R}{d\mu} + \frac{\partial f_g}{\partial \mu} = 0$$

becos we want f(g) to remain fixed

$$\frac{\partial f_\alpha}{\partial g_R}\frac{dg_R}{d\mu} + \frac{\partial f_\alpha}{\partial \alpha_R}\frac{d\alpha_R}{d\mu} + \frac{\partial f_\alpha}{\partial m_R}\frac{dm_R}{d\mu} + \frac{\partial f_\alpha}{\partial \mu} = 0$$

$$\frac{\partial f_m}{\partial g_R}\frac{dg_R}{d\mu} + \frac{\partial f_m}{\partial \alpha_R}\frac{d\alpha_R}{d\mu} + \frac{\partial f_m}{\partial m_R}\frac{dm_R}{d\mu} + \frac{\partial f_m}{\partial \mu} = 0$$

hence the physics doesn't change

(these are a set of linear eqns for $\frac{dg_R}{d\mu}$, $\frac{d\alpha_R}{d\mu}$ & $\frac{dm_R}{d\mu}$ & can solve for these)

Solve for $dg_R/d\mu$, $d\alpha_R/d\mu$, $\frac{dm_R}{d\mu}$

$\xrightarrow[\text{get}]{\text{we}}$

$$\mu\frac{dg_R}{d\mu} = \beta_g(g_R, m_R, \alpha_R, \mu)$$

$$\mu\frac{d\alpha_R}{d\mu} = \beta_\alpha(g_R, m_R, \alpha_R, \mu)$$

$$\mu\frac{dm_R}{d\mu} = \beta_m(g_R, m_R, \alpha_R, \mu)$$

can be found by solving the eqns ⟹ β-functions

(These are known as β-functions)

(g_R, α_R & m_R evolve as a fn. μ acc. to these diff. eqns)

μ, in some sense, is a redundant parameter, to make g_R dimensionless — change in μ can be compensated for by a change in α_R, m_R, g_R.

$\frac{\text{momentum}}{\text{mass}}$ ratio

Simple dimension analysis of the original th. breaks down for the renormalized theory
→ so we study these eqns

In stat mech., the cut-off is a physical cut-off, but nevertheless we want to remove the cut-off dep. by looking at diff. ~~sact~~ scales
→ Renorm. grp.

In QFT, the cut-off is not a phy. cut-off & eventually we want to remove it

Calculation of β-function in gauge theory coupled to fermions

$$g = Z_g \, g_R \, \mu^{\epsilon/2}$$

$$Z_g = \left(1 - \frac{A}{\epsilon} g_R^2\right), \text{ where } A = \frac{1}{8\pi^2} \frac{11 C_G - 4 T_R}{6}$$

Recall :-

$$f^{acd} f^{bcd} = C_G \, \delta_{ab}, \quad \mathrm{Tr}\left(R(T^a)\, R(T^b)\right) = T_R \, \delta_{ab}$$

We get $g = \left(g_R - \frac{A}{\epsilon} g_R^3\right) \mu^{\epsilon/2}$

We want $\mu \frac{dg}{d\mu} = 0$

$$\mu \frac{dg}{d\mu} = 0 \Rightarrow \left(1 - \frac{3A}{\epsilon} g_R^2\right) \mu \frac{dg_R}{d\mu} \mu^{\epsilon/2} + \left(g_R - \frac{A}{\epsilon} g_R^3\right) \frac{\epsilon}{2} \mu^{\epsilon/2} = 0$$

μ is the indep.-variable → all the parameters are fns of μ only

This gives

$$\mu \frac{dg_R}{d\mu} = -\frac{\epsilon}{2} g_R \left(1 - \frac{A}{\epsilon} g_R^2\right) \times \left(1 - \frac{3A}{\epsilon} g_R^2\right)^{-1}$$

$$\Rightarrow \mu \frac{dg_R}{d\mu} = -\frac{\epsilon}{2} g_R \left(1 + \frac{2A}{\epsilon} g_R^2\right)$$
$$= \underbrace{-\frac{\epsilon}{2} g_R - A g_R^3}_{\beta_g}$$

In the intermediate steps ϵ is kept fixed

$$\boxed{\beta_g = -\frac{\epsilon}{2} g_R - A g_R^3} \xrightarrow{\epsilon \to 0} -A g_R^3$$

Bcos $\epsilon \to 0$ is what we are interested in

~~though~~ the relations become singular as $\epsilon \to 0$, the relationship bet. μ & m_R, g_R, α_R we expect to be finite, bcos the amplitudes should remain finite (physics shouldn't change we change these)

Here, $g = (g_R - A/\epsilon\, g_R^3)\mu^{\epsilon/2}$ is infinite as $\epsilon \to 0$ but here we have a relation bet. g_R & μ which should be finite bcos physical amplitudes are finite fns of μ, g_R, α_R, m_R.

these $\underline{\beta\text{-fns}}$ are relations bet. finite quantities — none of are allowed to become ∞.

$\mu \frac{dg_R}{d\mu} = \beta_g$ has solution

$$\frac{1}{g_R^2} = 2A \ln\mu + c$$

c → integration constant

$$\Rightarrow g_R^2 = \frac{1}{2A\ln\mu + c}$$

If $A > 0$, then $g_R \to 0$ as $\mu \to \infty$

this is just an observation

later on we will become its significance

these theories are called → **Asymptotically free theories**

If $A < 0$, g_R^2 will become so large that we can no ~~longer~~ longer trust this — perturbation th. is no longer valid — take c +ve & then decrease A ... eventually ...

So provided we choose μ to be large, g_R can be chosen to be small → but it may seem to be a cheating — so clearly sth. else should go wrong when we try to increase μ arbitrarily

Pure gauge th. are always asym. free, bcos for no fermions, $A = \frac{1}{8\pi^2}\frac{11 C_A}{6}$

If you add a large no. of fermions, the th. is not asym. free → $A = \frac{11 C_A - 4T_R}{6}$ → for one fermion repn

Calculation of β_m :-

$$m = Z_m m_R = \left(1 - B\frac{g_R^2}{\epsilon}\right) m_R$$

where $B = \frac{3}{8\pi^2} C_R$, $\boxed{\sum_a R(T^a)R(T^a) = C_R \mathbb{1} \Rightarrow \text{defines } C_R}$

$$\mu \frac{d}{d\mu} m = 0$$

gives us → $\left(1-\frac{B}{\epsilon}g_R^2\right)\mu\frac{dm_R}{d\mu} - \frac{B}{\epsilon}2g_R\,\mu\frac{dg_R}{d\mu}m_R = 0$

$$\Rightarrow \mu\frac{dm_R}{d\mu} = \left(1-\frac{B}{\epsilon}g_R^2\right)^{-1}\frac{2B}{\epsilon}m_R\,\mu\frac{dg_R}{d\mu}$$

$$= \left(1+\frac{B}{\epsilon}g_R^2\right)\frac{2B}{\epsilon}m_R g_R\left(-\frac{\epsilon}{2}g_R\right)\left(1+\frac{2A}{\epsilon}g_R^2\right)$$

$$\Rightarrow \quad \mu \frac{dm_R}{d\mu} = -B m_R g_R^2 \left(1 + \frac{B}{\epsilon} g_R^2 + \frac{2A}{\epsilon} g_R^2\right)$$

$$\simeq -B m_R g_R^2$$

can't keep $O(g_R^4)$ terms bcos 2-loop renorm. contributes $O(g_R^4)$ terms

$$\Rightarrow \boxed{\beta_m = -B m_R g_R^2}$$

α is a g.f. parameter & phys. amplitudes should not depend on this gauge-fixing parameter — so calculation of α is not important.

Calculation of β_α

$$\alpha = Z_\alpha \alpha_R = \left(1 - \frac{K}{\epsilon} g_R^2\right) \alpha_R$$

where $K = \frac{1}{8\pi^2} \left\{ \frac{4}{3} T_R - \frac{1}{2} C_G \left(\frac{13}{3} - \alpha_R \right) \right\}$

must take derivative of this also

$$\mu \frac{d\alpha}{d\mu} = 0$$

gives us

$$\left(1 - \frac{K}{\epsilon} g_R^2\right) \frac{d\alpha_R}{d\mu} \mu + \alpha_R \left(-\frac{K}{\epsilon}\right) 2 g_R \mu \frac{dg_R}{d\mu} - \frac{1}{\epsilon} g_R^2 \alpha_R \left(\frac{1}{16\pi^2} C_G \frac{d\alpha_R}{d\mu}\mu\right) = 0$$

Call $\rightarrow \frac{1}{16\pi^2} C_G = L$

i.e., $$\left(1 - \frac{K}{\epsilon} g_R^2\right) \mu \frac{d\alpha_R}{d\mu} + \alpha_R \left(-\frac{K}{\epsilon}\right) 2 g_R \frac{dg_R}{d\mu} \mu - \frac{1}{\epsilon} g_R^2 \alpha_R \left(L \frac{d\alpha_R}{d\mu}\right) = 0$$

$$\Rightarrow \left(1 - \frac{K}{\epsilon} g_R^2 - \frac{L}{\epsilon} g_\alpha^2 \alpha_R\right) \mu \frac{d\alpha_R}{d\mu} = 2 \alpha_R \frac{K}{\epsilon} g_R \mu \frac{dg_R}{d\mu}$$

$$= 2\alpha_R \frac{K}{\epsilon} \left(-\frac{\epsilon}{2} g_R^2\right) \left(1 + \frac{2A}{\epsilon} g_R^2\right)$$

$$\mu \frac{d\alpha_R}{d\mu} = -K\alpha_R g_R^2\left(1+\frac{2A}{\epsilon} g_R^2\right) \times \left(1-\frac{K}{\epsilon}g_R^2 - \frac{L}{\epsilon}\alpha_R g_R^2\right)^{-1}$$

$$\simeq -K\alpha_R g_R^2$$

$$\Rightarrow \boxed{\beta_\alpha = -K\alpha_R g_R^2}$$

$O(g_R^4)$ terms have to be combined with higher loop contributions

$$\boxed{\mu \frac{d}{d\mu}\left\langle \prod_i \phi_i(x_i)\right\rangle = 0}$$

amplitude

where ϕ_i are unrenormalised fields (gauge fields, fermions, ghosts, etc.)

(Bcos m, g, α are unchanged)
This isn't true for ϕ_{iR}

The original action did not have any μ's so $\frac{d}{d\mu}\langle \prod_i \phi_i\rangle = 0$; now if we take unrenorm. parameters, but renorm. fields, that has hidden μ-dep. — the action acquires a μ-dep. & so $\frac{d}{d\mu}\langle \prod_i \phi_{iR}\rangle \neq 0$

$$\mu \frac{d}{d\mu}\left\langle \prod_i \phi_i(x_i)\right\rangle$$

$$\mu \frac{d}{d\mu}\left\langle \prod_{i=1}^n \tilde{\phi}_i(k_i)\right\rangle$$

$$f(\epsilon, g_R, m_R, \alpha_R, \mu, k_1, \dots, k_n)$$

$$F(\epsilon, g, m, \alpha, k_1, \dots, k_n)$$

∴ we have,

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g_R} + \beta_\alpha \frac{\partial}{\partial \alpha_R} + \beta_m \frac{\partial}{\partial m_R}\right) \left\langle \prod_{i=1}^{n} \tilde{\Phi}_i \right\rangle = 0$$

this follows from

$$\left(\mu \frac{\partial}{\partial \mu} + \mu \frac{d g_R}{d\mu} \frac{\partial}{\partial g_R} + \mu \frac{d \alpha_R}{d\mu} \frac{\partial}{\partial \alpha_R} + \mu \frac{d m_R}{d\mu} \frac{\partial}{\partial m_R}\right) \left\langle \prod_{i=1}^{n} \tilde{\Phi}_i(-k_i) \right\rangle = 0.$$

However we aren't interested in $\langle \prod_i \Phi_i \rangle$ bcos it is infinite — we want the renormalized corrln. fns, $\langle \prod_i \Phi_{iR} \rangle$

Convert this into an equation for

$$\left\langle \prod_{i=1}^{n} \tilde{\Phi}_{iR}(-k_i) \right\rangle.$$

β-fns have an expansion in powers of g_R — take higher loops & add more terms to the expression for β-fns.

17/11/06

Renormalization group: General set up

Take a quantum field theory with fields $\phi_1, \phi_2, \ldots, \phi_N$ and parameters $g_1, g_2, \ldots, g_M$.

$\{\phi_s\} : s = 1, 2, \ldots, N$

and $\{g_\alpha\} : \alpha = 1, 2, \ldots, M$

Renormalized fields & parameters :-

$\{\phi_{sR}\}, \{g_{\alpha R}\}$ & also μ.

$$\phi_s = \left(\tilde{Z}^{1/2}_{(\vec{g}_R, \mu, \epsilon)}\right)_{st} \phi_{tR}, \quad g_\alpha = \mu^{k_\alpha \epsilon} F_\alpha(\vec{g}_R, \mu, \epsilon)$$

~~$\tilde{Z}^{1/2}_{st}(\vec{g}_R, \mu, \epsilon)$~~ fn. of $(\vec{g}_R, \mu, \epsilon)$

A general renorm. can give a matrix valued renorm. of this kind

We define $\beta_\alpha(\vec{g}_R, \mu, \epsilon)$ such that under $\mu \to \mu + \delta\mu$,

$$g_{\alpha R} \to g_{\alpha R} + \beta_\alpha(\vec{g}_R, \epsilon, \mu)\mu^{-1}\delta\mu$$

g_α's remain unchanged.

(We can rewrite this as →)

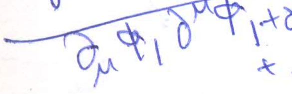
$$\left(\frac{\partial}{\partial\mu} + \beta_\alpha \mu^{-1} \frac{\partial}{\partial g_{\alpha R}}\right)\left\{\mu^{k_\gamma \epsilon} F_\gamma(\vec{g}_R, \mu, \epsilon)\right\} = 0$$

(γ is not summed over)

Relation bet. ϕ_s & ϕ_{tR} is linear, while those bet. coupling const. may be non-linear

$\tilde{Z}$ has to be diagonalized order by order

$\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial^\mu \phi_2 \partial_\mu \phi^2 + \ldots$



↳ suppose it is divergent

by allowing for $\phi_1 = Z_{11}\phi_{1R} + Z_{12}\phi_{2R}$ $Z \to$ matrix

$\phi_2 = Z_{21}\phi_{1R} + Z_{22}\phi_{2R}$

it will be possible to remove the div. (allowing for mixing

K_α 's are chosen in such a way that in each dim., g_α has the same dim.

→ e.g. → If in 4-D g_α is dim.-less, in $4-\epsilon$ dim. also it is dim.-less

If in 4-D g_α has 1 mass-dim., in $4-\epsilon$ ------ 1 ".

$$\left(\frac{\partial}{\partial\mu} + \beta_\alpha \mu^{-1} \frac{\partial}{\partial g_{\alpha R}}\right)\left\{\mu^{K_\gamma \epsilon} f_\gamma(\vec{g}_R, \mu, \epsilon) = 0\right.$$

$\gamma = 1, 2, \ldots, M$ [γ is not summed over]

known from renormalization

K_γ → known from dimension analysis

∴ we have M linear eqns for the M β_α's. (each eqn. is linear in β_α)

→ can be solved ~~for~~ to find β_α's.

Unrenorm. green's fns don't change bcos they depend on unren. parameters which don't change — the renorm. green's fns however do change

Define

$$G^{(n),R}_{s_1, s_2, \ldots, s_n}(k_1, \ldots, k_n, \vec{g}_R, \mu) \equiv \left\langle \prod_{i=1}^{n} \tilde{\Phi}_{s_i R}(k_i) \right\rangle$$

$$\left\langle \prod_{i=1}^{n} \phi_{s_i}(-k_i) \right\rangle = \prod_{i=1}^{n} \tilde{Z}^{1/2}_{s_i t_i} \left\langle \prod_{i=1}^{n} \tilde{\Phi}_{t_i R}(k_i) \right\rangle$$

$$= \sum_{t_1, t_2, \dots, t_n = 1}^{N} \prod_{j=1}^{n} \tilde{Z}^{1/2}_{s_j t_j} \left\langle \prod_{k=1}^{n} \tilde{\Phi}_{t_k R} \right\rangle$$

(Each t_i is summed over)

$$\Rightarrow \left\{ \mu \frac{\partial}{\partial \mu} + \beta_\alpha \frac{\partial}{\partial g_{\alpha R}} \right\} \left(\left(\prod_{i=1}^{n} \tilde{Z}^{1/2}_{s_i t_i} \right) \left\langle \prod_{i=1}^{n} \tilde{\Phi}_{t_i R}(k_i) \right\rangle \right) = 0$$

$$\Rightarrow \left(\prod_{i=1}^{n} \tilde{Z}^{-1/2}_{r_i s_i} \right) \left\{ \mu \frac{\partial}{\partial \mu} + \beta_\alpha \frac{\partial}{\partial g_{\alpha R}} \right\}$$

$$\times \left(\prod_{i=1}^{n} \tilde{Z}^{1/2}_{s_i t_i} \right) \left\langle \prod_{i=1}^{n} \tilde{\Phi}_{t_i R}(k_i) \right\rangle = 0$$

Sum over each s_i & each t_i

~~e.g.~~ For $n = 2$:-

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_\alpha \frac{\partial}{\partial g_{\alpha R}} \right) \left(\tilde{Z}^{1/2}_{s_1 t_1} \tilde{Z}^{1/2}_{s_2 t_2} \left\langle \tilde{\Phi}_{t_1 R}(k_1) \tilde{\Phi}_{t_2 R}(k_2) \right\rangle \right) = 0$$

$$\Rightarrow \sum_{s_1, s_2, t_1, t_2} \tilde{Z}^{-1/2}_{r_1 s_1} \tilde{Z}^{-1/2}_{r_2 s_2} \left(\mu \frac{\partial}{\partial \mu} + \beta_\alpha \frac{\partial}{\partial g_{\alpha R}} \right) \tilde{Z}^{1/2}_{s_1 t_1} \tilde{Z}^{1/2}_{s_2 t_2}$$

$$\times \left\langle \tilde{\Phi}_{t_1 R}(k_1) \tilde{\Phi}_{t_2 R}(k_2) \right\rangle = 0$$

Define $\gamma_{st}(\vec{g}_R, \epsilon, \mu)$ through the eqn

$$\tilde{Z}^{-1/2}_{rs} \left(\mu \frac{\partial}{\partial \mu} + \beta_\alpha \frac{\partial}{\partial g_{\alpha R}} \right) \tilde{Z}^{1/2}_{st} = \gamma_{rt}$$

think this as the inverse matrix multiplying

$\gamma_{st} \longrightarrow$ anomalous dimension matrix

Eigenvalues of γ_{st} are called <u>anomalous dimensions</u>.

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_\alpha \frac{\partial}{\partial g_{\alpha R}}\right) \left\langle \prod_{i=1}^{n} \tilde{\phi}_{r_i R}(k_i) \right\rangle$$

$$+ \gamma_{r_1 t_1} \left\langle \tilde{\phi}_{t_1 R}(k_1) \prod_{i=2}^{n} \tilde{\phi}_{r_i R}(k_i) \right\rangle$$

$$+ \gamma_{r_2 t_2} \left\langle \tilde{\phi}_{t_2 R}(k_2) \prod_{\substack{i=1 \\ i \neq 2}}^{n} \tilde{\phi}_{r_i R}(k_i) \right\rangle$$

$$\vdots$$

$$= 0$$

we had $\delta_{r_i t_i}$

from

$$\prod_{i=1}^{n} \tilde{Z}^{-1/2}_{r_i s_i} \times \prod_{i=1}^{n} \tilde{Z}^{1/2}_{s_i t_i}$$

$$\Rightarrow \left(\mu \frac{\partial}{\partial \mu} + \beta_\alpha \frac{\partial}{\partial g_{\alpha R}}\right) G^{(n)R}_{r_1 \ldots r_n}(k_1, \ldots, k_n; \vec{g}_R, \mu)$$

$$+ \sum_{j=1}^{n} \gamma_{r_j t_j} G^{(n)R}_{r_1, \ldots, r_{j-1} t_j r_{j+1} \ldots r_n}(k_1, \ldots, k_n; \vec{g}_R, \mu)$$

$$= 0$$

Physically it should mean that μ-dep. is redundant

for a diagonal matrice

If $\tilde{Z}^{1/2}_{st} = \tilde{Z}^{1/2}_{s} \delta_{st}$, then

$$\gamma_{st} = \gamma_s \delta_{st}$$

where $\gamma_s = \left(\mu \frac{\partial}{\partial \mu} + \beta_\alpha \frac{\partial}{\partial g_{\alpha R}}\right) \ln \tilde{Z}^{1/2}_{s}$

Calculation of γ_s's in gauge theory:

For gauge field A^a_μ,

$$\tilde{Z}_A = \left(1 - \frac{K}{\epsilon} g_R^2\right)$$

upto one loop

$$K = \frac{1}{8\pi^2}\left(\frac{4}{3}T_R - \frac{1}{2}C_G\left(\frac{13}{3} - \alpha_R\right)\right)$$

$$= \frac{1}{8\pi^2}\left(\frac{4}{3}T_R - \frac{13}{6}C_G\right) + L\alpha_R$$

$$\left[\text{where } L = \frac{C_G}{16\pi^2}\right]$$

Now, $\ln \tilde{Z}_A^{1/2} = \ln\left(1 - \frac{K}{2\epsilon}g_R^2\right)$

$$= -\frac{K}{2\epsilon}g_R^2 + O(g_R^4)$$

$$\tilde{Z}_A^{1/2} = 1 - \frac{K}{2\epsilon}g_R^2 + O(g_R^4)$$

$$\gamma_A = \left(\mu\frac{\partial}{\partial\mu} + \beta_g\frac{\partial}{\partial g_R} + \beta_m\frac{\partial}{\partial m_R} + \beta_\alpha\frac{\partial}{\partial\alpha_R}\right)\left(-\frac{K}{2\epsilon}g_R^2\right)$$

$$= \beta_g\left(-\frac{K}{2\epsilon}2g_R\right) - \beta_\alpha\frac{g_R^2}{2\epsilon}L$$

$$= \left(-\frac{1}{2}\epsilon g_R - A g_R^3\right)\left(-\frac{K}{2\epsilon}2g_R\right)$$

$$- \left(-K\alpha_R g_R^2\right)\frac{g_R^2}{2\epsilon}L$$

$$= \frac{1}{2}Kg_R^2 + O(g_R^4)$$

We are supposed to keep terms upto $O(g_R^2)$

Ex: Calculate γ_ψ, γ_b, γ_c

Notice → γ's (upto this order) are finite. In the $\epsilon \to 0$ limit, they should be finite bcos renorm. corr. fns are finite in whose diff. eqns they appear. (just like β's were finite)

$\mu\frac{\partial}{\partial\mu}G^{(n)} = \text{------}$ → involves γ_s

For simplicity we ~~will to~~ shall assume from now on that

$\tilde{Z}^{1/2}_{st}$ is diagonal.

$\therefore \quad \gamma_{st} = \gamma_s \, \delta_{st}$

$$\therefore \boxed{\left(\mu \frac{\partial}{\partial \mu} + \sum_{\alpha=1}^{M} \beta_\alpha \frac{\partial}{\partial g_{\alpha R}} + \sum_{i=1}^{n} \gamma_{s_i}\right) \times G^{(n)R}_{s_1 \ldots s_n}(k_1, \ldots, k_n; \{g_{\alpha R}\}, \mu) = 0} \quad \text{— (A)}$$

(Now, we will derive an eqn. for $G^{(n)R}$ from simple dimensional analysis) ⟶

suppose $\tilde{\Phi}_s(k)$ has dimension d_s

and $g_{\alpha R}$ has dimension Δ_α

these are all mass dim.

$\tilde{\Phi}_{sR}(k)$ also has dimension d_s

since $\tilde{Z}$'s are dimensionless ⟶ (by construction)

In pert. th. the first term is 1

$\cancel{G^{(n)R}_{s_1 \ldots s_n}(\lambda k_1, \ldots, \lambda k_n)}$

$$G^{(n)R}_{s_1 \ldots s_n}(\lambda k_1, \ldots, \lambda k_n, \{\lambda^{\Delta_\alpha} g_{\alpha R}\}, \lambda \mu)$$

$$= \lambda^{\sum_{i=1}^{n} d_{s_i}} G^{(n)R}_{s_1 \ldots s_n}(k_1, k_2, \ldots, k_n, \{g_{\alpha R}\}, \mu) \quad \text{— (B)}$$

from dimensional analysis

We have scaled everything by mass dim. or $\lambda^{\text{mass dim.}}$

$\sum d_s$ is the dim. of the quantity we are trying to calculate

Replace g_α by $\lambda^{-\Delta_\alpha} g_{\alpha R}$, μ by $\lambda^{-1}\mu$ everywhere.

Then,

scaled all the momenta with changing the coupling const.

$$G^{(n)R}_{s_1 \ldots s_n}(\lambda k_1, \ldots, \lambda k_n, \{g_{\alpha R}\}, \mu)$$

$$= \lambda^{\sum_{i=1}^{n} d_{s_i}} G^{(n)R}_{s_1 \ldots s_n}(k_1, \ldots, k_n, \{g_{\alpha R}\lambda^{-\Delta_\alpha}\}, \mu\lambda^{-1}) \quad \text{(C)}$$

Take $\lambda \frac{\partial}{\partial \lambda}$ on both sides to get

$$\lambda \frac{\partial}{\partial \lambda} G^{(n)R}_{s_1 \ldots s_n}(\lambda k_1, \ldots, \lambda k_n, \{g_\alpha\}, \mu)$$

Remove the μ-dep. with the 2 eqns

there would be no μ if renorm. weren't reqd.

$$= \left(\sum_{i=1}^{n} d_{s_i}\right) \lambda^{\sum_{i=1}^{n} d_{s_i}} G^{(n)R}_{s_1 \ldots s_n}(k_1, \ldots, k_n, \{g_{\alpha R}\lambda^{-\Delta_\alpha}\}, \mu\lambda^{-1})$$

Doing one theory at large mom. is equiv. to do the other other theory with small mom.

$$\cancel{+\lambda^{\sum d_s}\left(-\sum_\alpha \Delta_\alpha g_{\alpha R}\frac{\partial}{\partial g_{\alpha R}}\right)}$$

We want to trivialise the μ-dep bcos μ-dep. isn't completely indep.

$$+\lambda^{\sum_{i=1}^{n} d_{s_i}}\left(-\sum_\alpha \Delta_\alpha g_{\alpha R}\frac{\partial}{\partial g_{\alpha R}}\right) G^{(n)R}_{s_1 \ldots s_n}(k_1, \ldots, k_n, \{g_{\alpha R}\lambda^{-\Delta_\alpha}\}, \mu\lambda^{-1})$$

$$+\lambda^{\sum_{i=1}^{n} d_{s_i}}\left(-\mu\frac{\partial}{\partial \mu}\right) G^{(n)R}_{s_1 \ldots s_n}(k_1, \ldots, k_n, \{g_{\alpha R}\lambda^{-\Delta_\alpha}\}, \mu\lambda^{-1})$$

Note. $\rightarrow$ $\lambda \frac{\partial}{\partial \lambda}\left(g_{\alpha R}\lambda^{-\Delta_\alpha}\right)$ and $-\Delta_\alpha g_{\alpha R}\frac{\partial}{\partial g_R}(g_{\alpha R}\lambda^{-\Delta_\alpha})$ are identical

$= g_\alpha \lambda^{-\Delta_\alpha}(-\Delta_\alpha)$

$= -\Delta_\alpha g_{\alpha R}\lambda^{-\Delta_\alpha}$

Similarly with $\mu\frac{\partial}{\partial\mu}$ ....

$$= \left(\sum_{i=1}^{n} d_{s_i} - \sum_{\alpha} \Delta_\alpha g_{\alpha R} \frac{\partial}{\partial g_{\alpha R}} - \mu \frac{\partial}{\partial \mu} \right) \hat{G}^{(n)R}_{s_1 \ldots s_n} \left(\lambda k_1, \ldots, \lambda k_n, \{g_{\alpha R}\}, \mu \right)$$

(Ⓐ is true with k_i repaced by λk_i) &

this is obtd. by using Ⓑ

hence

$$= \left\{ \sum_{i=1}^{n} \left(d_{s_i} + \gamma_{s_i} \right) + \sum_{\alpha=1}^{M} \left(\beta_\alpha - \Delta_\alpha g_{\alpha R} \right) \frac{\partial}{\partial g_{\alpha R}} \right\} G^{(n)R}_{s_1 \ldots s_n} \left(\lambda k_1, \ldots, \lambda k_n, \{g_{\alpha R}\}, \mu \right)$$

[using Ⓐ]
replacing k_i by λk_i in Ⓐ

μ-dep. has dropped out

(μ can now be thought as a fixed variable on both sides)

[as if the dim. of the fields has changed from d_{s_i} to $(d_{s_i} + \gamma_{s_i})$ — that is why γ_s is called the anamolous dim.

As if the standard canonical dim of the coupling const. has changed by β_α]

(Of course the phys. dim. hasn't changed — it appears as if they have changed)

$$\underset{\sim}{\tilde G}_{\mu\nu} = \tilde G^{a}_{\mu\nu} T^a$$

$$= \partial_\mu \left(U T^b U^{-1} B_\nu^b - i \partial_\nu U U^{-1} \right) - \mu \leftrightarrow \nu$$

$$- i \left[U T^b U^{-1} B_\mu^b - i \partial_\mu U U^{-1}, U T^d U^{-1} B_\nu^d - i \partial_\nu U U^{-1} \right]$$

$$= \underline{U \partial_\mu B_\nu^b T^b U^{-1}} + (\partial_\mu U) T^b U^{-1} B_\nu^b + U T^b (\partial_\mu U^{-1}) B_\nu^b$$

$$- i \cancel{(\partial_\mu \partial_\nu U) U^{-1}} - i (\partial_\nu U)(\partial_\mu U^{-1}) - \underline{U \partial_\nu B_\mu^b T^b U^{-1}}$$

$$- (\partial_\nu U) T^b U^{-1} B_\mu^b - U T^b (\partial_\nu U^{-1}) B_\mu^b + i \cancel{(\partial_\nu \partial_\mu U) U^{-1}}$$

$$+ i (\partial_\mu U)(\partial_\mu U^{-1}) - i \left[U T^b U^{-1} B_\mu^b, U T^d U^{-1} B_\nu^d \right]$$

$$- i \left[U T^b U^{-1} B_\mu^b, -i \partial_\nu U U^{-1} \right] - \left[\partial_\mu U U^{-1}, U T^d U^{-1} B_\nu^d \right]$$

$$- \left[\partial_\mu U U^{-1}, -i \partial_\nu U U^{-1} \right]$$

$$= \underline{U \left(\partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu \right) U^{-1}} + (\partial_\mu U) T^b U^{-1} B_\nu^b + U T^b (\partial_\mu U^{-1}) B_\nu^b$$

$$- i (\partial_\nu U)(\partial_\mu U^{-1}) - (\partial_\nu U) T^b U^{-1} B_\mu^b - U T^b (\partial_\nu U^{-1}) B_\mu^b$$

$$+ i (\partial_\mu U)(\partial_\nu U^{-1}) - i \underline{U \left[\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu \right] U^{-1}}$$

$$- \left(U T^b U^{-1} B_\mu^b \partial_\nu U U^{-1} - \partial_\nu U U^{-1} U T^b U^{-1} B_\mu^b \right)$$

$$- \left(\partial_\mu U U^{-1} U T^d U^{-1} B_\nu^d - U T^d U^{-1} B_\nu^d \partial_\mu U U^{-1} \right)$$

$$+ i \left(\partial_\mu U U^{-1} \partial_\nu U U^{-1} - \partial_\nu U U^{-1} \partial_\mu U U^{-1} \right)$$

$$= U \underset{\sim}{G}_{\mu\nu} U^{-1} + (\partial_\mu U) T^b U^{-1} B_\nu^b + \cancel{U T^b (\partial_\mu U^{-1}) B_\nu^b}$$

$$\cancel{- i (\partial_\nu U)(\partial_\mu U^{-1})} - (\partial_\nu U) T^b U^{-1} B_\mu^b - U T^b (\partial_\nu U^{-1}) B_\mu^b$$

$$\cancel{+ i (\partial_\mu U)(\partial_\nu U^{-1})} + U T^b B_\mu^b U^{-1} U \partial_\nu U^{-1}$$

$$+ \partial_\nu U T^b U^{-1} B_\mu^b - \partial_\mu U T^d U^{-1} B_\nu^d \cancel{- U T^d B_\nu^d U^{-1} U \partial_\mu U^{-1}}$$

$$\cancel{- i \partial_\mu U U^{-1} U \partial_\nu U^{-1}} \cancel{+ i \partial_\nu U U^{-1} U \partial_\mu U^{-1}}$$

$$= U \underset{\sim}{G}_{\mu\nu} U^{-1} + \cancel{(\partial_\mu U) T^b U^{-1} B_\nu^b} \cancel{- (\partial_\nu U) T^b U^{-1} B_\mu^b} \cancel{- U T^b (\partial_\nu U^{-1}) B_\mu^b}$$

$$+ U T^b B_\mu^b \cancel{\partial_\nu U^{-1}} \cancel{+ \partial_\nu U T^b U^{-1} B_\mu^b} \cancel{- \partial_\mu U T^d U^{-1} B_\nu^d}$$

$$\therefore \boxed{\tilde{G}_{\mu\nu} = U\, G_{\mu\nu}\, U^{-1}} \quad \text{(proved)}$$

~~Che~~ # Check $\delta S = 0$ under an infinitesimal trs.

solⁿ: $\longrightarrow$ $S_{gauge} = -\frac{1}{2g^2}\int d^4x \, \mathrm{Tr}\left(G_{\mu\nu} G^{\mu\nu}\right)$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - i\left[B_\mu, B_\nu\right]$$

$$\therefore \tilde{G}_{\mu\nu} = \partial_\mu\left(B_\nu^a T^a - f^{cba}\epsilon^b B_\nu^c - \partial_\nu \epsilon^a\right)$$
$$- \partial_\nu\left(B_\mu^a T^a - f^{cba}\epsilon^b B_\mu^c - \partial_\mu \epsilon^a\right)$$
$$- i\left[B_\mu^a T^a - f^{cba}\epsilon^b B_\mu^c ,\; B_\nu^i T^i - f^{kji}\epsilon^j B_\mu^k\right]$$

$$= \left(\partial_\mu B_\nu - \partial_\nu B_\mu\right) - f^{cba}\partial_\mu\left(\epsilon^b B_\nu^c\right) - \cancel{\partial_\mu \partial_\nu \epsilon^a}$$
$$+ f^{cba}\partial_\nu\left(\epsilon^b B_\mu^c\right) + \cancel{\partial_\nu \partial_\mu \epsilon^a}$$
$$- i\left[B_\mu, B_\nu\right] + i\left[B_\mu^a T^a, f^{kji}\epsilon^j B_\mu^k\right]$$
$$+ \left[f^{cba}\epsilon^b B_\mu^c, B_\nu^i T^i\right] - i\left[f^{cba}\epsilon^b B_\mu^c, f^{kji}\epsilon^j B_\mu^k\right]$$

$$= G_\mu$$

$$\delta S_{gauge} = -\frac{1}{g^2}\int d^4x \, \mathrm{Tr}\left(\delta G_{\mu\nu} G^{\mu\nu}\right)$$

Should we add all dim. zero op.?

Problem Set 4: Date Due: Dec. 6, 2006

1. Consider a two dimensional free field theory of a scalar field ϕ and fermion field ψ with action:

$$S = \int d^2x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \right]$$

Now consider adding to this action a term proportional to

$$\int d^2x\, \phi^m \,(\bar{\psi}\psi)^n \,(\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi)^l$$

Determine the range of values of m, n and l for which the theory will be renormalizable.

2. Consider ϕ^3 theory in six dimensions, given by the action:

$$S = \int d^6x \Big[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 \Big] .$$

(a) Show that the theory is power counting renormalizable.

(b) Calculate all the renormalization constants to one loop order.

(c) Calculate the β-function for the coupling g to one loop order.

$\bar{u}(p_1)\gamma^\mu (k)_\mu v(p_2)$

$= \bar{u}(p_1)\gamma^\mu (p_1+p_2)_\mu v(p_2)$

$= \bar{u}(p_1)(\not{p}_1+\not{p}_2) v(p_2)$

$= \bar{u}(p_1)(m-m) v(p_2)$

$= 0$

3. Consider n complex scalar fields $(\phi_1, \ldots \phi_n)$ with action

$$S = \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\sum_{\alpha=1}^{n}\partial_\mu\phi^*_\alpha\partial_\nu\phi_\alpha - V(\phi_1, \ldots \phi_n) \right]$$

where

$$V(\phi_1, \ldots \phi_n) = -\mu^2\sum_{\alpha=1}^{n}\phi^*_\alpha\phi_\alpha + \lambda\left(\sum_{\alpha=1}^{n}\phi^*_\alpha\phi_\alpha\right)^2$$

for some real, positive constants μ and λ.

(a) Find the full symmetry group as well as the unbroken symmetry group of the theory.

(b) Find the masses of different scalar particles in the theory and show that the results are consistent with Goldstone's theorem.

(c) If we couple the system to gauge fields so as to make all the global symmetries into gauge symmetries, how many massless and how many massive gauge fields shall we get?