$$e^{\quad} \quad a(\vec{k})^{+}|0\rangle \longrightarrow e^{i(\vec{k}\cdot\vec{x} - Et)}$$

We know now to construct mom. eigenstate

08/04/2011

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$i\partial_0 \psi = -i\gamma^0\gamma^i\partial_i\psi + \gamma^0 m\psi = \hat{h}\psi$$

Action for the Schrödinger case

$$S = \int d^4x\, \psi^*(x,t)\,(i\partial_t - \hat{h})\,\psi(\vec{x},t)$$

Here instead of ψ^*, we will have $\psi^\dagger$ (~~or it~~ ψ is a column vector)

$$S = \int d^4x\, \psi^\dagger(\vec{x},t)\,(i\partial_t - \hat{h})\,\psi(\vec{x},t)$$

$$\psi^\dagger \to (\psi_1^* \;\; \psi_2^* \;\; \psi_3^* \;\; \psi_4^*)$$

$$\pi_\alpha = \frac{\delta S}{\delta(\partial_t \psi)} = \frac{\delta S}{\delta(\partial_t \psi_\alpha)} = i\psi_\alpha^*$$

index of ψ.

it will transformed by a 4×4 matrix

$$H = \int d^3x\,(\pi_\alpha \psi_\alpha - L)$$

$$= \int d^3x\, \psi^\dagger(\vec{x},t)\,\hat{h}\,\psi(\vec{x},t)$$

$$\{\psi_\alpha(\vec{x},t), \pi_\beta(\vec{y},t)\}_{PB}$$

$$= \delta_{\alpha\beta}\,\delta^3(\vec{x}-\vec{y})$$

Quantize :- not a standard quantization.

(as $\{\}_{P.B} \to \frac{1}{i}\{\}_{A.C}$).

$$\frac{1}{i}\left\{\psi_\alpha(\vec{x},t), \pi_\beta(\vec{y},t)\right\}.$$

↳ anticommutation

$$\left\{\psi_\alpha(\vec{x},t), \psi_\beta^+(\vec{y},t)\right\} = \delta_{\alpha\beta}\,\delta^3(\vec{x}-\vec{y})$$

↳ $\psi_\beta^\oplus, \psi_\alpha$ is an operator.

Let us go back to Schrödinger problem :-

$$\psi(\vec{x},t) = \sum_n a_n(t)\, u_n(x).$$

role of $a_n(t)$ is played here by $b(\vec{p},s,t)$ & $a^+(-\vec{p},s,t)$.

here are following the Schrödinger ~~eqn~~ step. (role of a_n played by b & c)

$$\sqrt{\frac{m}{\omega_p}}\, u(\vec{p},s)\, \frac{e^{ip.x}}{(2\pi)^{3/2}}, \quad \sqrt{\frac{m}{\omega_p}}\, v(-p,s)\, \frac{e^{ip.x}}{(2\pi)^{3/2}}.$$

$$u^+ u = \frac{\omega_{\vec{p}}}{m}\,\delta s s'$$

now

$$\psi_\alpha(\vec{x},t) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega_{\vec{p}}}}$$

$$\Big[b(\vec{p},s,t)\, u_\alpha(\vec{p},s)\, e^{i\vec{p}\cdot\vec{x}}$$

$$+ \underbrace{d^+(-\vec{p},s,t)\, v_\alpha(-\vec{p},s)\, e^{+i\vec{p}\cdot\vec{x}}}_{\text{here we will change } p \to -\vec{p}} \Big]$$

$$\psi_\alpha(\vec{x},t) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega_{\vec{p}}}}$$

$$\Big[b(\vec{p},s,t)\, u_\alpha(\vec{p},s)\, e^{i\vec{p}\cdot\vec{x}}$$

$$+ d^+(\vec{p},s,t)\, v_\alpha(\vec{p},s)\, e^{-i\vec{p}\cdot\vec{x}} \Big].$$

Ex:- using the anticommutation relations of b, d, b^+, d^+ show that

$$\{\psi_\alpha(\vec{x},t), \psi_\beta^+(\vec{y},t)\} = \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{y}).$$

$$\{\psi_\alpha(\vec{x},t), \psi_\beta(\vec{y},t)\} = 0$$

$$\{\psi_\alpha^+(\vec{x},t), \psi_\beta^+(\vec{y},t)\} = 0,$$

try both ways

There are certain symmetry which ~~were~~ are there in classical theory upon quantization gets broken in the Quantum theory. → Anomalous theory.

Here $\psi_\alpha(\vec{x},t)$ is not real.

<u>Some useful relations</u> :-

(Choose any representation of the γ matrices), find $u(\vec{p},s)$ & $v(\vec{p},s)$ and check this).

$$u^+(\vec{p},s)\,u(\vec{p},s') = \frac{\omega_{\vec{p}}}{m}\delta_{ss'}$$

$$v^+(\vec{p},s)\,v(\vec{p},s') = \frac{\omega_{\vec{p}}}{m}\delta_{ss'}$$

$$u^+(\vec{p},s)\,v(-\vec{p},s') = 0.$$

Define: $\bar{u} = u^+\gamma^0$, $\bar{v} = v^+\gamma^0$

$$\left.\begin{aligned} \bar{u}(\vec{p},s)\,u(\vec{p},s') &= \delta_{ss'} \\ \bar{v}(\vec{p},s)\,v(\vec{p},s') &= -\delta_{ss'} \\ \bar{v}(\vec{p},s)\,u(\vec{p},s') &= 0 \end{aligned}\right] \text{ new normalisation rel.}$$

<u>In scalar field</u>.

also $f_k = \dfrac{1}{\sqrt{\omega_k}}$

$f_k^* f_k = \dfrac{1}{\omega_k}$, but $\int f_k^* \overleftrightarrow{\partial_0} f_k \, d^3x$

doesn't name any $\omega_{\vec{k}}$.

$$\int f_k^{\dagger} \overleftrightarrow{\partial_0} f_k \, d^3k \rightarrow$$ now we have introduced ∂_0, there γ_0.

Completeness :-

$$\sum_{s=1}^{2} \left\{ u_\alpha(\vec{p},s)\, \bar{u}_\beta(\vec{p},s) - v_\alpha(\vec{p},s)\, \bar{v}_\beta(\vec{p},s) \right\} = \delta_{\alpha\beta}.$$

Dimension of spinor in D dim $2^{[D/2]}$, it comes from relativistic invariance.

$$\cancel{\int d^3x \; f_k^{\dagger} \overleftrightarrow{\partial_0} f_{k'}}.$$

$$\sum_{s=1}^{2} u_\alpha(\vec{p},s)\, \bar{u}_\beta(\vec{p},s) = \left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta} \Bigg|_{p^0=\omega_{\vec{p}}} \quad \text{--- (i)}$$

$$\sum_{s=1}^{2} v_\alpha(\vec{p},s)\, \bar{v}_\beta(\vec{p},s) = \left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta} \Bigg|_{p^0=\omega_{\vec{p}}} \quad \text{--- (ii)}$$

(i) − (ii) $= \delta_{\alpha\beta}$.

Ex :- check that

$$b(\vec{p}, s, t) = \sqrt{\frac{\omega_{\vec{p}}}{m}} \int \frac{d^3x}{(2\pi)^{3/2}} \bar{u}(\vec{p}, s)\, \psi(\vec{x}, t)\, e^{-i\vec{p}\cdot\vec{x}}$$

$$a^{+}(\vec{p}, s, t) = \sqrt{\frac{\omega_{\vec{p}}}{m}} \int \frac{d^3x}{(2\pi)^{3/2}} \bar{v}(\vec{p}, s)\, \psi\, e^{i\vec{p}\cdot\vec{x}}.$$

$$b^{+}(\vec{p}, s, t) = \sqrt{\frac{\omega_{\vec{p}}}{m}} \int \frac{d^3x}{(2\pi)^{3/2}} \left(\left(\bar{u}(\vec{p}, s)\, \psi(\vec{x}, t) \right)^{+} e^{i\vec{p}\cdot\vec{x}} \right)$$

$$= \int \frac{d^3x}{(2\pi)^{3/2}} \bar{\psi}(\vec{x}, t)\, u(\vec{p}, s)\, e^{i\vec{p}\cdot\vec{x}}$$

$$\bar{\psi} = \psi^{+} \gamma^{0}$$

$$H = \int \psi^{+}(\vec{x}, t)\, \hat{h}\, \psi(\vec{x}, t)\, d^3x$$

$\longrightarrow$ let us consider this as Hamiltonian in the Quantum theory.

$$i \frac{\partial \psi}{\partial t} = [\psi, H],$$

$\longrightarrow$ They must satisfy the Heisenberg rule.

The fact that it is Fermion is used during going to classical $\longrightarrow$ Quantum theory.

But once we have established the Quantum theory then

$$i\frac{\partial \psi}{\partial t} = [\psi, H].$$

Every operator will evolve as $i\frac{\partial O}{\partial t} = [O, H]$.

Once we have the Hamiltonian we follow the usual rule.

$$i\frac{\partial \psi_\gamma(\vec{y},t)}{\partial t} = [\psi_\gamma(\vec{y},t), H]$$

$$= \left[\psi_\gamma(\vec{y},t), \int \psi^+_\alpha(\vec{x},t)\, \hat{h}_{\alpha\beta}\, \psi_\beta(\vec{x},t)\, d^3x\right]$$

$$= \left[\psi_\gamma(\vec{y},t), \int \psi^+_\alpha(\vec{x},t)\, \hat{h}_{\alpha\beta}\, \psi_\beta(\vec{x},t)\, d^3x\right]$$

$$[A, BC] = [A,B]C + B[A,C]$$

$$[A, BC] = ABC - BCA$$

$$= ABC + BAC - BAC - BCA$$

$$= \{A,B\}C - B\{A,C\}.$$

$$= \int d^3x \; \underline{\delta_{\alpha\gamma}} \, \delta^3(\vec{x}-\vec{y}) \, \underline{\hat{h}_{\alpha\beta}} \, \psi_\beta(\vec{x},t).$$

$$\{\underline{\psi_\gamma}, \underline{\psi_\beta}\} = 0$$

$$= \hat{h}_{\gamma\beta} \, \psi_\beta(\vec{y},t).$$

$$i \frac{\partial \psi_\gamma(\vec{y},t)}{\partial t} = \hat{h}_{\gamma\beta} \, \psi_\beta(\vec{y},t).$$

even though we have modified the rules for going from classical to Quantum, but we obtained the same classical ψ eqn.

→ we have obtained the field theory version Quantum Heisenberg eqn ~~would~~ gives us the classical Dirac field eqn.

If we construct a ~~cla~~ conserved charge in the classical theory then that charge remains conserved in the Quantum theory

Operator eqn. of motion in the ~~cla~~ Q.T is same as dynamic variable eqn. of motion in C.T.

In Q. Theory

$$\psi \psi^{+} \neq \psi^{+} \psi$$

But in classical theory

$$\psi \psi^{*} = \psi^{*} \psi .$$

Whether charge will be conserved in the quantum theory

Conservation $\Rightarrow \frac{dQ}{dt} = 0$

when classical eqn. hold.

But $\frac{dQ}{dt} = 0$ Q. Theory ~~will~~ satisfies same eqn.

$$\frac{dQ}{dt} = 0 .$$

As long as it involved Quadratic field.

Whether charge will be conserved in the Quantum theory

$$\partial_\mu J^\mu = 0 .$$

Conserved charges :-

Space translation :-

$$\psi(\vec{x},t) \longrightarrow \psi(\vec{x}+\vec{a},t)$$

Infinitesimal version

$$\psi(\vec{x},t) \longrightarrow \psi(\vec{x},t) + \epsilon^i \partial_i \psi(\vec{x},t)$$

Conserved charge :-

$$P_i = \int d^3x \, \psi^\dagger(\vec{x},t)\,(-i\partial_i)\,\psi(\vec{x},t)$$

$$\underset{ex.}{=} \sum_{s=1,2} \int d^3p \; p^i \left\{ b^\dagger(\vec{p},s,t)\, b(\vec{p},s,t) + d^\dagger(\vec{p},s,t)\, d(\vec{p},s,t) \right\}.$$

Time translation $\longrightarrow$ Hamiltonian

Another symmetry $\qquad Q = \int J^0 d^3x$

$$\psi(\vec{x},t) \to e^{i\phi}\psi(\vec{x},t)$$

Infinitesimal version

$$\cancel{= \psi(\vec{x},t)} = \psi(\vec{x},t) + i\epsilon\,\psi(\vec{x},t)$$

$$= \int d^3x \, \psi^\dagger(\vec{x},t)\,\psi(\vec{x},t)$$

$\longmapsto$ for electrons

Would be electric charge

Charge conservation $\longrightarrow$ it is common ~~would~~ possible to find ~~the~~ eigenstate of

In terms of b, d.

$$\int d^3p \sum_s \left(b^+(\vec{p}, s, t)\, b(\vec{p}, s, t) - d^+(\vec{p}, s, t)\, d(\vec{p}, s, t) \right)$$

$\longrightarrow$ particles created by d^+ = -ve charge.

$$\cancel{Q|b} \quad Q\, b^+ |0\rangle = 1\, b^+ |0\rangle$$

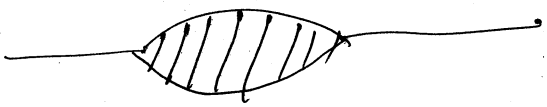
$$Q\, d^+ |0\rangle = -1\, d^+ |0\rangle$$

$\rightarrow$ Two kinds of single particle state. ~~has opp. charge~~ particles created by d^+ have ~~opp.~~ charge ~~d^+~~, ~~opp.~~ to ~~the~~ b^+.

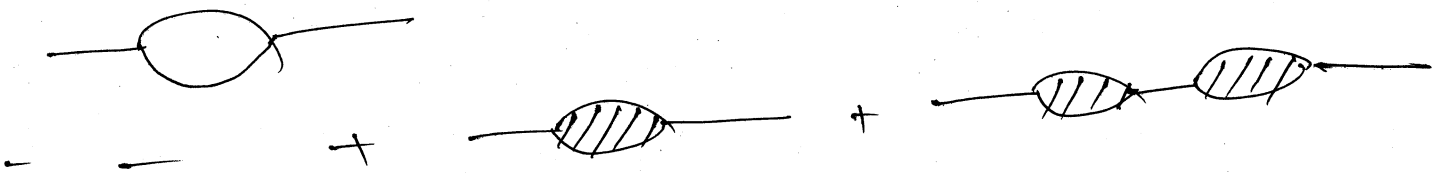
$b^+_c + d^+_c$ = same as vacuum.

Lecture 3. 13/04/2011

1PI



$$-i\Sigma(p)$$



$$= \frac{i}{-p^2-m^2+i\epsilon} + \frac{i}{-p^2-m^2+i\epsilon}(-i\Sigma)\frac{i}{-p^2-m^2+i\epsilon}$$

$$+ \quad .$$

$$= \frac{i}{-p^2-m^2+i\epsilon}\left\{\frac{1}{1-\frac{\Sigma}{-p^2-m^2+i\epsilon}}\right\} = \frac{i}{-p^2-m^2+i\epsilon-\Sigma}$$

m_p ~~defined through~~

$$-p^2-m^2-\Sigma(p)=0 \quad \text{at } p^2=-m_p^2$$

$$-p^2 = m^2+\Sigma(p^2)$$

0th order : $-p^2 = m^2 = m_p^2$

1st : $m_p^2 = m^2+\Sigma(-m_p^2)$

2nd :

$$-p^2-m^2-\Sigma(p^2) = 0 \quad (-p^2-m_p^2) \text{ near } p^2=-m_p^2$$

Dirac eqn :-

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

2nd Quantization

$$\psi_\alpha(\vec{x},t) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega_{\vec{p}}}} \Big[b(\vec{p},s,t)\, u_\alpha(\vec{p},s)\, e^{i\vec{p}\cdot\vec{x}} + d^+(\vec{p},s,t)\, v_\alpha(\vec{p},s)\, e^{-i\vec{p}\cdot\vec{x}} \Big]$$

$$\{ b(\vec{p},s,t),\ b^+(\vec{p}',s',t) \} = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}')$$

$$\{ d(\vec{p},s,t),\ d^+(\vec{p}',s,t) \} = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}')$$

$$H = \sum_{s=1}^{2} \int d^3p\, \omega_{\vec{p}} \Big[b^+(\vec{p},s,t)\, b(\vec{p},s,t) + d^+(\vec{p},s,t)\, d(\vec{p},s,t) \Big]$$

Equation of motion :-

$$i \frac{\partial \psi}{\partial t}(\vec{x},t) = [\psi(\vec{x},t), H]$$

$\longrightarrow$ gives us Dirac eqn

$$[H, b(\vec{p},s,t)] = -\omega_{\vec{p}}\, b(\vec{p},s,t)$$

Even though b satisfy ~~eigen~~ anticommutator ~~eq~~ reln., ~~eigen~~ Heisenberg eqn. don't change.

$$\cancel{[A, BC] = ABC - BCA}$$

$$\cancel{= ABC + ACB - ACB - BCA}$$

$$\cancel{= A\{B, C\}}.$$

$$[AB, C] = ABC - CAB$$

$$= ABC + ACB - ACB - CAB$$

$$= A\{B, C\} - \{A, C\}B.$$

$$[H, b(\vec{p}, s, t)] = -\omega_{\vec{p}}\, b(\vec{p}, s, t)$$

$$i\frac{\partial b(\vec{p}, s, t)}{\partial t} = [b, H] = \omega_{\vec{p}}\, b(\vec{p}, s, t)$$

$$\rightarrow b(\vec{p}, s, t) = e^{-i\omega_{\vec{p}} t}\, b(\vec{p}, s, 0)$$

$$a(\vec{p}, s, t) = e^{-i\omega_{\vec{p}} t}\, a(\vec{p}, s, 0)$$

$\longrightarrow$ Energy conserving operators.

To calculate explicite time dependence ~~of~~ on t use

we will substitute $b(\vec{p}, s, t)$ by $e^{-i\omega_p t}\, b(\vec{p}, s, 0)$

To find the propagator we have to express it at equal time

$$\psi_\alpha(\vec{x}, t) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega_{\vec{p}}}} \Big[b(\vec{p}, s, 0)\, e^{-i\omega_p t + i\vec{p}\cdot\vec{x}}\, u_\alpha(\vec{p}, s)$$

$$+ d^\dagger(\vec{p}, s, 0)\, e^{i\omega_{\vec{p}} t - i\vec{p}\cdot\vec{x}}\, v_\alpha(\vec{p}, s) \Big]$$

$$\cancel{[ABC, D] = [FC, D]}$$

$$\cancel{= F\{C, D\} - \{F, D\} C}$$

product of odd no. of spin ½ object can never make a scalar.

If $\{ABC, D\}$

$\longrightarrow$ all are anticommuting.

Time order product of field (info) is useful to calculate scattering amplitude.

$$T\left(A_1(t_1)\, A_2(t_2) \cdots A_n(t_n)\right)$$

$$= A_{i_1}(t_{i_1})\, A_{i_2}(t_{i_2}) \cdots A_{i_n}(t_{i_n})$$

$t_{i_1} > t_{i_2} > \cdots > t_{i_n}$.

largest time should be at left.

If A_i's are anticommuting.

Then the defn. is same except there is an extra factor of $(-1)^P$ = no. of exchanges need to bring $1, 2, \ldots n$ to $i_1, i_2, \ldots i_n$,

$$\psi_\alpha(\vec{x}_1, t_1)\, \psi_\beta^+(\vec{x}_2, t_2)$$

$$= \theta(t_1 - t_2)\, \psi_\alpha(x_1, t_1)\, \psi_\beta^+(\vec{x}_2, t_2)$$

$$+ \theta(t_2 - t_1)\, \psi_\beta^+(\vec{x}_2, t_2)\, \psi_\alpha(\vec{x}_1, t_1)\, (-1)$$

$$= \theta(t_1 - t_2)\, \psi_\alpha(\vec{x}_1, t_1)\, \psi_\beta^+(\vec{x}_2, t_2)$$

$$- \theta(t_2 - t_1)\, \psi_\beta^+(\vec{x}_2, t_2)\, \psi_\alpha(\vec{x}_1, t_1)$$

useful for Fermions.

$- \psi_\alpha(\vec{x}_1, t_1)\, \psi_\beta^+(\vec{x}_2, t)$ + Anticom.

$$\psi_\alpha(\vec{x}_1, t_1)\, \psi_\beta^+(\vec{x}_2, t_2)$$

$$= \theta(t_1 - t_2)\, \psi_\alpha(\vec{x}_1, t_1)\, \psi_\beta^+(\vec{x}_2, t_2)$$

$$- \theta(t_2 - t_2)\, \psi_\beta^+(\vec{x}_2, t_2)\, \psi_\alpha(\vec{x}_1, t_1)$$

$\cancel{\langle 0 | (b + d^+)}$

$$\langle 0 | T(\psi_\alpha(\vec{x}, t)\, \psi_\beta(\vec{y}, t')) | 0 \rangle$$

$$(\underline{b} + \cancel{d^+})(\cancel{b} + \underline{d^+})$$

$$b d^+ = - d^+ b = 0$$

$\psi_\alpha(\vec{x}, t)$ = produces an antiparticle at x.

$\psi^+(\vec{x}, t)$ = " a particle at x.

$$\psi_1 \psi_2 | 0 \rangle$$

$\longrightarrow$ creates an antiparticle

$\longrightarrow$ can annihilate a particle or create an antiparticle

$\langle 0 |$ 2 antiparticle $| 0 \rangle = 0$

$$\langle 0 | T\left(\bar{\psi}_\alpha(\vec{x},t)\, \bar{\psi}_\beta(\vec{y},t) \right) | 0 \rangle = 0 .$$

$$\downarrow$$

$$\leftarrow (\underline{b}^+ + \underline{d})\,(\underline{b}^+ + \underline{d}) \rightarrow \underline{0} .$$

$$\{ d, b^+ \} = 0$$

Only non zero

$$S_{F\alpha\beta}(\vec{x},t\,;\,\vec{y},t')$$

$$= \langle 0 | T\left(\psi_\alpha(\vec{x},t)\, \bar{\psi}_\beta(\vec{y},t') \right) | 0 \rangle$$

$$= \theta(t-t') \langle 0 | \psi_\alpha(\vec{x},t)\, \bar{\psi}_\beta(\vec{y},t') | 0 \rangle$$

$$(\underline{b} + \cancel{d^+})\,(\underline{b}^+ + \cancel{d})$$

$$- \theta(t'-t) \langle 0 | \bar{\psi}_\beta(\vec{y},t')\, \psi_\alpha(\vec{x},t) | 0 \rangle$$

$$= \theta(t-t') \langle 0 | (\underline{b} + d^+)(b^+ + d) | 0 \rangle$$

$$- \theta(t'-t) \langle 0 | (b^+ + d)(\underline{b} + d^+) | 0 \rangle$$

either we can take out γ^0

or rewrite u, v in terms of $\bar{u}, \bar{v}$

$$= \theta(t-t') \text{ anticommutator of } b \,\&\, b^+$$

$$- \theta(t'-t) \text{ anticommutator of } d, d^+ .$$

Ex:-

$$S_{F\alpha\beta}(\vec{x}, t, \vec{y}, t')$$

$$= \sum_{s=1}^{2} \sum_{s'=1}^{2} \int \frac{d^3p}{(2\pi)^{3/2}} \int \frac{d^3p'}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega_{\vec{p}}}} \sqrt{\frac{m}{\omega_{\vec{p}'}}}$$

$$\Big[\theta(t-t') \langle 0 | \underbrace{b(\vec{p}, s, 0)\, b^{\dagger}(\vec{p}', s', 0)}_{\delta^3(\vec{p}-\vec{p}')\delta_{ss'}}$$

$$u_\alpha(\vec{p}, s)\, \bar{u}_\beta(\vec{p}', s')\, e^{i(\vec{p}\cdot\vec{x} - \vec{p}'\cdot\vec{y})}\, e^{-i(\omega_{\vec{p}} t - \omega_{\vec{p}'} t')}$$

$$- \theta(t'-t)\, e^{-i(\vec{p}\cdot\vec{x} - \vec{p}'\cdot\vec{y})}\, e^{i(\omega_{\vec{p}} t - \omega_{\vec{p}'} t')}$$

$$\langle 0 | \underbrace{a(\vec{p}', s', 0)\, a^{\dagger}(\vec{p}, s, 0)}_{\delta^3(\vec{p}-\vec{p}')\delta_{ss'}} | 0 \rangle$$

$$v_\alpha(\vec{p}, s)\, \bar{v}_\beta(\vec{p}', s') \Big]$$

$$= \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{m}{\omega_{\vec{p}}} \Big[\theta(t-t')\, e^{i\{\vec{p}\cdot(\vec{x}-\vec{y}) - \omega_{\vec{p}}(t-t')\}}$$

$$u_\alpha(\vec{p}, s)\, \bar{u}_\beta(\vec{p}, s)$$

$$- \theta(t'-t)\, e^{-i\{\vec{p}\cdot(\vec{x}-\vec{y}) - \omega_{\vec{p}}(t-t')\}}$$

$$v_\alpha(\vec{p}, s)$$

$$\bar{v}_\beta(\vec{p}, s) \Big]$$

$$\sum_{s=1}^{2} u_\alpha(\vec{p}, s)\, \bar{u}_\beta(\vec{p}, s) = \left(\frac{\not{p} + m}{2m} \right) \Big|_{p_0 = \omega_{\vec{p}}}$$

$$\sum_s u_\alpha(\vec{p}, s)\, \bar{u}_\beta(\vec{p}, s) = \left(\frac{\not{p} - m}{2m}\right)_{\alpha\beta}\Bigg|_{p_0 = \omega_{\vec{p}}}$$

$$= S_{F\alpha\beta}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \Big[\theta(t - t')\, e^{i\{\vec{p}\cdot(\vec{x}-\vec{y})\}} (\not{p} + m)_{\alpha\beta}$$

$$- \theta(t' - t)\, e^{-i(p\cdot(x-y))} (\not{p} - m)_{\alpha\beta} \Big]$$

$$x^0 = t, \quad y^0 = t'$$

$$p\cdot(x-y) = \vec{p}\cdot(\vec{x}-\vec{y}) - p^0(x^0 - y^0)$$

second integral change

$$\vec{p} \to -\vec{p}$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p}\, \theta(t' - t)\, e^{-i\left[\vec{p}\cdot(\vec{x}-\vec{y}) + p^0(x^0 - y^0)\right]} (\not{p} + m)_{\alpha\beta} \Big|_{p^0 = \omega_{\vec{p}}}$$

$$\vec{y}\cdot\vec{p} - y^0 p^0$$

$$p^0 \Big|_{p^0 = -\omega_p}$$

$$S_{F\alpha\beta} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}}$$

$$\Big[\theta(t - t')\, e^{i\vec{p}(x-y)} (\not{p} + m) \Big|_{p^0 = \omega_{\vec{p}}}$$

$$- \theta(t' - t)\, e^{ip\cdot(x-y)} (-\not{p} - m)_{\alpha\beta} \Big|_{p^0 = -\omega_{\vec{p}}} \Big]$$

$$S_{F\,\alpha\beta} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}}$$

$$\Big[\theta(t-t')\, e^{ip\cdot(x-y)} (\not{p}+m)_{\alpha\beta} \Big|_{p^0=\omega_{\vec{p}}}$$

$$+ \theta(t'-t)\, e^{ip\cdot(x-y)} (\not{p}+m)_{\alpha\beta} \Big|_{p^0=-\omega_{\vec{p}}} \Big]$$

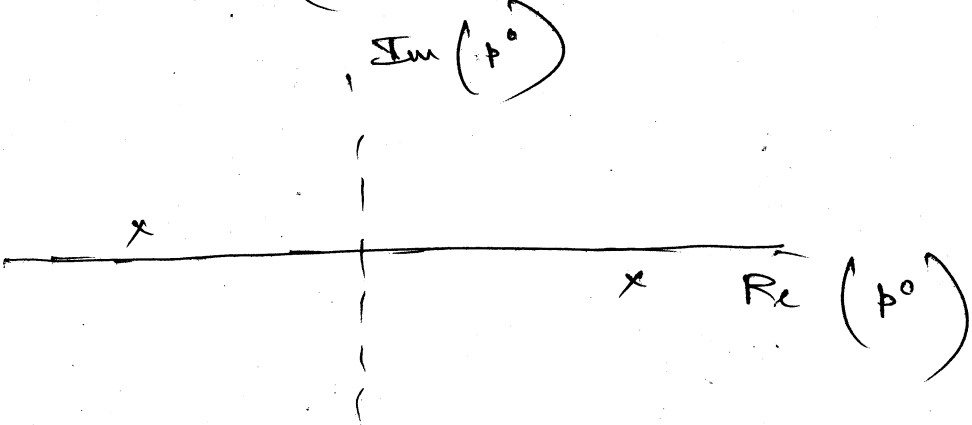
$$S_{F\,\alpha\beta} = -\int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi i}\, e^{ip\cdot(x-y)} \frac{(\not{p}+m)_{\alpha\beta}}{-p^2-m^2+i\epsilon}$$

integration variable.

$$e^{ip\cdot(x-y)} = e^{i\vec{p}\cdot(\vec{x}-\vec{y}) - ip^0(x^0-y^0)}$$

Feynmann

$$(p^0)^2 - \vec{p}^{\,2} - m^2 + i\epsilon .$$



If $x^0 - y^0 > 0$

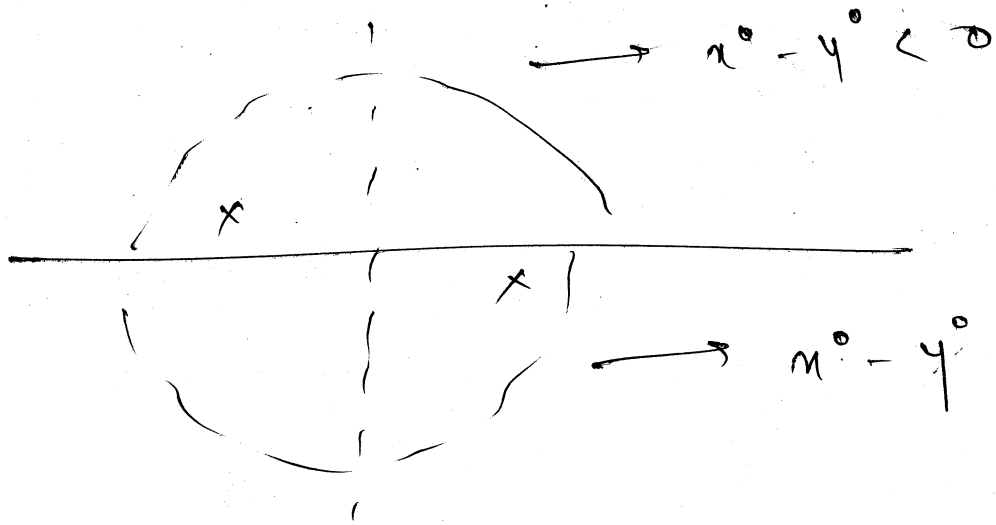
if $x^0 - y^0 > 0$ ~~pease~~ the contour below

$$\frac{1}{(p^0)^2 - m^2 - p^2 + i\epsilon} = -\cancel{2\pi i} \times \frac{1}{+2\omega_{\vec{p}}} \times \frac{-1}{\cancel{2\pi i}}$$

$$\frac{1}{(p^0 + \sqrt{p^2 + m^2 - i\epsilon})(p^0 - \sqrt{p^2 + m^2 - i\epsilon})}$$

$$p^0 = -\sqrt{p^2 + m^2 - i\epsilon}$$

$$p^0 = \sqrt{p^2 + m^2 - i\epsilon}$$



If $x^0 - y^0 < 0$ $\quad p^0 = -\sqrt{p^2 + m^2 - i\epsilon}$

$$2\pi i \times \frac{1}{-2\omega_{\vec{p}}} \times -\frac{1}{2\pi i} = -\frac{1}{2\omega_{\vec{p}}}$$

$$S_{F\alpha\beta} = -\int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{(2\pi)} e^{ip(x-y)} \frac{(\not{p} + m)_{\alpha\beta}}{-p^2 - m^2 + i\epsilon}$$

$$S_{F\alpha\beta} = \int \frac{d^4 p}{(2\pi)^4} e^{ip(x-y)} \frac{i(\not{p}+m)_{\alpha\beta}}{-p^2-m^2+i\epsilon}$$

$$\psi \longrightarrow e^{i\alpha}\psi$$

$$\psi^+ \longrightarrow e^{-i\alpha}\psi^+$$

for complex scalar field same procedure

$$\phi \longrightarrow e^{i\alpha}\phi$$

$$\phi^* \longrightarrow e^{-i\alpha}\phi^*$$

$$\langle 0 | \psi\psi | 0 \rangle = 0$$

$$\langle 0 | \phi\phi | 0 \rangle = 0$$

$$\langle T(\psi_{\alpha_1}(x_1) \cdots \psi_{\alpha_n}(x_n)\bar{\psi}_{\beta}(y_1) \cdots \bar{\psi}_{\beta_n}(y_m)) | 0 \rangle$$

= sum of products of S_F's

equal no. of ψ & ψ^+

Lecture 4. 14/04/2011

$x^0 > y^0$

In order to find

$(x^0 - y^0)$ changes sign

such L.T exist only when

$(x^0 - y^0)$ space like

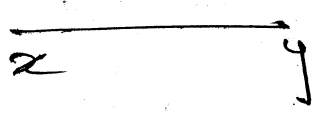
If $(x - y)$ is space like

then $[\phi(x), \pi(y)] = 0$

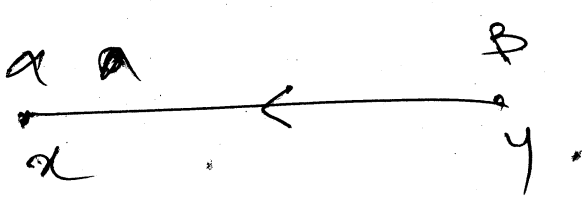
for time like $x^0 - y^0 > 0$.

$S_{F\alpha\beta}(x,y) = \langle 0 | T(\psi_\alpha(x)\bar{\psi}_\beta(y)) | 0 \rangle$

propagator from x to y.



we have to compute which one is ψ end & which one is $\bar{\psi}$ end.



Arrow is towards y.

$S_{F\alpha\beta}(x,y)$

Suppose we want to calculate

$$\langle 0 | T(\psi_{\alpha_1}(x_1) \cdots \psi_{\alpha_n}(x_n)\, \bar{\psi}_{\beta_1}(y_1) \cdots \bar{\psi}_{\beta_n}(y_n)) | 0 \rangle$$

$$\cancel{= S_F(\alpha\ \beta)(x_1, y_1)\, S_F(\alpha}$$

$$= S_{F\alpha_1\beta_1}(x_1, y_1)\, S_{F(\alpha_2,\beta_2)}(x_2, y_2) \cdots S_{F\alpha_n\beta_n}(x_n, y_n) \times \text{sign}$$

+ all permutations of $(\beta_1, y_1) \cdots (\beta_n, y_n)$.

$(-1)^{\text{no. of exchanges needed}}$ to bring $(\alpha_1, \ldots \alpha_n, \beta_1, \ldots, \beta_n)$ to $(\alpha_1 \beta_1\ \alpha_2 \beta_2 \cdots \alpha_n \beta_n)$

The operators restoring Time' Ordering are always commuting.

To determine the sign.

$$(\psi_{\alpha_1} \psi_{\alpha_2} \cdots \psi_{\alpha_n}\ \bar{\psi}_{\beta_1} \bar{\psi}_{\beta_2} \cdots \bar{\psi}_{\beta_n})$$

Join each line in $S_{F\alpha_1\beta_n}$

$$\psi_{\alpha_1}\ \psi_{\alpha_2} \cdots \psi_{\alpha_n}\ \bar{\psi}_{\beta_1}\ \bar{\psi}_{\beta_2} \cdots \bar{\psi}_{\beta_n}$$

sign $= (-1)^{\text{no. of crossing}}$

Ex:-

$$\langle 0 | T\left[\psi_{\alpha_1}(x_1)\, \psi_{\alpha_2}(x_2)\, \bar{\psi}_{\beta_1}(y_1)\, \bar{\psi}_{\beta_2}(y_2)\right] | 0 \rangle$$

$\psi_{\alpha_1}\ \psi_{\alpha_2}\ \bar{\psi}_{\beta_1}\ \bar{\psi}_{\beta_2}$

no. of crossing = 1.

$$S_{F\alpha_1\beta_1}(x_1, y_1)\ S_{F(\alpha_2,\beta_2)}(x_2, y_2)\ (-1)$$

$$+\ S_{F\alpha_1\beta_2}(x_1, y_2) + S_{F(\alpha_2,\beta_1)}(x_2, y_2)$$

$\psi_{\alpha_1}\ \psi_{\alpha_2}\ \bar{\psi}_{\beta_1}\ \bar{\psi}_{\beta_2}$

no. of crossing = 0

$\psi_{\alpha_1}\ \psi_{\alpha_2}\ \bar{\psi}_{\beta_1}\ \bar{\psi}_{\beta_2}$

no. of crossing = 2

~~Quantum Space Formulation~~

Interactions with Electrodynamics

Quantization of electromagnetic fields.

QED

Classical Electrodynamics

$$S = \int d^4x \, \mathcal{L}.$$

(Source free).

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad A_\mu = \text{vector pot.}$$

$$\delta S = -\frac{1}{2} \int d^4x \, F^{\mu\nu} \, \delta F_{\mu\nu}$$

$$\delta F_{\mu\nu} = \delta\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$$

$$= \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu.$$

$$\delta S = -\frac{1}{2} \int d^4x \, F^{\mu\nu} \left(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu\right)$$

$$= -\frac{1}{\cancel{2}} \cancel{2} \int d^4x \, F^{\mu\nu} \partial_\mu \delta A_\nu$$

$$= -\int d^4x \, F^{\mu\nu} \partial_\mu \delta A_\nu$$

$$\delta S = + \int d^4x \, \partial_\mu F^{\mu\nu} (\delta A_\nu)$$

$$\delta S = 0$$

for arbitrary δA_ν. (excluding B. contribution)

Thus

$$\partial_\mu F^{\mu\nu} = 0$$

$\longrightarrow$ Maxwell's eqn.

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

Lagrangian

$$L = \int d^3x \, \mathcal{L}.$$

$$= \int d^3x \left\{ \frac{1}{2} (\partial_0 A_i - \partial_i A_0)(\partial_0 A_i - \partial_i A_0) - \frac{1}{4} \sum_{i,j=1}^{3} (\partial_i A_j - \partial_j A_i)(\partial_i A_j - \partial_j A_i) \right\}$$

$$\dot{A}_\mu = \partial_0 A_\mu .$$

Euler Lagrange eqn :-

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{A}_\mu}\right) - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0 .$$

Hamiltonian formulation

$$\Pi_\mu(\vec{x}, t) = \frac{\delta L}{\delta \dot{A}_\mu(\vec{x}, t)}$$

$$= \frac{\delta L}{\delta \dot{A}_\mu(\vec{x}, t)}$$

$$\Pi_i(\vec{x}, t) = \partial_0 A_i - \partial_i A_0 = F_{0i}$$

$$\Pi_0 = 0 .$$

Normal procedure :- Eliminate $\dot{A}_\mu \longrightarrow \Pi_\mu$.

In ~~the~~ Hamiltonian formulation we treat Π & ϕ are independent,

But $\Pi_0 = 0$: (constraint)

so standard formulation will not work.

Need techniques for dealing with constrained system in the Hamiltonian formulation

~~If~~ $\mathcal{L}(q_i, \dot{q}_i)$, if some $\dot{q}_i$

doesn't enter the Lagrangian
Then the system is constrained

$$L = a\dot{q}_1 f(q_1, q_2)$$

$$\cancel{\pi_1 = \frac{\partial L}{\partial \dot{q}_1}} \quad \pi_1 = f(q_1, q_2)$$

As π depends on q_1 & q_2.
Thus no ~~to have~~ Hamiltonian formulation.

Need Techniques for dealing with constrained system in the Hamiltonian formulation

Cyclic Coordinates :- If q is missing in L,
q can take arbitrary value (independent)

But here

$$\pi_0 = 0$$

$\longrightarrow$ not independent.

To define $\pi_1 = f_1(\vec{q}_i, \dot{\vec{q}}_i)$

$\vdots$

$\pi_n =$

Routhian formalism :-
Combn. of Hamiltonian & Lagrangian formalism.

$$H = (p_i, q_i)$$

$L = (q_i, \dot{q}_i)$

In Routhian formalism we ~~can~~ have to follow the usual method for all the variables

Suppose we have some dynamical system with coordinates:

$\{q_a\}, \quad a = 1, \ldots m$

$\{q_\alpha\}, \quad \alpha = m+1 \ldots n$.

Split is arbitrary

Define $\quad p_a = \dfrac{\partial L}{\partial \dot{q}_a}$

& assume that we can use these eqns to find $\dot{q}_a$'s in terms p_a's.

We will choose set $\{q_a\}$ in such a way that we can solve p_a in terms of $\dot{q}_a$. $(p_a, q_a) \rightarrow$ independent

If for some a p_a is not independent:

Then we will include a in α.

& assume that we can use these eqns to find $\dot{q}_a$'s in terms of p_a's.

$$R(\{q_a\}, \{p_a\}, \{q_\alpha\}, \{\dot{q}_\alpha\})$$

$$= \sum_a p_a \dot{q}_a - L$$

Eqns. of motion :-

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_\alpha} - \frac{\partial R}{\partial q_\alpha} = 0.$$

$$\dot{q}_a = \frac{dq_a}{dt} = \frac{\partial R}{\partial p_a}, \quad \frac{dp_a}{dt} = -\frac{\partial R}{\partial q_a}$$

Check this.

fixing the $\dot{q}_a$'s is not same as fixing p_a.

For electrodynamics we treat the A_i's as q_a's & A_0 as q_α.

$$\pi_i(\vec{x}, t) = \frac{\partial R}{\partial \dot{A}_i(\vec{x}, t)} = \partial_0 A_i - \partial_i A_0,$$

$$= E_i(\vec{x}, t)$$

$\longrightarrow$ electric field

Routhian $\sum_i \int d^3x \, \pi^i(\vec{x}, t) \, \dot{A}_i(\vec{x}, t)$.

$$\dot{A}_i = \pi_i + \partial_i A_0$$

$$E_x = \frac{1}{2}\int d^3x \sum_1^3 \left(\pi_i^2 + B_i^2\right)$$

$$- \int d^3x \, A_0 \, \partial_i \pi^{ij}$$

$$B_i = \sum_{j,k=1}^{3} \epsilon^{ijk} \partial_j A_k .$$

ϵ^{ijk} = ~~total~~ antisymmetric matrix

If we rewrite E.L. eqn. for A_0 & Hamiltonian eqn. for A_i, then we will get complete Maxwell's eqn.

π_i & A_i's are still independent

A_0 eqn.

$$\frac{d}{dt}\left(\frac{\delta R}{\delta \dot{A}_0}\right) - \frac{\delta R}{\delta A_0} = 0 ,$$

$$\frac{\delta R}{\delta \dot{A}_0} = 0$$

$$\partial_i \pi^i = 0 , \Rightarrow \nabla \cdot \vec{E} = 0 .$$

In Hamiltonian eqn. of motion, we ~~result get~~

$$\partial_i \pi^i = 0 .$$

$$\frac{dA_i}{dt} = \frac{\delta R}{\delta \pi_i} = \pi_i + \partial_i A_0 \quad ; \quad q_i = \frac{\partial R}{\partial \dot{A}_i}$$

$$\frac{d\pi_i}{dt} = -\frac{\delta R}{\delta A_i} = -\epsilon^{ijk}\partial_j B_k$$

There is no eqn. that determine time evolution of A_0. eqn. of motion are the soln. to initial value prob.

Time evolution of A_0 is not determined $\longleftrightarrow$ related to gauge invariance

$$A_\mu \longleftrightarrow A_\mu + \partial_\mu \Lambda(\vec{x}, t)$$

$\longrightarrow$ gives the same $F_{\mu\nu}$ & same eqn. of motion.

Given any soln. $A_\mu(\vec{x},t)$ for Maxwell's eqn., $A_\mu(\vec{x},t) + \partial_\mu \Lambda(\vec{x},t)$ is also a soln.

We can chose $\partial_\mu \Lambda(\vec{x},t)$ in such way that $\partial_\mu \Lambda(\vec{x},0) = 0$.

$\cancel{\partial_\mu (A \quad \partial_\mu \Lambda(\vec{x},}$

claim :- given any soln. for which $A_0(\vec{x},t)$ is given, as $\hat{A}_0(\vec{x},t)$ we can make a gauge transf. to make A_0 into an arbitrary fn. $\hat{A}_0(\vec{x},t)$

~~choose~~ $\Lambda(\vec{x},t) = \int_0^t dt' \left(\tilde{A}_0(\vec{x},t') - \hat{A}_0(\vec{x},t') \right)$

$q^0 \to t$

$$\tilde{A}_0(\vec{x},t) \longrightarrow \hat{A}_0(\vec{x},t) + \partial_0 \Lambda(\vec{x},t)$$

$$= \hat{A}_0(\vec{x},t)$$

$\longrightarrow$ thats why we ~~didn't~~ get time evolution A_0.

~~Thus A_0~~

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

Lagrangian eqns are not designed to give a soln. for initial value problem always.

We will use this freedom

$\longrightarrow$ gauge fixing.

Gauge fixing :-

One possible choice of gauge is

$$A_0 = 0$$

Coulomb gauge :-

$$\vec{\nabla} \cdot \vec{A} = 0$$

i.e $\partial_i A^i = 0$.

Is it possible

Imagine

$A^i \longrightarrow$ arbitrary func.

given that config. can we find

Λ such that

$$\cancel{\nabla_i (A^i +} \quad \cancel{\nabla_i (}$$

$$\partial_i (A_i + \partial_i \Lambda) = 0$$

$$\cancel{\partial_i \partial_i \Lambda = 0} \quad \partial_i \partial_i \Lambda = -\partial_i A_i(\vec{x}, t)$$

$$\Lambda(\vec{x}, t) = \int d^3x' \, G(\vec{x}, \vec{x}') \left(-\partial_i' A(\vec{x}', t)\right)$$

$$G(\vec{x}, \vec{x}') = \frac{1}{4\pi |\vec{x} - \vec{x}'|}$$

$$\nabla^2 G(\vec{x}, \vec{x}') = \delta^{(3)}(\vec{x} - \vec{x}')$$

We assume that it falls of at ∞.

Suppose we have chose

$$\partial_i A^i = 0 .$$

Then time evolution

$$\partial_i \left(\frac{\partial A_i}{\partial t} \right) = \partial_i \pi_i + \nabla^2 A_0$$

$\partial_i \pi_i \to 0$; $\nabla^2 A_0 \to \nabla \cdot [\hat{i} + \hat{j} + \hat{k}]$

$$\frac{\partial}{\partial t} (\partial_i A_i) = 0$$

Thus $\vec{\nabla}^2 A_0 = 0$

$A_0 = 0$, if A_0 goes to 0 at ∞.

$$\frac{\partial A_i(\vec{x},t)}{\partial t} = \frac{\partial R}{\partial \pi_i} = \{ A_i(\vec{x},t), R \}_{P.B}$$

$$\frac{\partial \pi_i}{\partial t} = -\frac{\partial R}{\partial A_i} = \{ \pi_i(\vec{x},t), R \}_{P.B}$$

$$\partial_i \pi^i = 0$$

$\partial_i A^i = 0 \longrightarrow$ 'Gauge cond.

$$\frac{d}{dt}(\partial_i \pi^i) = 0 .$$

$\longrightarrow \partial_i \pi^i = K.$

The constraint actually restrict ~~the is satisfy~~ the initial value'

$\longrightarrow$ keeps the value same,

If I chose $\partial_i \pi^i = 0$

$\longrightarrow$ Initial cond.

$\longrightarrow$ then it ensures that all the time $\partial_i \pi^i = 0$

we will find a Hamiltonian which will give Maxwells eqn then we will Quantize.

Classically we can work with FMN but for Euler Lagrange eqn. we have to ~~make~~ use AH.

Lecture - 5 19/04/2011

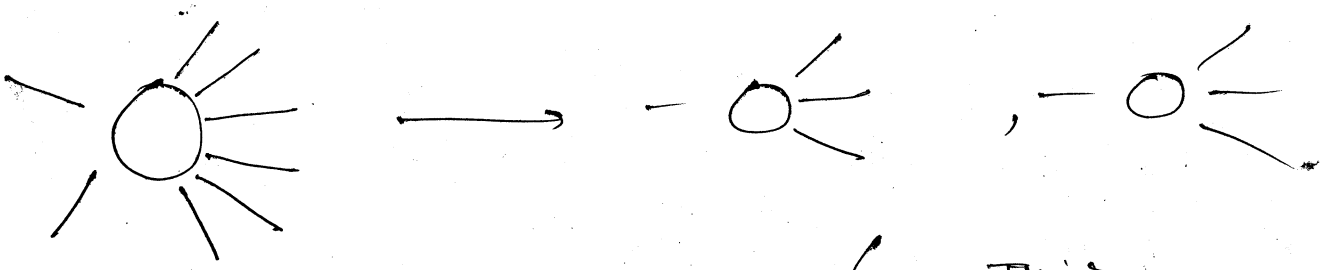
n - particle final state

3n - momentum.

(3n - 4) parameters.

4 momentum conservation

(3n - 8) subspace.



~~(3n - 8~~ This processes belong to 3N - 8

$p + e + \bar{\nu}_e + \nu_\mu$

μ — — p , $p + e + \bar{\nu}_e + \nu_\mu$

$$\mu \longrightarrow \begin{cases} e \\ \bar{\nu}_e \\ \nu_\mu \end{cases}$$

we can always measure

$$(E_e + E_{\bar{\nu}_e} + E_{\nu_\mu})^2 - (\vec{p}_e + \vec{p}_{\bar{\nu}_e} + \vec{p}_{\nu_\mu})^2$$

> 0

So these type of subprocesses will not be considered here.

R = Hamiltonian

$$= \frac{1}{2}\int d^3x \sum_{i=1}^{3}\left(\pi_i^2 + B_i^2\right)$$

$$+ \int d^3x \, A_0 \, \partial_i \pi_i$$

$$B_i = \epsilon^{ijk}\partial_j A_k$$

Treat A_i, π_i in Hamiltonian formalism.

A_0 as in the Lagrangian formalism.

A_0 eqn. $\frac{d}{dt}\frac{\partial R}{\partial \dot{A}_0} = \frac{\delta R}{\delta A_0}$

$$\Rightarrow \partial_i \pi_i = 0$$

$$\frac{dA_i}{dt} = \frac{\delta R}{\delta \pi_i} = \pi_i + \partial_i A_0 .$$

$$\frac{d\pi_i}{dt} = -\frac{\delta R}{\delta A_i} = \cancel{-\epsilon^{ijk}\partial_j A_k} - \epsilon^{ijk}\partial_j B_k$$

$$\frac{d}{dt}\left(\partial_i \pi_i\right) = 0 .$$

$\Rightarrow$ Once we have set $\partial_i \pi_i = \text{const}$ then it will remain same.

Gauge fixing :- Coulomb gauge $\partial_i A_i = 0$

Gauge parameters are arbitrary func. of x & t.

Local symmetry $\equiv f(x,t)$.

Use Fourier transformation variables:

$$A_i(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{A}_i(\vec{k},t) e^{i\vec{k}\cdot\vec{x}}$$

$$\pi_i(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{\pi}_i(\vec{k},t) e^{i\vec{k}\cdot\vec{x}}$$

$$\dot{R} = \int d^3k \Big[\frac{1}{2} \hat{\pi}^i(-\vec{k},t) \tilde{\pi}^i(\vec{k},t)$$

$$+ \frac{1}{4} \Big(k_i A_j(-\vec{k},t) - k_j \tilde{A}_i(-\vec{k},t) \Big) \Big(k_i \hat{A}_j(\vec{k},t) - k_j \hat{A}_i(\vec{k},t) \Big)$$

$$- i \int d^3k \, \hat{A}_0(-\vec{k},t) \, k_i \hat{\pi}^i(\vec{k},t)$$

$$\epsilon^{ijk} \epsilon^{imn} = \left(\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \right).$$

Poisson Bracket.

$$\{A_i(\vec{x},t), \pi_j(\vec{y},t)\}_{PB}$$

$$= \delta^{(3)}(\vec{x}-\vec{y}) \, \delta_{ij}$$

$$\Rightarrow \{\hat{A}_i(\vec{x},t), \hat{\pi}_j(\vec{x}',t)\}_{PB} = \delta^{(3)}(\vec{x}+\vec{x}')\,\delta_{ij}$$

We can define can. of motion standard Hamiltonian eqn. is for $\hat{A}_i$ & $\hat{\pi}_j$

& Euler Lagrangian equation for A_0 & π_0.

We will take linear comb. of A_i among themselves & π_i among themselves.

Comp. of $\vec{A} \parallel \vec{k}$ has a special role.

& $k_i \hat{A}_j - k_j \hat{A}_i = 0$

$k_i \hat{\pi}_i$ contributes reason

$\vec{k} \perp \vec{\pi}$.

So we will express it such that for any $\vec{k}$, define an orthonormal basis.

$$\left(\vec{\epsilon}^{(1)}(\vec{k}), \vec{\epsilon}^{(2)}(\vec{k}), \frac{\vec{k}}{|\vec{k}|}\right)$$

$$\sum_a \epsilon^{(a)}_i \epsilon^{(a)}_j + \frac{k_i k_j}{|\vec{k}^2|} = \delta_{ij}.$$

Define

$$\hat{A}^{(a)}(\vec{k}) = \epsilon_i^{(a)}(\vec{k})\,\hat{A}_i(\vec{k})$$

Transverse comp.

$$\hat{A}_{\parallel}(\vec{k}) = \frac{i k_i \hat{A}_i(\vec{k},t)}{|\vec{k}|}$$

all of these depend on time.

$$\hat{\pi}^{(a)}(\vec{k},t) = \epsilon_i^{(a)}\,\hat{\pi}_i(\vec{k},t)$$

$$\hat{\pi}_{\parallel}(\vec{k},t) = \frac{i k_i \hat{\pi}_i(\vec{k},t)}{|\vec{k}|}$$

rewrite the Hamiltonian in terms of $\hat{A}^{(a)}$ & $\hat{A}_{\parallel}$

Ex:-

Hamiltonian

$$H = \int d^3k \Big[\frac{1}{2} \sum_{a=1}^{2} \hat{\pi}^{(a)}(-\vec{k},t)\,\hat{\pi}^{(a)}(\vec{k},t)$$

$$+ \frac{1}{2} k^2 \sum_{a=1}^{2} \hat{A}^{(a)}(-\vec{k},t)\,\hat{A}^{(a)}(\vec{k},t)$$

$$+ \frac{1}{2} \hat{\pi}_{\parallel}(-\vec{k},t)\,\hat{\pi}_{\parallel}(\vec{k},t)$$

$$- \hat{A}_0(-\vec{k},t)(\vec{k})\,\hat{\pi}_{\parallel}(\vec{k},t) \Big].$$

ϵ depends on k. (Arbitrary choice)

We will assume

$$\left.\begin{aligned} \vec{\epsilon}^{(1)}(\vec{k}) &= \vec{\epsilon}^{(1)}(-\vec{k}) \\ \vec{\epsilon}^{(2)}(\vec{k}) &= \vec{\epsilon}^{(2)}(-\vec{k}) \end{aligned}\right].$$

if k_x is +ve, we will choose ϵ in such a way that it ~~of~~ compensates the effect ~~of might have~~ negativeness.

$$A_\mu(\vec{x},t)^* = +A_\mu(\vec{x},t)$$

$$\pi_i(\vec{x},t)^* = \pi_i(\vec{x},t)$$

$$\tilde{A}_\mu(\vec{k},t)^* = \tilde{A}_\mu(-\vec{k},t)$$

$$\left[\tilde{\pi}_\mu(\vec{k},t)\right]^* = \tilde{\pi}_\mu(-\vec{k},t)$$

$F^{\mu\nu}$ real.

Imaginary part of A_μ must be pure gauge.

So Im. part of $A_\mu = i\partial_\mu \Lambda$

which we can remove it always by a suitable

gauge transformation,

$$A^{(a)}(\vec{x},t)^* = \hat{A}^{(a)}(-\vec{x},t).$$

$$\pi^{(a)}(\vec{x},t)^* = \hat{\pi}^{(a)}(-\vec{x},t)$$

$$\hat{A}_{\parallel}(\vec{x},t)^* = \hat{A}_{\parallel}(-\vec{x},t)$$

$$\hat{\pi}_{\parallel}(\vec{x},t)^* = \hat{\pi}_{\parallel}(-\vec{x},t)$$

lex:-

$$\{\hat{A}^{(a)}(\vec{x},t), \hat{\pi}^{(b)}(\vec{x}',t)\} = \delta_{ab}\, \delta^{(3)}(\vec{x}+\vec{x}')$$

$$\{\hat{A}_{\parallel}(\vec{x},t)\, \hat{\pi}_{\parallel}(\vec{x}',t)\} = \delta^{(3)}(\vec{x}+\vec{x}')$$

$\hat{A}_i$ & $\hat{\pi}_i$ ~~are~~ should be treated in Hamiltonian formulation but

$\hat{A}_0$ & $\hat{\pi}_0$ in ~~to~~ ~~Ro~~ Lagrangian formulation.

$$R = \int d^3x \Big[\frac{1}{2} \sum_{a=1}^{2} \hat{\pi}^{(a)}(\vec{x},t)\, \hat{\pi}^{(a)}(-\vec{x},t)$$

$$+ \frac{1}{2} k^2 \sum_{a=1}^{2} \hat{A}^{(a)}(\vec{x},t)\, \hat{A}^{(a)}(-\vec{x},t)$$

$$+ \frac{1}{2} \hat{\pi}_{\parallel}(-\vec{x},t)\, \hat{\pi}_{\parallel}(\vec{x},t)$$

$$- \hat{A}_0(-\vec{x},t)\, (\vec{k})\, \hat{\pi}_{\parallel}(\vec{x},t) \Big]$$

Consider gauge cond.

$$k^i \hat{\underline{A}}_i = 0 .$$

$$\Longrightarrow k^i \left[\epsilon \hat{\underline{A}}^{\parallel} + \hat{\underline{\underline{A}}}_{\perp} \right]$$

$$\Longrightarrow \hat{A}^{\parallel} = 0 .$$

$$\Longrightarrow \hat{A}^{\parallel}(\vec{x},t) = 0 .$$

Eqn. of motion of $\hat{A}_{\parallel}(\vec{x},t)$.

$$\frac{d}{dt} \hat{A}_{\parallel}(\vec{x},t) = \{ \hat{A}_{\parallel}(\vec{x},t), R \}_{PB}$$

$$= \hat{\pi}_{\parallel}(\vec{x},t) - |\vec{k}|\, \hat{A}_0(\vec{x},t)$$

Eqn. of motion for $\hat{\pi}_{\parallel}$

$$\frac{d}{dt} \hat{\pi}_{\parallel}(\vec{x},0) = 0$$

$$\frac{\partial \hat{A}_0}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial R}{\partial \hat{A}_0(\vec{k},t)}\right) = \frac{\delta R}{\delta A_0(\vec{k},t)} = 0$$

$$\Longrightarrow \quad |\vec{k}|\, \hat{\pi}_{\parallel}(-\vec{k},t) = 0$$

Coulomb gauge cond :-

$$\hat{A}_{\parallel}(\vec{k},t) = 0$$

$$\Longrightarrow \quad \hat{\pi}_{\parallel}(\vec{k},t) = 0$$

$$\hat{A}_0(\vec{k},t) = 0$$

$$\frac{\partial}{\partial t}\hat{\pi}_{\parallel}(\vec{k},t) = 0$$

$$\Longrightarrow \quad \hat{\pi}_{\parallel}(\vec{k},t) = \text{const} = 0$$

$$\hat{A}_{\parallel}(\vec{k},t) = 0$$

$$\hat{A}_0(\vec{k},t) = 0$$

$\Longrightarrow$ electromagnetic field contains two independent polarisation

$\Rightarrow$ The equations of motion can be derived from

$$R = \frac{1}{2}\int d^3K \sum_{a=1}^{2} \left\{ \hat{H}^{(a)}(\vec{k},t)\hat{H}^{(a)}(\vec{k},t) + \frac{1}{2} k^2 \hat{A}^{(a)}(-\vec{k},t)\, \hat{A}^{(a)}(-\vec{k},t) \right\}.$$

equivalent two Klein Gordon field, the variables which is on left can be treated in Hamiltonian formalism, to Routhian can be treated as Hamiltonian

two massless particles,

$S_z = 1, 0, -1$

Lecture - 6. 20/04/2011

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$$

$$A_i \longrightarrow A_i + \partial_i \Lambda$$

$$\nabla \cdot \vec{A} \longrightarrow \nabla \cdot \vec{A} + \nabla^2 \Lambda$$

$$\nabla^2 \Lambda = 0 \Rightarrow \Lambda = 0$$

$$\Lambda = C,$$

Check always that the gauge transformation parameter field goes to zero.

Equations of motion of $\hat{A}_0$, $\hat{A}_{\parallel}$ & $\hat{\pi}_{\parallel}$.

$$\Rightarrow \quad \hat{A}_0 = 0, \ \hat{A}_{\parallel} = 0 \ \& \ \hat{\pi}_{\parallel} = 0$$

Sometimes it may happen that eqn. of motion of one field gives some other eqn. involving other field. Then we can not set it to 0

$$R = \int d^3k \left[\frac{1}{2} \sum_{a=1}^{2} \hat{\pi}^{(a)}(\vec{k},t)\, \hat{\pi}^{(a)}(-\vec{k},t) + \frac{1}{2}\vec{k}^2 \hat{A}^{(a)}(\vec{k},t)\, \hat{A}^{(a)}(-\vec{k},t) \right]$$

$$\left\{ \hat{A}^{(a)}(\vec{k},t), \hat{\pi}^{(b)}(\vec{k}',t) \right\}_{PB} = \delta_{ab}\, \delta^3(\vec{k} - \vec{k}')$$

same as K.G field. mass = 0.

Quantize as in case of K.G. field.

with $\omega_{\vec{k}} = |\vec{k}|$

a describes the oscillation whose dirnc. is along ϵ^a.

<u>Quantization</u> :-

i) $$\left[\hat{A}^{(a)}(\vec{k},t), \tilde{\pi}^{(b)}(\vec{k}',t)\right] = i\,\delta_{ab}\,\delta^{(3)}(\vec{k}+\vec{k}')$$

<u>Define</u> :-

$$a^{(c)}(\vec{k},t) = \frac{1}{\sqrt{2}}\left\{\omega_{\vec{k}}^{1/2}\,\hat{A}^{(c)}(\vec{k},t) + i\,\omega_{\vec{k}}^{-1/2}\,\hat{\pi}^{(c)}(\vec{k},t)\right\}$$

$$a^{(c)+}(\vec{k},t) = \frac{1}{\sqrt{2}}\left\{\omega_{\vec{k}}^{1/2}\,\tilde{A}^{(c)}(-\vec{k},t) - i\,\omega_{\vec{k}}^{-1/2}\,\tilde{\pi}^{(c)}(-\vec{k},t)\right\}.$$

Ex:- $$\left[a^{(c)}(\vec{k},t), a^{(d)+}(\vec{k}',t)\right] = \delta_{cd}\,\delta^{(3)}(\vec{k}-\vec{k}')$$

$$\left[a^{(c)}(\vec{k},t), a^{(d)}(\vec{k}',t)\right] = 0$$

$$\left[a^{(c)+}(\vec{k},t), a^{(d)+}(\vec{k}',t)\right] = 0.$$

Ex:- Hamiltonian same as Routhian.

As remaining degrees of freedom satisfy Hamiltonian's eqn

$$H(R) = \int d^3k \; \omega_{\vec{k}} \sum_{c=1}^{2} a^{(c)}(\vec{k},t)^{+} a^{(c)}(\vec{k},t)$$

$+$ const

we will drop the const term

$$[H, a^{(c)}(\vec{k},t)] = -\omega_{\vec{k}} \, a^{(c)}(\vec{k},t)$$

$$[H, a^{+(c)}(\vec{k},t)] = \omega_{\vec{k}} \, a^{+(c)}(\vec{k},t)$$

reduces energy. (annihilation operator.)

creation operator

$$a^{(c)}(\vec{k},t)|0\rangle = 0.$$

for all $\vec{k}$ & $c = 1,2$.

Then $a^{+(c)}(\vec{k},t)|0\rangle$

energy eigenstate of eigenvalue.

$\omega_{\vec{k}} = |\vec{k}|$ $\longrightarrow$ Thus it describes a massless particle.

$\longrightarrow$ show that $\vec{k}$ is the momentum of this state.

Instead of 1 particle, we have
& massless particle $\longrightarrow$ two different
polarisation

Or we can take linear comb.
left circular / right circular.

Two types of particle

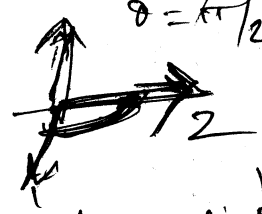
$\longrightarrow$ One kind of photon that exist in 2 diff. ~~particle~~ polarization

$\longrightarrow$ Similar to Dirac particle.

$\longrightarrow$ As it is a massless particle.
so we can not direct measure
spin But we can measure
helicity.

$$J^{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} a$$

$\theta = \pi/2$



$$\vec{k} = (0, 0, k_3)$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$J^{12} = J$$

$$J^{12} a^{(1)+}(\vec{k}, t) = a^{(2)+}(\vec{k}, t)$$

$$J^{12} a^{(2)+}(\vec{k}, t) = -i\, a^{(1)+}(\vec{k}, t)$$

$$J^{12}(a^{(1)+} + i a^{(2)+}) = i(a^{(2)+} - i a^{(1)+}) = -i(a^{(1)+} + i a^{(2)+})$$

$$J^{12}(a^{(1)+} - i a^{(2)+}) = i(a^{(1)+} - i a^{(2)+})$$

For massless particle $\longrightarrow$ helicity

$\hookrightarrow$ degenerate.

Small mass. $\longrightarrow$ Infrared divergence.

for massive particle $\frac{1}{k^2+m^2}$

$\hookrightarrow$ k - small finite

for massless particle $\frac{1}{k^2}$

for $k \to 0$ $\longrightarrow \infty$

Physical interpretation :- There is a cut off for low mom.

If we fix $\vec{k} = (0, 0, k_3)$

J^{13} we want to know com. of angular mom. along mom

$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{J_{12}}$

Calculate :-

$$\langle 0 | A_\mu(\vec{x}, t) A_\nu(\vec{x}', t) | 0 \rangle$$

$$D_{\mu\nu}(\vec{x}, t, \vec{x}', t')$$

$$= \langle 0 | T\left[A_\mu(\vec{x}, t) A_\nu(\vec{x}', t')\right] | 0 \rangle$$

Before Quantize :- $D_{00} = 0$,

in mom. space

|| comp. $= 0$

$D_{\mu\nu}(\vec{x}, t, \vec{x}', t')$

$$= \langle 0 | T\left(A_\mu(\vec{x}, t)\, A_\nu(\vec{x}', t')\right) | 0 \rangle$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d^3k'}{(2\pi)^{3/2}}\, e^{i\vec{k}\cdot\vec{x}}\, e^{i\vec{k}'\cdot\vec{x}'}$$

$$\langle 0 | T\left(\hat{A}_\mu(\vec{k}, t)\, \hat{A}_\nu(\vec{k}', t')\right) | 0 \rangle$$

we will rewrite $\hat{A}_\mu(\vec{k}, t)$ in terms of $\hat{A}_0$, $\hat{A}_{\parallel}$, $\hat{A}^{(a)}$

Define :-

$\eta_\mu = (1, 0, 0, 0)$ — lower comp. is $+1$.

$\bar{k}_\mu = (0, \vec{k})$, ~~$\bar{k}_\mu = (0, \vec{k})$~~

$\vec{k} = (k_1, k_2, k_3)$ ~~$\bar{k}^\mu = (0,$~~

$$\bar{\eta}^{(a)}_\mu(\vec{k}) = (0, \vec{\epsilon}^{(a)}(\vec{k}))$$

we are now making a basis of 4 vectors.

$$\delta_\mu^{\ \nu} = -\eta_\mu \eta^\nu + \frac{\vec{k}_\mu \vec{k}^\nu}{k^2} + \sum_a \bar{\eta}^{(a)}_\mu \bar{\eta}^{(a)\nu}$$

$\delta^0_{\ 0} = 1$ $\quad -1, -1$

$\delta^0_{\ i} = 0 = \delta^i_{\ 0}$.

$\delta^j_{\ i} = \delta_i^{\ j}$

$$\hat{A}_\mu(k) = S_\mu{}^\nu \hat{A}_\nu(k)$$

$$= -\eta_\mu \eta^\nu \hat{A}_\nu(\vec{k}) + \frac{\vec{k}_\mu \vec{k}^\nu}{|\vec{k}|^2} \hat{A}_\nu(\vec{k})$$

$$+ \sum_a \eta^a_\mu \eta^{a\nu} \hat{A}_\nu(\vec{k})$$

$$= \hat{A}_0 + \hat{A}_{\parallel} + \sum_a \eta^{(a)}_\mu \hat{A}^{(a)}(\vec{k})$$

$$= 0 + 0 + \sum_a \eta^{(a)}_\mu \hat{A}^{(a)}(\vec{k}).$$

$$D_{\mu\nu}(\vec{x}, t, \vec{x}', t')$$

$$= \sum_a \sum_b \eta^{(a)}_\mu(\vec{k}) \eta_\nu^{(b)}(\vec{k}')$$

$$\hat{A}^{(a)}(\vec{k}, t) \hat{A}^{(b)}(\vec{k}', t).$$

$$D_{\mu\nu}(\vec{x}, t, \vec{x}', t')$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d^3k'}{(2\pi)^{3/2}} e^{i(\vec{k}\cdot\vec{x} + \vec{k}'\cdot\vec{x}')}$$

$$\sum_{a,b} \eta_\mu(\vec{k}) \eta_\nu^{(b)}(\vec{k}')$$

$$\langle 0 | T(\hat{A}^{(a)}(\vec{k}, t) \hat{A}^{(b)}(\vec{k}', t')) | 0 \rangle.$$

rewrite $\hat{A}^{(a)}$ & $\hat{A}^{(b)}$ in terms of a & $a^{\dagger}$.

$$\langle 0 | \hat{A}(\vec{x},t)\, \hat{A}(\vec{x}',t') | 0 \rangle$$

$\longrightarrow$ probability amplitude of a photon $(\vec{x}',t') \to (\vec{x},t)$

~~$D_{\mu\nu}(\vec{x},t,\vec{x}',t')$~~

Recall for KG:-

$$\langle 0 | T\left(\hat{\phi}(\vec{x},t)\, \hat{\phi}(\vec{x}',t')\right) | 0 \rangle$$

$$= \delta^{(3)}(\vec{x}+\vec{x}') \int \frac{dk_0}{2\pi} e^{-ik_0(t-t')} \frac{i}{-k^2-m^2+i\epsilon}$$

Thus

$$\langle 0 | \hat{A}^{(a)}(\vec{x},t)\, \hat{A}^{(b)}(\vec{x}',t') | 0 \rangle$$

$$= \delta_{ab}\, \delta^{3}(\vec{x}+x') \int \frac{dk_0}{(2\pi)} \frac{e^{-ik_0(t-t')}\, i}{-k^2+i\epsilon},$$

$$D_{\mu\nu}(\vec{x},t,\vec{x}',t')$$

$$= \int \frac{d^3k\, dk_0}{(2\pi)^4} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \sum_a \hat{\eta}_{\mu}^{(a)}(\vec{k})\, \hat{\eta}_{\nu}^{(a)}(-\vec{k}) \frac{i}{-k^2+i\epsilon}$$

$\hat{\eta}_{\nu}^{(a)}(\vec{k})$

$$= \int \frac{d^3k\, dk_0}{(2\pi)^4} e^{ik\cdot(x-x')} \frac{i}{-k^2+i\epsilon}$$

$$\sum_a \hat{\eta}^{(a)}_\mu(\vec{k})\, \hat{\eta}^{(a)}_\nu(\vec{k}).$$

$$\delta^\mu_\nu = -\eta_\mu k^\nu + \frac{k_\mu k^\nu}{\vec{k}^2} + \sum_a \eta^a_\mu \eta^{(a)\nu}$$

$$\eta_{\mu\nu} = -\eta_\mu \eta_\nu + \frac{\vec{k}_\mu \vec{k}_\nu}{\vec{k}^2} + \sum_a \eta^a_\mu \eta^a_\nu,$$

$\longrightarrow$ we want to write it ~~in terms of~~ k as independent of $\epsilon^{(a)}$, $\epsilon^{(b)}$.

$$D_{\mu\nu}(\vec{x}, t, \vec{x}', t')$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-x')} \frac{i}{-k^2+i\epsilon} \left(\eta_{\mu\nu} + \eta_\mu \eta_\nu - \frac{\vec{k}_\mu \vec{k}_\nu}{\vec{k}^2}\right)$$

$\longrightarrow$ explicite ϵ dependence has gone away.

propagator for the photon

$$\bar{k} = (0, \vec{k})$$

$$n_\mu = (1, 0, 0, 0)$$

$$n \cdot \bar{k} = 0$$

$$\bar{k} = k$$

$$\bar{k} = k - k^0 (1, 0, 0, 0)$$

$$= k + (k \cdot n) n$$

Thus

$$D_{\mu\nu}(\vec{x}, t, \vec{x}', t')$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-x')} \frac{i}{-k^2 + i\epsilon}$$

$$\left[\eta_{\mu\nu} + n_\mu n_\nu - \frac{(k_\mu + (k\cdot n) n_\mu)(k_\nu + (k\cdot n) n_\nu)}{k^2} \right]$$

$$= \eta_{\mu\nu} + n_\mu n_\nu \left(1 - \frac{(k\cdot n)^2}{k^2}\right) - \frac{k_\mu k_\nu}{k^2}$$

$$- \frac{(k\cdot n)(k_\nu n_\mu + k_\mu n_\nu)}{k^2}$$

$$= \eta_{\mu\nu} + n_\mu n_\nu \frac{k^2}{\vec{k}^2} - \frac{\left(k_\mu k_\nu + \overset{(n\cdot k)}{\boxed{n_\mu k_\nu + n_\nu k_\mu}}\right)}{k^2}$$

$$D_{\mu\nu}(\vec{x}, t, \vec{x}', t')$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-x')} \left\{ \frac{\eta_{\mu\nu}}{-k^2+i\epsilon} - \frac{n_\mu n_\nu}{\vec{k}^2} - \frac{k_\mu k_\nu + \overset{(n\cdot k)}{\boxed{n_\mu k_\nu + n_\nu k_\mu}}}{\vec{k}^2(-k^2+i\epsilon)} \right\}$$

$i\epsilon \rightarrow$ helps us to pick the pole

A 4 vector eqn. is valid both as covariant & contravariant eqn.

$\rightarrow$ Instead of $\vec{k}$, k is used because when we are trying to calculate S matrix.

~~propagator of k~~

$k_\mu k_\nu + n_\mu k_\nu + n_\nu k_\mu$

$\longrightarrow$ vanishes

$\rightarrow$ we can use it more simply.

In the interaction theory $\frac{n_\mu n_\nu}{k^2}$ → gets cancelled.

To create a particle we have to start with momentum eigenstate

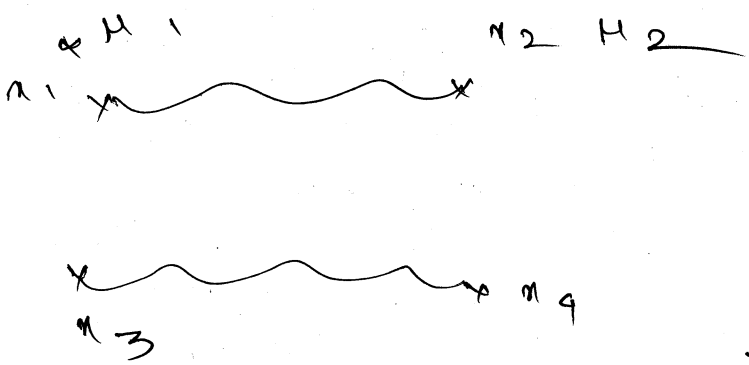
$D_{\mu\nu}$ is not same in all gauge.

But S matrix is independent of gauge choice. We have to specify μ, ν.

$$\langle 0 | T\left(A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n)\right) | 0 \rangle$$

$= D_{\mu\nu}(x, y)$.

To calculate it



$$D_{\mu_1\mu_2}(x_1, x_2)\, D_{\mu_3\mu_4}(x_3, x_4)$$
$$+ D_{\mu_1\mu_3}(x_1, x_3)\, D_{\mu_2\mu_4}(x_2, x_4)$$
$$+ D_{\mu_1\mu_4}(x_1, x_4)\, D_{\mu_2\mu_3}(x_2, x_3)$$

We have to keep track over μ,

exchange x, y & μ, ν simultaneously

Suppose we have a free Maxwell theory

+ free Dirac fermion

~~+ e.-fie~~

$$S = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi \right)$$

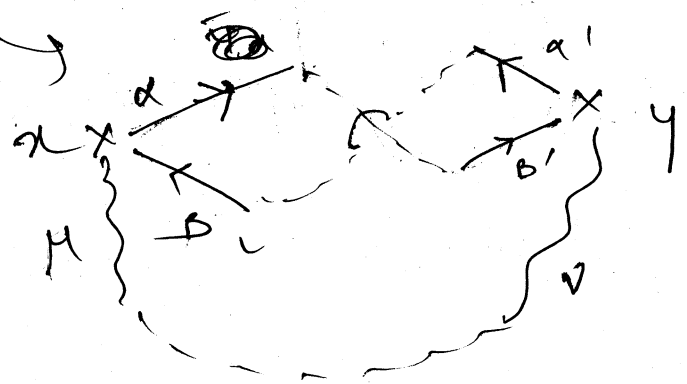
vacuum all b, a & a-annihilated

$$\langle 0 | T \left(: \bar{\psi}(x) \gamma^\mu A_\mu \psi(x) : : \bar{\psi}(y) \gamma^\nu A_\nu \psi(y) : \right) | 0 \rangle$$

$$= \langle 0 | T \left(: \bar{\psi}(x) \gamma^\mu A_\mu(x) \psi(x) : : \bar{\psi}(y) \gamma^\nu A_\nu(y) \psi(y) : \right) | 0 \rangle$$

$$\langle 0 | T \left(: \bar{\psi}^\alpha(x) \left(\gamma^\mu \right)_{\alpha\beta} A_\mu(x) \psi_\beta(x) : \right.$$

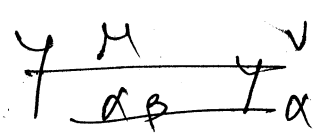
$$\left. : \bar{\psi}^{\alpha'}(y) \gamma^\nu_{\alpha'\beta'} A_\nu(y) \psi_{\beta'}(y) : \right) | 0 \rangle$$



$S_{F\alpha\beta}$

α refers to ψ

$$\langle 0 | \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle = S_{F\alpha\beta}(x, y)$$



$$\gamma^\mu_{\alpha\beta} \gamma^\nu_{\alpha'\beta'} S_{F\beta'\alpha}(y, x) S_{F\beta\alpha'}(x, y) D_{\mu\nu}(x, y).$$

$$\bar{\psi}_{\alpha'}(x)\,\psi_{\beta}(x)\,\bar{\psi}_{\alpha'}(y)\,\psi_{\beta'}(y)$$

L.H.S R.H.S $\longrightarrow$ count the no. of crossing.

(-1)

After counting no. of crossing see name to make sure that in each pair is left. ~~right~~ of $\bar{\psi}$

$$\langle 0|\,T\left(\psi_{\alpha}(x)\,\bar{\psi}_{\beta}(y)\right)|0\rangle$$

$$\langle 0|\,T\left(\bar{\psi}_{\beta}(y)\,\psi_{\alpha}(x)\right)|0\rangle$$

$$\cancel{\langle 0|} = -S_{F\alpha\beta}(x,y).$$

Lecture 7 21/04/2014

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi \right]$$

It describes a completely e.m. system.

neutral fermions

Now if we want to describe e^-.

coupling with e. magnetism

Eqn. of motion for A_μ

$$\partial_\mu F^{\mu\nu} = 0$$

If there is e^-.

then

$$\partial_\mu F^{\mu\nu} = -J^\mu \longrightarrow \text{4 vector current for charge.}$$

$\nu = 0$,

$$\partial_i F^{i0} = -J^0$$

$$\Rightarrow -\partial_i F_{i0} = -J^0$$

$$\Rightarrow -\nabla \cdot E = -\rho$$

$$\Rightarrow \nabla \cdot E = \rho.$$

$$E = \partial_i A_0 - \partial_0 A_i$$

Modify the action

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi + A_\mu J^\mu \right]$$

Assume J^μ doesn't depend on A^μ.

$$\frac{\delta S}{\delta A_\nu} = 0 .$$

$$\partial_\mu F^{\mu\nu} + J^\nu = 0 ,$$

$$J^\mu = \bar{\psi}\gamma^\mu\psi$$

$\bar{\psi}\gamma^\mu\psi$ = measures no. of e^-.

$$J^\mu = -e\bar{\psi}\gamma^\mu\psi \qquad (e > 0)$$

e → electron charge

→ consequence of Gauge Invariance.

Therefore we interpret F_{i0}/F^{i0} as electric field. ~~It matters~~

Extra term $= -e\int \bar{\psi}\gamma^\mu\psi A_\mu d^4x$

Eqn. of motion for fermion now will change.

$$\psi \longrightarrow e^{i\alpha}\psi .$$

⟹ is a symmetry of the action

$S\psi$ ~~to~~ will remain same

$$\frac{\delta L}{\delta(\partial_\mu \psi)} \delta\psi \longrightarrow \text{conserved current}$$

$\Longrightarrow$ J^μ is conserved.

But for K.G field their involve a term involving $\partial_\mu \psi$. Thus conserved current changes.

We should first check $\longrightarrow$ if it has gauge Invariance

$$\delta A_\mu = \partial_\mu \Lambda$$

$$\cancel{\partial_\mu \bar\psi \gamma^\mu \psi + \bar\psi \gamma^\mu \partial_\mu \psi}$$

$$\cancel{\delta \int d^4x}$$

$$\delta \cdot \int A_\mu J^\mu d^4x = \int d^4x \, J^\mu \partial_\mu \Lambda.$$

We will modify ψ, so that the additional gauge transformation will compensate the change.

$A_0 =$ ~~doesn't have any K terms~~

So the theory is not sensible.

$$\psi(x) = e^{-e\Lambda(x)} \psi$$

$$\bar\psi(x) = e^{ie\Lambda(x)} \bar\psi$$

$$\bar{\psi}\left(i\gamma^\mu \partial_\mu e^{-ie\Lambda(x)}\psi\right)$$

$$= \bar{\psi} i\gamma^\mu \left[-ie\,\partial_\mu\Lambda\,\psi e^{-ie\Lambda(x)} + e^{-ie\Lambda(x)}\partial_\mu\psi\right]$$

$$= e\bar{\psi}\gamma^\mu(\partial_\mu\Lambda)\psi e^{-ie\Lambda} + \bar{\psi} i\gamma^\mu\partial_\mu\psi$$

4:00 - 6:00

$$\delta\int A_\mu J^\mu d^4x = \int d^4x\, J^\mu \partial_\mu\Lambda$$

$$= \int d^4x\left(-e\bar{\psi}\gamma^\mu\psi\right)\partial_\mu\Lambda.$$

As A_0 has no time derivative. That can not be quantized using the standard Quantization procedure.

Gauge invariance is present. But gauge transformation laws are different

Let us consider a add another field $\longrightarrow -\frac{1}{2}\partial_\mu\phi\,\partial^\nu\phi + A_\mu\bar{\psi}\gamma^\mu\psi\phi^2$,

$$\partial_\mu F^{\mu\nu} + J^\nu = 0$$

$$\Rightarrow \quad \partial_\nu J^\nu = 0$$

But

$$J^\mu = A_\mu \bar{\psi} \gamma^\mu \psi \phi$$

$$\partial_\mu J^\mu \neq 0$$

Let us proceed with the same method.

Construct Position:

Treat A_i, ψ_α in the canonical formulation.

A_0 in the Lagrangian formulation.

$\cancel{\Rightarrow R(A_i, \phi_i, \pi}$ $\quad \psi$

$$R(A_i, \pi_i, A_0, \psi_\alpha, \pi_\alpha = i\psi_\alpha^*)$$

$$R(A_i, \pi_i, A_0, \psi_\alpha, \pi_\alpha = -i\psi_\alpha^*)$$

$$= \frac{1}{2}\int \sum_{i=1}^{3} d^3x \left[(\pi_i(x))^2 + B_i(x)^2 - 2A_0 \partial_i \pi_i - 2A_\mu J^\mu \right]$$

$+ H$ Dirac

$\rightarrow = \bar{\psi} \gamma_\mu \psi$

$\Psi \gamma^\mu \psi \longrightarrow$ is given ☺ It is same to Kl

We will treat A_μ's as classical field.

Fourier transform variable :-

$$A_i \,,\ \Pi_i \,,\ A_0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\tilde{A}_i \qquad \tilde{\Pi}_i \qquad \tilde{A}_0$$

$$J^\mu(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \tilde{J}^\mu(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}$$

Introduce $\hat{A}_{\parallel}$, $\hat{A}^{(a)}$ as before

$\tilde{\Pi}_{\parallel}$ & $\tilde{\Pi}^{(a)}$ in the same way

$\tilde{J}_{\parallel}$ & $\tilde{J}^{(a)}$ in "

$$\tilde{J}_{\parallel}(\vec{k},t) = \frac{i k_i}{|\vec{k}|} \tilde{J}^i(\vec{k},t)$$

$$\tilde{J}^{(a)}(\vec{k},t) = \epsilon_i^{(a)} \tilde{J}_i(\vec{k},t)$$

Ex :-

$$R = \int d^3k \left[\frac{1}{2} \sum_{a=1}^{2} \tilde{\Pi}^{(a)}(\vec{k},t)\, \tilde{\Pi}^{(a)}(-\vec{k},t) \right.$$

$$+ \frac{1}{2}|\vec{k}|^2 \sum_{a=1}^{2} \tilde{A}^{(a)}(\vec{k},t)\, \tilde{A}^{(a)}(-\vec{k},t)$$

$$+ \frac{1}{2} \tilde{\Pi}_{\parallel}(\vec{k},t)\, \tilde{\Pi}_{\parallel}(-\vec{k},t)$$

$$- \tilde{A}_0(-\vec{k},t)\, |\vec{k}|\, \tilde{\Pi}_{\parallel}(\vec{k},t)$$

$$- \tilde{A}^{(a)}(-\vec{k},t)\, \tilde{J}^{(a)}(\vec{k},t) - \tilde{A}_{\parallel}(-\vec{k},t)\, \tilde{J}_{\parallel}(\vec{k},t)$$

$- \hat{A}_0(-\vec{k},t)\,\hat{J}^0(\vec{k},t)\Big] + H_{Dirac}$

find eqn. of motion $\hat{A}_0$, $\hat{A}_{\parallel}$ & $\hat{\pi}_{\parallel}$.

Coulomb gauge cond.

$\hat{A}_0$ is not determined.

$\nabla \cdot \hat{A} = 0$, $\hat{A}_{\parallel} = 0$.

$\hat{A}_0$ eq: $\dfrac{\delta R}{\delta \hat{A}_0(-\vec{k},t)} = 0$

$$-i|\vec{k}|\,\hat{\pi}_{\parallel}(\vec{k},t) - \hat{J}^0 = 0$$

$$\Longrightarrow \hat{J}^0(\vec{k},t) = -i|\vec{k}|\,\hat{\pi}_{\parallel}(\vec{k},t)$$

$$\frac{d\hat{A}_{\parallel}(\vec{k},t)}{dt} = \Longrightarrow \left\{\hat{A}_{\parallel}(\vec{k},t), R\right\}_{PB}$$

$$= -|\vec{k}|\,\hat{A}_0(\vec{k},t) + \hat{\pi}_{\parallel}(\vec{k},t).$$

$\hat{\pi}_{\parallel}$ eq.

$$\frac{d\hat{\pi}_{\parallel}(\vec{k},t)}{dt} = \left\{\hat{\pi}_{\parallel}(\vec{k},t), R\right\}$$

$$= J_{\parallel}(\vec{k},t),$$

$$\hat{\pi}_{11} = -\frac{\hat{J}^0(\vec{k},t)}{|\vec{k}|}$$

$$\frac{\partial \hat{A}_{11}}{\partial t} = 0 = -|\vec{k}|\,\hat{A}_0(\vec{k},t) + \hat{\pi}_{11}(\vec{k},t)$$

$$\hat{A}_0(\vec{k},t) = \frac{1}{|\vec{k}|}\,\hat{\pi}_{11}(\vec{k},t)$$

$$= \frac{-1}{|\vec{k}|^2}\,\hat{J}^0(\vec{k},t).$$

In the earlier case

$$\hat{A}_0(\vec{k},t) = -\frac{\hat{J}^0(\vec{k},t)}{k^2}$$

In case of free electromagnetism

$$\nabla\cdot\vec{A} = 0, \quad \hat{A}_0 = 0.$$

But in this case

$$\nabla\cdot\vec{A} = 0 \Rightarrow \hat{A}_0 = -\frac{\hat{J}_0(\vec{k},t)}{k^2}$$

$$\Rightarrow -\frac{1}{|\vec{k}|}\,\partial_0\hat{J}^0(\vec{k},t) = \hat{J}_{11}(\vec{k},t)$$

$$\Rightarrow \partial_0\hat{J}^0(\vec{k},t) + |\vec{k}|\,\hat{J}_{11}(\vec{k},t) = 0$$

In position space

$$\partial_0 J^0 + \nabla\cdot\vec{J} = 0$$

As long as 4 satisfy eqn. of motion.

$$\partial_0 \hat{J}^0 + |\vec{k}| \hat{J}_{\parallel} = 0$$

So now we will eliminate

$\hat{J}^0$, $\hat{J}^{\parallel}$ & $\hat{A}_0$.

Thus we will set the values of $\hat{J}^0$, $\hat{J}^{\parallel}$, $\hat{A}_0$

Redefine fields

$$\hat{B}_0 = \hat{A}_0 + \frac{1}{k^2}\hat{J}^0 = 0$$

$$\hat{B}_{\parallel} = \hat{A}_{\parallel} = 0$$

$$\hat{B}^{(a)} = \hat{A}^{(a)}$$

Just substitute them & renormalize.

Ex:-

$$R = \int d^3k \left[\frac{1}{2} \sum_{a=1}^{2} \left\{ \hat{\pi}^{(a)}(-\vec{k},t)\, \hat{\pi}^{(a)}(\vec{k},t) + \frac{1}{2}|\vec{k}|^2 \hat{B}^{(a)}(-\vec{k},t)\, \hat{B}^{(a)}(\vec{k},t) \right\} + \frac{1}{2}\hat{J}^0(-\vec{k},t)\frac{\hat{J}^0(\vec{k},t)}{k^2} - \hat{B}^{(a)}(-\vec{k},t)\, \hat{J}^{(a)}(\vec{k},t) \right]$$

hang mang

Columns in (input + output + 1 dim a.s)

Now we can work with this. Instead of calling it ~~Lagrangian~~ Hamiltonian we can call it.

$$R_{free} + R_{int}$$

$$R_{free} = \int d^3k \left[\frac{1}{2} \sum_{a=1}^{2} \tilde{\pi}^{(a)}(-\vec{k},t)\tilde{\pi}^{(a)}(\vec{k},t) + \frac{1}{2} k^2 \tilde{B}^{(a)}(-\vec{k},t)\tilde{B}^{(a)}(\vec{k},t) \right]$$

$$+ H_{Dirac}$$

$$= \int d^3k \left[\frac{1}{2} \sum_{a=1}^{2} \tilde{\pi}^{(a)}(-\vec{k},t)\tilde{\pi}^{(a)}(\vec{k},t) + \frac{1}{2} k^2 \tilde{B}^{(a)}(-\vec{k},t)\tilde{B}^{(a)}(\vec{k},t) \right]$$

$$+ H_{Dirac}$$

$\frac{M_{\mu\nu}}{\vec{k}^2} \longrightarrow$ In the photon propagator it cancels $\frac{\tilde{J}^{(0)}(\vec{k},t)\,\tilde{J}^{0}(-\vec{k},t)}{|\vec{k}|^2}$.

$$R_{int} = \int d^3k \left[\frac{1}{2} \tilde{J}^{0}(-\vec{k},t)\frac{1}{\vec{k}^2}\tilde{J}^{0}(\vec{k},t) - \tilde{B}_{\mu}(-\vec{k},t)\tilde{J}^{\mu}(\vec{k},t) \right]$$

Let us calculate ~~T(opera~~ time order product of operators.

Suppose we want to calculate

$$\langle \Omega | T \left(\prod_i A_{\mu i}(x_i) \prod_j \psi_{\alpha j}(y_j) \prod_k \bar{\psi}_{\beta k}(w_k) \right) | \Omega \rangle$$

We will follow exactly same procedure.

$$\frac{\langle 0 | T \left(\prod_i A_{\mu i}^{I}(x_i) \prod_j \psi_{\alpha j}^{I}(y_j) \prod_k \bar{\psi}_{\beta k}^{I}(w_k) \exp\left(-i \int R_{int}^{I} d^4x\right) \right) | 0 \rangle}{\langle 0 | T \exp\left(-i \int R_{int}^{I} d^4x\right) | 0 \rangle}$$

$\longrightarrow$ same as in ϕ^4 case.

Lecture - 8. 27/04/2011

If there is a pole in two pt. func:
- single particle state.
when the wavefnc. start overlapping
effect of interaction gets switched
on

$|in\rangle$, $|out\rangle$ → form a complete basis.

it is a superposition of many in state.

A_μ, ψ_α - we started with

$\hat{A}_\mu$, $\hat{\psi}_\alpha$

$J^\mu = -e\bar{\psi}\gamma^\mu\psi$

$\hat{J}^\mu =$

→ Fourier transform → we will work with $\hat{B}$.

$$\bar{B}_0 = \hat{A}_0 + \frac{\hat{J}^0}{\vec{k}^2}, \quad \bar{B}_{\parallel} = \hat{A}_{\parallel}$$

$$\& \ \bar{B}(a) = \hat{J}(a)$$

Eqn. of motion for $\hat{B}_{\parallel}, \hat{\pi}_{\parallel}, \bar{B}_0$

$\Longrightarrow \bar{B}_0 = 0$.

$\bar{B}_{\parallel} = 0, \ \hat{\pi}_{\parallel} = 0$.

When we consider interaction

$\Rightarrow$

$$R = R_{free} + R_{int}.$$

$$R_{free} = \int d^3k \left[\frac{1}{2} \sum_{a=1}^{2} \hat{\pi}^{(a)}(\vec{k},t)\, \hat{\pi}^{(a)}(-\vec{k},t) + \vec{k}^2 \sum_{a=1}^{2} \hat{B}^{(a)}(\vec{k},t)\, \hat{B}^{(a)}(-\vec{k},t) + \bar{\psi}(-\vec{k},t)\,(\gamma^i k_i + m)\,\hat{\psi}(\vec{k},t) \right] \rightarrow \text{Quadratic}$$

$\rightarrow$ free Dirac Hamiltonian.

$$R_{int} = \int d^3k \left[\frac{1}{2} \hat{J}^{(0)}(-\vec{k},t) \frac{1}{\vec{k}^2} \hat{J}^0(\vec{k},t) - \bar{B}_\mu(-\vec{k},t)\, \hat{J}^\mu(\vec{k},t) \right]$$

$\rightarrow$ no distinction between Routhian & Hamiltonian

$\tilde{B}_{11} = 0$

$\tilde{B}_0 = 0.$

$\rightarrow$ Cubic

$\rightarrow$ free e.m. field + free Dirac field

We want to study the vacuum expectation value?

Q:- How to calculate :-

$$\langle \Omega | T \left(\prod_{i=1}^{n} B_{\mu_i}(x_i) \prod_{j=1}^{m} \psi_{\alpha_j}(y_j) \prod_{k=1}^{m} \bar{\psi}_{\beta_k}(z_k) \right) | \Omega \rangle$$

unequal no. of $\bar{\psi}$ will give me 0,

no. of ψ = no. of $\bar{\psi}$

~~A~~ ~~If~~ Q is ~~the~~ generator ~~of the~~ symmetry

$$\langle \Omega | A | \Omega \rangle$$

$$\langle \Omega | [Q, A] | \Omega \rangle = 0 .$$

$$Q = \int \bar{\psi} \gamma^0 \psi \, d^3x .$$

$$Q|\psi\rangle = -|\psi\rangle$$

$$\downarrow$$

$$\psi^+ \psi .$$

$$\langle \psi | Q$$

$$[Q, A] = qA$$

$\longrightarrow$ no. of $\bar{\psi}$ − no. of ψ.

$$q \neq 0$$

$$q \langle \Omega | A | \Omega \rangle = 0 \Longrightarrow \langle \Omega | A | \Omega \rangle = 0 ,$$

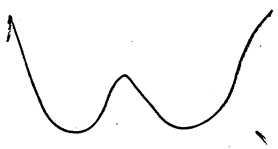
$$[Q, A_1 A_2 A_3] \qquad [Q, A] = qA$$

$$= [Q, A_1] A_2 A_3 + A_1 [Q, A_2] A_3 + A_1 A_2 [Q, A_3]$$

$$= q A_1 A_2 A_3 + q A_1 A_2 A_3 + q A_1 A_2 A_3 .$$

unless q ~~is~~ 0, ~~a~~ $\langle \Omega | A | \Omega \rangle$ must ~~wa~~ vanish.

~~If~~ vacuum doesn't annihilates q.



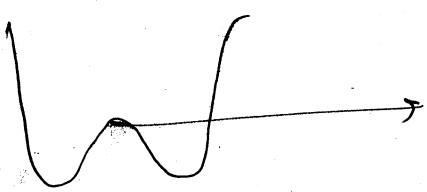
in Q.M, ~~the~~ ground state is ~~superposition of~~ two

of two ground state.

$$H = \begin{pmatrix} \langle \psi_1 | H | \psi_1 \rangle & \overset{0}{\overset{\shortparallel}{\langle \psi_1 | H | \psi_2 \rangle}} \\ \underset{0}{\underset{\shortparallel}{\langle \psi_2 | H | \psi_1 \rangle}} & \langle \psi_2 | H | \psi_2 \rangle \end{pmatrix}$$

Parity operation doesn't leave the system invariant.

In $\phi^4 \rightarrow H = \int d^3x \left[K.E + \phi^4 \right]$.



energy $= \infty$. (infinite vol. integral)

If the symmetry is spontaneously broken, then we will get the charge which is not conserved.

vacuum is not invariant under symmetry.

$$\langle \Omega | T \left(\prod_{i=1}^{n} B_{\mu_i}(x_i) \prod_{j=1}^{m} \psi_{\alpha_j}(y_j) \prod_{k=1}^{m} \overline{\psi}_{\beta_k}(z_k) \right) | \Omega \rangle$$

$$= \langle 0 | T \left(\prod_{i=1}^{n} B^{I}_{\mu_i}(x_i) \prod_{j=1}^{m} \psi^{I}_{\alpha_j}(y_j) \right)$$

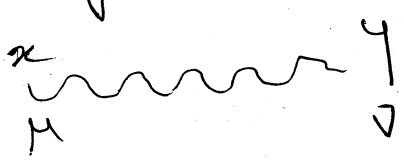
$$\prod_{k=1}^{m} \bar{\psi}^{I}_{Bk}(x_k) \exp\left(-i \int R^{I}_{int}(t')\,dt'\right)|0\rangle$$

$$\langle 0 | \exp\left(-i \int R^{I}_{int}(t')\,dt'\right)|0\rangle$$

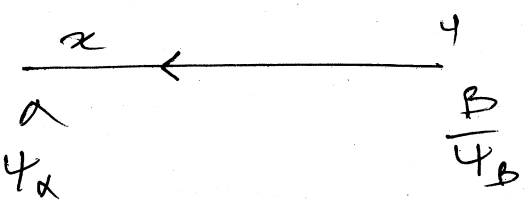
→ vacuum of the free field.

→ expand it & calculate it in terms of Feynmann diagrams.

Feynmann rules :-



$D_{\mu\nu}(x,y)$.

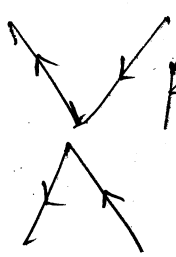


$S_{F\alpha\beta}(x,y)$.

$J^0 J^0 \longrightarrow$ two $\bar{\psi}$'s & two ψ's.

from 4 vertex. 2 $\bar{\psi}$ will come out & two will go in.

In position space



$$R_{int} = \frac{1}{2}\int \frac{d^3x}{(2\pi)^{3/2}} \int \frac{d^3y}{(2\pi)^{3/2}} \int e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \frac{d^3k}{k^2}$$

$$-i \int dt\, R_{int}(t)$$

$J^0(\vec{x},t)\, J^0(\vec{y},t)$

$$\frac{-i}{2}\int \frac{d^4x\, d^4y}{(2\pi)^4}\int \frac{d^4k}{k^2}\, e^{ik\cdot(x-y)}\, J^0(x)J^0(y)$$

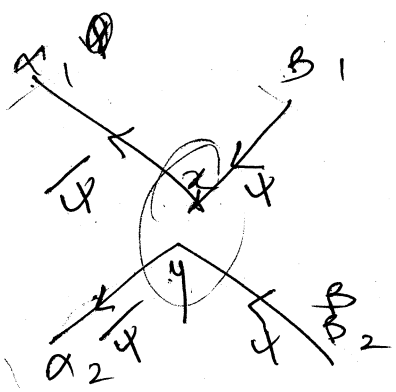
$$\delta(t-t')$$

$$e^2$$

$$\bar\psi_{\alpha_1}\gamma^0\psi_{\beta_1}\,\bar\psi_{\alpha_2}\gamma^0\psi_{\beta_2}$$

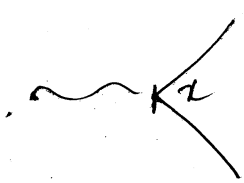
$$S_{F\alpha_1\beta_1}(x,x)$$

$$S_{F\alpha_2\beta_2}(y,y)$$



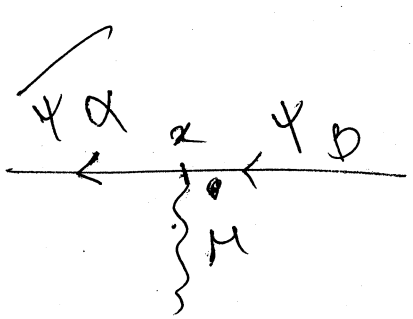
one which is going out is $\bar\psi$ & ~~goin~~ coming in ψ.

$$\frac{-i}{2}\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2}\, e^{ik(x-y)}\,\gamma^0_{\alpha_1\beta_1}\,\gamma_{\alpha_2\beta_2}$$



$$-i\int dt\, R_{int}^{(2)}$$

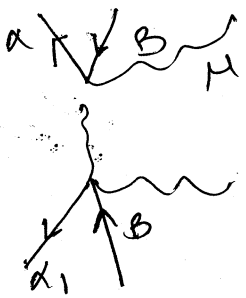
$$= +i\int d^4x\, B_\mu(x)\, J^\mu(x)$$



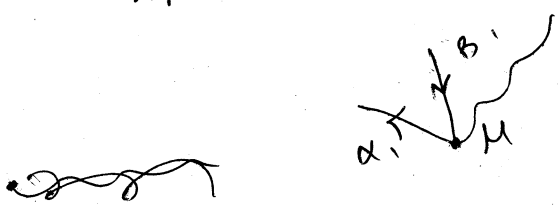
$$-e\,\bar\psi_\alpha\,\gamma_{\alpha\beta}\,\psi_\beta$$

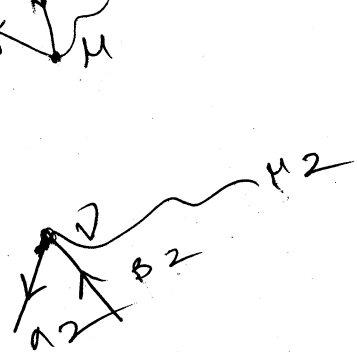
$-ei\,\gamma^{\mu}_{\alpha\beta}$

$-i\int d^4x\, d^4$



$(-i)\,e^2\,\gamma^{\mu_1}_{\alpha_1\beta_1}\,\gamma^{\mu_2}_{\alpha_2\beta_2}\,f(x-y)$





~~first we have to fix the center.~~

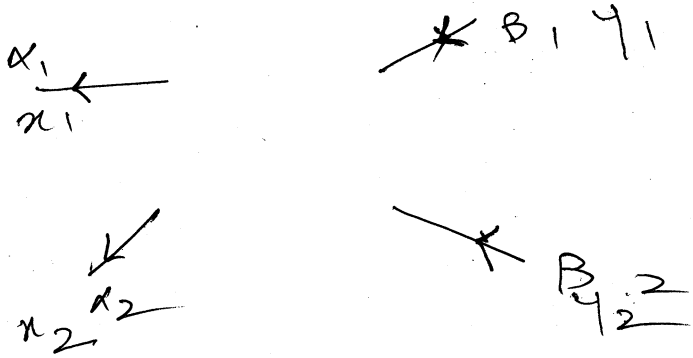
~~Then the indices~~ ~~must be~~ contracted.

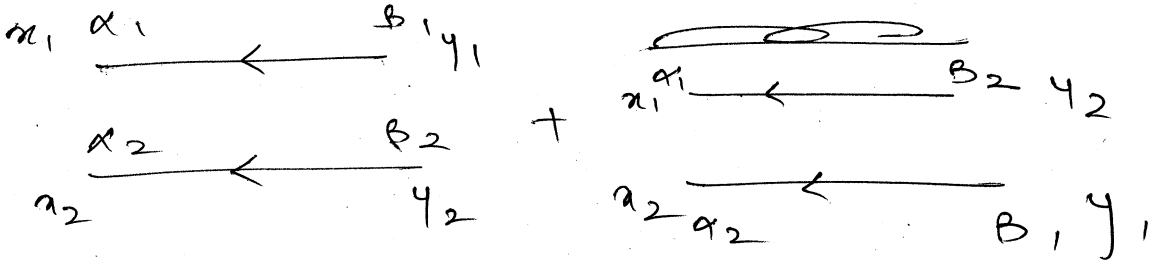
Effect of the denominator :

bubbles have to be excluded.

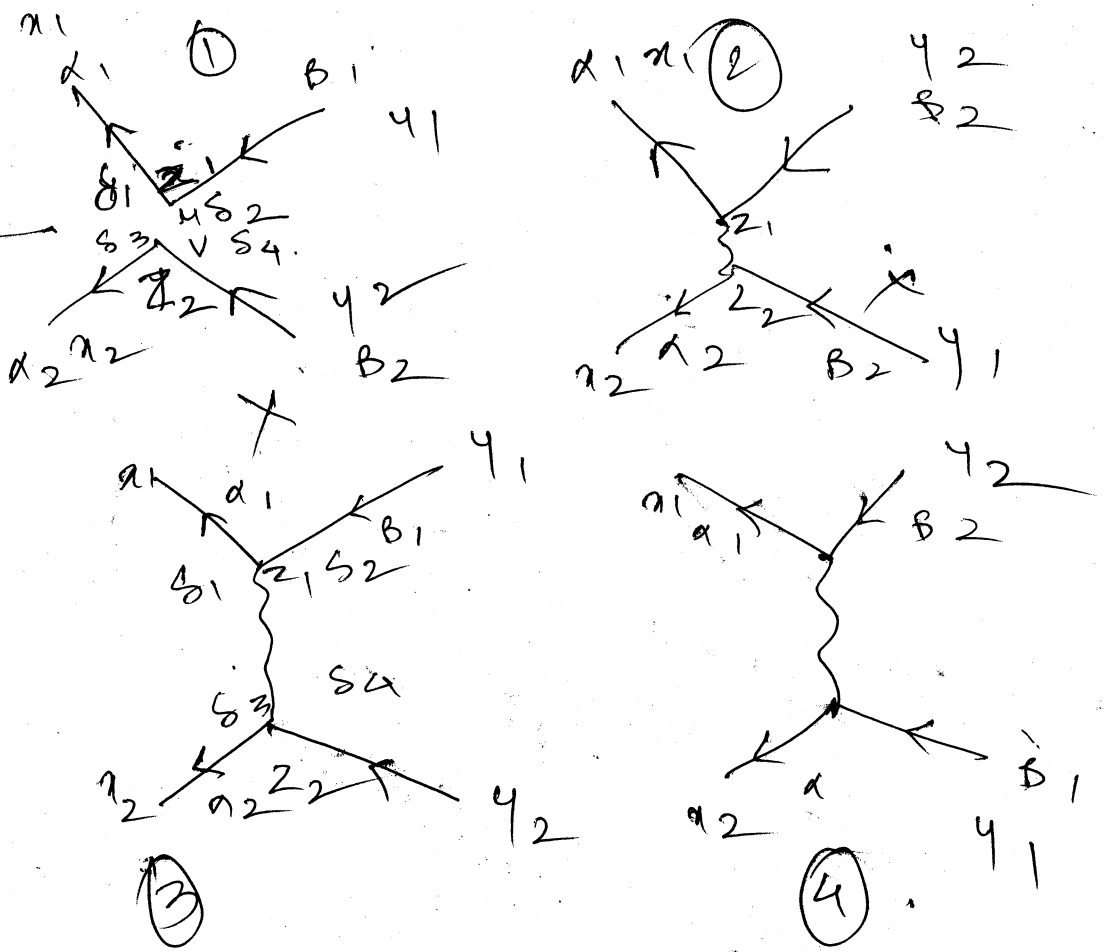
$\longrightarrow$ ignore bubbles

$$\langle\Omega|T\left(\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\bar{\psi}_{\beta_1}(y_1)\bar{\psi}_{\beta_2}(y_2)\right)|\Omega\rangle$$





$$\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\,\bar{\psi}_{\beta_1}(y_1)\,\bar{\psi}_{\beta_2}(y_2)$$



$$\int d^4z_1 \int d^4z_2\; S_{F\,\alpha_1\delta_1}(x_1,z_1)\, S_{F\,\delta_2\beta_1}(z_1,y_1)$$

$$S_{F\,\alpha_2\delta_3}(x_2,z_2)\, S_{F\,\delta_4\beta_2}(z_2,y_2).$$

$$-\frac{ie^2}{2}\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2}\, e^{ik(z_1-z_2)}\, \gamma^0_{\delta_1\delta_2}\, \gamma^0_{\delta_3\delta_4}$$

② exchange β_1 β_2

$$\frac{1}{2!}\left[\left(\frac{-ie^2}{2}\right)\int\frac{d^4}{(2\pi)^4}\frac{1}{\vec{k}^2}e^{ik(z_1-z_2)}(\gamma^0)_{\delta_1\delta_2}(\gamma^0)_{\delta_3\delta_4}\right.$$

$$\left.+\frac{1}{2}D_{\mu\nu}(z_1-z_2)(-ie)^2\gamma^\mu_{\delta_1\delta_2}\gamma^\nu_{\delta_3\delta_4}\right]$$

$$D_{\mu\nu}(z_1-z_2)$$

$$=\int\frac{d^4k}{(2\pi)^4}e^{ik(z_1z_2)}$$

$$\left[\frac{i}{-k^2+i\epsilon}\left\{g_{\mu\nu}-\frac{k_\mu k_\nu+k_\mu(n\cdot k)n_\mu+k_\nu(n\cdot k)n_\mu}{\vec{k}^2}\right\}\right.$$

$$\left.-\frac{i}{\vec{k}^2}n_\mu n_\nu\right]$$

$n_\nu\gamma^\nu$ & $n_\mu\gamma^\mu$,

this term cancels

we have a term that looks like action at a dist. this is because we have chosen Coulomb gauge.

But last of all it should be Lorentz invariant. thus in the $D_{\mu\nu}$ actually it cancels the contribution of $n_\mu n_\nu$ (from 1).

General result:- we can drop the vertex V and the $n_\mu n_\nu$ terms from internal propagates in all Feynmann diagrams.

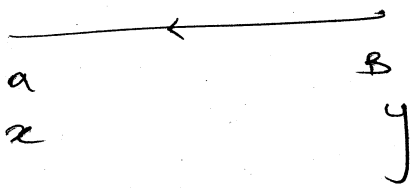
$$T(AB)\,T(CD)$$

$$T(ABCD)$$

$$T(\Phi_A\,\Phi_B\,.$$

28/04/2011

New Feynman rules :-



$S_{F\alpha B}(x,y)$

$D_{\mu\nu}(x,y)$ for external propagator.

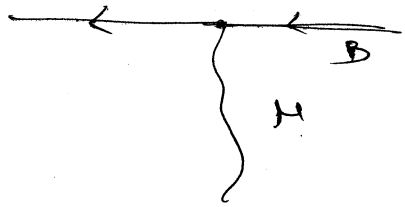
$D_{\mu\nu}'(x,y)$ for internal propagator

$$D_{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)} \left[\eta_{\mu\nu} - \frac{k_\mu k_\nu + (n\cdot k)(k_\nu n_\mu + k_\mu n_\nu)}{\vec{k}^2} \right\} \frac{i}{-k^2+i\epsilon}$$

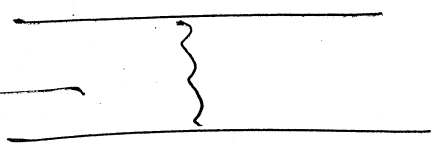
$$- \frac{i\, n_\mu n_\nu}{\vec{k}^2} \Bigg]$$

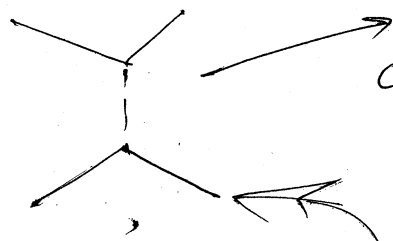
$$D'_{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-y)} \frac{i}{-k^2+i\epsilon} \left\{ \eta_{\mu\nu} - \frac{k_\mu k_\nu + (n\cdot k)(k_\mu n_\nu + k_\nu n_\mu)}{\vec{k}^2} \right.$$

Vertex :-



$-ie\gamma^{\mu}_{\alpha\beta}$.





I will not consider this graph.

effect of this vertex is to cancel it

$$\int \bar{\psi}\gamma^{\mu}\psi(\vec{x},t)\,\bar{\psi}\gamma_{\mu}\psi(\vec{y},t)\,d^3x\,d^3y.$$

interaction of this kind is action at a distance

$D_{\mu\nu}$ → instantaneous action of the propagator.

Any type of propagator $\propto k_{\mu}$.

→ doesn't contribute to the S-matrix.

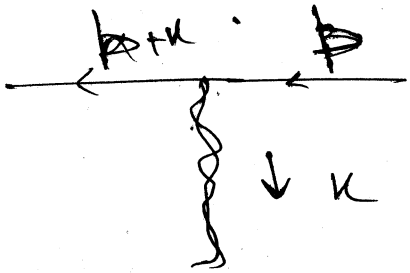
Especially for S-matrix calculation

we will use this.

$\longrightarrow$ Feynman. gauge

for S - matrix calculation we will use

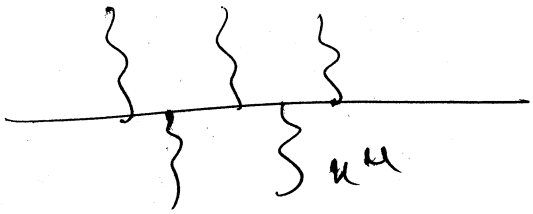
$$D_{\mu\nu}'(x,y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{i}{-k^2+i\epsilon}$$



$$\bar{u}(p+k)(-ie\gamma^\mu k_\mu)u(p)$$

$$\bar{u}(p+k)(p^\mu + k^\mu)\gamma_\mu - m \to 0.$$

$$-(p^\mu \gamma_\mu - m)u(p) \to 0$$



$\longrightarrow$ terms cancel pair wise

we can put the gauge cond. beforehand. $k^\mu A_\mu = 0$.

- Average over μ.

$e^{-\frac{1}{2\alpha}b^2} \longrightarrow$ weight factor.

for a specific value α).

Averaging over the many gauge choice.

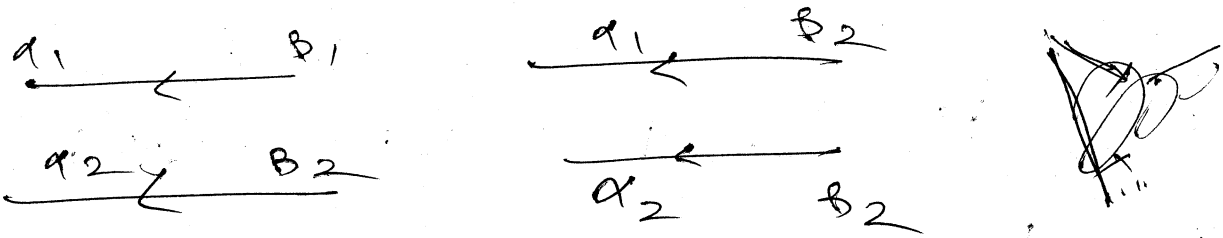
S matrix independent of the gauge choice. ~~As for~~ Green's

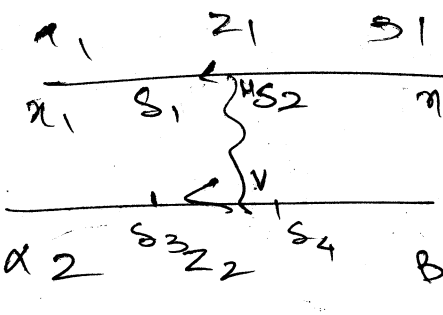
~~Gauge depend upon gauge choice.~~

Physical process $\sum_n \frac{|\langle 0|\mathcal{H}_1|n\rangle|^2}{E_0 - E_n}$

$\cancel{\langle \psi_{\alpha_1}(x_1)\, \psi_{\alpha_2}}$

$$\langle \Omega | T\left(\psi_{\alpha_1}(x_1)\, \psi_{\alpha_2}(x_2)\, \bar{\psi}_{B_1}(y_1)\, \bar{\psi}_{B_2}(y_2)\right) | \Omega \rangle$$





$+ \; B_1 \leftrightarrow B_2$

exchange y_1, y_2, B_1, B_2

check in $\alpha_1 B_1$ comp. of $S_F \gamma_\mu S_F$ we can write it as matrix prod

$$\frac{2\times}{2!} \; S_{F\,\alpha_1 B_1}(x_1, z_1)\; S_{F\,S_2 B_1}(z_2, x_2)$$

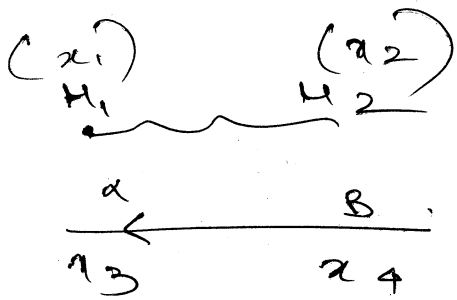
$$S_{F\,\alpha_2 S_3}(y_1, z_2)\; S_{F\,S_4 B_2}(z_2, y_2)$$

$$D'_{\mu\nu}(z_1, z_1)\,(-ie\,\gamma^\mu_{S_1 S_2})\,(-ie\,\gamma^\nu_{S_3 S_4}).$$

First ψ can go into two vertices and then the 2nd ψ can go to one vertex. But $\bar{\psi}$ can go into 2 vertex.

Out of this 4 diagrams, two diagrams are same

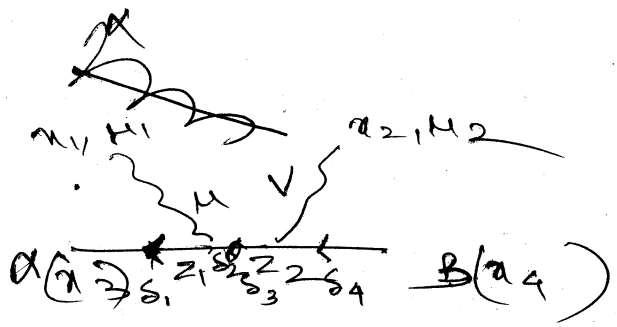
$$\langle \Omega | T\left(B_{\mu_1}(x_1)\, B_{\mu_2}(x_2)\, \psi_\alpha(x_3)\, \bar{\psi}_\beta(x_4) \right) | \Omega \rangle$$

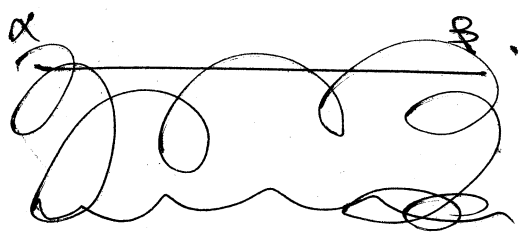


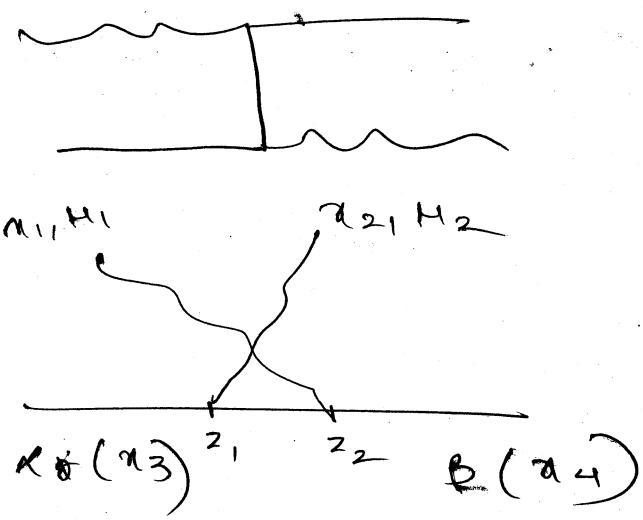
$D_{\mu_1 \mu_2}(x_1, x_2)$

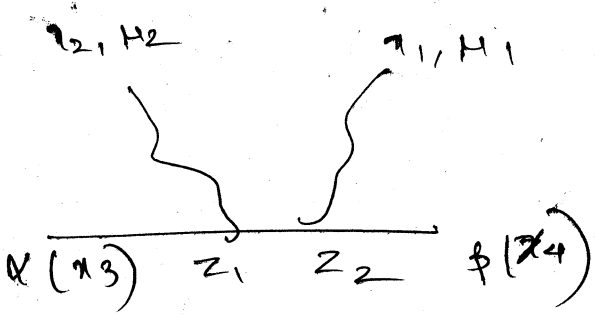
$S_{F\alpha\beta}(x_3, x_4)$.

Fermion lines can go straight









exchanging.

~~x_3, x_4~~

x_1, x_2

μ_1, μ_2.

$$\frac{1}{2! \cdot 2!} S_{F\alpha\delta_1}(x_3, z_1)\, S_{F\,\delta_2\delta_3}(z_1, z_2)\, S_{F\,\delta_4\beta}(z_2, x_4)\, D_{\mu_1\mu}(x_1, z_1)\, D_{\mu_2\nu}(x_2, z_2)\, (-ie)(\gamma^\mu)_{\delta_1\delta_2}\, (-ie)\, \gamma^\mu_{\delta_3\delta_4} \longrightarrow$$

Sign factor

$\cancel{\psi_\alpha(x_3)\,\bar\psi_\beta(x_4)\,\bar\psi(z_1)\,\psi^\mu\,\psi(z_2)}$

$\psi_\alpha(x_3)\,\bar\psi_\beta(x_4)\,\bar\psi\,\psi^\mu\,\psi(z_1)$

$\bar\psi\,\psi^\mu\,\psi(z_2)$

$$\langle 0|T\{\psi_\alpha(x_3)\,\bar\psi_\beta(x_4)\,\bar\psi\,\psi^\mu\,\psi(z_1)\,\bar\psi\,\psi^\mu\,\psi(z_2)\}|0\rangle$$

For

$$\langle 0|T\{\psi_\alpha(x_3)\,\bar\psi_\beta(x_4)\,\bar\psi\,\psi^\mu\,\psi(z_1)\,\bar\psi\,\psi^\mu\,\psi(z_2)\}|0\rangle$$

(1 crossing) $\longrightarrow$ one sign.

~~Sign~~ $\bar\psi$ first $\longrightarrow$ do - sign.

Momentum Space Feynman's rule :-

" " " "

useful, ~~give~~ need directly in Δ_F,

$\hat{G}(n\,m)\,(k_1, \ldots, k_n, p_1, \ldots p_m, l_1, \ldots l_m)$

$$\langle \Omega | T \left(\prod_{i=1}^{n} B_{\mu i}(x_i) \prod_{j=1}^{m} \psi_{\alpha j}(y_j) \prod_{k=1}^{m} \bar{\psi}_{\beta k}(z_k) \right) | \Omega \rangle$$

$$= \int \prod_{i=1}^{n} \frac{d^4 k_i}{(2\pi)^4} \prod_{j=1}^{m} \frac{d^4 p_j}{(2\pi)^4} \prod_{k=1}^{m} \frac{d^4 q_k}{(2\pi)^4} \, e^{i\left(\sum_i k_i x_i + \sum_j p_j y_j + \sum_k q_k z_k\right)} \cdot \vec{G}^{(n,m)}(k_1, \ldots k_n; p_1, \ldots p_m; q_1, \ldots q_m)$$

How to Calculate.

q_m, q_2, q_1 — m external ψ

k_1, k_2, k_n — n external photons.

p_1, p_2, p_m — m external ψ's

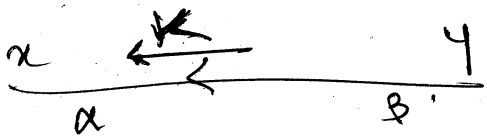
dim. of ψ's.

dim. of mom. entering the vertex.

$$S_{F\,\alpha\beta}(x,y) = \int \frac{d^4 k}{(2\pi)^4} \, e^{ik\cdot(x-y)} \frac{i}{-k^2 - m^2 + i\epsilon} (\not{k} + m)_{\alpha\beta}$$

$$\left(\frac{i}{\not{k} - m + i\epsilon} \right)_{\alpha\beta}$$

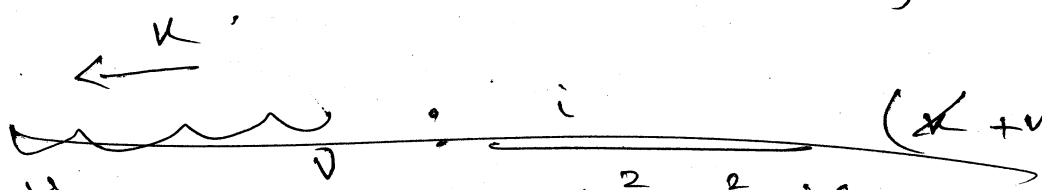
$$(\not{k} + m)(\not{k} - m) = -k^2 - m$$



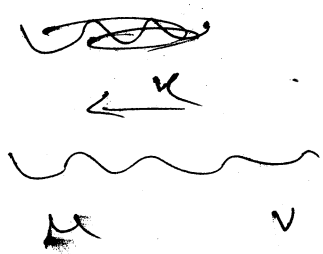
$$: \frac{i}{-k^2-m^2+i\epsilon}\left(\not{k}+m\right)_{\alpha\beta}.$$

$e^{ik\cdot x} \longrightarrow$ k entering x

$e^{-k\cdot y} \longrightarrow$ k leaving y.



$$: \frac{i}{-k^2-m^2+i\epsilon}\left(\not{k}+m\right)_{\alpha\beta}$$

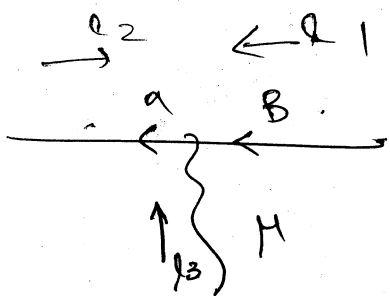


$$: \underbrace{\frac{i}{-k^2+i\epsilon}\left(\eta^{\mu\nu} - k_\mu k_\nu + n\cdot k\,\frac{(n_\mu k_\nu + n_\nu k_\mu)}{k^2}\right)}_{\hat{D}'_{\mu\nu}}$$

$$\hat{D}_{\mu\nu}$$

$$\rightarrow -i\,\frac{\eta_{\mu\nu}}{\vec{k}^2}$$

$\longrightarrow$ only for external legs.



$$\gamma_{\alpha\beta}^{\mu}\,(2\pi)^4\,\delta^{(4)}(l_1+l_2+l_3)$$

every internal propagator with momentum l see same.

$\int \frac{d^4 l}{(2\pi)^4}$ → Take the expression for Green's func & take Fourier transform

line of k & l same if opp then $k \to -k$

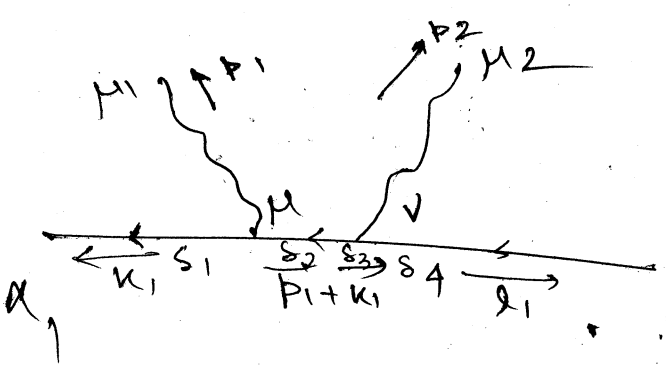
use the mom. conservation at the vertices to cancel the momentum

→ net effect of δ^4 & $\int$

(a) $(2\pi)^4 \delta^{(4)}\left(\sum p_i + \sum k_i + \sum l_i\right)$

(b) every time there is a loop! there is an integration left over

$\int \frac{d^4 l}{(2\pi)^4}$ for every loop.



$(2\pi)^4 (p_1+p_2+k_1+l_1)$

$$\frac{i(\not{k}_1+m)_{\alpha_1\delta_1}}{-k_1^2-m^2+i\epsilon}$$

$$\cdot\, B_1\, \tilde{D}_{\mu_1\mu}(p_1)\, \tilde{D}_{\mu_2\nu}(p_2)$$

$$\frac{i(\not{p}_1+\not{k}+m)_{\delta_2\delta_3}}{-(\not{p}_1+\not{k}_1)-m^2+i\epsilon}$$

$$\frac{i}{-l_1^2-m^2+i\epsilon}(-\not{l}_1+m)_{\delta_4}\, B_1\, (-il\gamma^\mu{}_{\delta_1\delta_2})(-il\gamma^\nu{}_{\delta_3\delta_4})$$

$\times \frac{1}{2} \times 2 (+1)$

~~S matrix for~~

How to rewrite S matrix in terms of Green's func.

$\Delta_F(x,y) =$

$\frac{1}{\sqrt{x}} \longrightarrow$ diverges at $x=0$.

$\longrightarrow \int \frac{dx}{\sqrt{x}}$ - finite.

$\longrightarrow$ if integral diverges.

$\longrightarrow$ ultraviolet.

$x \to y$ (short range)

loop divergent.

$\int \frac{d^4p}{p^2} \longrightarrow$ finite

$p \to \infty$.

29/04/2011

How to find S-matrix in terms of $\langle 0 | T(\phi_I(x)\,\phi_I(y)) | 0 \rangle$

Derivation follows similarly acc. to ϕ^4 but with more indices.

Momentum space Feynman propagator

$$\frac{i}{-k^2 - m^2 + i\epsilon}(\not{k} + m)_{\alpha\beta},$$

$$\not{k}\not{k} = \frac{1}{2}\left[k_\mu k_\nu \gamma^\mu \gamma^\nu + k_\nu k_\mu \gamma^\nu \gamma^\mu\right]$$

$$= \frac{i}{\not{k} - m + i\epsilon}$$

$$(\not{k} + m)(\not{k} - m) = \gamma^\mu \underline{k_\mu} \gamma^\nu \underline{k_\nu} - m^2$$

$$= -\eta^{\mu\nu} \underline{k_\mu} \underline{k_\nu} - m^2$$

$$= \underline{k_0}^2 - k^2 - m^2$$

$$= -k^2 - m^2.$$

$$(\not{k} - m)^{-1} = \frac{\not{k} + m}{-k^2 - m^2}$$

if we replace $m \longrightarrow m - i\epsilon$

$$m^2 = m^2 - 2i\epsilon m \, \epsilon^2$$

$$= m^2 - i\epsilon'$$

$$\epsilon' = 2\epsilon m$$

sign of ϵ is same (matters)
$\epsilon \approx \epsilon'$.

$$\frac{i\,(\not{k}+m)_{\alpha\beta}}{-k^2-m^2+i\epsilon} = \left(\frac{i}{\not{k}-m+i\epsilon}\right)_{\alpha\beta},$$

In numerator adding $i\epsilon$ won't make any difference.

$$\langle \Omega | T[\psi_\alpha(x)\bar{\psi}_\beta(y)] | \Omega \rangle$$
$$= \int \frac{d^4k}{(2\pi)^4}\, \tilde{G}^{(2)}_{\alpha\beta}(k)\, e^{ik\cdot(x-y)}$$

→ what information does it have.

location of pole of the ~~pole~~ ~~gives us the~~ two pt. Θ gives us physical mass

Just using Lorentz invariance we can prove
$$\tilde{G}^{(2)}_{\alpha\beta} = A(k^2)\not{k}_{\alpha\beta} + B(k^2)\delta_{\alpha\beta}$$

$\psi_\alpha \bar{\psi}_\beta \longrightarrow$ transforms in a certain way.

That ~~contract~~ shows that
$$\tilde{G}^{(2)}_{\alpha\beta} = A(k^2)\not{k}_{\alpha\beta} + B(k^2)\delta_{\alpha\beta}.$$

$\not{k}_{\alpha\beta}$ & $\delta_{\alpha\beta}$ - Lorentz covariant.

As k is the only momentum.

if there is a Fermion with mass m_f, the single particle state contributes

$$\tilde{G}^{(2)}_{\alpha\beta} = Z_f \frac{i(\not{k} + m_f)_{\alpha\beta}}{-k^2 - m_f^2 + i\epsilon} + \text{non singular.}$$

m_f = physical mass of the Fermion.

Z_f = const.

→ same as propagator (in momentum space) with m replaced by m_f + a const. Z_f.

For electrodynamics series is asymptotic (never converges).

for suff. small e, there is a certain order up to which the series is convergent toward the actual value

for Q.E.D, e^2 → coupling const.

$1/e^2$ → pole can go

$e^{-1/x} \longrightarrow$ Do perturbation ~~eq~~ expansion around $x \to 0$ discontinuous at $x=0$.

if $e^2 \to$ negative

theory is unstable. But as an analytic fn. it ~~does~~ is discontinuous at $e^2 = 0$, (as it cannot be $-\infty$).

$$\frac{d}{dx} e^{-1/x} \cdot x = \frac{+e^{-1/x}}{x^2}$$

Taylor series of $e^{-1/x}$ ~~goes to 0~~ is 0 for all x.

but $e^{-1/0.1} \neq 0$, $\frac{1}{e^{1/0.1}} = \frac{1}{e^{10}}$

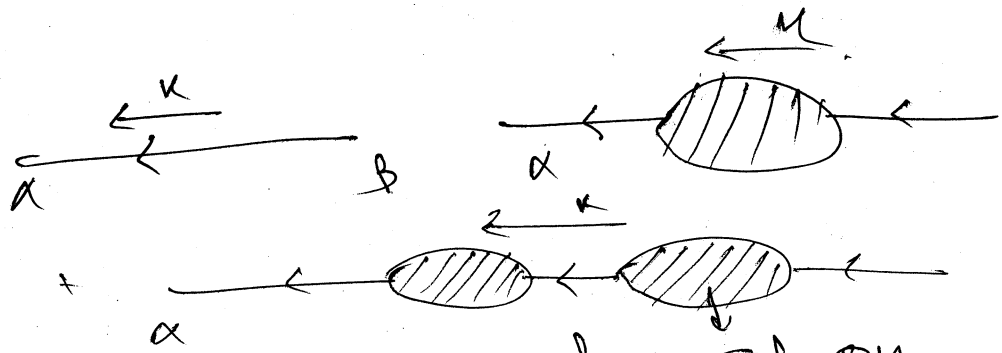
$$\frac{1}{1-x} + e^{-1/x} \simeq 1 + x + x^2 + x^3 + \ldots$$

$\longrightarrow$ it won't give the correct value as there is no contribution from $e^{-1/x}$.

For sufficiently small x, it doesn't give correct value.

For QED, upto 137th order will give us correct value.

$\hat{G}^{(2)}_{AB}$



sum of on particle irreducible graphs

one particle irreducible.

$\cancel{-i\Sigma_{\alpha\beta}(k)}$ $\quad -i\Sigma_{\gamma\delta}(k)$.

$$\hat{G}(2) = i(\not{k} - m + i\epsilon)^{-1}$$

$$+ i(\not{k} - m + i\epsilon)^{-1} - i\Sigma\, i(\not{k} - m + i\epsilon)^{-1}$$

$$+ i(\not{k} - m + i\epsilon)^{-1}(-i\Sigma)\, i(\not{k} - m + i\epsilon)^{-1}(-i\Sigma)\, i(\not{k} - m + i\epsilon)$$

$$= i(\not{k} - m + i\epsilon)^{-1}$$

$$\left\{ 1 + \Sigma(\not{k} - m + i\epsilon)^{-1} + \left\{\Sigma(\not{k} - m + i\epsilon)^{-1}\right\}^2 \right\}$$

$$= i(\not{k} - m + i\epsilon)^{-1}\left(1 - \Sigma(\not{k} - m + i\epsilon)^{-1}\right)^{-1}$$

$$1 + M + M^2 + \cdots = (1 - M)^{-1}. \qquad \Sigma(\not{k} - m + i\epsilon)^{-1}$$

$$\cancel{= i(\not{k} - m + i\epsilon - \Sigma)}. \qquad = (\not{k} - m + i\epsilon)$$

$$\cancel{= i\left[(\not{k} - m + i\epsilon)\right.}$$

$$= \left[\left(1 - \Sigma(\not{k} - m + i\epsilon)^{-1}\right)(\not{k} - m + i\epsilon)\right]^{-1}$$

$$= \cancel{1 - \Sigma}\,(\not{k} - m + i\epsilon - \Sigma)^{-1}.$$

$$= i\left[(\not{k} - m + i\epsilon) - A(k^2)\not{k} - B(k^2)\right]$$

from Lorentz invariance

$$\tilde{G}^{(2)}(k) = i\left[\not{k}\left(1 - A(k^2)\right) - \left(m - i\epsilon + B(k^2)\right)\right]^{-1}$$

$$= \frac{\not{k}\left(1 - A(k^2)\right) + m - i\epsilon + B(k^2)}{-k^2\left(1 - A(k^2)\right)^2 - \left(m - i\epsilon + B(k^2)\right)^2}$$

A & B are coming from Z. (order e^2)

$m_f^2 =$ where the denominator has a pole.

$$= \frac{i}{\left(1 - A(k^2)\right)} \times \frac{\not{k} + \dfrac{m + B(k^2) - i\epsilon}{1 - A(k^2)}}{-k^2 - \dfrac{\left(m + B(k^2) - i\epsilon\right)^2}{\left(1 - A(k^2)\right)^2}}$$

m_f determined from

$$+ m_f^2 - \frac{\left(m + B(-m_f^2)\right)^2}{\left(1 - A(m_f)^2\right)^2} = 0$$

$$m_f^2 = \frac{\left(m + B(-m_f^2)\right)^2}{\left(1 - A(-m_f^2)\right)^2}$$

$$\cancel{m_f^2 = m + B(}$$

$$m_f^2 = \frac{\left(m + B(-m_f^2)\right)^2}{\left(1 - A(-m_f^2)\right)^2}$$

1st iteration

$$m_f^2 = m^2 .$$

$\longrightarrow$ expand out in powers of e^2.

$$m_f^2 = \frac{\left(m + B(-m^2)\right)^2}{\left(\left(1 - A(-m^2)\right)\right)^2}$$

$$\frac{i\cancel{k}}{\left(1 - A(k^2)\right) F(k^2)}$$

$F(k^2)$ vanishes at $k^2 = m^2$

$$F'(-m_p^2)\,(+k^2 + m_p^2)$$

$$Z = \frac{1}{\left(1 - A(-m_p^2)\right)} \left(\frac{1}{-F'(-m_p^2)}\right);$$

We have to look at the coeff. of k making sure that the term denominator has $\frac{1}{-k^2 - m_p^2}$.

<u>Photons</u>

$$\langle \Omega | T\left(B_\mu(x) B_\nu(y)\right) | \Omega \rangle$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \tilde{G}^{(2)}_{\mu\nu}(k).$$

$\longrightarrow$ Two pt. func. of photon.

We will pick up the contribution from single photon state.

We can show that

$$\tilde{G}^{(2)}_{\mu\nu}(k) = \frac{i Z_\gamma}{-k^2 + i\epsilon} \underbrace{\sum_a \eta^{(a)}_\mu(k)\, \eta^{(a)}_\nu(k)}$$

$\longrightarrow$ easy singular part.

$$\eta_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} + \eta_\mu \eta_\nu$$

m_f remains 0 in the perturbation theory

$\Rightarrow$ pole doesn't drift to 0.

All the photon propagator can do is to have extra coeff. multiplied to it.

thus $Z_\gamma \longrightarrow$ identify one particles

$$\frac{1}{-k^2 - F(k^2) + i\epsilon}$$

$\longmapsto F(k^2) \propto k^2$

$\longrightarrow$ pole still remains at k^2, but residue will change.

Calculate S-matrix as fn. over m & ϵ.

Express S-matrix in terms of m_f, $e_p \longrightarrow$ physically measurable quantity $\longrightarrow$ finite.

we have to express the results in terms of m_f.

use the relation between m_f & ϵ_{p_1}'.

we have to rewrite everything in terms m_f & ϵ_p.

$$m_f = m + C_1(m) e^2 + C_2(m) e^4 + \cdots$$

for photons

$$\tilde{G}^2(k) = A(k^2) \sum_a \eta_\mu^{(a)}(k) \eta_\nu^{(a)}(k).$$

general S-matrix element :-

Defn.

$$\langle \Omega | T\left(\prod_{i=1}^{n} B_{\mu_i}(x_i) \prod_{j=1}^{m} \psi_{\alpha_j}(y_j) \prod_{k=1}^{m} \bar{\psi}_{\beta_j}(z_k) \right) | \Omega \rangle$$

$$= \int \prod_{i=1}^{n} \frac{d^4 k}{(2\pi)^4} \prod_{j=1}^{m} \frac{d^4 p_j}{(2\pi)^4} \prod_{k=1}^{m} \frac{d^4 q_k}{(2\pi)^4}$$

$$e^{i\left(\sum_i k_i \cdot x_i + \sum_j p_j \cdot y_j + \sum_k q_k \cdot z_k\right)}$$

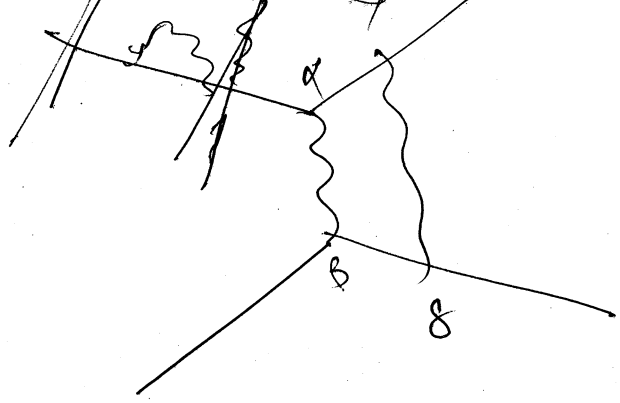
$$\tilde{G}^{(n,m)}_{\mu_1 \cdots \mu_n \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m}(k_1, \cdots k_n; p_1, \cdots p_m; q_1 \cdots q_m)$$

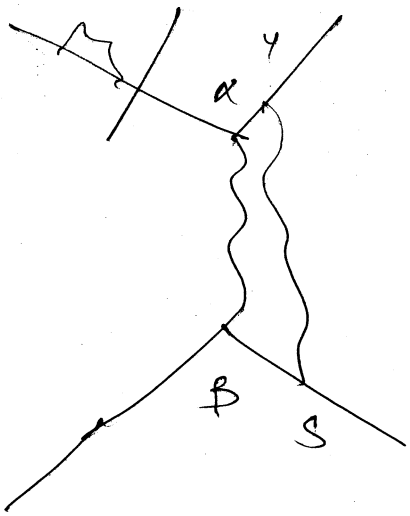
$\alpha_1 \cdots \alpha_n \to \psi$, $\beta_1 \cdots \beta_m \to \bar{\psi}$

$\mu_1 \cdots \mu_n$ ↑ photon

$$\hat{G}^{(n,m)}_{\mu_1, \cdots \mu_n, \alpha_1, \cdots \alpha_m, \beta_1, \cdots \beta_m}(k_1, \cdots k_n, p_1, \cdots p_m, q_1, \cdots q_m)$$

$\longrightarrow$ Truncated Green's func (external propagators removed)





most general in state
(particle, antiparticle, photons)

" " out "

" " " "

" "

→ particle - Antiparticle
→ conserve.

photons

$\{\vec{k}_1, a_1, \ldots \vec{k}_{n_1}, a_{n_1}, \vec{p}_1, s_1, \ldots \vec{p}_{n_2}, s_{n_2}, \vec{l}_1, \mathbf{f}_1, \ldots \vec{l}_{n_3}, \mathbf{f}_{n_3}\}$ out

e^-

'~~electron~~ momentum

polarisation
momentum

spin

→ general

spin

'outstate'

$$\langle \vec{k}_1 a_1, \ldots \vec{k}_{n_1} a_{n_1}, \vec{p}_1 s_1, \ldots \vec{p}_{n_2} s_{n_2}, \vec{q}_1 r_1, \ldots \vec{q}_{n_3}, r_{n_3} \,|\, \vec{k}_1' a_1', \ldots \vec{k}_{n_1'}' a_{n_1'}', \vec{p}_1' s_1', \ldots \vec{p}_{n_2}' s_{n_2}', \vec{q}_1' r_1', \ldots \vec{q}_{n_3'}' r_{n_3'} \rangle$$

$n_2 = \overline{4}$

$n_3 = \overline{4}$

$n_2 - n_3 = n_2' - n_3'$ no. of positrons to create + no. of e^- to destroy $\longrightarrow$ Change conservation

$$= \hat{G}^{(n_1+n_1',\, n_2+n_3')} (k_1, \ldots k_{n_1}, -k_1', \ldots -k_{n_1'}')$$

$\longrightarrow$ no. of photons

$\longrightarrow$ no. of 4's.

$\longrightarrow$ Truncated Green's fne.

$$|\Psi\rangle = \psi^+ \psi^0$$

$$= \langle \overline{4} | = \psi^0 \underline{\langle + |}$$

$\longrightarrow$ going out (as the out state)

$$\hat{G}^{(n_1+n_1',\, n_2+n_3')} (k_1, \ldots k_{n_1}, -k_1', \ldots -k_{n_1'}', p_1, \ldots p_{n_2}, -q_1, \ldots -q_{n_3'}', q_1, \ldots q_{n_3}, -p_1' \ldots -p_{n_2'}')$$

for the incoming state mom. is coming in to $\vec{k}_1' = -\vec{k}_1$

$$= G^{n}(n_1+n_1', n_2+n_3')$$

$$\Big(\overbrace{\mu_1 \cdots \mu_{n_1}, \mu_1' \cdots \mu_{n_1}'}^{B\mu_i}, \overbrace{\alpha_1, \cdots \alpha_{n_2}, \beta_1' \cdots \beta_{n_3}'}^{\psi}, \underbrace{\beta_1, \cdots \beta_{n_3}, \alpha_1' \cdots \alpha_{n_2}'}_{B_\theta \cdot \ \overline{\psi}}\Big)$$

There is a manifest ~~new~~ renormalization

$\longrightarrow$ for every external leg $\longrightarrow \sqrt{Z}$

$$\left(\sqrt{Z_\psi}\right)^{n_1+n_1'} \left(\sqrt{Z_f}\right)^{n_2+n_3'+n_3+n_2'}$$

$$\prod_{i=1}^{n_1}\left[\frac{1}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}_i}}}\right] \prod_{i=1}^{n_1}\left[\frac{1}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}_i}}}\right]$$

$$\prod_{i=1}^{n_2}\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m_f}{\omega_{\vec{p}_i}}} \prod_{i=1}^{n_3'}\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m_f}{\omega_{\vec{q}_i}}}$$

$$\prod_{i=1}^{n_2'} _____ \prod_{i=1}^{n_3} _____$$

$$\prod_{i=1}^{n_1} \eta^{(a_i)\mu_i} \prod_{i=1}^{n_1'} \eta^{(a_i')\mu_i'} \prod_{i=1}^{n_2} \bar{u}_{\alpha_i}(\vec{p}_i, s_i)$$

$$\prod_{i=1}^{n_2'} u_{\alpha_i'}(p_i', s_i') \prod_{i=1}^{n_3'} \bar{v}_{\beta_i}(q_i', s_i)$$

for photon $\rightarrow \frac{1}{\sqrt{2\omega_k}}$

for fermion $\rightarrow \sqrt{\frac{m_f}{\omega_p}}$.

polarisation factor it should contract.

for $e^- \longrightarrow u, \bar{u}$

for $e^+ \longrightarrow v, \bar{v}$

if e^- is outgoing $\bar{u}$

if e^- is incoming u.

if e^+ is outgoing v

" " incoming $\bar{v}$.

$$\Phi_r'(x) = M_{rs}\,\Phi_s(\Lambda x)$$

$$\Lambda\eta\Lambda^T = \eta, \quad \Lambda^T\eta\Lambda = \eta.$$

Lorentz transformation matrix.

$$\Lambda^\mu{}_\nu$$

$$(\Lambda x)^\mu = \Lambda^\mu{}_\nu x^\nu.$$

Scalar $\quad M = 1.$

for vector

$$A_\mu'(x) = \Lambda^\nu{}_\mu A_\nu(\Lambda x).$$

what is the Lorentz transformation law for a Dirac field.

$\psi_\alpha \longrightarrow$ spinor.

$$\psi_\alpha'(x) = M_{\alpha\beta}\,\psi_\beta(\Lambda x).$$

what is this matrix.

$$= M_{\alpha\beta}(\Lambda)\,\psi_\beta(\Lambda x).$$

Q:- what is M?

Action must be invariant under Lorentz transf.

Ex:- check that the ~~electrons~~

$\int d^4x \, F_{\mu\nu} F^{\mu\nu}$ is same

under $F_{\mu\nu} \longrightarrow F_{\mu\nu}'$.

We will now try to determine M.

$$S = \int d^4x \, \bar{\psi} \left(i \gamma^\mu \partial_\mu - m \right) \psi$$

$\longrightarrow$ Lorentz invariance for every m.

~~$\bar{\psi} m \psi = \bar{\psi}' M$~~ $\qquad \Lambda x \longrightarrow y$

$$\int \bar{\psi}' m \psi' d^4x = \int \psi^{+\prime}(x) \gamma^0 m \psi'(x) d^4x$$

$$\cancel{= \int \psi^+ M^+} = m \int \psi^+ (\Lambda x) M^+ \gamma^0 M \psi(\Lambda x) d^4x$$

$$= m \int d^4y \, \psi^+(y) M^+ \gamma^0 M \psi(y).$$

Original action was

$$m \int d^4y \, \psi^+(y) \gamma^0 \psi(y),$$

$$\boxed{M^+ \gamma^0 M = \gamma^0.} \longrightarrow \text{constraint over } M$$

$i \bar{\psi} \gamma^\mu \partial_\mu \psi \longrightarrow$ is invariant.

$$i \int d^4x \, \bar{\psi}'(x) \gamma^\mu \partial_\mu \psi'(x) d^4x = i \int \psi^+ (\Lambda x) M^+ \gamma^0 \gamma^\mu M \partial_\mu \psi(\Lambda x)$$

$$= i \int \psi^+ (\Lambda x) M^+ \gamma^0 \gamma^\mu M \frac{\partial \psi}{\partial x^\mu} (\Lambda x) d^4x.$$

$\Lambda x \longrightarrow y. \quad \Longrightarrow \quad y^\mu = \Lambda^\mu{}_\nu x^\nu$

$$\Longrightarrow \frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial y^\mu}\frac{\partial y^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu \frac{\partial}{\partial y^\mu}.$$

$$= i\int \psi^+(y)\, M^+ \gamma^0 \gamma^\mu M\, \Lambda^\mu{}_\nu \frac{\partial}{\partial y^\mu}.$$

$$= i\int \psi^+(y)\, M^+ \gamma^0 \gamma^\mu M\, \Lambda^\nu{}_\mu \frac{\partial \psi(y)}{\partial y^\nu}\, d^4y.$$

1 is set

$$= i\int \psi^+(y)\, M^+ \gamma^0 \gamma^\mu M\, \Lambda^\nu{}_\mu\, \partial_\nu \psi(y)\, d^4y$$

$$= i\Lambda^\nu{}_\mu \int d^4y\, \psi^+(y)\, M^+ \gamma^0 \gamma^\mu M \frac{\partial \psi(y)}{\partial y^\nu}.$$

For Lorentz invariance :-

$\longrightarrow$ should be equal to

(i) $$i\int d^4y\, \psi^+(y)\, \gamma^0 \gamma^\mu \frac{\partial}{\partial y^\nu}\psi(y).$$

On comparing, we will get.

$$\Lambda^\nu{}_\mu M^+ \gamma^0 \gamma^\mu M = \gamma^0 \gamma^\nu$$

$$\Lambda^\nu{}_\mu M^+ \gamma^0 \gamma^\mu M = \gamma^0 \gamma^\nu \longrightarrow$$ allows us to find relation between M & Λ.

Now let us work with infinitesimal version.

(We are considering cont. transformation).

Aim
No relation for cont. (1.7).

Infinitesimal transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon\omega^{\mu}{}_{\nu}$$

$\epsilon \longrightarrow$ infinitesimal parameter.

$\omega^{\mu}{}_{\nu} \longrightarrow$ matrix.

$$\Lambda\eta\Lambda^{T} = \eta \Longrightarrow \omega_{\mu\nu} = -\omega_{\nu\mu}$$

(antisymmetry).

$$\omega_{\mu\nu} = \eta_{\mu\rho}\omega^{\rho}{}_{\nu}.$$

Infinitesimal Lorentz transformations are generated by Antisymmetric matrix.

If we take infinitesimal L.T, then M should be close to identity

Take $M = \mathbb{1} + \epsilon\Sigma \longrightarrow$ Aim is to find it.

$$M^{+} = \mathbb{1} + \epsilon\Sigma^{+}$$

$$(1 + \epsilon\Sigma^{+})\gamma^{0}(1 + \epsilon\Sigma) = \gamma^{0}$$

$$\Rightarrow \cancel{\gamma^{0}} + \epsilon(\Sigma^{+}\gamma^{0} + \gamma^{0}\Sigma^{+}) = \cancel{\gamma^{0}}$$

$$\Longrightarrow \boxed{\{\Sigma^{+}, \gamma^{0}\} = 0.}$$

$$\cancel{A^{\nu}{}_{\mu}}\ (\delta^{\nu}{}_{\mu} + \epsilon\omega^{\nu}{}_{\mu})(\mathbb{1} + \epsilon\Sigma^{+})\gamma^{0}\gamma^{\mu}(\mathbb{1} + \epsilon\Sigma) = \gamma^{0}\gamma^{\mu}$$

$$= (\delta^{\nu}{}_{\mu} + \epsilon\omega^{\nu}{}_{\mu})(\gamma^{0}\gamma^{\mu} + \epsilon\Sigma^{+}\gamma^{0}\gamma^{\mu} + \gamma^{0}\gamma^{\mu}\epsilon\Sigma)$$

$$= \gamma^{0}\gamma^{\mu}.$$

$$\epsilon \omega^{\mu} \gamma^0 \gamma^{\mu} + \epsilon \Sigma^{+} \gamma^0 \gamma^0 + \gamma^0 \gamma^0 \epsilon \Sigma = 0$$

$$\Rightarrow \omega^{\mu} \gamma^0 \gamma^{\mu} + \Sigma^{+} \gamma^0 \gamma^0 + \gamma^0 \gamma^0 \Sigma = 0$$

find a Σ that satisfies this eqn.

Σ must be some linear func. in ω.

Try $\Sigma = c\,\omega_{\rho\sigma} \gamma^{\rho} \gamma^{\sigma}$ (ρ & σ should be contracted).

$\omega_{\rho\sigma} \gamma^{\rho} \gamma^{\mu} \gamma^{\sigma} \gamma^{\mu}$, $\longrightarrow \alpha\, \omega_{\rho\sigma} \gamma^{\rho} \gamma^{\sigma}$.

const.

$(\gamma^0)^{+} = \gamma^0$

$(\gamma^i)^{+} = -\gamma^i$

$(\gamma^0)^2 = 1$ & $(\gamma^i)^2 = -1$.

$\cancel{c\, \gamma^{\sigma} \gamma^{\rho} \omega_{\rho\sigma} \gamma^0 + \gamma^0 \omega_{\rho\sigma} \gamma^{\rho} \gamma^{\sigma}, = (\gamma^0 \gamma^{\rho})^{+}}$

$$\Sigma^{+} \gamma^0 + \gamma^0 \Sigma$$

$$= c^{*} \omega_{\rho\sigma} (\gamma^{\sigma})^{+} (\gamma^{\rho})^{+} \gamma^0 + c\, \omega_{\rho\sigma} \gamma^0 \gamma^{\rho} \gamma^{\sigma}.$$

$$= c^{*} \omega_{\rho\sigma} \left[(\gamma^{\sigma})^{+} (\gamma^{\rho})^{+} \gamma^0 + \gamma^0 \gamma^{\rho} \gamma^{\sigma} \right].$$

for $(\gamma^{\rho})^{+} \gamma^0 = \gamma^0 \gamma^{\rho}$ for $\rho = 0$

$= \gamma^0 \gamma^{\rho}$ for $\rho = i$

$$= \omega_{\rho\sigma} \left[(\gamma^\sigma)^\dagger \gamma^0 \gamma^\rho + c\, \gamma^0 \gamma^\rho \gamma^\sigma \right]$$

$$= \omega_{\rho\sigma} \left[\gamma^0 \gamma^\sigma \gamma^\rho + \gamma^0 \gamma^\rho \gamma^\sigma \right].$$

$$= c^* \omega_{\rho\sigma} \gamma^0 \gamma^\sigma \gamma^\rho + c\, \omega_{\rho\sigma} \gamma^0 \gamma^\rho \gamma^\sigma$$

$$= -c^* \omega_{\rho\sigma} \gamma^0 \gamma^\rho \gamma^\sigma + c\, \omega_{\rho\sigma} \gamma^0 \gamma^\rho \gamma^\sigma$$

$$= (c - c^*)\, \omega_{\rho\sigma} \gamma^0 \gamma^\rho \gamma^\sigma.$$

$$= 0 \qquad \Sigma = c\, \omega_{\rho\sigma} \gamma^\rho \gamma^\sigma.$$

$$\Longrightarrow c = c^* \text{ (real)}.$$

$$\omega^\nu{}_\mu \gamma^0 \gamma^\mu + \Sigma^\dagger \gamma^0 \gamma^\nu + \gamma^0 \gamma^\nu \Sigma = 0.$$

$$\omega^\nu{}_\mu \gamma^0 \gamma^\mu + c^* \omega_{\rho\sigma} (\gamma^\sigma)^\dagger (\gamma^\rho)^\dagger \gamma^0 \gamma^\nu$$

$$+ c\, \omega_{\rho\sigma} \gamma^0 \gamma^\nu \gamma^\rho \gamma^\sigma$$

$$= \omega^\nu{}_\mu \gamma^0 \gamma^\mu + c\, \omega_{\rho\sigma} (\gamma^\sigma)^\dagger \gamma^0 \gamma^\rho \gamma^\nu.$$

$$+ c\, \omega_{\rho\sigma} \gamma^0 \gamma^\nu \gamma^\rho \gamma^\sigma$$

$$= \omega^\nu{}_\mu \gamma^0 \gamma^\mu + c\, \omega_{\rho\sigma} \gamma^0 \underline{\gamma^\sigma \gamma^\rho \gamma^\nu}$$

$$+ c\, \omega_{\rho\sigma} \gamma^0 \gamma^\nu \gamma^\rho \gamma^\sigma$$

$$\cancel{= \omega^\nu{}_\mu + c\, \omega_{\rho\sigma} \left(-\gamma^0 \gamma^\sigma \gamma^\nu \gamma^\rho - \gamma^0 2 \eta^{\nu\rho} \right)}$$

$\cancel{+c\omega_{\rho\sigma}\gamma^0\gamma^\nu\gamma^\rho\gamma^\sigma}$ $\qquad \gamma^\sigma\gamma^\nu + \gamma^\nu\gamma^\sigma = -2\eta^{\nu\sigma}$

$$= \omega^\nu{}_\mu \gamma^0\gamma^\mu + c\omega_{\rho\sigma}\gamma^0\gamma^\sigma\gamma^\rho\gamma^\nu + c\omega_{\rho\sigma}\gamma^0\gamma^\nu\gamma^\rho\gamma^\sigma.$$

$$= \omega^\nu{}_\mu \gamma^0\gamma^\mu - c\omega_{\rho\sigma}\gamma^0\gamma^\rho\gamma^\sigma\gamma^\nu + c\omega_{\rho\sigma}\gamma^0\gamma^\nu\gamma^\rho\gamma^\sigma.$$

$$\cancel{= \omega^\nu{}_\mu \gamma^0\gamma^\mu - c\omega_{\rho\sigma}(\gamma^0\gamma^\rho\gamma^\sigma\gamma^\nu}$$

$$= \omega^\nu{}_\mu \gamma^0\gamma^\mu - c\omega_{\rho\sigma}\gamma^0\gamma^\rho(-2\eta^{\nu\sigma} - \gamma^\nu\gamma^\sigma) + c\omega_{\rho\sigma}\gamma^0\gamma^\nu\gamma^\rho\gamma^\sigma$$

$$= \omega^\nu{}_\mu \gamma^0\gamma^\mu + c\omega_{\rho\sigma}\gamma^0\gamma^\rho 2\eta^{\nu\sigma} + c\omega_{\rho\sigma}\gamma^0\gamma^\rho\gamma^\nu\gamma^\sigma + c\omega_{\rho\sigma}\gamma^0\gamma^\nu\gamma^\rho\gamma^\sigma.$$

$$= \omega^\nu{}_\mu \gamma^0\gamma^\mu + c\omega_{\rho\nu}\gamma^0\gamma^\rho + c\omega_{\rho\sigma}\left(\gamma^0\left[-\gamma^\nu\gamma^\rho - 2\eta^{\nu\rho}\right]\right)\gamma^\sigma + c\omega_{\rho\sigma}\gamma^0\gamma^\nu\gamma^\rho\gamma^\sigma$$

$$= \omega^\nu{}_\mu \gamma^0\gamma^\mu + 2c\omega^\nu{}_\rho\gamma^0\gamma^\rho.$$

$$\cancel{c\omega_{\rho\sigma}\gamma^0\gamma^\nu\gamma^\rho\gamma^\sigma} - 2c\omega^\nu{}_\sigma\gamma^0\gamma^\sigma + \cancel{c\omega_{\rho\sigma}\gamma^0\gamma^\nu\gamma^\rho\gamma^\sigma}.$$

$$= \omega^{\nu}{}_{\mu}\gamma^{0}\gamma^{\mu} - 2c\left(\omega^{\nu}{}_{\rho}\gamma^{0}\gamma^{\rho} + \omega^{\nu}{}_{\sigma}\gamma^{0}\gamma^{\sigma}\right)$$

$$= \omega^{\nu}{}_{\mu}\gamma^{0}\gamma^{\mu} - 4c\left(\omega^{\nu}{}_{\rho}\gamma^{0}\gamma^{\rho}\right)$$

$$= 0$$

$$\Longrightarrow \quad 1 - 4c = 0$$

$$\Longrightarrow \quad c = \frac{1}{4}.$$

Thus

$$\boxed{M = I + \frac{1}{4}\epsilon\,\omega_{\mu\nu}\gamma^{\mu}\gamma^{\nu}.}$$

final version
exponential of the matrix M.

Ex:- Find expressions for the conserved charges

Sum of 2 pieces → one piece

$x \longrightarrow \Lambda x$ (orbital part)

Then due to matrix

$M. \longrightarrow$ spin part.

Intrinsic spin.

Anyons - arbitrary fraction $\alpha \rightarrow e^{i\alpha}$.

Abelian group → $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Ex:- (2) Express the Lorentz generators $J^{\mu\nu}$ in terms of creation/annihilation operators. & show that $J^{\mu\nu}|0\rangle = 0$.

(3) How they act on the 1 particle states

Ex. Show that J^{12}

$\cancel{J^3 \sum b^\dagger(\vec{p}=0,s)|0\rangle}$

$J^3 b^\dagger(\vec{p}=0,s)|0\rangle$

$J^3 d^\dagger(\vec{p}=0,s))|0\rangle$

are determined in terms of 1.

, Prove that s index is a spin index.

J^3 eigenvalues $= \pm \hbar/2$.

Express (Noether charge in terms of b & $b^\dagger$).

Coupling to electrodynamics

$-\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}$ is L. invariant.

Dirac Action is L.I.

QED Lorentz invariant?

Interaction Terms

$$\mathcal{L} = \int d^4x\, \bar{\psi}\gamma^\mu\psi\, A_\mu(x)$$

$$A_\mu \longrightarrow \Lambda^\nu{}_\mu A_\nu(\Lambda x).$$

$$\cancel{A^\nu{}_\mu M^\dagger \gamma^0 A^\mu}$$

$$\cancel{= \gamma^0 A}$$

$$\Lambda^\nu{}_\mu M^\dagger \gamma^0 \gamma^\mu M = \gamma^0\gamma^\nu.$$

Lorentz trans.

$$\int \bar{\psi}'\gamma^\mu\psi' A_\mu'\, d^4x.$$

$$= \int \psi^\dagger(\Lambda x) M^\dagger \gamma^0 \gamma^\mu M \psi(\Lambda x)\, \Lambda^\nu{}_\mu A_\nu(\Lambda x)$$

$$= \int \psi^\dagger(x')\, \gamma^0\gamma^\nu\psi\, A_\nu(x')\, d^4x'.$$

Ex:- Construct $M^{\mu\nu}$ in free electrodynamics.

Have $M_{\mu\nu}|0\rangle$.

Gauge Choice $\longrightarrow$: (if ~~[illegible]~~ gauge choice breaks gauge

Lorentz invariance) then

M^{ij}

$$J^3 a^{(c)+}(\vec{k})|0\rangle$$

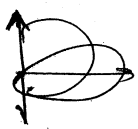
we can not go to a rest frame.

Case in our frame mom. is along a specific axis.

$$J^3 a^{(c)+}(\vec{k}=(0,0,k^3))|0\rangle.$$

$\longrightarrow$ eigen value of J_3 to ± 1

& it rotates



$$J^3 a^{(1)+}|\vec{k}=(0,0,k^3))|0\rangle$$

$$= i\, a^{(2)+}(\vec{k}=0,0,k^3)|0\rangle$$

$$J^3 a^{(2)+}(\vec{k}=0,0,k^3)|0\rangle$$

$$= -i\, a^{(1)+}(\vec{k}=0,0,k^3)|0\rangle$$

J^3 takes

$1 \longrightarrow 2.$

$2 \longrightarrow -1$

final 2×2 matrix

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \longrightarrow \text{eigenvalues } \pm\lambda.$$

eigenvector $= \cancel{\frac{1}{2}} \pm i \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$.

Circularly polarized. $\longrightarrow$ eigenstate J^3,

$\epsilon^1, \epsilon^2 \longrightarrow$ plane polarized

photon has 2 states $\longrightarrow J_3 = \pm 1$