Lessons from QFT I

In a field theory of a single scalar field ϕ, the physical information about the spectrum & S-matrix contained in

$$G(x_1, \cdots x_n) \equiv \langle \Omega | T\Big(\prod_{i=1}^{n} \hat{\phi}(\underbrace{x_i}_{\to (\vec{x}_i,\, t_i)})\Big) |\Omega\rangle \Big/ \langle \Omega | \Omega \rangle$$

$\hat{\phi}$: operator representing ϕ.

$|\Omega\rangle$: ground state

T: Time ordering.

Lowest t_i: extreme right.

Highest t_i: extreme left.

$G(x_1, x_2)$: gives ~~spe~~ mass spectrum.

$G(x_1, \cdots x_n)$ for $n \geq 2$ gives S-matrix.

$$\hat{\phi}(\vec{x}_i, t_i) = e^{i \underbrace{\hat{H}}_{\text{full Hamiltonian}} t_i} \hat{\phi}(\vec{x}_i, 0)\, e^{-i\hat{H}t_i}$$

Similar formula exists for more complicated theories including fermions and gauge fields.

Perturbation theory:

$$\mathcal{H} = \underbrace{\mathcal{H}_0}_{\text{free}} + \lambda \underbrace{\mathcal{H}_{int}}_{\text{interaction.}}$$

We have a well defined procedure for computing $G(x_1, \cdots x_n)$ as a power series expansion in λ.

$\rightsquigarrow$ Feynman diagrams.

We shall now describe a new method for computing $G(x_1, \cdots x_n)$ based on path integral approach which leads to the same Feynman rules.

Why should we do this?

① The method based on canonical procedure breaks manifest Lorentz invariance although the final results are Lorentz invariant.

⤳ we treat space and time differently.

Path integral method maintains manifest Lorentz invariance all along.

② Path integral approach is naturally suited for quantizing non-abelian gauge theories

⤳ Theory of strong, weak and electromagnetic interactions.

Path integral method in quantum mechanics.

final time

$$\mathcal{H} = \frac{p^2}{2m} + V(q)$$

$$L = \frac{1}{2} m \dot{q}^2 - V(q) \qquad S = \int_{t'}^{t''} L \, dt$$

initial time

Consider the quantity:

$$K(q', t'; q'', t'') = \langle q'' | e^{-i\hat{\mathcal{H}}(t''-t')} | q' \rangle$$

$\hbar = 1$

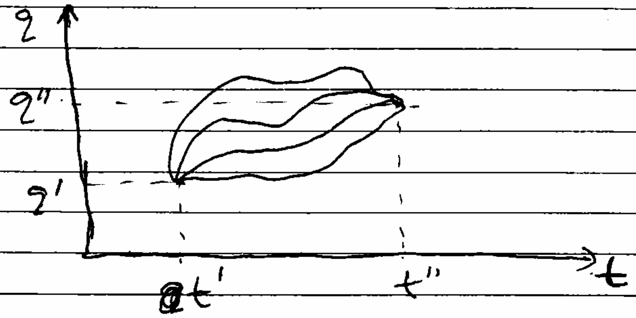
$|q'\rangle, |q''\rangle$: Eigenstates of $\hat{q}$ with eigenvalues q', q''.

Note: We are using Heisenberg representation.

Path integral description:

$$K(q',t';\,q'',t'') = \int [\mathcal{D}q]\, e^{iS}$$

"Sum over all paths beginning at q' at time t' and ending at q'' at time t'.



To make sense of this we discretize time:

$$\tau_i = t' + (i-1)\Delta \qquad i = 1 \cdots N$$

$$\Delta = \frac{t''-t'}{N}$$

$\tau_1 = t',\ \tau_{N+1} = t''$.

$q_i = q(\overset{\tau_i}{\cancel{t_i}})$ for a given path.

$$S = \Delta \sum_{i=1}^{N} \left[\frac{1}{2} m \left(\frac{q_{i+1}-q_i}{\Delta}\right)^2 - \underset{\downarrow}{V(q_i)}\right]$$

or $\dfrac{V(q_i)+V(q_{i+1})}{2}$

$K(q',t';\, q'',t'')$

$$= \lim_{N\to\infty} \mathcal{N} \int dq_2 \cdots dq_N \, \exp\left[i\Delta \sum_{i=1}^{N} \left\{ \frac{1}{2} m \left(\frac{q_{i+1}-q_i}{\Delta}\right)^2 - V(q_i) \right\}\right]$$

$\mathcal{N}$: A normalization constant which is independent of q' & q''.

We can prove the equivalence between path integral formulation and canonical formulation by straightforward manipulation.

More general quantity:

$$K(q', t'; q'', t''; t_1, \cdots t_n)$$

$$= \langle q''| e^{-i\hat{H} t''} T\left(\prod_{i=1}^{n} \hat{q}(t_i)\right) e^{i\hat{H} t'} |q'\rangle$$

$$t'' > (t_1, \cdots t_n) > t'$$

~~For any~~ Assume that

$$t_n > t_{n-1} > \cdots\cdots t_1.$$

$$\langle q''| e^{-i\hat{H} t''} \hat{q}(t_n)\, \hat{q}(t_{n-1}) \cdots \hat{q}(t_1)\, e^{i\hat{H} t'} |q'\rangle$$

$$\hat{q}(t) = e^{i\hat{H} t'} \hat{q}(0)\, e^{-i\hat{H} t'}$$

$$\Rightarrow K(q', t'; q'', t''; t_1, \cdots t_n)$$

$$= \langle q''| e^{-i\hat{H}(t''-t_n)} \hat{q}(0)\, e^{-i\hat{H}(t_n - t_{n-1})} \hat{q}(0) \cdots\cdots e^{-i\hat{H}(t_2 - t_1)} \hat{q}(0)\, e^{i\hat{H}(t_1 - t')} |q'\rangle$$

Path integral description:

~~Suppose~~

$$\tau_i = t' + (i-1)\,\Delta$$

$$\Delta = (t''-t')/N.$$

Suppose $t_1 = \tau_{k_1}$, $t_2 = \tau_{k_2}$, ...etc.

$$q_{k_i} = q(\tau_{k_i}).$$

Then.

$$K(q',t';q'',t'';t_1,\dots t_n)$$

$$\lim_{N\to\infty} N \int dq_2 \cdots dq_N \; e^{iS} \; q_{k_1} \cdots q_{k_n}$$

$$\to \int(\mathcal{D}q)\; e^{iS} \; q(t_1)\cdots q(t_n)$$

order is not important.

Path integral always produces matrix elements of time ordered products.

Our goal: Compute:

$$\langle \Omega | T\left(\prod_{i=1}^{n} \hat{q}(t_i)\right) | \Omega \rangle$$

└ ground state

Set $t' = -T(1-i\epsilon)$, $t'' = T(1-i\epsilon)$

$$K(q', -T(1-i\epsilon), q'', T(1-i\epsilon); t_1, \cdots t_n)$$

$$= \langle q'' | e^{-i\hat{H}T - \hat{H}\epsilon T} T\left(\prod_{i=1}^{n} \hat{q}(t_i)\right) e^{i\hat{H}(-T + i\epsilon T)} | q' \rangle$$

$$= \sum_{m,n} \langle q'' | m \rangle \langle m | e^{-iE_m T - \epsilon E_m T} T\left(\prod_{i=1}^{n} \hat{q}(t_i)\right) e^{-iE_n T - \epsilon E_n T} | n \rangle \langle n | q' \rangle$$

$$\underset{T \to \infty}{\sim} \langle q'' | \Omega \rangle \langle \Omega | q' \rangle e^{-2iE_\Omega T - 2\epsilon E_\Omega T} \langle \Omega | T\left(\prod_{i=1}^{n} \hat{q}(t_i)\right) | \Omega \rangle$$

Thus

$$\langle \Omega | T\left(\prod_{i=1}^{n} \hat{q}(t_i)\right) | \Omega \rangle$$

$$= \lim_{T \to \infty} \frac{K(q', -T(1-i\epsilon); q'', T(1-i\epsilon); t_1, t_n)}{K(q', -T(1-i\epsilon), q'', T(1-i\epsilon))}$$

$\rightsquigarrow$ Can be calculated using the path integral approach.

Note: The results can be easily generalized for multiple variables.

$$\mathcal{H}(q^{(1)}, \ldots q^{(s)}, p^{(1)}, \ldots p^{(s)})$$

$$\leftrightarrow L(q^{(1)}, \ldots q^{(s)}, \dot{q}^{(1)}, \ldots \dot{q}^{(s)})$$

QM → QFT

$$\underbrace{\{q^{(1)}(t), q^{(2)}(t), \cdots\}}_{QM} \to \underbrace{\phi^{(1)}(\vec{x},t), \phi^{(2)}(\vec{x},t)}_{\text{different fields.}}, \cdots$$

For simplifying notation we consider the case of a single scalar field $\phi(\vec{x},t)$.

$S[\phi(\vec{x},t)]$ is a functional of ϕ.

|| e.g.

$$\int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4\right]$$

} regard this as QM of ∞ # of variables by discretizing space.

$$\vec{x} = \underset{\text{a small \#}}{h}\,\underbrace{(i_1, i_2, i_3)}_{\text{integers}}$$

$-\infty < i_1, i_2, i_3 < \infty$.

$$\phi(\vec{x},t) \longrightarrow \phi_{\vec{i}}(t)$$

$$\vec{i} \; \shortparallel \; (i_1, i_2, i_3)$$

$$S = h^3 \int dt \sum_{i_1,i_2,i_3} \Big[\frac{1}{2} \frac{\partial \phi_{\vec{i}}(t)}{\partial t} \frac{\partial \phi_{\vec{i}}(t)}{\partial t}$$

$$- \frac{1}{2} \left(\frac{\phi_{i_1+1,i_2,i_3}(t) - \phi_{i_1,i_2,i_3}(t)}{h} \right)^2$$

$$- \frac{1}{2} \left(\frac{\phi_{i_1,i_2+1,i_3}(t) - \phi_{i_1,i_2,i_3}(t)}{h} \right)^2$$

$$- \frac{1}{2} \left(\frac{\phi_{i_1,i_2,i_3+1}(t) - \phi_{i_1,i_2,i_3}(t)}{h} \right)^2$$

$$- \frac{1}{2} m^2 \phi_{\vec{i}} \phi_{\vec{i}} - \frac{\lambda}{4!} \left(\phi_{\vec{i}}\right)^4 \Big]$$

$$\langle \Omega | T \Big(\prod_k \hat{\phi}(x_k) \Big) | \Omega \rangle$$

$$\rightsquigarrow \langle \Omega | T \Big(\hat{\phi}_{\vec{i}(k)}(t_k) \Big) | \Omega \rangle$$

$$\vec{x}_k = h \left(i_{(k)1}, i_{(k)2}, i_{(k)3} \right)$$

$$= \int \prod_{i_1,i_2,i_3} \mathcal{D}\phi_{\vec{i}}(t)\; e^{iS} \prod_{k=1}^{n} \phi_{\vec{i}_{(k)}}(t_k)$$

To define the path integral we have to discretize time:

$$t = \Delta\, i_0. \qquad -\infty < i_0 < \infty.$$

$$S = h^3 \Delta \sum_{i_0,i_1,i_2,i_3} \Big[\frac{1}{2} \left(\frac{\phi_{i_0+1,i_1,i_2,i_3} - \phi_{i_0,i_1,i_2,i_3}}{\Delta} \right)^2$$

$\to$ Same with $\phi_{i_1,i_2,i_3}(t)$

$\to \phi_{i_0,i_1,i_2,i_3}$

$$\langle \Omega | T\Big(\prod_k \phi(x_k) \Big) | \Omega \rangle$$

$$\to \mathcal{N} \int \prod_{i_0,i_1,i_2,i_3} d\phi_{i_0,i_1,i_2,i_3}\; e^{iS}$$

$$\prod_k \phi_{i_{(k)0},\, i_{(k)1},\, i_{(k)2},\, i_{(k)3}}.$$

Euclideanization:

$$\langle q'', t'' | T\left(\prod_i q(t_i)\right) | q', t'\rangle$$

✓

Required us to choose:

$$t'' = T(1-i\epsilon), \qquad t' = -T(1-i\epsilon)$$

& taking $T \to \infty$.

We could achieve the same result by taking

$$t'' = -iT, \qquad t' = iT$$

& taking $T \to \infty$.

i.e. we take

$$t \to -i\tau$$

$$\Delta \to -i\delta, \quad t_i \to -i\tau_i$$

$$iS = -h^3 \delta \sum_{i_0, \cdots i_3} \Big[\frac{1}{2} \left(\frac{\phi_{i_0+1, i_1, i_2, i_3} - \phi_{i_0, i_1, i_2, i_3}}{\delta} \right)^2$$

$$+ \frac{1}{2} \left(\frac{\underbrace{\phi_{i_0, i_1+1, i_2, i_3} - \phi_{i_0, i_1, i_2, i_3}}}{h} \right)^2$$

$$+ \ 2 \ \& \ 3$$

$$+ \frac{1}{2} m^2 \phi^2_{i_0, i_1, i_2, i_3} + \frac{\lambda}{4!} \phi^4_{i_0, i_1, i_2, i_3} \Big]$$

$$= - S_E$$

↳ Euclidean action.

$$S_E \xrightarrow[h \to 0, \delta \to 0]{} \int d\tau \, d^3x \Big[\frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2$$

$$+ \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \Big]$$

∼ Positive definite

→ Well defined functional integra

$$\langle \Omega | T_\tau \left(\prod_{k=1}^{n} \phi(\vec{x}_k, -i\tau_k) \right) | \Omega \rangle$$

$$= \int [d\phi_{i_0, i_1, i_2, i_3}] \, e^{-S_E}$$

$$\prod_k \phi_{i_{(k)0}, i_{(k)1}, i_{(k)2}, i_{(k)3}}$$

$$\tau_k = \delta \, i_{(k)0}, \qquad \vec{x}_k = h(i_{(k)1}, i_{(k)2}, i_{(k)3})$$

Once we calculate this we can get the result for real time by analytic continuation.

An alternative to analytic continuation:

$$\langle q''| e^{-iHt''} T\left(\prod_{k=1}^{n} q(t_k)\right) e^{iHt'} |q'\rangle$$

$$= \langle q''|m\rangle \langle m| T\left(\prod_{i=1}^{n} q(t_k)\right) |n\rangle$$

$$\langle n|q'\rangle \; e^{-iE_m t'' + iE_n t'}$$

We used $t' = -T(1-i\epsilon)$, $t'' = T(1-i\epsilon)$

to generate $e^{-\epsilon E_m T - \epsilon E_n T}$ terms.

Instead suppose we manage to get:

$$E_m = E_{m,real} - i\epsilon_m$$

$\rightarrow$ vanishes for $|\Omega\rangle$

+ve for $|m\rangle \neq |\Omega\rangle$.

Set $t'' = T$, $t' = -T$

$$e^{-(iE_{m,real} + iE_{n,real})T - (\epsilon_m + \epsilon_n)T}$$

In this case again as $T \to \infty$

only $|\Omega\rangle$ will contribute

→ serves the purpose.

$$E(\vec{p}_1 \cdots \vec{p}_n) = \sum_i \sqrt{\vec{p}_i^{\,2} + m_{phys}^2}$$

If we can change $m_{phys}^2 \to m_{phys}^2 - i\epsilon$

then it serves the purpose.

$$m_{phys}^2 = m^2 + \text{corrections.}$$

Change $m^2 \to m^2 - i\epsilon$

$$iS = i\int d^4x \left[\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2}(m^2 - i\epsilon)\phi^2 - \tfrac{1}{4!}\lambda\phi^4\right)$$

$$= i\int d^4x \left[\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\phi^4\right]$$

$$- \tfrac{1}{2}\epsilon\int d^4x\,\phi^2$$

→ damping term makes the path integral well defined.

Functional differentiation:

$F[\phi]$: A functional of $\phi(\underset{\vec{x},t}{\underset{\downarrow}{x}})$.

In discrete form $\phi(\vec{x},t) \to \phi_{\underset{t/\Delta}{\underset{\downarrow}{i_0}},\underset{\vec{x}/h}{\underset{\searrow}{\vec{i}}}}$

$F(\phi)$: A function of $\phi_{i_0,\vec{i}}$ for all $i_0, \vec{i}$.

Define : $$\frac{\delta}{\delta\phi(\vec{x})} = \lim_{h,\Delta\to 0} \frac{1}{h^3\Delta} \frac{\partial}{\partial\phi_{t/\Delta,\,\vec{x}/h}}$$

$$\frac{\delta F}{\delta\phi(x)} = \frac{1}{h^3\Delta}\frac{\partial F}{\partial\phi_{i_0,\vec{i}}}$$

Example:

$$F[\phi] = \int d^4x \; \phi(\vec{x},t)\; \underset{\text{some given function}}{\underset{\downarrow}{J(\vec{x},t)}}$$

$$= h^3\Delta \sum_{i_0,\vec{i}} \phi_{i_0,\vec{i}}\; J_{i_0,\vec{i}}$$

$$\frac{\delta F}{\delta \phi(x)} = \frac{\partial}{\partial \phi_{\frac{t}{\Delta},\frac{\vec{x}}{h}}} \sum_{i_0,\vec{i}} \phi_{i_0,\vec{i}} \, J_{i_0,\vec{i}}$$

$$= J_{\frac{t}{\Delta},\frac{\vec{x}}{h}} = J(\vec{x},t)$$

If $J(x) = \delta^{(4)}(x-x')$

$$\ni F = \int d^4x \; \phi(x) \; \delta^{(4)}(x-x') = \phi(x')$$

$$\ni \frac{\delta \phi(x')}{\delta \phi(x)} = \delta^{(4)}(x-x')$$

We shall keep using this rule.

Also:

$$\frac{\delta}{\delta \phi(x)} (F G) = \frac{\delta F}{\delta \phi(x)} G + F \frac{\delta G}{\delta \phi(x)}$$

F, G: Two functionals.

Now consider a field theory with fields $\phi^{(1)}(x)$, $\phi^{(2)}(x)$, --- $\phi^{(s)}(x)$ and action S

$\downarrow$

A functional of $\phi^{(1)}, \phi^{(2)}, \cdots \phi^{(s)}$.

Define:

$$Z[J^{(1)}, \cdots J^{(s)}] = \int (\mathcal{D}\phi^{(1)} \cdots \mathcal{D}\phi^{(s)}) \; e^{iS + i \sum_{\ell=1}^{s} \int d^4x \, J(x) \, \phi^{(\ell)}(x)}$$

$$\left(-i \frac{\delta}{\delta J^{(k_1)}(x_1)}\right) \cdots \left(-i \frac{\delta}{\delta J^{(k_n)}(x_n)}\right) Z$$

$$= \int [\mathcal{D}\phi^{(1)} \cdots \mathcal{D}\phi^{(s)}] \; e^{iS} \; \phi^{(k_1)}(x_1) \cdots \phi^{(k_n)}(x_n)$$

$$\Rightarrow \langle \Omega | \, T\left(\prod_{i=1}^{n} \phi^{(k_i)}(x_i)\right) | \Omega \rangle$$

→ desired information.

Thus knowing $Z[J^{(1)}, \cdots J^{(s)}]$ we can learn all relevant information about field theory.

Goal: Develop perturbation theory for computing $Z[J^{(1)}, \cdots J^{(s)}]$

Simple Case: Free scalar field

$$S = \int d^4x \left[-\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right]$$

$$Z_{free}[J] = \int [\mathcal{D}\phi] \, e^{iS + i\int d^4x \, J(x)\, \phi(x)}$$

$$= \int \mathcal{D}\phi \, \exp\left[i \int d^4x \left\{ \frac{1}{2} \phi(x) (\Box_x - m^2) \phi(x) + J(x)\, \phi(x) \right\} \right]$$

$$= \int [\mathcal{D}\phi] \exp\Big[\frac{i}{2} \int d^4x \Big\{ \phi(x) + \int d^4x' \, G(x,x')\, J(x') \Big\} (\Box_x - m^2)$$
$$\Big\{ \phi(x) + \int d^4x'' \, G(x,x'')\, J(x'') \Big\} - \frac{i}{2} \int d^4x \, d^4x' \, J(x)\, G(x,x')\, J(x') \Big]$$

$$(\Box_x - m^2)\ G(x,x') = \delta^{(4)}(x-x')$$

Define $\chi(x) = \phi(x) + \int d^4x'\, G(x,x')\, J(x')$

$$Z[J] = \exp\left[-\frac{i}{2}\int d^4x\ d^4x'\ G(x,x')\ \phi(x)\phi(x')\right]$$

$$\underbrace{\int[\mathcal{D}\chi]\ e^{\frac{i}{2}\int \chi(x)(\Box_x - m^2)\chi(x)\, d^4x}}_{\mathcal{N}}$$

Explicit form of $G(x,x')$:

$$G(x,x') = \int e^{ik(x-x')}\,\tilde{G}(k)\,\frac{d^4k}{(2\pi)^4}$$

$$(\Box_x - m^2 + i\epsilon)\ G(x,x')$$

$$= \int e^{ik\cdot(x-x')}\,(-k^2 - m^2 + i\epsilon)\,\tilde{G}(k)\,\frac{d^4k}{(2\pi)^4}$$

$$\delta^{(4)}(x-x') = \int e^{ik\cdot(x-x')}\,\frac{d^4k}{(2\pi)^4}$$

$$\Rightarrow \quad \tilde{G}(k) = \frac{1}{-k^2 - m^2 + i\epsilon}$$ ~~$\Rightarrow G(x,x') =$~~

~~Feynman prop~~

$$Z[J] = \mathcal{N} \exp\left[-\frac{1}{2} \int d^4x \, d^4x' \; \Delta(x,x') \; \phi(x) \; \phi(x') \right]$$

$$\Delta(x,x') = i \, G(x,x')$$

→ Feynman propagator.

$$\langle \Omega | T(\phi(x_1) \; \phi(x_2)) | \Omega \rangle / \langle \Omega | \Omega \rangle$$

$$= \left(-i \frac{\delta}{\delta J(x_1)} \right) \left(-i \frac{\delta}{\delta J(x_2)} \right) \frac{Z[J]}{Z[0]}$$

$$= \Delta(x_1, x_2)$$

$$\langle \Omega | T(\phi(x_1) \; \phi(x_2) \; \phi(x_3) \; \phi(x_4)) | \Omega \rangle / \langle \Omega | \Omega \rangle$$

$$= \left(-\frac{i\delta}{\delta J(x_1)} \right) \cdots \left(-i \frac{\delta}{\delta J(x_4)} \right) \frac{Z[J]}{Z[0]} \Bigg|_{J=0.}$$

$$= \Delta(x_1, x_2) \; \Delta(x_3, x_4) + \Delta(x_1, x_3) \; \Delta(x_2, x_4)$$

$$+ \Delta(x_1, x_3) \; \Delta(x_2, x_4)$$

→ Wick's theorem.

Systematic procedure for computing 2n-point function in KG theory

$$G^{(2n)}(x_1, \cdots x_{2n})$$

$$\equiv \langle \Omega | T\left(\prod_{i=1}^{2n} \hat{\phi}(x_i) \right) | \Omega \rangle / \langle \Omega | \Omega \rangle$$

$$= \frac{1}{Z[0]} \left(-i \frac{\delta}{\delta J(x_1)} \right) \cdots \left(-i \frac{\delta}{\delta J(x_{2n})} \right) Z(J) \Big|_{J=0}$$

$$Z[J] = \sum_{m=0}^{\infty} \frac{1}{m!} \left\{ -\frac{1}{2} \int d^4x \, d^4x' \, \Delta(x, x') \, J(x) \, J(x') \right\}^m$$

Only $m = n$ term is relevant.

Symbol: $-i \frac{\delta}{\delta J(x)}$: $\overset{x}{\times}$ $\{\Delta(x, x')\}$ $x \rule[0.5ex]{3em}{0.4pt} x'$

$\downarrow$ propagator

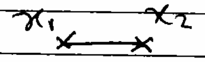
Rule: ① Every $\times$ must be attached to the end of a line.

$\left(-i \frac{\delta}{\delta J(x)}\right)$ must act on some $J(x)$ to given non-zero result

② Each end of — must be attached to a ×.

(No left-over $J(x)$ at the end)

Sum over all diagrams.

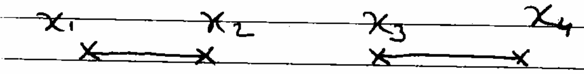
$2n = 2$: 

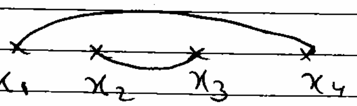
$\Delta(x_1, x_2)$

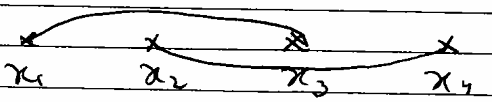
$\downarrow$

$$\left(-\tfrac{1}{2}\Delta(x_1,x_2)\right) \times (-i)(-i) \times \underset{\substack{\downarrow \\ x_1 \leftrightarrow x_2 \\ \text{always there.}}}{2}$$

$$= \Delta(x_1, x_2)$$

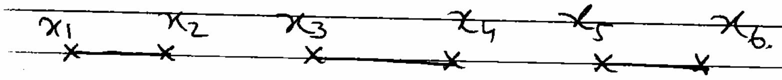






$$(-i)^4\,(-\Delta(x_1,x_2))\,(-\Delta(x_1,x_2))$$

$$\times \frac{1}{2!} \times \frac{1}{2} \times \frac{1}{2} \times \underset{\substack{\downarrow \\ x_1 \leftrightarrow x_2}}{2} \times \underset{\substack{\downarrow \\ x_3 \leftarrow x_4}}{2} \times \underset{\substack{\downarrow \\ \text{perm. of the} \\ \text{two } \Delta\text{'s.}}}{2}$$



$$(-i).(-i).(-i)(-i)(-i)(-i)$$

$$\left(-\Delta(x_1,x_2)\right)\left(-\Delta(x_3,x_4)\right)\left(-\Delta(x_5,x_6)\right)$$

$$= \Delta(x_1,x_2)\,\Delta(x_3,x_4)\,\Delta(x_5,x_6)$$

+ other diagrams.

We can generalize these rules to interacting theory.

$$S = \underset{\checkmark}{S_{free}} + S_{int} \rightarrow \int d^4x \left(-\frac{\lambda}{4!}\phi^4\right)$$

$$\int d^4x \left\{-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right\}$$

$$Z[J] = \int[\mathcal{D}\phi]\, e^{iS_{free}[\phi] + iS_{int}[\phi] + i\int d^4x\, \phi(x)J(x)}$$

$$\int[\mathcal{D}\phi]\sum_{n=0}^{\infty}\frac{\{iS_{int}[\phi]\}^n}{n!}\, e^{iS_{free}[\phi] + i\int d^4x\, \phi(x)J(x)}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \left\{ i S_{int}\left[-\cancel{\phi}\, i \frac{\delta}{\delta J(x)}\right] \right\}^{m}$$

$$\underbrace{\int [\mathcal{D}\phi] \; e^{i S_{free}[\phi] + i \int d^4x \, J(x) \, \phi(x)}}_{Z_{free}[J]}$$

$$\parallel$$

$$\exp\left[-\frac{1}{2} \int d^4x \, d^4x' \, \Delta(x,x') \, J(x) \, J(x')\right]$$

$$\left(i S_{int}\left[-i \frac{\delta}{\delta J(x)}\right]\right)^{m}$$

$$= \left(\frac{-i\lambda}{4!}\right)^{m} \int d^4y_1 \left(-i \frac{\delta}{\delta J(y_1)}\right)^{4}$$

$$\cdots \int d^4y_m \left(-i \frac{\delta}{\delta J(y_m)}\right)^{4}.$$

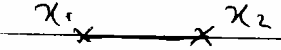
Note: If we want to pick up contribution up to a given power of λ, we only have a finite no. of terms.

Modify the Feynman rules by adding a new type of vertex:

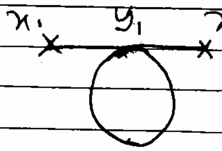
$\times \quad \left(\frac{-i\lambda}{4!}\right)$

Each of the 4 short legs out of the vertex must attach to one end of a propagator

Draw all possible Feynman diagrams and add up their contribution. G(x_1, x_2) to order λ



$\Delta(x_1, x_2)$

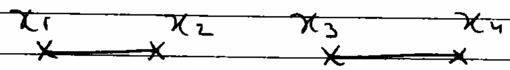


$$\int d^4y_1 \; \Delta(x_1, y_1) \cdot \Delta(x_2, y_1) \; \Delta(y_1, y_1) \times \left(-i\frac{\lambda}{4!}\right) \times 4 \times 3$$

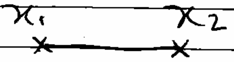
$4 \downarrow$ x_1 can connect to any of the 4-lines

$3 \to$ x_2 can connect to the 3 left over lines

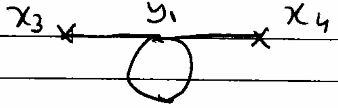
$G(x_1, x_2, x_3, x_4)$



+ 2 other terms

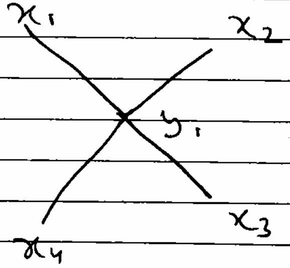


+ 5 other terms.



$$\Delta(x_1,x_2) \cdot \int d^4y_1 \left(-\frac{i\lambda}{4!}\right) \Delta(x_1, y_1)\, \Delta(x_2,y_1)$$

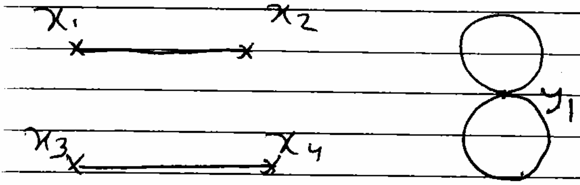
$$\Delta(y_1, y_1) \times 4 \times 3$$



$$\int d^4y_1 \left(-\frac{i\lambda}{4!}\right) \Delta(x_1, y_1)$$

$$\Delta(x_2, y_1)\ \Delta(x_3, y_1)\ \Delta(x_4, y_1)$$

$$\times 4 \times 3 \times 2.$$



$$\Delta(x_1, x_2)\ \Delta(x_3, x_4)$$

$$\int d^4y_1 \left(-\frac{i\lambda}{4!}\right)$$

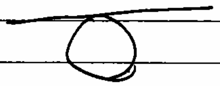
$$\Delta(y_1, y_1)\ \Delta(y_1, y_1)$$

$$\times 3$$

+ 2 more diagrams.

$\Delta(y,y)$: divergent.

In canonical quantization, normal ordering removes contraction of two lines from the same vertex.



not allowed.

We can follow the same prescription here.

However we shall see later that there are other divergences which are not removed by normal ordering.

$\leadsto$ need renormalization.

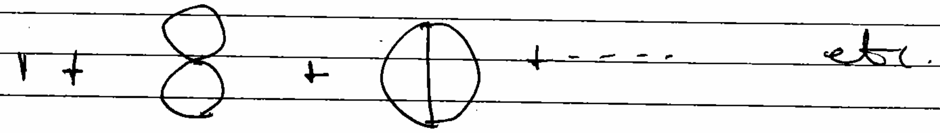
Also cures.

..

Effect of dividing by Z[0]:

$$Z[0] = \exp\left(i S_{int}\left[-i\frac{\delta}{\delta J}\right]\right) Z_{free}[J]\Big|_{J=0}$$

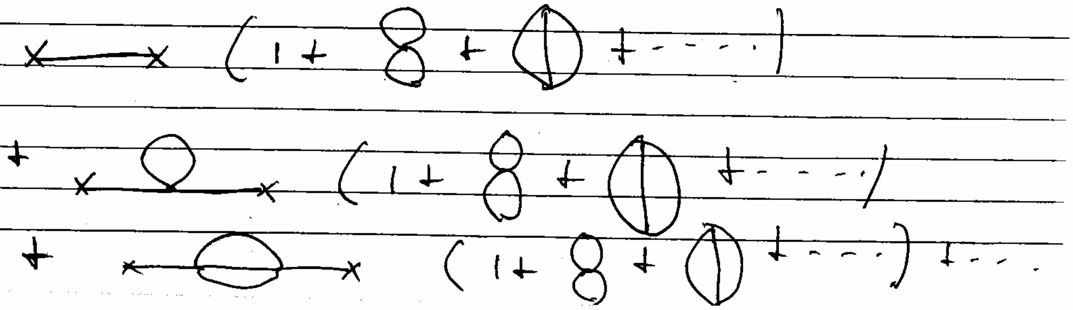
$$\Downarrow$$



~ vacuum bubbles.

In the numerator the graphs may be organised to have this as a factor.

e.g. 2 point function:



Non-trivial result (Prove it)

Combinatoric factors inside each ( ) are the same as that in the denominator.

⇒ The effect of dividing by Z[0] is to remove the vacuum graphs.

Draw only ~~connect~~ those diagrams for which each vertex can be continuously connected to at least one external line.

Momentum space Green's function:

$$\tilde{\phi}(x) = \int e^{-ik.x}\,\phi(x)\,d^4x$$

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik.x}\,\tilde{\phi}(k)$$

$$k.x = \eta_{\mu\nu}\,k^{\mu}\,x^{\nu} = -k^0x^0 + \vec{k}.\vec{x}$$

$$\left\langle \prod_{j=1}^{n} \tilde{\phi}(k_j) \right\rangle$$

$$\equiv \int d^4x_1 \cdots d^4x_n \, \langle\Omega| T\left(\prod_{i=1}^{n} \hat{\phi}(x_i)\right)|\Omega\rangle$$

→ relevant for S-matrix.

Feynman rules for this can be derived from those of position space Greens function, but we shall follow a more direct approach.

$$(1c)\quad \int[\mathcal{D}\phi(x)] \rightarrow \underset{\swarrow}{\int[\mathcal{D}\tilde{\phi}(k)]}$$

Can be defined in the same way by discretizing momentum space.

$$(k^0, k^1, k^2, k^3) = \underset{\substack{\swarrow \\ \text{Small no.}}}{\tilde{\Delta}}\;\underset{\substack{\Downarrow \\ \text{integers}}}{(n^0, n^1, n^2, n^3)}$$

(Corresponds to periodic b.c. on $\phi(x)$: $\phi(x^0+\hat{\Delta}, \vec{x}) = \phi(x^0, \vec{x})$ etc.)

$$\int[\mathcal{D}\tilde{\phi}] = \tilde{\mathcal{N}} \int \prod_{n^0, n^1, n^2, n^3} d\tilde{\phi}_{n^0, n^1, n^2, n^3}$$

Functional differentiation can be defined in the same way.

$$\frac{\delta\tilde{\phi}(k)}{\delta\tilde{\phi}(k')} = \delta^{(4)}(k-k') \quad \text{etc.}$$

$$Z[J] = \int [\mathcal{D}\tilde{\phi}] \; e^{iS + i\int \frac{d^4k}{(2\pi)^4} \tilde{J}(-k)\tilde{\phi}(k)}$$

$$\equiv \tilde{Z}[\tilde{J}]$$

Use $\int d^4x \; \phi(x) J(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{J}(-k)\tilde{\phi}(k)$

$$\left\langle \prod_{i=1}^{n} \tilde{\phi}(k_i) \right\rangle$$

$$= \left(-i \frac{\delta}{\delta \tilde{J}(-k_1)}\right) \cdots \left(-i \frac{\delta}{\delta \tilde{J}(-k_n)}\right) \tilde{Z}[\tilde{J}]$$

Example: Free scalar field theory:

$$S = \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \right]$$

$$= -\int \frac{d^4k}{(2\pi)^4} \; \frac{1}{2} \; \tilde{\phi}(-k) \; (k^2+m^2) \; \tilde{\phi}(k)$$

$$\tilde{Z}[\tilde{J}] = \int [\mathcal{D}\tilde{\phi}(k)]. \exp\Big[-\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}$$

$$\big[\tilde{\phi}(-k)\,(k^2+m^2)\,\tilde{\phi}(k) + \tilde{J}(-k)\,\tilde{\phi}(k) + \tilde{J}(k)\,\tilde{\phi}(-k)\big]$$

$$= \int [\mathcal{D}\tilde{\phi}(k)]\ \exp\Big[-\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}$$

$$\big\{\tilde{\phi}(-k) + (k^2+m^2)^{-1}\,\tilde{J}(-k)\big\}\,(k^2+m^2)$$

$$\big\{\tilde{\phi}(k) + (k^2+m^2)^{-1}\,\tilde{J}(k)\big\}$$

$$+\frac{i}{2}\,\tilde{J}(-k)\,(k^2+m^2)^{-1}\,\tilde{J}(k)\Big]$$

Can be ignored

$$= \int [\mathcal{D}\tilde{\chi}(k)]\ \exp\Big[-\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\ \tilde{\chi}(-k)\,(k^2+m^2)\,\tilde{\chi}(k)\Big]$$

$$\times\ \exp\Big[\frac{i}{2}\,\tilde{J}(-k)\,(k^2+m^2)^{-1}\,\tilde{J}(k).\Big]$$

$$= N\ \exp\Big[-\frac{1}{2}\int \frac{d^4k_1}{(2\pi)^4}\,\frac{d^4k_2}{(2\pi)^4}\ \tilde{\Delta}(k_1,k_2)$$

$$\tilde{J}(-k_1)\ \tilde{J}(-k_2)\Big]$$

$$\tilde{\Delta}(k_1,k_2) = -\frac{i}{k_1^2+m^2-i\epsilon}\,(2\pi)^4\,\delta^{(4)}(k_1+k_2)$$

$$< \prod_{i=1}^{2n} \tilde{\phi}(k_i) >$$

$$= \frac{1}{Z[0]} \left(-i \frac{\delta}{\delta \tilde{J}(-k_1)}\right) \cdots \left(-i \frac{\delta}{\delta \tilde{J}(-k_2)}\right) Z[J]$$

$$= \tilde{\Delta}(k_1, k_2)\, \tilde{\Delta}(k_3, k_4) \cdots \tilde{\Delta}(\tilde{k}_{2n-1}, \tilde{k}_{2n})$$

+ permutations (inequivalent)

Effect of interaction terms:

$$e^{iS_{int}} = \sum_{n=0}^{\infty} \frac{1}{n!} (iS_{int})^n$$

$$\int [\mathcal{D}\tilde{\phi}]\, e^{i(S_0 + S_{int}) + i\int \frac{d^4k}{(2\pi)^4} \tilde{J}(-k)\tilde{\phi}(k)} \prod_{i=1}^{n} \tilde{\phi}(k_i)$$

$$= \left(-i \frac{\delta}{\delta \tilde{J}(-k_1)}\right) \cdots \left(-i \frac{\delta}{\delta \tilde{J}(-k_n)}\right)$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left[iS_{int}\left(-i \frac{\delta}{\delta \tilde{J}(-k)}\right)\right]^n \tilde{Z}[\tilde{J}]$$

$$S_{int}[\phi(x)] = -\frac{\lambda}{4!}\int d^4x\,(\phi(x))^4$$

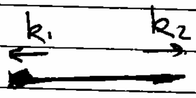
$$= -\frac{\lambda}{4!}\int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_4}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}(k_1 + \cdots k_4)$$

$$\tilde{\phi}(k_1) \cdots\cdots \tilde{\phi}(k_n)$$

$$S_{int}\left[-i\frac{\delta}{\delta\tilde{J}(-k)}\right]$$

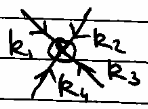
$$= -\frac{\lambda}{4!}\int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_4}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}(k_1 + \cdots k_4)$$

$$\left(-i\frac{\delta}{\delta\tilde{J}(-k_1)}\right) \cdots\cdots \left(-i\frac{\delta}{\delta\tilde{J}(-k_4)}\right).$$

Propagator: 

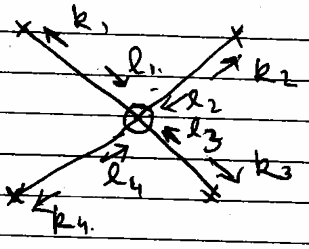
$\tilde{\Delta}(k_1, k_2)$

$$\left(-i\frac{\delta}{\delta\tilde{J}(-k)}\right)$$ 

$$iS_{int}\left[-i\frac{\delta}{\delta\tilde{J}(-k)}\right]$$ 

$$\rightarrow -i\frac{\lambda}{4!}\int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_4}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}(k_1 + \cdots k_4)$$

Example: $\langle \tilde{\Phi}(k_1) \cdots \tilde{\Phi}(k_n) \rangle$



$$\int \frac{d^4 l_1}{(2\pi)^4} \cdots \frac{d^4 l_4}{(2\pi)^4} \; (2\pi)^4 \delta^{(4)}(l_1 + \cdots l_4)$$

$$\left(-i \frac{\lambda}{4!}\right) \; \tilde{\Delta}(k_1, l_1) \; \tilde{\Delta}(k_2, l_2) \; \tilde{\Delta}(k_3, l_3)$$
$$\tilde{\Delta}(k_4, l_4)$$

$$\swarrow$$

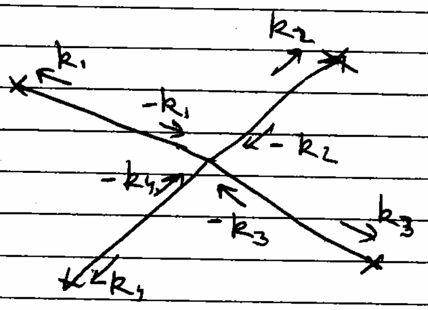
$$\prod_{j=1}^{4} \frac{i}{-k_j^2 + m^2 - i\epsilon} \; (2\pi)^4 \delta^{(4)}(k_1 + l_1) \cdots (2\pi)^4$$
$$\delta^{(4)}(k_4 + l_4)$$

$$\times \; 4 \times 3 \times 2$$

$$= (2\pi)^4 \delta^{(4)}(k_1 + k_2 + k_3 + k_4) \left(-\frac{i\lambda}{4!}\right)$$
$$\prod_{j=1}^{4} \left(\frac{i}{-k_j^2 + m^2 - i\epsilon} \right) \times 4 \times 3 \times 2.$$

Note: many of the δ-functions and momentum integrals are trivial.

⇒ implement diagramatically.

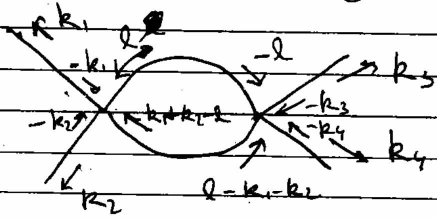


$$(2\pi)^4\,\delta^{(4)}(k_1+k_2+k_3+k_4)$$

$$\left(-i\frac{\lambda}{4!}\right) \times 4\times3\times2$$

$$\times \prod_{j=1}^{4} \frac{i}{-k_j^2-m^2+i\epsilon}$$

Any unfixed momentum must be integrated $\int \frac{d^4k}{(2\pi)^4}$

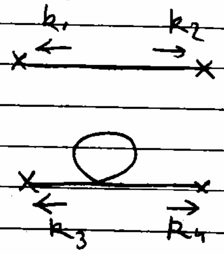


$$\int \frac{d^4 l}{(2\pi)^4}\,(\;\cdots\;)$$

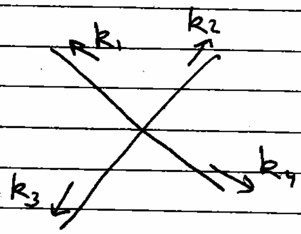
$$\times(2\pi)^4\,\delta^{(4)}(k_1+k_2+k_3+k_4)$$

Connected vs. disconnected diagrams:

Compare:



vs.



$(2\pi)^4 \, \delta^{(4)}(k_1+k_2)$

$\times (2\pi)^4 \, \delta^{(4)}(k_3+k_4)$

$\times \ldots$

$(2\pi)^4 \, \delta^{(4)}(k_1+k_2+k_3+k_4)$

Overall 4-momentum conservation is a must.

However the first diagram puts additional conditions.

→ vanishes for generic momenta satisfying momentum conservation.

From now on we shall work with generic momentum & hence ignore disconnected diagrams.

→ Okay for particle physics example where the detectors are placed at generic angles, avoiding 'forward scattering'.

However in general, for other applications, disconnected diagrams may be important.

S-matrix

Define $G^{(n)}(-k_1, \cdots -k_n) = \underset{\text{connected}}{\langle \prod_{i=1}^{n} \tilde{\phi}(k_i) \rangle_c}$

Define $G^{(2)}(k)$ via:

$$G^{(2)}(k_1, k_2) = (2\pi)^4 \delta^{(4)}(k_1 + k_2)\, G^{(2)}(k_1).$$

Typically $G^{(2)}(k)$ has pole at some value of $k^2 = -\underset{\substack{\downarrow \\ \text{Physical mass of particle.}}}{m^2_{phys}}$

Near $k^2 \simeq -m^2_{phys}$,

$$G^{(2)}(k) \simeq \frac{-i \overset{\text{Some constant}}{Z}}{k^2 + m^2_{phys} - i\epsilon}$$

$$S(\overbrace{\vec{p}_1, \cdots \vec{p}_n}^{\text{outgoing}}; \overbrace{\vec{k}_1, \cdots \vec{k}_m}^{\text{incoming}}) = (-\sqrt{Z})^{m+n} \prod_{i=1}^{m} \frac{1}{\sqrt{(2\pi)^3 2\omega_{\vec{k}_i}}} \prod_{j=1}^{n} \frac{1}{\sqrt{(2\pi)^3 2\omega_{\vec{p}_j}}} \prod_{i=1}^{m} \{G^{(2)}(k_i)\}^{-1} \prod_{i=1}^{n} \{G^{(2)}(p_i)\}^{-1}$$

$$G^{(m+n)}(k_1, \cdots k_m; -p_1, \cdots -p_n)\Big|_{\substack{k_i^0 = \omega_{\vec{k}_i} \\ p_i^0 = \omega_{\vec{p}_i}}}$$

One advantage of path integral quantization over canonical quantization → easy to deal with interactions containing derivatives.

e.g. $S_{int} = K \int d^4x \, \eta^{\mu\nu} \phi \, \partial_\mu \phi \, \partial_\nu \phi$

In canonical quantization this will require defining new momentum Π, eliminate $\partial_0 \phi$ in terms of Π, define new $\mathcal{H}_{int}$ etc.

→ Cumbersome.

In path integral formulation we wri

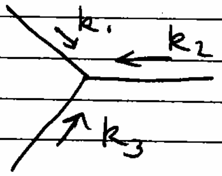
$$i S_{int} = i K \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(k_1+k_2+k_3)$$

$$\tilde{\Phi}(k_1) \, \tilde{\Phi}(k_2) \, \tilde{\Phi}(k_3) \times \quad i k_2 \cdot i k_3$$

$$\Downarrow$$

$$-\frac{1}{3}(k_2 \cdot k_3 + k_1 \cdot k_2 + k_1 \cdot k_3)$$

Vertex:



$$-\frac{2\kappa}{3}\left(k_1\cdot k_2 + k_2\cdot k_3 + k_3\cdot k_1\right)$$
$$\times\ (2\pi)^4\ \delta^{(4)}(k_1+k_2+k_3)$$

Multiple bosonic fields:

$$\phi^{(1)}(x), \cdots\cdots \phi^{(s)}(x)$$
$$\rightsquigarrow \tilde{\phi}^{(1)}(k), \cdots \tilde{\phi}^{(s)}(k)$$

$$S = S_{free} + S_{int}$$

$\downarrow$ general form

$$\frac{1}{2}\int d^4x\ \phi^{(\ell)}(x)\ D_{\ell\ell'}\ \phi^{(\ell')}(x)$$

$\downarrow$

Some differential operator, e.g.

$$\left(\delta_{\ell\ell'}\ \square - C_{\ell\ell'}\right)$$

Momentum space:

$$S_{free} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{\phi}^{(l)}(-k)\, M_{ll'}(k)\, \tilde{\phi}^{(l')}(k)$$

$$\Downarrow$$

by $\frac{\partial}{\partial x^\mu} \to i k_\mu$ in $D_{ll'}$.

We can choose:

$$M_{ll'}(k) = M_{l'l}(-k).$$

ϕ

Proof: $$S_{free} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{\phi}^{(l)}(-k)\, \tilde{\phi}^{(l')}(k)$$

$$\underbrace{\left\{ \frac{1}{2} M_{ll'}(k) + \frac{1}{2} M_{l'l}(-k) \right\}}_{\text{New } M_{ll'}(k)}$$

$$\tilde{Z}_{free}[\tilde{J}] = \int [\mathcal{D}\tilde{\phi}]\, e^{iS_{free} + i\int \frac{d^4k}{(2\pi)^4} \tilde{J}^{(l)}(-k)\, \tilde{\phi}^{(l)}(k)}$$

~~$\int \frac{d^4k}{(2\pi)}$~~

$$S_{free} + 2\int \frac{d^4k}{(2\pi)^4} \tilde{J}^{(l)}(-k)\, \tilde{\Phi}^{(l)}(k)$$

$$= \int \frac{d^4k}{(2\pi)^4} \Big[\frac{1}{2} \tilde{\Phi}^{(l)}(-k)\, M_{ll'}(k)\, \tilde{\Phi}^{(l')}(k)$$

$$+ \tilde{J}^{(l)}(-k)\, \tilde{\Phi}^{(l')}(k)\Big]$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{1}{2} \Big[\big\{\tilde{\Phi}^{(l)}(-k) + \tilde{J}^{(q)}(-k)\, M^{-1}_{ql}(k)\big\}$$

$$M_{ll'}(k) \big\{\tilde{\Phi}^{(l')}(k) + M^{-1}_{l'q'}(k)\, \tilde{J}^{(q')}(k)\big\}$$

$$- \tilde{J}^{(q)}(-k)\, M_{qq'}(k)\, \tilde{J}^{(q')}(k)\Big]$$

$$M^{-1}_{ql}(k)\, M_{ll'}(k) = \delta_{ql'} \rightarrow \text{defines } M^{-1}_{ql}(k)$$

~~Note~~: Ex: $M^{-1}_{ql}(k) = M^{-1}_{lq}(-k)$

(Inverse of transpose = transpose of inverse)

$$\tilde{\chi}^{(l)}(k) = \tilde{\Phi}^{(l)}(k) + M^{-1}_{lq}(k)\, \tilde{J}^{(q)}(k)$$

$$\tilde{Z}[J] = \int [\mathcal{D}\tilde{\chi}(k)]$$

$$\exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{\chi}^{(l)}(-k)\, M_{ll'}(k)\, \tilde{\chi}^{(l')}(k)\right]$$

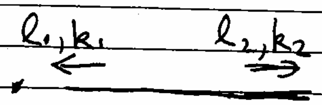
$$\exp\left(-\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{J}^{(q)}(-k)\, M^{-1}_{qq'}(k)\, \tilde{J}^{(q')}(k)\right]$$

$$\Downarrow$$

$$\exp\left[-\frac{1}{2}\int \frac{d^4k_1}{(2\pi)^4}\,\frac{d^4k_2}{(2\pi)^4}\, \Delta_{l_1l_2}(k_1,k_2)\, \tilde{J}^{(l_1)}(-k_1)\, \tilde{J}^{(l_2)}(-k_2)\right]$$

$$\Delta_{l_1l_2}(k_1,k_2) = i\, M^{-1}_{l_1l_2}(k_1)\, (2\pi)^4\, \delta^{(4)}(k_1+k_2)$$

Propagator:



$\Delta_{l_1l_2}(k_1,k_2)$

Note: If Δ_{l_1,l_2} is block diagonal then we can use different symbols for each block.

——— ⇒ for Δ in one block

〰〰 ⇒ for Δ in another block.

Interactions.

General form:

$$S_{int} = \sum_{n=3}^{\infty} \frac{1}{n!} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4}$$

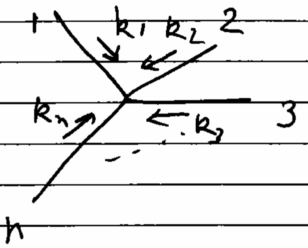
$$\tilde{\phi}^{(\ell_1)}(k_1) \cdots \tilde{\phi}^{(\ell_n)}(k_n) \; V^{(n)}_{\ell_1 \cdots \ell_n}(k_1, \cdots k_n)$$

$$(2\pi)^4 \, \delta^{(4)}(k_1 + \cdots k_n)$$

$\downarrow$

symmetric under
$\ell_i \leftrightarrow \ell_j$, $k_i \leftrightarrow k_j$

Vertex:



$$\frac{i}{n!} \, V^{(n)}(k_1, \cdots k_n)$$

$$(2\pi)^4 \, \delta^{(4)}(k_1 + \cdots k_n)$$

Note: If the propagator is block diagonal we could ~~we~~ divide the vertex into different terms.

Path integral with fermion fields:

Canonical anti-commutation relations:

$[illegible] = 0 = \{\hat{\psi}^{\dagger}_{\alpha}(\vec{x},t), \hat{\psi}^{\dagger}_{\beta}(\vec{y},t)\}$

$$\{\hat{\psi}_{\alpha}(\vec{x},t), \hat{\psi}^{\dagger}_{\beta}(\vec{x},t)\} = \hbar\, \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{y})$$

$\hat{\mathcal{H}}$: given in terms of $\hat{\psi}_{\alpha}(\vec{x},t)$, $\hat{\psi}^{\dagger}_{\alpha}(\vec{x},t)$.

→ Perfectly sensible quantum system.

→ but it is not the quantization of a classical system in a standard sense.

Cannot get from $[q_i, p_j] = \delta_{ij}\, i\hbar$

In the same sense, path integral over fermions will be a formal integral.

~~$\prod \psi^{+}_{\beta}(y_j))$~~

$$\equiv \frac{\langle \Omega | T(\prod_i \hat{\psi}_{\alpha_i}(x_i) \prod_j \hat{\psi}^{+}_{\beta_j}(y_j)) | \Omega \rangle}{\langle \Omega | \Omega \rangle}$$

$\downarrow$

well defined

$$= \frac{\int [\mathcal{D}\psi][\mathcal{D}\psi^{+}]\, e^{iS} \prod_{i=1}^{n} \psi_{\alpha_i}(x_i) \prod_{j=1}^{n} \psi^{+}_{\beta_j}(y_j)}{\int [\mathcal{D}\psi][\mathcal{D}\psi^{+}]\, e^{iS}}$$

$$S_{free} = \int d^4x\, \overline{\Psi}(x) \left(i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - m\right) \psi(x)$$

$\sim$ gives Dirac eq. if we treat ψ^{+}_{α}, ψ_{α} as independent classical variables and require $\delta S_{free} = 0$. under arbitrary variation of ψ_{α}, ψ^{+}_{α}.

Do not treat these as ordinary path integral but follow the following steps:

Step I Discretize

$$\psi_\alpha(\vec{x},t) \to \psi_{\alpha, t_0, \vec{t}}$$

or momentum space

$$\tilde{\psi}_\alpha(k) \to \tilde{\psi}_{\alpha,(n_0,\vec{n})}.$$

$$[\mathcal{D}\psi][\mathcal{D}\tilde{\psi}] = \prod_{\alpha} \prod_{n_0,\vec{n}} d\tilde{\psi}_{\alpha, n_0, \vec{n}} \, d\tilde{\psi}^{+}_{\alpha, n_0, \vec{n}}$$

$$S = \int \frac{d^4k}{(2\pi)^4} \bar{\psi}(-k)\left(-\gamma^\mu k_\mu - m\right)\psi(k)$$

$$\to \frac{1}{(2\pi)^4} \tilde{\Delta}^4 \sum_{n_0,\vec{n}} \bar{\psi}_{\beta, -n_0, \vec{n}} \left\{-\gamma^\mu n_\mu \tilde{\Delta} - m\right\}_{\alpha\beta} \psi_{\alpha, n_0, \vec{n}}$$

Step 2

Instead of regarding $\psi_{\alpha;n_0,\vec{n}}$, $\psi^{+}_{\alpha;n_0,\vec{n}}$ as ordinary numbers we treat them as some abstract objects — grassman numbers — following different rules.

Rules: Suppose we have n grassmann variables $\theta_1, \cdots \theta_n$.

$$\Leftrightarrow \psi_{\alpha;n_0,\vec{n}}, \quad = \chi_{\alpha;n_0,\vec{n}} + i\,\phi_{\alpha;n_0,\vec{n}}$$

(Complex variables $\theta_1 + i\theta_2$ etc.

$$\left(\theta_1 + i\theta_2\right)^{+} = \theta_1 - i\theta_2)$$ → same rules.

a) $\theta_i\ \theta_j = -\theta_j\ \theta_i \quad \Rightarrow \theta_i^2 = 0.$

b) A function $f(\theta_1, \cdots \theta_n)$ is defined by Taylor series expansion:

$$f(\theta_1, \cdots \theta_n) = A^{(0)} + \sum_i A^{(1)}_i \theta_i$$
$$+ \sum_{i,j} A^{(2)}_{ij} \theta_i \theta_j + \cdots \sum_{i_1, \cdots i_n} A^{(n)}_{i_1 \cdots i_n} \theta_{i_1} \cdots \theta_{i_n}$$

$A^{(k)}_{i_1 \cdots i_k}$ can be taken to be totally anti-symmetric in $i_1, \cdots i_k$.

$A^{(m)}_{i_1 \cdots i_m} = 0$ if $m > n$.

↳ finite no. of terms in the Taylor series.

Specifying f ↔ specifying all the $A^{(k)}_{i_1 \cdots i_k}$.

e.g. $\sin\theta_1 = \theta_1 - \frac{\theta_1^3}{3!} + \cdots = \theta_1$.

$e^{i\theta_1} = 1 + i\theta_1$ etc.

This tells us how to interprete

$$e^{iS}$$

$\rightsquigarrow$ Taylor series expansion in

$\chi_{\alpha;n_0,\vec{n}}$ and $\phi_{\alpha;n_0,\vec{n}}$

Integration rules:

$$\int d\theta_i = 0. \quad \int d\theta_i \, \theta_j = \delta_{ij}$$

$\rightsquigarrow$ ordinary numbers.

$\Rightarrow$ Tells us how to define

$\int [\mathcal{D}\psi, \mathcal{D}\bar{\psi}] \, e^{iS}$ as ordinary numbers.

$$d\theta_1 \, d\theta_2 = - d\theta_2 \, d\theta_1,$$

$$\theta_i \, d\theta_j = - d\theta_j \, \theta_i, \quad \theta_i \theta_j = - \theta_j \theta_i$$

$$f = \sum_k A^{(k)}_{i_1 \cdots i_k} \theta_{i_1} \cdots \theta_{i_k}$$

If only even k's appear in the expansion f is even.

If only odd k's appear in the expansion f is odd.

e.g. $\sin(\theta_1+\theta_2+\theta_3) = \theta_1 + \theta_2 + \theta_3 + \theta_1\theta_2\theta_3$ is odd.

$\cos(\theta_1+\theta_2+\theta_3)$ is even. etc.

Differentiation:

Rules: $\frac{\partial}{\partial\theta_i}(\theta_j) = \delta_{ij}$ $\frac{\partial}{\partial\theta_i}(\text{constant}) = 0$

$$\frac{\partial}{\partial\theta_i}(FG) = \frac{\partial F}{\partial\theta_i} G + (-1)^F F \frac{\partial G}{\partial\theta_i}$$

$(-1)^F \equiv 1$ if F is even

$= -1$ if F is odd.

It follows from this that

$$\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} = -\frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i}$$

Proof:

$$F(\theta_1, \cdots \theta_n)$$

$$= F_1 + \theta_i F_2 + \theta_j F_3 + \theta_i \theta_j F_4$$

functions of other θ_k's.

$$\left.\begin{aligned} \frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} F &= -F_4 \\ \frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i} F &= F_4 \end{aligned}\right\} \text{ if } i \neq j.$$

$$\frac{\partial^2}{\partial\theta_i^2} = \frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_i} = 0.$$

Useful notation: $\overleftarrow{\frac{\partial}{\partial\theta_i}}$

$$\theta_i \overleftarrow{\frac{\partial}{\partial\theta_j}} = \delta_{ij} \qquad (\text{Constant}) \overleftarrow{\frac{\partial}{\partial\theta_j}} = 0.$$

$$FG \overleftarrow{\frac{\partial}{\partial\theta_i}} = F\left(G\overleftarrow{\frac{\partial}{\partial\theta_i}}\right) + (-1)^G F\overleftarrow{\frac{\partial}{\partial\theta_i}}\, G.$$

Ex. $\frac{\partial}{\partial\theta_i} F = (-1)^{F+1} F \frac{\overleftarrow{\partial}}{\partial\theta_i}$

Why define $\frac{\partial}{\partial\theta_i}$ with these properties?

→ allows us to do integration by parts.

$$\frac{\partial}{\partial\theta_i} \underset{\overset{\parallel}{A+\theta_i B}}{F(\vec{\theta})} = \int d\theta_i \, F(\vec{\theta})$$

$$\int d\theta_i \, F(\vec{\theta}) \frac{\partial}{\partial\theta_i} G(\vec{\theta})$$

$$= \frac{\partial}{\partial\theta_i}\left(F(\vec{\theta}) \frac{\partial}{\partial\theta_i} G(\vec{\theta})\right)$$

$$= \frac{\partial F(\vec{\theta})}{\partial\theta_i} \frac{\partial}{\partial\theta_i} G(\vec{\theta})$$

$$\int d\theta_i \, \frac{\partial F(\vec{\theta})}{\partial\theta_i} G(\vec{\theta}) = \frac{\partial}{\partial\theta_i} \frac{\partial F}{\partial\theta_i} G(\vec{\theta})$$

$$= (-1)^{F+1} \frac{\partial F}{\partial\theta_i} \frac{\partial G}{\partial\theta_i}$$

$$\int d\theta_i \left(F(\vec{\theta}) \frac{\partial}{\partial \theta_i} G(\vec{\theta}) \right)$$

$$= (-1)^{F+1} \int d\theta_i \frac{\partial F(\vec{\theta})}{\partial \theta_i} G(\vec{\theta})$$

$$= \int d\theta_i \; F(\vec{\theta}) \frac{\overleftarrow{\partial}}{\partial \theta_i} G(\vec{\theta})$$

For even function we can use the usual rules of integration by parts.

For odd functions we have extra - sign.

chain rule:

If $G(\vec{\theta})$ is an even function then

$$\frac{\partial}{\partial \theta_i} F(G(\vec{\theta})) = F'(G(\vec{\theta})) \frac{\partial G}{\partial \theta_i}$$

Proof: $F(x) = \sum_n a_n x^n$

$$\frac{\partial}{\partial\theta_i} F(\vec{\theta}) = \frac{\partial}{\partial\theta_i} \sum_n a_n G(\vec{\theta})^n$$

$$= \sum_n a_n \Big\{ \frac{\partial G(\vec{\theta})}{\partial\theta_i} (G(\vec{\theta}))^{n-1}$$

$$+ G(\vec{\theta}) \frac{\partial G(\vec{\theta})}{\partial\theta_i} (G(\vec{\theta}))^{n-2} + \cdots \Big\}$$

$$= \sum_n a_n \frac{\partial G(\vec{\theta})}{\partial\theta_i} n (G(\vec{\theta}))^{n-1}$$

$$= \frac{\partial G(\vec{\theta})}{\partial\theta_i} F'(G(\vec{\theta}))$$

$$= F'(G(\vec{\theta})) \frac{\partial G}{\partial\theta_i}$$

This does not work for odd f.

e.g. $\underset{\overset{\parallel}{0}}{\frac{\partial}{\partial\theta}(\cos\theta)} \neq \underset{\overset{\parallel}{\theta}}{\sin\theta}$

Rules for shifting variables:

$$\int d\theta_i \; F(\underset{\substack{\downarrow \\ \text{fn. of other } \theta_k\text{'s.}}}{\theta_i + c}) = \int d\theta_i \; F(\theta_i)$$

Proof: $F(\theta_i) = \underbrace{A + \theta_i B}_{\text{fn. of other } \theta_k\text{'s.}}$

$$\int d\theta_i \; F(\theta_i) = B$$

$$\int d\theta_i \; F(\theta_i + c)$$

$$= \int d\theta_i \{ A + \theta_i B + c B \} = B.$$

Scaling of variables:

For ordinary variables $d(\lambda x) = \lambda dx$

Const. or fn. of other variables

For grassman variables $d(\lambda \theta_i) = \lambda^{-1} d\theta_i$

even fn of other θ_k's.

Proof:

$$F(\vec{\theta}) = A(\vec{\theta}') + \theta_i B(\vec{\theta}')$$

other θ_k's.

$$\int d\theta_i \, F(\vec{\theta}) = B(\vec{\theta}')$$

Define $\theta_i = \lambda(\vec{\theta}')\phi_i$

→ even fn.

$$F(\vec{\theta}) = A(\vec{\theta}') + \phi_i \lambda(\vec{\theta}') B(\vec{\theta}')$$

$$\int d\phi_i \, F(\vec{\theta}) = \lambda(\vec{\theta}') B(\vec{\theta}')$$

$$\Rightarrow \quad d\phi_i = \lambda(\vec{\theta}') \, d\theta_i$$

Generalization:

If $\theta_i = A_{ij}\,\phi_j$

Then $d\theta_1 \cdots d\theta_n = (\det A)^{-1}\, d\phi_1 \cdots d\phi_n$.

Proof:

$$F = \sum_{k=0}^{n} \frac{1}{k!} A^{(k)}_{i_1\cdots i_k}\, \theta_{i_1} \cdots \theta_{i_k}$$

$$\int d\theta_1 \cdots d\theta_n\, F = A^{(n)}_{n\cdots 1}$$

$$F = \sum_{k=0}^{n} \frac{1}{k!} A^{(k)}_{i_k\cdots i_1}\, A_{i_k j_k}\phi_{j_k} \cdots A_{i_1 j_1}\phi_{j_1}$$

$$\int d\phi_1 \cdots d\phi_n\, F$$

$$= A_{i_n n} \cdots A_{i_1 1}\, A^{(n)}_{i_n \cdots i_1}$$

$$\Downarrow$$

$$A^{(n)}_{n\cdots 1}\, \epsilon_{i\, i_1 \cdots i_n}$$

$$\epsilon_{i_1 \cdots i_n} A_{i_n n} \cdots A_{i_1 1} = (\det A|\, A^{(n)}_{n\cdots 1}$$

$$d\theta_1 \cdots d\theta_n = (\det A)^{-1} \, d\phi_1 \cdots d\phi_n.$$

For ordinary variables:

$$\int \prod_{i=1}^{n} dx_i \; \exp\Big(-\frac{1}{2} \underset{\longleftrightarrow}{A_{ij}} \, x_i x_j\Big)$$

symmetric real

$$\propto (\det A)^{-1/2}$$

Proof: diagonalize A

$$A = R^T A_d \underbrace{R}_{\text{orthogonal}}$$

$$\Rightarrow \quad y_i = R_{ik} x_k \qquad \det R = 1$$

$$\Rightarrow \quad \int \prod_i dy_i \; \exp\Big(-\frac{1}{2} \underbrace{y^T A_d y}_{\sum_i a_i y_i^2}\Big)$$

$$\propto \prod_i (a_i)^{-1/2} = (\det A)^{-1/2}$$

For grassman variables:

$$\int \prod_{i=1}^{n} d\theta_i \; \exp\left(-\tfrac{1}{2} \underleftrightarrow{A_{ij}} \; \theta_i \theta_j\right)$$

anti-symmetric

$$\propto \; (\det A)^{1/2}$$

Proof: $A = R^T \underset{\swarrow}{A_{SK}} R$ — orthogonal

$$\begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & 0 & a_2 & \\ & & -a_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

Change variables: $\phi_i = R_{ik} \theta_k$

$$\int \prod_i d\phi_i \; \exp\left[-\tfrac{1}{2} \phi_i (A_{SK})_{ij} \phi_j\right]$$

$$= \int d\phi_1 \cdots d\phi_n \; \exp\left[-a_1 \phi_1 \phi_2 - a_2 \phi_3 \phi_4 \cdots\right]$$

$$= (-a_1)(-a_2) \cdots \; \propto \; (\det A_{SK})^{1/2} = (\det A)^{1/2}$$

δ-function:

$$\int dx\, \delta(x)\, F(x) = F(0)$$

We want

$$\int d\theta_i\, \delta(\theta_i)\, F(\vec{\theta}) = F\Big|_{\theta_i = 0}$$

$$F(\vec{\theta}) = A(\vec{\theta}') + \theta_i B(\vec{\theta}')$$

$$F(\vec{\theta})\Big|_{\theta_i = 0} = A(\vec{\theta}')$$

What fr. of θ_i gives

$$\int d\theta_i\, \delta(\theta_i)\, \left(A(\vec{\theta}') + \theta_i B(\vec{\theta}')\right) = A(\vec{\theta}')?$$

$$\delta(\theta_i) = \theta_i$$

$$\delta^{(n)}(\vec{\theta}) = \delta(\theta_n) \cdots \delta(\theta_1)$$

If $\phi_i = \sum_{j=1}^{n} A_{ij}\, \theta_j$

then

$$\delta^{(n)}(\vec{\phi}) = (\det A)\ \delta^{(n)}(\vec{\theta}).$$

Proof:

$$\delta^{(n)}(\vec{\phi}) = \phi_n \cdots \phi_1$$

$$= A_{n i_n} A_{(n-1) i_{n-1}} \cdots A_{1 i_1} \underbrace{\theta_{i_n} \cdots \theta_{i_1}}_{\epsilon_{i_1 \cdots i_n} \theta_n \cdots \theta_1}$$

$$= (\det A)\ \theta_n \cdots \theta_1$$

Complex grassman variables:

$$\theta_i = \frac{1}{\sqrt{2}}(\phi_i + i\psi_i)$$

$$\theta_i^\dagger = \frac{1}{\sqrt{2}}(\phi_i - i\psi_i)$$

$$d\theta_i \, d\theta_i^\dagger = \left\{ \det \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \right\}^{-1} d\phi_i \, d\psi_i$$

$$= i \, d\phi_i \, d\psi_i$$

Thus up to a normalization we can take θ_i, $\theta_i^\dagger$ as independent grassman variables and proceed.

~~$\int d\theta_i = 0 \qquad \int d\theta_i \, \theta_i$~~

~~$= \int \sqrt{2}(d\phi_i + i d\psi_i) \frac{1}{\sqrt{2}}(\phi_i + i\psi_i)$~~

$$\int d\theta_i \, d\theta_i^\dagger \, \theta_i^\dagger \theta_i = \overset{\cancel{2}}{i} \int d\phi_i \, d\psi_i \; \hat{i} \; \phi_i \psi_i = 1$$

Quantity of interest:

$$\int [\mathcal{D}\psi][\mathcal{D}\bar\psi]\, e^{iS}\, \psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\, \bar\psi_{\beta_1}(k_1)\,\cdots\,\bar\psi_{\beta_m}(k_m)$$

$$\Rightarrow \int [\mathcal{D}\tilde\psi(k)][\mathcal{D}\bar{\tilde\psi}(k)]\, e^{iS}\, \tilde\psi_{\alpha_1}(k_1)\cdots\tilde\psi_{\alpha_n}(k_n)\, \bar{\tilde\psi}_{\beta_1}(p_1)\cdots\bar{\tilde\psi}_{\beta_m}(p_m)$$

Note:

$$\psi_\alpha(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\, \tilde\psi_\alpha(k)$$

$$\bar\psi_\alpha(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x}\, \underset{\underset{\tilde\psi(k)^\dagger\gamma^0}{\|}}{\overline{\tilde\psi_\alpha(k)}}$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\, \underset{\underset{\bar{\tilde\psi}(k)}{\longleftrightarrow}}{\overline{\tilde\psi(-k)}}$$

$$\tilde\psi_\alpha(k) \leftrightarrow \tilde\psi_{\alpha;\, n^0,\cdots n^3}\;;\quad \bar{\tilde\psi}_\alpha(k) \longrightarrow \bar{\tilde\psi}_{\alpha;\, n^0,\cdots n^3}.$$

Define: New grassman nos.

$$\tilde{Z}[\tilde{J},\bar{\tilde{J}}] = \int [\mathcal{D}\tilde{\Psi}(k)][\mathcal{D}\bar{\tilde{\Psi}}(k)]$$

$$\exp\left[iS + i\int \left(\bar{\tilde{J}}(-k)\,\tilde{\Psi}_\alpha(k) + \bar{\tilde{\Psi}}(-k)\,\tilde{J}_\alpha(k) \right) \frac{d^4k}{(2\pi)^4} \right]$$

$$\langle \tilde{\Psi}_{\alpha_1}(k_1) \cdots \tilde{\Psi}_{\alpha_n}(k_n)\, \tilde{\Psi}_{\beta_1}(p_1) \cdots \tilde{\Psi}_{\beta_m}(p_m) \rangle$$

$$= \frac{1}{\tilde{Z}[0]} \left\{ (-i) \frac{(2\pi)^4\,\delta}{\delta \bar{\tilde{J}}_{\alpha_1}(-k_1)} \cdots (-i) \frac{(2\pi)^4\,\delta}{\delta \bar{\tilde{J}}_{\alpha_n}(-k_n)} \right\}$$

$$\cdot \left\{ (+i)(2\pi)^4 \frac{\delta}{\delta \tilde{J}_{\beta_1}(-p_1)} \right\} \cdots \left\{ +i(2\pi)^4 \frac{\delta}{\delta \tilde{J}_{\beta_m}(-p_m)} \right\}$$

$$\tilde{Z}[\tilde{J},\bar{\tilde{J}}] \Big|_{\substack{\tilde{J}=0 \\ \bar{\tilde{J}}=0}}$$

Computation of $Z_{free}[\tilde{J}, \bar{\tilde{J}}]$

$$S_{free} = \int d^4x \, \bar{\psi}(x) \left(i \gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \psi$$

$$= \int \frac{d^4k}{(2\pi)^4} \, \tilde{\bar{\psi}}(-k) \, (\underbrace{- k_\mu \gamma^\mu}_{\not{k}} - m) \, \tilde{\psi}(k)$$

$$\not{k} = - k_\mu \gamma^\mu = k^0 \gamma^0 - \vec{k} \cdot \vec{\gamma}$$

$$\tilde{Z}_{free}[\tilde{J}(k), \bar{\tilde{J}}(k)]$$

$$= \int [\mathcal{D}\tilde{\psi}][\mathcal{D}\bar{\tilde{\psi}}] \; \exp\Big[i \int \frac{d^4k}{(2\pi)^4} \, \bar{\tilde{\psi}}_\alpha(-k) (\not{k} - m)_{\alpha\beta} \tilde{\psi}_\beta(k)$$

$$+ i \int \bar{\tilde{J}}_\alpha(-k) \tilde{\psi}_\alpha(k) + \bar{\tilde{\psi}}_\alpha(-k) \tilde{J}_\alpha(k) \Big]$$

$$= \int [\mathcal{D}\tilde{\psi} \; \mathcal{D}\bar{\tilde{\psi}}]$$

$$\exp\Big[i \int \frac{d^4k}{(2\pi)^4} \left\{ \bar{\tilde{\psi}}_\alpha(-k) + \tilde{J}_\gamma(-k) (\not{k} - m)^{-1}_{\gamma\alpha} \right\}$$

$$(\not{k} - m)_{\alpha\beta} \left\{ \tilde{\psi}_\beta(k) + (\not{k} - m)^{-1}_{\beta\delta} \tilde{J}_\delta(k) \right\}$$

$$- i \int \frac{d^4k}{(2\pi)^4} \, \bar{\tilde{J}}_\beta(-k) \, (\not{k} - m)^{-1}_{\beta\delta} \, \tilde{J}_\delta(k) \Big]$$

$$\tilde{\chi}_\beta(k) = \tilde{\Psi}_\beta(k) + (\not{k}-m)^{-1}_{\beta\delta}\,\tilde{J}_\delta(k)$$

$$\tilde{\bar{\chi}}_\alpha(-k) = \tilde{\bar{\Psi}}_\alpha(-k) + \tilde{\bar{J}}_\gamma(-k)\,(\not{k}-m)^{-1}_{\gamma\delta}$$

$$\tilde{Z}_{free}\left[\tilde{J}(k), \tilde{\bar{J}}(k)\right]$$

$$= N \exp\left[-i\int \frac{d^4k}{(2\pi)^4}\,\tilde{\bar{J}}_\beta(-k)(\not{k}-m)^{-1}_{\beta\delta}\,\tilde{J}_\delta(k)\right]$$

$$\langle \Psi_\alpha(\overset{k_1}{\ })\ \bar{\Psi}_\beta(\overset{k_2}{\ })\rangle$$

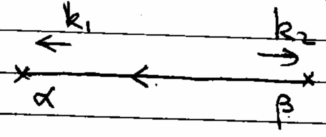
$$= \frac{1}{Z_{free}[0]}\left(-(2\pi)^4\, i\,\frac{\delta}{\delta\tilde{\bar{J}}_\alpha(-k_1)}\right)\left(+(2\pi)^4\, i\,\frac{\delta}{\delta\tilde{J}_\beta(-k_2)}\right)\tilde{Z}_{free}\Big|_{\tilde{J}=\tilde{\bar{J}}=0}$$

$$= S_{\alpha\beta}(k_1; k_2)$$

$$= i\,(\not{k}_1-m)^{-1}_{\alpha\beta}\,(2\pi)^4\,\delta^{(4)}(k_1+k_2)$$

$$\left[\frac{i}{\not{k}_1-m+i\epsilon}\right]_{\alpha\beta}(2\pi)^4\,\delta^{(4)}(k_1+k_2)$$

Notation:

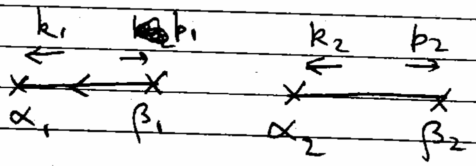


The arrow is directed towards

$$\Psi \quad \left(\frac{\delta}{\delta \bar{J}_\alpha}\right)$$

$$\left(\frac{i}{\not k_1 - m}\right)_{\alpha\beta} (2\pi)^4 \, \delta^{(4)}(k_1+k_2)$$

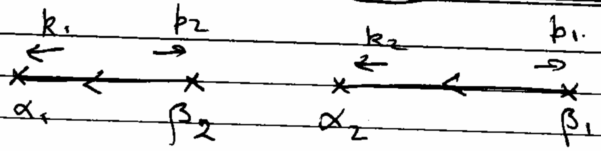
$$\langle \tilde{\Psi}_{\alpha_1}(k_1) \, \tilde{\Psi}_{\alpha_2}(k_2) \, \bar{\tilde{\Psi}}_{\beta_1}(p_1) \, \bar{\tilde{\Psi}}_{\beta_2}(p_2) \rangle$$



$$-S_{F\alpha_1\beta_1}(k_1, p_1)$$

$$S_{F\alpha_2\beta_2}(k_2, p_2)$$

Need to exchange the position of $\tilde{\Psi}_{\alpha_2}(p_2)$ & $\bar{\tilde{\Psi}}_{\beta_1}(p_1)$.



$$+\,S_{F\alpha_1\beta_2}(k_1,p_2)\;S_{F\alpha_2\beta_1}(k_2,p_1)$$

General rule:

For any graph draw ~~lines connecting the two edges in the~~ the propagators in the original diagram.

of − signs = # of X-ings + # of

(−1) one X-ing $\bar{\psi}\ \psi$ order.

$$\tilde{\psi}_{\alpha_1}(k_1)\quad \tilde{\psi}_{\alpha_2}(k_2)\quad \tilde{\bar{\psi}}_{\beta_1}(p_1)\quad \tilde{\bar{\psi}}_{\beta_2}(p_2)$$

$$\tilde{\psi}_{\alpha_1}(k_1)\quad \tilde{\psi}_{\alpha_2}(k_2)\quad \tilde{\bar{\psi}}_{\beta_1}(p_1)\quad \tilde{\bar{\psi}}_{\beta_2}(p_2)$$

No crossing. +1

All pairs are $\psi\bar{\psi}$ → no − sign from there.

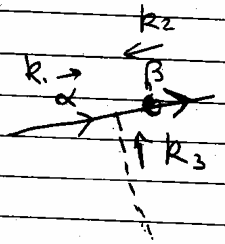
Interactions:

Example: $S = g\int \bar{\Psi}\,\Psi\,\phi\;d^4x$

scalar

$$= \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_3}{(2\pi)^4}\; \tilde{\bar{\Psi}}_\alpha(k_1)\; \tilde{\Psi}_\alpha(k_2)\; \phi(k_3)$$

$$(2\pi)^4\; \delta^{(4)}(k_1 + k_2 + k_3)$$



$$ig\,\delta_{\alpha\beta}\quad (2\pi)^4\; \delta^{(4)}(k_1 + k_2 + k_3)$$

How to check if we have got all diagrams and combinatoric factors.

Consider a scalar field theory and a given order diagram.

Count total no. of lines coming out of external legs and vertices.

$\rightsquigarrow 2N$.

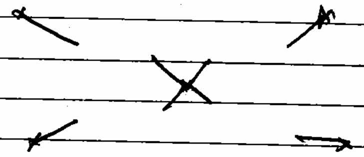
Total # of ways of joining these to each other

$= (2N-1)(2N-3) \cdots 1 = (2N-1)!!$

$\rightsquigarrow$ must be equal to sum over {~~all diagrams~~ x combinatorial factors of all diagrams} at this order.

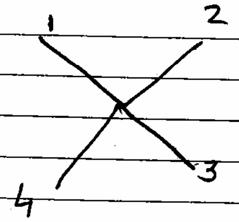
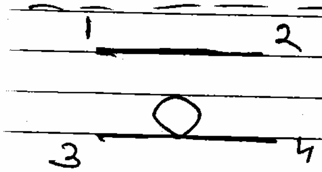
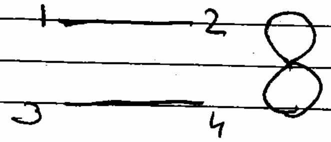
Note: Ignore vertex factors, $\frac{1}{n!}$ for n-th order perturbation theory etc.

Example: Order λ contribution to $\langle \prod_{i=1}^{4} \phi(x_i) \rangle$ in ϕ^4 theory.



Total no. $2N$ $= 4+4 = 8$.

$(2N-1)!! = 7 \times 5 \times 3 \times 1 = 105$

Diagrams	Similar diagrams.	Combinatorial factor.
	1	× $4 \times 3 \times 2$.
	6	4×3
	3	3

$\boxed{24 + 72 + 9 = 105}$

Ex. Calculate similar consistency tests for scalar + Dirac fermion with $\bar{\psi}\psi\phi$ interaction.

Non-abelian gauge theories.

~ Will be developed in analogy with QED.

Recall QED action:

$$S = \int d^4x \, \mathcal{L}.$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \{ i \gamma^\mu (\partial_\mu - i e A_\mu) - m \} \psi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \qquad \partial_\mu = i \frac{\partial}{\partial x^\mu}$$

$$\eta_{\mu\nu} = \mathrm{diag.}(-1, 1, 1, 1).$$

Dirac indices suppressed.

Symmetries of $\mathcal{L}$:

① $\psi(x) \to e^{ie\lambda}\psi(x)$, $A_\mu \to A_\mu$

λ: arbitrary constant.

→ Global symmetry: exists even in absence of e.m. field.

② $\psi \to e^{ie\lambda(x)}\psi(x)$, $A_\mu \to A_\mu + \partial_\mu \lambda(x)$

$\lambda(x)$: arbitrary function of space-time coordinates x.

This symmetry exists only in the presence of gauge fields. Free Dirac action does not have this symmetry. But free Maxwell action has this symmetry.

Define: $D_\mu \psi \equiv \partial_\mu \psi - i e A_\mu \psi$

→ Covariant derivative.

$$[D_\mu, D_\nu]\psi = D_\mu D_\nu \psi - D_\nu D_\mu \psi$$

$$= -ie F_{\mu\nu} \psi$$

$$\Rightarrow [D_\mu, D_\nu] = -ie F_{\mu\nu}$$

Now consider a theory of N Dirac fields $\psi^1, \cdots \psi^N$

(Note: Dirac indices are still suppressed)

$$S_{free} = \int d^4x \, \bar{\psi}^k (i\gamma^\mu \partial_\mu - m) \psi^k$$

(k summed over from 1 to N)

Global symmetry:

$$\psi^k \to U_{kl}\,\psi^l \qquad \bar{\psi}^k \to U^*_{km}\,\bar{\psi}^m.$$

U : unitary matrix i.e. $U_{kl}\,U^*_{km} = \delta_{lm}$.

$$U^\dagger U = \mathbb{1}.$$

U : an element of U(N) group.

A subset of U(N) matrixes with

$$\det U = 1$$

⇒ SU(N) group.

Goal: Try to make either the whole of U(N) or SU(N) subgroup of U(N) into a local symmetry.

$$\psi^k(x) \to U_{kl}(x)\;\psi^l(x)$$

$$\bar{\psi}^k(x) \to U^*_{km}(x)\,\psi^m(x)$$

$\forall x$ U(x) is a U(N) or SU(N) matrix.

Check how the free action transforms

$$S_{free} = \bar{\psi}^k(x)\,(i\gamma^\mu\partial_\mu - m)\,\psi^k(x)$$

$$\to U^*_{kl}(x)\,\bar{\psi}^l(x)\,(i\gamma^\mu\partial_\mu - m)\{U_{kr}(x)\,\psi^r(x)\}$$

$$= \bar{\psi}^l(x)\,(i\gamma^\mu\partial_\mu - m)\,\psi^l(x)$$

$$+ \;\bar{\psi}^l(x)\; i\gamma^\mu\,\psi^r(x)\;\underbrace{U^*_{kl}(x)\,\partial_\mu U_{kr}(x)}_{(U^\dagger(x)\,\partial_\mu U(x))_{lr}}$$

$$\downarrow$$

We have to find ways of cancelling the second term.

Add to the action

$$S_{int} = \bar{\psi}^l(x)\,\gamma^\mu\,\psi^r(x)\,S_{\mu l r}(x).$$

~~$S_{\mu l m}(x) \to -(U^\dagger(x)\,\partial_\mu U(x))_{lm}$~~

~~$+(U S_\mu U^\dagger)_{lr}$~~

$$S_{\mu \ell r} \to \{U(-iU^\dagger \partial_\mu U)U^\dagger\}_{\ell r}$$
$$+ (U S_\mu U^\dagger)_{\ell r}$$

~~Check~~ → some function of new fields.

Check:

$$\bar{\Psi}^\ell \gamma^\mu \psi^r(x) \quad S_{\mu \ell r}(x)$$

$$\to U^*_{\ell \ell'}(x)\bar{\Psi}^{\ell'} \gamma_\mu U_{r r'}(x)\psi^{r'}(x)$$

$$\{U(-iU^\dagger \partial_\mu U + S_\mu)U^\dagger\}_{\ell r}$$

$$= \{U^\dagger U(-iU^\dagger \partial_\mu U + S_\mu)U^\dagger U\}_{\ell' r'}$$
$$\bar{\Psi}^{\ell'} \gamma_\mu \psi^{r'}$$

$$= -i(U^\dagger \partial_\mu U)_{\ell' r'} \bar{\Psi}^{\ell'} \gamma_\mu \psi^{r'}$$ ← cancels extra term.

$$+ \bar{\Psi}^{\ell'} \gamma_\mu \psi^{r'} (S_\mu)_{\ell' r'} \to S_{int}$$

$$(S_\mu)_{\ell r} \to -i(\partial_\mu U\, U^\dagger)_{\ell r} + (U S_\mu U^\dagger)_{\ell r}$$

Note:

① $i\partial_\mu U\, U^\dagger$ is hermitian if U is unitary.

$$UU^\dagger = 1 \Rightarrow \partial_\mu U\, U^\dagger + U\,\partial_\mu U^\dagger = 0$$

$$\Rightarrow (\partial_\mu U\, U^\dagger) + (\partial_\mu U\, U^\dagger)^\dagger = 0$$

$\Rightarrow \partial_\mu U\, U^\dagger$ is anti-hermitian

$\Rightarrow i\,\partial_\mu U\, U^\dagger$ is hermitian.

② $i\,\partial_\mu U\, U^\dagger$ is traceless if $\det U = 1$.

Proof: $\ln \det U = \mathrm{Tr} \ln U$.

$$\Rightarrow 0 = \mathrm{Tr} \ln U$$

$$0 = \partial_\mu \mathrm{Tr}(\ln U) = \mathrm{Tr}(U^{-1} \partial_\mu U)$$

$$= \mathrm{Tr}(\partial_\mu U\, U^\dagger)$$

Suppose $T^1, T^2, \ldots$ denote the basis of linearly independent $N \times N$ hermitian (also traceless for SU(N)) matrices.

Take $S_{\mu l m} = \sum_a B_\mu^a (T^a)_{lm}$.

How many such matrices are there?

N^2 for Hermitian

N^2-1 for traceless hermitian.

Any hermitian matrix can be expressed as a linear combination of T^a's.

$$-i(\partial_\mu U\, U^\dagger)_{lm} = \sum_a \alpha_\mu^a T^a$$

$$(U S_\mu U^\dagger)_{lm} = \sum_a B_\mu^a \underbrace{(U T^a U^\dagger)}_{R^{(x)}_{ba} T^b}$$

$$\sum_a B_\mu^a(x) T^a \to \sum_a \alpha_\mu^a(x) T^a + \sum_{a,b} R_{ab}(x) B_\mu^b(x) T^a$$

$$\Rightarrow \quad B_\mu^a(x) = \alpha_\mu^a(x) \cancel{T^a} + R_{ab}(x) B_\mu^b(x)$$

Note: Both $\alpha_\mu^a(x)$ and $R_{ab}(x)$ are determined in terms of $U(x)$.

Thus we have a gauge invariant action:

$$S = \int d^4x \Big[\bar\psi^k(x) \left(i\gamma^\mu \partial_\mu - m \right) \psi^k(x) + \bar\psi^l(x) \gamma^\mu \psi^m(x) B_\mu^a(x) T^a_{lm} \Big]$$

$B_\mu^a(x)$: new fields. (N^2 for $U(N)$, N^2-1 for $SU(N)$)

for U(2)

Example: U(2)

$\Uparrow$

$$T^a : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

U(3)

$$T^a : \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\Downarrow$

for U(3).

Notation:

$$\underset{\sim}{B}_\mu = B_\mu^a T^a \rightarrow \text{an } N\times N \text{ hermitian matrix.}$$

$$(\underset{\sim}{B}_\mu)_{\ell r} = S_{\mu \ell r}.$$

$$S = \int d^4x \left[\bar\Psi^k(x) \left(i\gamma^\mu \partial_\mu - m\right) \psi^k(x) \right.$$

$$+ \bar\Psi^\ell(x)\, \gamma^\mu\, \psi^m(x)\, \underbrace{B_\mu^a\, T^a_{\ell m}}_{(\underset{\sim}{B}_\mu)_{\ell m}}$$

$$= \int d^4x \left[\bar\Psi \left(i\gamma^\mu \partial_\mu - m\right) \psi \right.$$

$$\left. + \bar\Psi \gamma^\mu \underset{\sim}{B}_\mu \psi \right]$$

$$= \int d^4x \left[\bar\Psi \left(i\gamma^\mu D_\mu - m\right) \psi \right.$$

$$D_\mu \psi = \left(\partial_\mu - i\underset{\sim}{B}_\mu\right)\psi.$$

Note: ψ now denotes a $4k$ dimensional column vector $\rightarrow \psi^k_\alpha$

γ^μ and B_μ are both matrices but commute

$$(\gamma^\mu \psi)_\alpha^k = (\gamma^\mu)_{\alpha\beta}\, \psi_\beta^k$$

$$(\underset{\sim}{B}_\mu \psi)_\alpha^k = (B_\mu)_{kl}\, \psi_\alpha^l$$

Ex: $(\gamma^\nu \underset{\sim}{B}_\mu \psi)_\alpha^k = (\underset{\sim}{B}_\mu \gamma^\nu \psi)_\alpha^k$

$\Downarrow$ same.

$$(\gamma^\nu)_{\alpha\beta} (\underset{\sim}{B}_\mu \psi)_\beta^k = (\gamma^\nu)_{\alpha\beta} (\underset{\sim}{B}_\mu)_{kl}\, \psi_\beta^l$$

Transformation Laws:

$$\underset{\sim}{B}_\mu \to U \underset{\sim}{B}_\mu U^\dagger - i\partial_\mu U\, U^\dagger$$

equivalent to

$$B_\mu^a \to \alpha_\mu^a(x) + R_{ab} B_\mu^b$$

with:

$$\alpha_\mu^a T^a = -i\, \partial_\mu U\, U^\dagger, \quad R_{ab} T^a = U T^b U^\dagger$$

Note: $\underset{\sim}{B}_\mu$ and B_μ^a contains the same information.

Transformation law of $D_\mu \psi$:

$$D_\mu \psi \to \left(\partial_\mu - i U \underset{\sim}{B}_\mu U^+ - \partial_\mu U U^+\right) U \psi$$

$$= \left(\partial_\mu U \ \psi + U \partial_\mu \psi - i U \underset{\sim}{B}_\mu \psi - \partial_\mu U \psi\right)$$

$$= U \left(\partial_\mu - i \underset{\sim}{B}_\mu \right) \psi = U D_\mu \psi$$

~~Eq~~ so it transforms in the same way as ψ (covariantly)

$$D_\nu D_\mu \psi \to U D_\nu D_\mu \psi$$

$$\Rightarrow [D_\mu, D_\nu] \psi \to U [D_\mu, D_\nu] \psi$$

$$[D_\mu, D_\nu] \psi = \left(\partial_\mu - i \underset{\sim}{B}_\mu\right) \left(\partial_\nu - i \underset{\sim}{B}_\nu\right) \psi - (\mu \leftrightarrow \nu)$$

$$= \partial_\mu \partial_\nu \psi - i (\partial_\mu \underset{\sim}{B}_\nu) \psi - i \underset{\sim}{B}_\nu \partial_\mu \psi$$

$$- i \underset{\sim}{B}_\mu \partial_\nu \psi - \underset{\sim}{B}_\mu \underset{\sim}{B}_\nu \psi \quad - (\mu \leftrightarrow \nu)$$

$$= - i \left((\partial_\mu \underset{\sim}{B}_\nu) - (\partial_\nu \underset{\sim}{B}_\mu) - i [\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu] \right) \psi$$

$$\underset{\sim}{G}_{\mu\nu} = \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu - i[\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu)$$

Claim: $\underset{\sim}{G}_{\mu\nu}$ is hermitian:

$$\underset{\sim}{G}^{\dagger}_{\mu\nu} = \partial_\mu \underset{\sim}{B}^{\dagger}_\nu - \partial_\nu \underset{\sim}{B}^{\dagger}_\mu + i[\underset{\sim}{B}^{\dagger}_\nu, \underset{\sim}{B}^{\dagger}_\mu]$$

$$= \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu - i[\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu]$$

$$\Rightarrow \quad \underset{\sim}{G}_{\mu\nu} = G^a_{\mu\nu} T^a$$

Transformation law of $\underset{\sim}{G}_{\mu\nu}$:

$$[D_\mu, D_\nu]\psi \to U[D_\mu, D_\nu]\psi$$

$$\Rightarrow \quad G'_{\mu\nu}\psi' = U G_{\mu\nu}\psi$$

$$\psi' = U\psi$$

$$\Rightarrow \quad G'_{\mu\nu} U = U G_{\mu\nu} \Rightarrow G'_{\mu\nu} = U G_{\mu\nu} U^{\dagger}$$

Gauge invariant term:

$\mathrm{Tr}(\underset{\sim}{G}_{\mu\nu} \underset{\sim}{G}^{\mu\nu})$ → generalizes Maxwell action.

Some Conventions:

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta_{ab}$$

→ normalization of T^a.

General form of ~~the~~ ~~action~~ $\mathcal{L}$:

$$-\frac{1}{2g^2}\,\mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu}\,\underset{\sim}{G}^{\mu\nu}\right) + \bar{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi$$

↓

arbitrary

constant

Component form:

$$-\frac{1}{4g^2}\,G^a_{\mu\nu}\,G^{a\mu\nu} + \bar{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi$$

$$D_\mu\psi = \partial_\mu\psi - i B^a_\mu T^a \psi$$

$G^a_{\mu\nu}$ in terms of B_μ:

$$G^a_{\mu\nu} T^a = \partial_\mu (B^a_{\cancel{\mu}\nu} T^a) - \partial_\nu B^a_\mu T^a - i[B^b_\mu T^b, B^c_\nu T^c]$$

$$= (\partial_\mu B^a_\nu - \partial_\nu B^a_\mu)\,T^a - i B^b_\mu B^c_\nu [T^b, T^c]$$

Note: $[T^b, T^c]^\dagger = +[(T^c)^\dagger, (T^b)^\dagger]$

$= [T^c, T^b] = -[T^b, T^c]$

$\Rightarrow$ $[T^b, T^c]$ must be a linear combination of iT^a's.

$[T^b, T^c] = i f^{bca} T^a$

$\Downarrow$

structure constants.

For $SU(2)$, $f^{bca} = \epsilon^{bca}$

$$\Rightarrow G_{\mu\nu}^a T^a = [\partial_\mu B_\nu^a - \partial_\nu B_\mu^a + g f^{bca} B_\mu^b B_\nu^c] T^a$$

$$\Rightarrow G_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + g f^{bca} B_\mu^b B_\nu^c$$

Transformation law:

$$G_{\mu\nu}^a T^a \to U G_{\mu\nu}^a T^a U^\dagger = G_{\mu\nu}^a T^b R_{ba}$$

$$= G_{\mu\nu}^b R_{ab} T^a$$

$\Rightarrow G_{\mu\nu}^a \to R_{ab} G_{\mu\nu}^b \to$ covariantly.

Role of g:

$$\mathcal{L}_{gauge} = -\frac{1}{4g^2} G^a_{\mu\nu} G^a_{\mu\nu}$$

$$= -\frac{1}{4g^2} \left(\partial_\mu B_\nu{}^a - \partial_\nu B_\mu{}^a + f^{bca} B_\mu^b B_\nu^e \right)$$

$$\left(\partial_\mu B_\nu{}^a - \partial_\nu B_\mu^a + f^{dea} B_\mu^d B_\nu^e \right)$$

The quadratic term looks like sum of many Maxwell terms except for $1/g^2$.

Define $B_\mu^a = g \underbrace{A_\mu^a}_{\text{new field.}}$

$$\mathcal{L}_{gauge} = -\frac{1}{4} \left(\partial_\mu A_\nu{}^a - \partial_\nu A_\mu^a + g f^{bca} A_\mu^b A_\nu^c \right)$$

$$\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{dea} A_\mu^d A_\nu^e \right)$$

Note: Cubic interaction $\propto g$

Quartic interaction $\propto g^2$

$$\mathcal{L}_{\text{fermion}} = \bar{\Psi}\left(i\gamma^\mu(\partial_\mu - igA_\mu^a T^a) - m\right)\Psi$$

Note $\bar{\Psi}\gamma^\mu\Psi A_\mu$ term $\propto g$.

g will play the role of small parameter in perturbation theory.

For now we proceed with the $\underset{\sim}{B}_\mu$ variables.

Note: Pure non-abelian gauge theory is interacting unlike pure Maxwell theory.

Coupling to scalars:

Consider n complex scalar fields:

$$\phi^k = \frac{1}{\sqrt{2}}\left(\phi^k_R + i\,\phi^k_I\right)$$

ordinary scalars.

Free action: $= \int d^4x\,\mathcal{L}$

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \phi^k_I\, \partial^\mu \phi^k_I + \partial_\mu \phi^k_R\, \partial^\mu \phi^k_R\right) - \frac{m^2}{2}\left(\phi^k_I \phi^k_I + \phi^k_R \phi^k_R\right)$$

$$= -\partial_\mu (\phi^k)^* \partial^\mu \phi^k - m^2 (\phi^k)^* \phi^k$$

$$= -\partial_\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi$$

~~How to couple it to~~ $\phi \to U(x)\phi$

$\mathcal{L}_{free}$ is not invariant.

Simple modification: $\partial_\mu \phi \to D_\mu \phi$

$$= \left(\partial_\mu \phi - i\,\underset{\sim}{B}_\mu\,\phi\right)$$

$$\mathcal{L}_{\text{scalar}} = -(D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi$$

$$\to -(D_\mu \phi)^\dagger U^\dagger U (D^\mu \phi) - m^2 \phi^\dagger U^\dagger U \phi$$

$$= -(D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi.$$

To this we could add other gauge invariant terms e.g.

$$-\lambda (\phi^\dagger \phi)^2$$

A general gauge invariant Lagrangian density with scalars, fermions gauge fields

= sum of gauge invariant terms with arbitrary coefficients.

Note : Not all fields need to transform under gauge transformation

e.g. there may be fermions χ such that

$$\chi \to \chi \quad \text{under gauge trs.}$$

Then $\bar{\chi}\left(i\gamma^\mu \partial_\mu - m_\chi\right)\chi$

is gauge invariant.

We could also have fields transforming in non-trivial representation

e.g. for SU(2) gauge theory we could have isospin $1, \frac{3}{2}, 2, \cdots$

($\frac{1}{2}$ has been discussed.)

~ Requires group representation theory to couple them.

Another generalization: product gauge groups.

$$\begin{tikzcd} SU(N) \arrow[d] & \times & SU(M) \arrow[d] & \text{(say)} \\ \underset{\sim}{B}_\mu & & \underset{\sim}{C}_\mu & \end{tikzcd}$$

$$\mathcal{L}_{SU(N)} = -\frac{1}{2g_1^2} \mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu} \underset{\sim}{G}^{\mu\nu} \right)$$

$$\mathcal{L}_{SU(M)} = -\frac{1}{2g_2^2} \mathrm{Tr}\left(\underset{\sim}{H}_{\mu\nu} \underset{\sim}{H}^{\mu\nu} \right)$$

$$\underset{\sim}{H}_{\mu\nu} = \partial_\mu \underset{\sim}{C}_\nu - \partial_\nu \underset{\sim}{C}_\mu - i\left[\underset{\sim}{C}_\mu , \underset{\sim}{C}_\nu \right]$$

g_1, g_2 : arbitrary constants.

A field may be charged under both, or one of them or none.

$$\psi_\alpha^{k,s} \to \underbrace{U_{kk'}}_{SU(N)\ \text{matrix}} \underbrace{V_{ss'}}_{SU(M)\ \text{matrix}} \psi_\alpha^{k',s'}$$

$$D_\mu \psi = \left(\partial_\mu - i \underset{\sim}{B}_\mu - i \underset{\sim}{C}_\mu\right) \psi$$

$$\left(D_\mu \psi\right)^{k,s} = \partial_\mu \psi^{k,s} - i \left(\underset{\sim}{B}_\mu\right)_{kl} \psi^{l,s} - i \left(\underset{\sim}{C}_\mu\right)_{sr} \psi^{k,r}$$

$\bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi$ is invariant under SU(N) × SU(M).

This can be generalized to product of more gauge groups.

Infinitesimal gauge trs.

$\mathbb{1}_N \in U(N)$ & $SU(N)$

Look for unitary matrices close to $\mathbb{1}$:

$$U = \mathbb{1} - i\underset{\text{small}}{\epsilon}\,\Omega \text{ — } N\times N \text{ matrix}$$

$$U^\dagger U = \mathbb{1} \Rightarrow \Omega^\dagger - \Omega = 0. \Rightarrow \epsilon\Omega = \sum_a \underset{\text{small, real}}{\epsilon^a} T^a$$

For $SU(N)$, $\det U = 0 \Rightarrow \mathrm{Tr}\,\Omega = 0$.

$$\psi^k \to (U\psi)^k = \psi^k \underbrace{- i\epsilon_a (T^a)_{kl}\psi^l}_{\delta\psi^k}$$

$$\delta\psi = -i\epsilon_a T^a \psi$$

$$\delta\bar\psi = i\epsilon_a \bar\psi T^a$$

Infinitesimal trs. on gauge fields:

$$B_\mu^a \to R_{ab} B_\mu^b + \alpha_\mu^a$$

Need to find R_{ab} and α_μ^a

$$U T^a U^\dagger = R_{ba} T^b$$

$$(1 - i\epsilon_b T^b) T^a (1 + i\epsilon^c T^c) = T^a - i\epsilon^c [T^c, T^a]$$

$$= T^a + f^{cab}\epsilon^c T^b$$

$$R_{ba} = \delta_{ab} + f^{cab}\,\epsilon^c$$

$$-i\,\partial_\mu U\,U^\dagger = \alpha_\mu^a T^a$$

$$\Downarrow$$

$$-i\,(-i\,\partial_\mu\epsilon^a T^a) = -\partial_\mu\epsilon^a T^a$$

$$\Rightarrow \alpha_\mu^a = -\partial_\mu\epsilon^a$$

$$\Rightarrow \delta B_\mu^a \Rightarrow B_\mu^a + f^{cba}\epsilon^c B_\mu^b - \partial_\mu\epsilon^a$$

$$\delta B_\mu^a = -\partial_\mu\epsilon^a + f^{cba}\epsilon^c B_\mu^b$$

$$= -\partial_\mu\epsilon^a - f^{bca}\epsilon^c B_\mu^b$$

Invariance of S under finite gauge trs.

$\Rightarrow$ invariance under infinitesimal gauge trs.

$\Rightarrow$ invariance under finite gauge trs.

Ex. Check $\delta S = 0$ explicitly to order ϵ^a.

Need: $f^{abc}f^{cde} + f^{cae}f^{bdc} + f^{dac}f^{cbe} = 0$

$\rightsquigarrow$ follows from

$$[[T^a,T^b],T^d] + [[T^b,T^d],T^a] + [[T^d,T^a],T^b] = 0.$$

$\rightsquigarrow$ Bianchi

Other representations.

A K dimensional representation of U(K) (or SU(K)) is a collection of K×K matrices $R_G(U)$ such that

$$R_G(U_1 U_2) = R_G(U_1)\, R_a(U_2) \quad \forall\, U_1, U_2 \in U(N)$$

$$R_G(U^{-1}) = (R_G(U))^{-1}$$

$$R_G(\mathbb{1}_N) = \mathbb{1}_M$$

Infinitesimal trs.

$$R_G(\mathbb{1}_N) = \mathbb{1}_M \Rightarrow R_G(\mathbb{1}_N - i\sum_a \epsilon_a T^a)$$

$$= \prod_a R_G(\mathbb{1}_M - i\,\epsilon_a T^a) = \prod_a (\mathbb{1}_M - i\,\epsilon_a \underset{\substack{\downarrow \\ \text{defn}}}{R_A(T^a)})$$

$$= \mathbb{1}_M - i \sum_a \epsilon_a R_A(T^a)$$

Ex. $[R_A(T^a), R_A(T^b)] = i\,\epsilon^{abc} R_A(T^c)$

Now suppose we have an M component Dirac field ψ^a $(a = 1, \ldots M)$ which transforms under gauge trs. as

$$\psi \to R_G(U)\, \psi$$

Can we write down a gauge invariant action for these fermions?

Ans: Define

$$D_\mu \psi = \partial_\mu \psi - i B_\mu^a R_A(T^a)\, \psi$$

Then

$$\bar{\Psi}(i\gamma^\mu D_\mu - m)\, \psi$$

is invariant under gauge trs.

Ex. check this for infinitesimal gauge transformations.

$\Rightarrow$ proves invariance under finite gauge trs.

Similar construction can be done for scalars.

Example: For SU(2) we can have spin j representation of dimension (2j+1)

Ex. $R_{ab}(U_1)\, R_{bc}(U_2) = R_{ac}(U_1 U_2)$ $\to$ forms a representation

$\Rightarrow$ Adjoint representation. (N^2-1 dim for SU(N))

So far we have discussed Dirac fermions.

What about Weyl fermions?

Left handed: ψ_L : $\gamma_5 \psi_L = -\psi_L$

Right handed χ_R $\gamma_5 \chi_R = \chi_R$.

Can we write down a gauge invariant action?

$$\psi_L^k \to R(U)_{kl} \psi_L^l$$

Same action will work:

$\bar{\psi}_L (i\gamma^\mu D_\mu - m) \psi_L$ is gauge invariant.

However $\bar{\psi}_L \psi_L = 0$ ⇒ no mass term

Ex. prove this.

Thus for Weyl fermions we can have a gauge invariant action but no mass term.

11.6

What about Majorana fermions?

ψ is real in Majorana representation.

$$\psi^k \to R(U)_{kl}\, \psi^l$$

makes sense only for real $R(U)$.

Thus $R(U) = U$ will not work for $U(N)$ or $SU(N)$ (will work for $SO(N)$).

We need to choose real representation

Example: Adjoint representation.

$R_{ab}(U)$ is always a real matrix

Standard model: Based on gauge group

$$\begin{tikzcd}
SU(3) \arrow[d] & \times & \underbrace{SU(2)\times U(1)}_{} \arrow[d] \\
\text{Strong interaction} \arrow[d, Rightarrow] & & \text{weak and e.m. interaction} \arrow[dl] \arrow[d, dash] \\
\underset{\sim}{B_\mu} & \underset{\sim}{C_\mu} & \underset{\sim}{A_\mu}
\end{tikzcd}$$

Left handed quarks: fundamental of $SU(2)$ ~~[illegible]~~ and $SU(3)$ and carries $U(1)$ charge. $\psi_L^{i,s}$ — i: $SU(2)$, s: $SU(3)$

$$D_\mu = \partial_\mu - i g_{(3)} B_\mu^a T^a - i g_{(2)} C_\mu^k L^k - i g_{(3)} A_\mu$$

Quantization of non-abelian gauge theories.

Since even pure gauge theories are interacting we shall begin with them.

$$S_{gauge} = -\frac{1}{2g^2}\int \mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu}\,\underset{\sim}{G}^{\mu\nu}\right) d^4x$$

$$= -\frac{1}{4g^2}\int d^4x\; G^a_{\mu\nu}\, G^{a\mu\nu}$$

$$G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + f^{bca} B^b_\mu B^c_\nu$$

$$B^a_\mu = g A^a_\mu \,,$$

$$\Rightarrow S_{gauge} = -\frac{1}{4}\int d^4x\; F^a_{\mu\nu}\, F^{a\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{bca} A^b_\mu A^c_\nu$$

$$\Rightarrow S_{gauge} = -\frac{1}{4}\int d^4x \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right) + O(g)$$

$\Rightarrow$ The free ~~theories~~ action is a sum of N^2 independent electrodynamics action.

Gauge invariance:

$$B_\mu^a \to R_{ab} B_\mu^b + \alpha_\mu^a$$

$$\Rightarrow \quad A_\mu^a \to R_{ab} A_\mu^b + \frac{1}{g}\alpha_\mu^a$$

$$U T^a U^{-1} = T^b R_{ba}, \quad -iU\partial_\mu U^{-1} = \alpha_\mu^a T^a$$

Infinitesimal gauge trs:

$$V = \mathbb{1} - ig\,\epsilon^a(x)\, T^a$$

↳ Note g.

$$\Rightarrow \quad A_\mu^a \to A_\mu^a + \delta A_\mu^a$$

$$\delta A_\mu^a = -\partial_\mu \epsilon^a + g f^{bca} \epsilon^b A_\mu^c$$

Leading term: $\delta A_\mu^a = -\partial_\mu \epsilon^a$

⇒ Same as in the case of abelian gauge trs.

Ex. Prove that f^{abc} is totally anti-symmetric if $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta_{ab}$

Begin by quantizing free theory.

$$S_{free} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{A}^a_\mu(-k)\, \underbrace{\left(-k^2\eta^{\mu\nu} + k^\mu k^\nu\right)}_{M^{\mu\nu}(k)}\, \tilde{A}^a_\nu(k)$$

Naive prescription:

Propagator: $\langle A^a_\mu(k_1)\; A^b_\nu(k_2)\rangle$

$$= i\,\delta_{ab}\,(2\pi)^4\,\delta(k_1+k_2)\,\left(M(k)\right)^{-1}_{\mu\nu}$$

Problem: M has zero eigenvalues.

$$M^{\mu\nu}(k)\,k_\nu = 0$$

⇒ Propagator is not defined.

Origin of zero eigenvalue:

$$\delta S_{\text{free}} = 0 \quad \text{under} \quad \delta A_\mu^a = -\partial_\mu \epsilon^a$$

$$\Rightarrow \delta \tilde{A}_\mu^a(k) = -i k_\mu \tilde{\epsilon}^a(k)$$

~~On the other hand~~

$$\delta S = \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu^a(-k) \, M^{\mu\nu}(k) \, (-i k_\nu \tilde{\epsilon}^a(k))$$

$$\|$$

$$0 \text{ (gauge invariance)}$$

$$\Rightarrow \quad M^{\mu\nu}(k) \, k_\nu = 0$$

Thus zero e.v. of M arises due to gauge invariance.

Why does this cause problem?

Consider a finite dimensional integral:

$$I = \frac{\int \prod_i du_i \; \exp(-\tfrac{1}{2}A_{ij} u_i u_j) \; u_k u_l}{\int \prod_i du_i \; \exp(-\tfrac{1}{2}A_{ij} u_i u_j)}$$

$u_i \Leftrightarrow \tilde{A}^a_\mu(k)$.

↘ μ, a, k ↳ discretized.

We can evaluate this by diagonalizing A.

$$A = W^T A_d W \qquad A_d = \mathrm{diag}(a_1, a_2, \ldots a_n)$$

If $v_i = W_{ij} u_j$ then

$$I = \frac{\int \prod_i dv_i \; \exp(-\tfrac{1}{2} \sum a_k v_k^2) \; W_{mk} W_{nl} v_m v_n}{\int \prod_i dv_i \; \exp(-\tfrac{1}{2} \sum a_k v_k^2)}$$

If all a_k's are non-zero then the numerator & denominator are finite.

(Even if $a_k < 0$ for some k we can define the integral by analytic continuation)

e.g. $\int da_k \exp(-\frac{1}{2} a_k v_k^2) = \sqrt{\frac{2\pi}{a_k}}$

→ makes sense even for -ve a_k as an analytic function.

Problem: What happens if $a_k = 0$ for one or more k?

→ The integrals are ill defined.

Note: If we take approprite linear combination of $u_k u_\ell$ so that when expressed in terms of ϑ_k it is independent of the ϑ_ℓ's for which $a_\ell = 0$.

In that case we can factor out the ~~unwanted~~ divergent ϑ_ℓ integrals from the numerator and the denominator and get a finite result.

In the language of symmetry:

The numerator and the denominator has certain symmetries:

$$\vartheta_k \to \vartheta_k + \epsilon_k \quad \text{for } a_k = 0.$$

We can divide by this symmetry factor to get finite results.

Now consider a more general problem:

$$\langle F \rangle \equiv \frac{\int \prod_s du_s \, e^{iS(\vec u)} F(\vec u)}{\int \prod_s du_s \, e^{iS(\vec u)}}$$

Suppose $S(\vec u)$ has certain symmetries

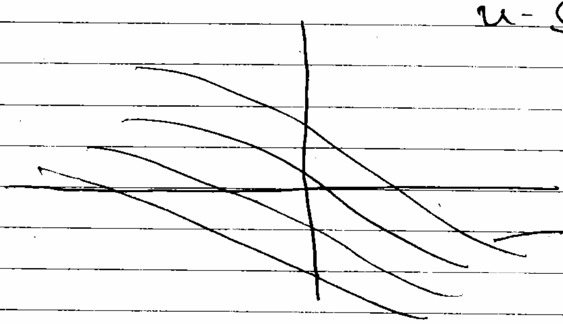
$$\vec u \to \vec f(\vec u, \theta)$$

↳ a continuous parameter.

$$\vec f(\vec u, \theta = 0) = \vec u .$$

$$S(\vec f(\vec u, \theta)) = S(\vec u)$$

u-Space



parametrized by $\vec f(\vec u, \theta)$ with different θ.

~~Integral of~~ e^{iS} is independent of these directions.

⇒ No suppression ~~along~~ of the integrand along this direction.

⇒ Both the numerator and the denominator are divergent and the result is ill defined.

If $F(\vec{u})$ is also invariant under this transformation then there is a way out.

The ~~integra~~ numerator and the denominator are both invariant under θ-deformation of u.

⇒ Factor out the integral over this direction

Then the result will be finite.

In the context of gauge theory invariance of $F(\vec{u})$ ~~to~~ under $\vec{u} \to \vec{\xi}(\vec{u}, \theta)$ translates to $F(\vec{u})$ being gauge invariant.

Conclusion: As long as we compute correlation function of gauge invariant operators, then in principle the result is well defined.

Since S-matrix is gauge invariant, this is okay.

Problem: To calculate S-matrix & other gauge invariant quantities in perturbation theory, we need to compute correlation fn of non-gauge invariant operators in the intermediate steps.

e.g. in QED to calculate

$$\langle F_{\mu\nu}(x) \quad F_{\rho\sigma}(y) \rangle$$

we shall calculate

$$\partial_\mu \partial_\rho \quad \langle A_\nu(x) \quad A_\rho(y) \rangle$$

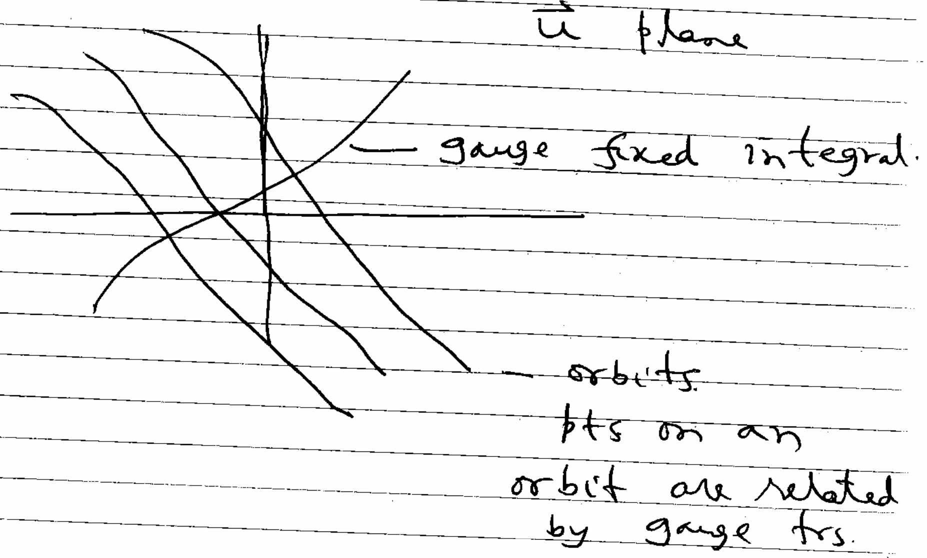
and then anti-symmetrize.

gauge non-invariant

Goal : Find a ~~method~~ prescription ~~for~~ that computes $\langle A^a_\mu(x) \ A^b_\nu(y) \rangle$ at intermediate stages even though it has no relation to $\langle A^a_\mu(x) \ A^b_\nu(y) \rangle$ in the original theory.

The prescription must be such that when we use this to compute correlators of gauge invariant operator, the result should agree with what we would have gotten in the original theory.

Go back the the finite dimensional example:



We have to find a prescription of integrating only over a subspace ⊥ the orbit.

Naive guess: $\mathcal{H}(\vec u) = 0$

some function of u

$\mathcal{H}(\vec f(\vec u, \theta)) \neq \mathcal{H}(\vec u)$

$\Rightarrow$ For small θ, $\frac{\partial \mathcal{H}(\vec u)}{\partial u^k} \frac{\partial f^k}{\partial \theta}\Big|_{\theta=0} \neq 0$

Insert $\delta(H(\vec{u}))$ into the integral.

Problem:

$$I = \frac{\int \prod_i du_i \; e^{iS(\vec{u})} \; F(\vec{u})}{\int \prod_i du_i \; e^{iS(\vec{u})}}$$

Gauge ~~tran~~ transformation:

$$u_i \to f_i(\vec{u}, \vec{\theta})$$

$$S(\vec{u}) = S(\vec{f}(\vec{u}, \vec{\theta})), \quad F(\vec{u}) = F(\vec{f}(\vec{u}, \vec{\theta}))$$

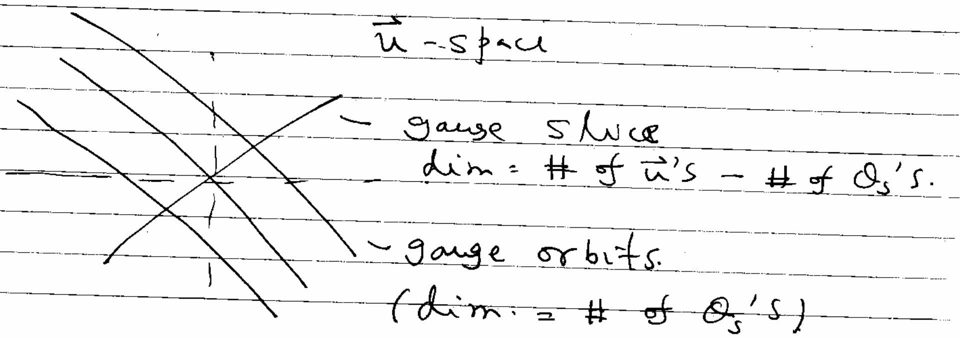
gauge trs. parameters.

$$u_i \to A_\mu^a(x)$$

(μ, a, x)

$$\theta_s \to \theta^a(x)$$

(a, x)



Naive approach: Choose a set of functions $\mathcal{H}_s(\vec{u})$ and insert into the integrals

$$\prod_s \delta(\mathcal{H}_s(\vec{u}))$$

If $K =$ # of u_i's and $L =$ # of θ_s's. we choose L functions $\mathcal{H}_s$.

$\Rightarrow$ The resulting space has dimension $K-L$.

How should we choose the $\mathcal{H}_s(\vec{u})$'s?

If we make a gauge trs. then $\mathcal{H}_s(\vec{u})$ should change.

Suppose $f_i(\vec{u}, \vec{\theta}) = u_i + g_{is}(\vec{u})\, \theta_s$ for small θ_i

$$\delta \mathcal{H}_s = \frac{\partial \mathcal{H}_s}{\partial u_i} g_{ir}\, \theta_r = M_{sr}(\vec{u})\, \theta_r$$

If $\mathcal{M}_{sr}(\vec{u})$ has a zero eigenvalue with eigenvector n_r then

$$\mathcal{M}_{sr}(\vec{u})\, n_r = 0.$$

⇒ If $\theta_r \propto n_r$ then $\delta \mathcal{H}_s = 0$.

Thus none of the $\mathcal{H}_s$'s change along this gauge orbit & hence the ~~surface~~ gauge slice $\{\mathcal{H}_s(\vec{u}) = 0\}$ is not transverse to the gauge orbit.

⇒ $\mathcal{M}_{sr}(\vec{u})$ should not have any zero eigenvalue.

⇒ $\det \mathcal{M}(\vec{u}) \neq 0$..

Given a set of $\mathcal{H}_s(\vec{u})$ satisfying this condition, we replace I by

$$J = \frac{\int \prod_i du_i \, e^{iS(\vec{u})} \prod_s \delta(\mathcal{H}_s(\vec{u})) \, F(\vec{u})}{\int \prod_i du_i \, e^{iS(\vec{u})} \prod_s \delta(\mathcal{H}_s(\vec{u}))}$$

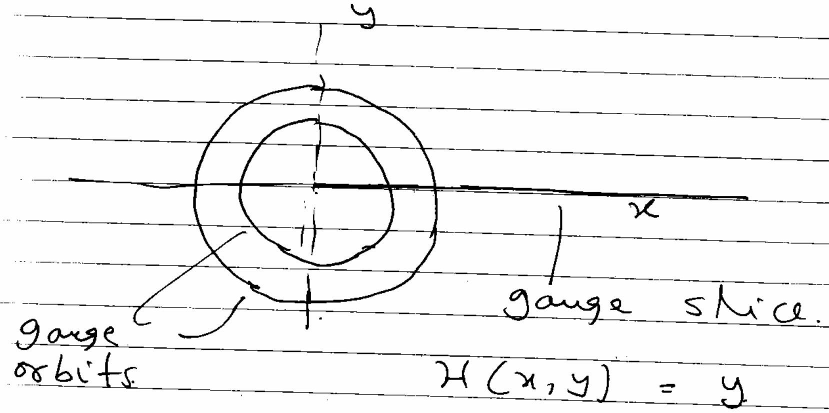
Is this enough?

→ Check if it works in a case where I and J can be both computed.

$$I = \underbrace{\int dx\, dy\, e^{-\alpha(x^2+y^2)}\,(x^2+y^2)}_{\int dx\, dy\, e^{-\alpha(x^2+y^2)}}$$

"Gauge invariance"

$$x \to x\cos\theta + y\sin\theta,\quad y \to -x\sin\theta + y\cos\theta$$



$$\mathcal{H}(x,y) = y$$

$$I' = \underbrace{\int dx\, dy\, e^{-\alpha(x^2+y^2)}\,(x^2+y^2)\,\delta(y)}_{\int dx\, dy\, e^{-\alpha(x^2+y^2)}\,\delta(y)}$$

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$$I = \frac{\int_0^\infty 2\pi r\,dr\, e^{-ar^2} r^2}{\int_0^\infty 2\pi r\,dr\, e^{-ar^2}}$$

$$I' = \frac{\int_0^\infty dx\, e^{-ax^2} x^2\, dx}{\int_0^\infty dx\, e^{-ax^2}\, dx}$$

$$I = -\frac{d}{da} \ln \int_0^\infty 2r\,dr\, e^{-ar^2}$$

$$= -\frac{d}{da} \ln 1/a = \frac{1}{a}$$

$$I' = -\frac{d}{da} \ln \int_0^\infty dx\, e^{-ax^2} dx$$

$$= -\frac{d}{da} \ln \sqrt{\frac{\pi}{a}} = \frac{1}{2a}$$

Thus $I \neq I'$

The naive replacement does not work.

We need to manipulate I and factor out integration along the gauge orbit.

Identity:

$$I = \frac{\mathcal{N}}{\mathcal{D}}$$

$$\mathcal{N} = \int \left(\prod_k du_k\right) e^{iS(\vec{u})} F(\vec{u})$$

$$\mathcal{D} = \int \left(\prod_k du_k\right) e^{iS(\vec{u})}$$

Use the identity:

$$\int \left(\prod_r d\theta_r\right) \prod_s \delta\left(\mathcal{H}_s(\vec{f}(\vec{u},\vec{\theta}))\right) \det\left(\frac{\partial \mathcal{H}_p(\vec{f}(\vec{u},\vec{\theta}))}{\partial \theta_q}\right) = 1$$

Insert this into $\mathcal{N}$ and $\mathcal{D}$.

$$\mathcal{N} = \int \prod_k du_k \; e^{iS(\vec{u})} \; F(\vec{u})$$

$$\int \prod_r d\theta_r \;\; \prod_s \delta(\mathcal{H}_s(\vec{f}(\vec{u},\vec{\theta})))$$

$$\det\left(\frac{\partial \mathcal{H}_p(\vec{f}(\vec{u},\vec{\theta}))}{\partial \theta_q} \right).$$

Change order of integrations over θ_r's and u_k's.

For fixed θ_r's, change variables from u_k's to v_k's.

$$v_k = f_k(\vec{u},\vec{\theta})$$

$$S(\vec{u}) = S(\vec{v}), \quad F(\vec{u}) = F(\vec{v})$$

$$\prod_k du_k = \prod_k dv_k$$

⇝ invariance of the measure.

$$\mathcal{N} = \int \prod_r d\theta_r \int \prod_k d\vartheta_k \; e^{iS(\vec\vartheta)} \; F(\vec\vartheta)$$

$$\prod_s \delta(H_s(\vec\vartheta)) \; \det\left(\frac{\underbrace{\partial H_p(\vec f(\vec u, \vec\theta))}}{\partial \theta_q} \right)$$

should be expressed as fn. of $\vec\vartheta, \vec\theta$.

Goal: Factor out the $\vec\theta$ integral from the $\vec\vartheta$ integral.

Then it can be cancelled between $\mathcal{N}$ and $\mathcal{D}$.

$$H_p(\vec f(\vec u, \vec\theta + \delta\vec\theta)) = H_p(\vec\vartheta) + \frac{\partial H_p(\vec f(\vec u, \vec\theta))}{\partial\theta_q} \delta\theta_q$$

$\vec\theta + \delta\vec\theta$ can be represented by a trs. by $\vec\theta$ followed by an infinitesimal trs. $\overrightarrow{\delta_W\theta}$

$$(\delta_W\theta)_r = S_{rs}(\vec\theta)\,\delta\theta_s$$

→ group composition law.

Important point: $S_{rs}(\vec{\theta})$ does not know anything about $\vec{u}$.

$$f_k(\vec{u}, \vec{\theta}+\delta\vec{\theta}) = f_k(\vec{v}, \overrightarrow{\delta_N\theta})$$

$$= f_k(\vec{v},0) + \frac{\partial f_k(\vec{v},\vec{\phi})}{\partial \phi_r}\Big|_{\vec{\phi}=0} (\delta_N\theta)_r$$

$$= v_k + \frac{\partial f_k(\vec{v},\vec{\phi})}{\partial \phi_r}\Big|_{\vec{\phi}=0} S_{rs}(\vec{\theta})\,\delta\theta_s$$

$$\mathcal{H}_p(\vec{f}(\vec{u}, \vec{\theta}+\delta\vec{\theta}))$$

$$= \mathcal{H}_p\left(\vec{v} + \frac{\partial \vec{f}(\vec{v},\vec{\phi})}{\partial \phi_r}\Big|_{\vec{\phi}=0} S_{rs}\,\delta\theta_s\right)$$

$$= \mathcal{H}_p(\vec{v}) + \frac{\partial \mathcal{H}_p(\vec{v})}{\partial v_k} \frac{\partial f_k(\vec{v},\vec{\phi})}{\partial \phi_r}\Big|_{\vec{\phi}=0} S_{rs}\,\delta\theta_s$$

Compare with

$$\mathcal{H}_p(\vec{v}) + \frac{\partial H_p(\vec{f}(\vec{u},\vec{\theta}))}{\partial \theta_r}\,\delta\theta_r$$

$$\ni \frac{\partial H_p(\vec{f}(\vec{u},\vec{\theta}))}{\partial \theta_q}$$

$$= \frac{\partial H_p(\vec{v})}{\partial v_k} \frac{\partial f_k(\vec{v},\vec{\phi})}{\partial \phi_r}\bigg|_{\vec{\phi}=0} S_{rj}(\vec{\theta})$$

$$\det \frac{\partial H_p(\vec{f}(\vec{u},\vec{\theta}))}{\partial \theta_q} \qquad \vec{\theta} \text{ indep.}$$

$$= \det\left(\frac{\partial H_p(\vec{v})}{\partial v_k} \frac{\partial f_k(\vec{v},\vec{\phi})}{\partial \phi_r}\bigg|_{\vec{\phi}=0} \right)$$

$$\times \det(S(\vec{\theta})) \qquad \underbrace{\qquad}_{\equiv\, M_{pr}(\vec{v})}$$

$$N = \int \prod_r d\theta_r \det S(\vec{\theta}).$$

$$\times \int \prod_k dv_k \, e^{iS(\vec{v})} \prod_s \delta(H_s(\vec{v})) \det M(\vec{v})$$

Now we can factor out $\vec{\theta}$ integral and cancel with the denominator.

Note: $M_{pr}(\vec{v}) = \dfrac{\partial H_p(\vec{f}(\vec{v},\vec{\phi}))}{\partial \phi_r}\bigg|_{\vec{\phi}=0}.$

$$I = \frac{\mathcal{N}'}{\mathcal{D}'}$$

$$\mathcal{N}' = \int \prod_k dv_k \, e^{iS(\vec{v})} \prod_s \delta(H_s(\vec{v})) \det(M(\vec{v})) \, F(\vec{v})$$

$\mathcal{D}'$ = Same with $F(\vec{v}) \to 1$.

This form is not amenable to perturbation theory.

① Quadratic part of $S(\vec{v})$ remains the same

→ propagator ill-defined.

② $F(\vec{v})$ is polynomial in $\vec{v}$ but neither $\prod_s \delta(H_s(\vec{v}))$ nor $\det(M(\vec{v}))$ is a low order polynomial in v remains finite even when # of v_k's → ∞.

Begin with $\det(M(\vec{v}))$

Can we represent it in a way that is amenable to perturbation theory?

$$\det M(\vec{v}) = (\text{Const}) \times \int \prod_{r} (dB_r \, dC_r) \exp\left(-i \, M_{rs}(\vec{v}) \, B_r \, C_s\right)$$

b_r, C_r : grassmann variables.

$\rightsquigarrow$ Fadeev-Popov ghosts.

We can treat B_r, C_r as new anti-commuting field degrees of freedom and $-M_{rs}(\vec{v}) B_r C_s$ as new term in the action.

$\rightsquigarrow$ Polynomial in $\vec{v}, \vec{b}, \vec{c}$ if $M_{rs}(\vec{v})$ is a polynomial in $\vec{v}$.

What about $\prod_s \delta(\mathcal{H}_s(\vec{\sigma}))$?

Note: Both $\mathcal{N}'$ and ∞' are independent of the choice of the fr.s $\mathcal{H}_s(\vec{\sigma})$.

($\mathcal{N}$ and ∞ were independent of $\mathcal{H}_s(\vec{\sigma})$

and $\mathcal{N} = \delta\mathcal{N}' \int \prod_{\alpha} d\theta_s \det S(\vec{\theta})$)

$\uparrow$ independent of $\mathcal{H}_s(\vec{\sigma})$'s.

~~Thus~~ ~~we~~ Choose $\mathcal{H}_s(\vec{\sigma}) = (\underset{\text{fixed functions}}{\tilde{\mathcal{H}}_s(\vec{\sigma})} - \underset{\text{constant}}{a_s})$

$\mathcal{N}', \infty'$ are a_s independent.

$$\mathcal{N}' \rightarrow \left(\int e^{\underset{\text{fixed no.}}{-\frac{i}{2\alpha_0}\sum_s a_s^2}} \prod_s da_s \, \mathcal{N}'\right)$$

$$\infty' \rightarrow \left(\int e^{-\frac{i}{2\alpha_0}\sum_s a_s^2} \prod_s da_s \; \delta\infty'\right)$$

$$\mathcal{N}' = \int \prod_s da_s \, e^{-\frac{i}{2\alpha_0}\sum_s a_s^2} \int \Big(\prod_k d\upsilon_k \Big|$$

$$\prod_r dB_r \, dC_r \quad e^{iS(\vec{\upsilon})} \quad F(\vec{\upsilon})$$

$$\exp\big(-i B_r \underset{\text{independent of } a_s}{\mathcal{M}_{rs}(\vec{\upsilon})} C_s\big) \;\prod_s \delta(\tilde{H}_s(\vec{\upsilon}) - a_s)$$

since a_s does not depend on $\vec{\upsilon}$.

Changed name to original

$$\mathcal{N}' = \int \prod_k du_k \prod_r dB_r \, dC_r \quad F(\vec{u})$$

$$\exp\Big[i \Big(\underbrace{S(\vec{u}) - B_r \mathcal{M}_{rs}(\vec{u}) C_s - \frac{1}{2\alpha_0}\sum_s \tilde{H}_s(\vec{u})^2}_{\text{new action}} \Big) \Big]$$

$$S_{\text{gauge-inv}} + S_{\text{ghost}} + S_{\text{gauge-fixing}}$$

Both depend on the gauge fixing term.

Note: The constraint $\tilde{\mathcal{H}}_s(\vec{\sigma}) - a_s = 0$ may locally intersect the gauge orbits transversally, but there may be global problems, e.g. it may intersect some orbits more than once or may not intersect some orbits.

⇝ Gribov ambiguity.

→ Does not affect perturbation theory.

Now back to gauge theory.

Example: $$\mathcal{H}_{\underset{(a,x)}{s}}(\vec{u}) = \partial^\mu A^a_\mu(x)$$

$\hookrightarrow$ discrete $\Delta(i^0, i^1, i^2, i^3)$

$$S_{g.f.} = -\frac{1}{2\alpha_0}\sum_s \mathcal{H}_s(\vec{u})\,\mathcal{H}_s(\vec{u})$$

$$= -\frac{1}{2\alpha_0}\sum_a \underbrace{\sum_{i^0,\dots i^3}}_{\frac{1}{\Delta^4}\int d^4x} \partial^\mu A^a_\mu(x)\;\partial^\nu A^a_\nu(x)$$

$$S_{g.f.} = -\frac{1}{2\alpha}\int d^4x \; \partial^\mu A_\mu^a(x) \;\; \partial^\nu A_\nu^a(x)$$

(sum over a implied)

$$\alpha \equiv \alpha_0 \Delta^4$$

Note: The new quadratic term in $S_{g.f.}$ is not gauge invariant.

$\Rightarrow$ Makes the propagator ~~invertible~~ well-defined.

Ghost action

$$\mathcal{H}_x(\vec{f}(\vec{u},\underset{\text{small}}{\vec{\phi}})) = \partial^\mu A_\mu^a(x)\Big|_{\underset{\text{gauge trs. by } \phi}{\phi}}$$

$$= \partial^\mu(-\partial_\mu \phi^a(x) + g f^{dca} \phi^d(x) A_\mu^c(x))$$

$$\frac{\partial \mathcal{H}_x(\vec{f}(\vec{u},\vec{\phi}))}{\partial \phi_s} = \Delta^4 \frac{\delta}{\delta\phi^b(y)} \left(-\partial^\mu \partial_\mu \phi^a(x) + g f^{dca} \partial_\mu(\phi^d(x) A_\mu^c(x)\right)$$

$$= \Delta^4 \left(-\partial^\mu_x \partial^x_\mu \delta^{(4)}(x-y)\, \delta_{ab} + g f^{bca} \partial_x^\mu(\delta^{(4)}(x-y) A_\mu^c(x)\right)$$

$$S_{ghost} = - B_r \left. \frac{\partial H_r(\vec{u},\vec{\phi})}{\partial \phi_s} \right|_{\vec{\phi}=0} C_s$$

$$= -\frac{1}{\Delta^8} \int d^4x \int d^4y \quad B_a(x). \; \Delta^4$$

$$\Big(-\partial_x^\mu \partial_\mu^x \delta^{(4)}(x-y)\, \delta_{ab} + g f^{bca} \frac{\partial}{\partial x^\mu} \big(\delta^{(4)}(x-y) \, A_\mu^c(x) \big) \Big) \; C^b (y)$$

$$= -\frac{1}{\Delta^4} \int d^4x \; \Big(-(\Box B_a(x))\, C_a(x)$$

$$- g f^{bca} (\partial_\mu B_a(x))\, C^b(x)\, A_\mu^c(x) \Big)$$

Absorb Δ^{-4} in B_a.

$$S_{ghost} = \int d^4x \, \Big(- \partial^\mu B_a \, \partial_\mu C_a$$

$$+ g f^{bca} \partial_\mu B_a \, C^b \, A_\mu^c \Big)$$

From this we can now derive the Feynman rules of gauge theory.

Total gauge fixed action:

$$S = S_{\text{gauge-inv.}} + S_{\text{gauge-fixing}} + S_{\text{ghost}}$$

$$S_{\text{gauge-inv.}} = -\frac{1}{4} \int d^4x \; F^a_{\mu\nu} \, F^{a\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{bca} A^b_\mu A^c_\nu$$

$$S_{\text{gauge-fixing}} = -\frac{1}{2\alpha} \int d^4x \; \partial_\mu A^{a\mu} \, \partial_\nu A^{a\nu}$$

$$S_{\text{ghost}} = \int d^4x \Big(-\partial^\mu B_a \, \partial_\mu C_a$$
$$+ g f^{bca} \partial_\mu B^a \, C^b \, A^{c\mu} \Big)$$

We shall now derive the Feynman rules for this theory.

Gauge field kinetic term

$$\int \left[-\frac{1}{4} F^{a}_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\alpha} \partial^{\mu} A^{a}_{\mu} \partial^{\nu} A^{a}_{\nu} \right] d^4x$$

$$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}^{a}_{\mu}(-k) \left\{ -k^2 \eta^{\mu\nu} + k^{\mu} k^{\nu} - \frac{1}{\alpha} k^{\mu} k^{\nu} \right\} \tilde{A}^{a}_{\nu}(k)$$

$$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}^{a}_{\mu}(-k) M^{\mu\nu}(k) \delta^{ab} \tilde{A}^{b}_{\nu}(k)$$

Propagator:

$$i \delta_{ab} (2\pi)^4 \delta^{(4)}(k_1 + k_2) (M^{-1})_{\mu\nu}(k_1)$$

$$\left(-k^2 \eta^{\mu\nu} + k^{\mu} k^{\nu} - \frac{1}{\alpha} k^{\mu} k^{\nu} \right) (M^{-1})_{\nu\rho}(k) = \delta^{\nu}_{\rho}$$

Take $(M^{-1})_{\mu\nu}(k) = f(k) \eta_{\mu\nu} + g(k) k_{\mu} k_{\nu}$

$$- k^2 f(k) \delta^{\mu}_{\rho} + k^{\mu} k_{\rho} f(k) \left(1 - \frac{1}{\alpha}\right)$$

$$+ g(k) \left\{ -k^2 k^{\mu} k_{\rho} + k^2 k^{\mu} k_{\rho} \left(1 - \frac{1}{\alpha}\right) \right\} = \delta^{\nu}_{\rho}$$

$$\Rightarrow -k^2 f(k) = \cancel{\delta^{\nu}_{\rho}}\, 1 \qquad f(k) \left(1 - \frac{1}{\alpha}\right) + k^2 g(k) \left\{ -\frac{1}{\alpha} \right\} = 0$$

$$g(k) = f(k) (\alpha - 1) \frac{1}{k^2}$$

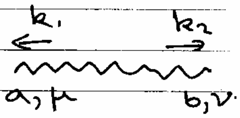
$$\langle \tilde{A}^a_\mu(k_1)\,\tilde{A}^b_\nu(k_2)\rangle$$

$$= i\,\delta_{ab}\,(2\pi)^4\,\delta^{(4)}(k_1+k_2)\left(-\frac{1}{k_1^2}\right)$$

$$\left\{\eta_{\mu\nu} + (\alpha-1)\,\frac{k_{1\mu}k_{1\nu}}{k_1^2}\right\}$$

$\alpha = 0$: Lorentz gauge

$\alpha = 1$: Feynman gauge.



Note : The result depends on α.

Gauge invariant correlation fn. :

$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ is gauge invariant for $g \to 0$.

$$\delta A_\mu = -\partial_\mu \alpha.$$

$$\tilde{F}^a_{\mu\nu}(k) = i\left(k_\mu \tilde{A}^a_\nu(k) - k_\nu \tilde{A}^a_\mu(k)\right)$$

$$\langle \tilde{F}^{a}_{\mu\rho}(k_1)\ \tilde{F}^{b}_{\nu\sigma}(k_2)\rangle$$

$$= i\,\delta_{ab}\,(2\pi)^4\,\delta^{(4)}(k_1+k_2)\,\left(-\frac{1}{k^2}\right)\,(i\times i)$$

$$\Big[k_{1\mu}\left(\eta_{\rho\sigma}+(\alpha-1)\,\frac{k_{1\rho}\,k_{1\sigma}}{k^2}\right)k_{2\nu}$$

+ ~~3 more~~ $-(\mu\leftrightarrow\rho)-(\nu\leftrightarrow\sigma)+(\mu\leftrightarrow\rho,\ \nu\leftrightarrow\sigma)\Big]$

$k_{1\mu}\,k_{1\rho}\to 0$ under anti-symmetrization in μ,ρ.

$k_{1\sigma}\,k_{2\nu} = -\,k_{1\sigma}\,k_{1\nu}\to 0$ under anti-symmetrization in ν,ρ.

$\Rightarrow$ Only $k_{[\mu}\,\eta_{\rho\sigma}\,k_{2\nu]}$ survives

$\to$ α-independent.

Physical States:

~ Supposed to be determined from the poles of the 2-point function.

In Feynaman gauge

$$\langle \tilde{A}^a_\mu (k_1) \; \tilde{A}^b_\nu (k_2) \rangle \propto -\frac{i}{k_1^2} \delta_{ab} \eta_{\mu\nu}$$

∀ a and ∀ μ we have a pole at $k^2 = 0$?

~ 4 massless states for each a?

From QED experience we know that the correct answer to be 2.

How does it come about?

~ Need to examine which poles contribute to gauge inv. correlation fn-s.

$$\langle \tilde{F}^{a}_{\mu\rho}(k_1)\ \tilde{F}^{b}_{\nu\sigma}(k_2)\rangle$$

$$\propto -\frac{i}{k^2}\,\delta_{ab}\,\{k_{[\mu}\,\eta_{\rho][\sigma}\,k_{\nu]}\}$$

$k = k_1$

We shall now express η as sum over polarizations:

$$\bar{k}^{\mu} = (k^0, -\vec{k})$$

$$\epsilon^{(\lambda)\mu} = (0, \vec{\epsilon}^{(\lambda)}) \qquad \vec{\epsilon}^{(\lambda)}\cdot\vec{k} = 0.$$

$\lambda = 1, 2.$

Claim:

$$\eta_{\mu\nu} = \frac{k_\mu \bar{k}_\nu + k_\nu \bar{k}_\mu}{k\cdot\bar{k}} + \sum_{\lambda=1}^{2} \epsilon^{(\lambda)}_{\mu}\,\epsilon^{(\lambda)}_{\nu}$$

$$+\ O(k^2).$$

Proof: Go to a frame in which

$$k_\mu = (k_0, k_1, 0, 0), \quad \bar{k}_\mu = (k_0, -k_1, 0, 0)$$

$$\epsilon^{(1)}_{\mu} = (0, 0, 1, 0), \quad \epsilon^{(2)}_{\mu} = (0, 0, 0, 1)$$

Proof: Can be done by inspection.

$$k \cdot \bar{k} = -(k_0)^2 - (k_1)^2$$

$$00: \quad r.h.s. = -\frac{(k_0)^2 + (k_0)^2}{(k_0)^2 + (k_1)^2}$$

$$= -1 - \underbrace{\frac{(k_0)^2 - (k_1)^2}{(k_0)^2 + (k_1)^2}}_{G(k^2)}$$

Similarly check other components.

Thus

$$\langle \tilde{F}^{a}_{\mu\rho}(k_1) \, \tilde{F}^{b}_{\nu\sigma}(k_2) \rangle$$

vanishes

$$\propto -\frac{i}{k^2} \delta_{ab} \, k_{[\mu} \Big[\underbrace{k_{\rho]} \bar{k}_{[\sigma} + \bar{k}_{\rho]} k_{[\sigma}}_{k \cdot \bar{k}}$$

$$+ \sum_{i=1}^{2} \epsilon^{(\lambda)}_{\rho]} \epsilon^{(\lambda)}_{[\sigma} + G(k^2) \Big] k_{\nu]}$$

no pole at $k^2 = 0$

$$\sim -\frac{i}{k^2} \delta_{ab} \sum_{i=1}^{2} k_{[\mu} \epsilon^{(\lambda)}_{\rho]} \epsilon^{(\lambda)}_{[\sigma} k_{\nu]}$$

Conclusion: Only transversely polarized states contribute to gauge invariant correlation fn.s.

⇒ 2 massless states ∀ a (as in QED)

~~Ghost~~ ~~B~~ Ghost action as $g \to 0$.

$$-\int d^4x\, \partial^\mu B^a(x)\, \partial_\mu C^a(x)$$

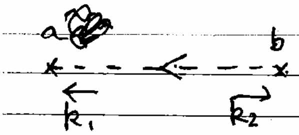
$$= \int \frac{d^4k}{(2\pi)^4}\, \tilde{B}^a(-k)\,(-k^2)\,\tilde{C}^a(k)$$

The propagator can be derived in the same way as that of a fermion field.

C^a → plays the role of ψ

B^a → plays the role of $\bar{\psi}$

$-k^2$ → plays the role of $\not{k}-m$.



$$\langle \tilde{C}^a(k_1)\ \tilde{B}^b(k_2) \rangle$$

$$= (2\pi)^4\ \delta^{(4)}(k_1 + k_2)\ \delta_{ab}\ \frac{i}{(-k_1^2)}$$

Note: The arrow flows out from a $\tilde{B}^b$ and flows into a $\tilde{C}^a$.

$$S_{int} = -\frac{g}{2}\int d^4x\ f^{abc} A^a_\mu A^b_\nu \left(\eta^{\nu\rho}\partial^\mu A^c_\rho - \eta^{\mu\rho}\partial^\nu A^c_\rho\right)$$

$$-\frac{g^2}{4}\int d^4x\ f^{abc} f^{dec} A^a_\mu A^b_\nu A^d_\rho A^e_\sigma \eta^{\mu\rho}\eta^{\nu\sigma}$$

$$+ g f^{bca}\int d^4x\ \partial^\mu B_a(x) A^c_\mu(x) C_b(x)$$

$$S_{int} = -\frac{g}{2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\,\tilde{A}^a_\mu(k_1)\,\tilde{A}^b_\nu(k_2)$$

$$\tilde{A}^c_\rho(k_3)\quad(2\pi)^4\quad\delta^{(4)}(k_1+k_2+k_3)$$

$$\times\ \frac{1}{3}\{\,f^{abc}(\eta^{\nu\rho}\,ik_3^\mu-\eta^{\mu\rho}\,ik_3^\nu)$$

$$+\,f^{bca}(\eta^{\rho\mu}\,ik_1^\nu-\eta^{\nu\mu}\,ik_1^\rho)$$

$$+\,f^{cab}(\eta^{\mu\nu}\,ik_2^\rho-\eta^{\rho\nu}\,ik_2^\mu)\,\}$$

$$-\frac{g^2}{4}\int\frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_4}{(2\pi)^4}\quad(2\pi)^4\quad\delta^{(4)}(k_1+k_2+k_3+k_4)$$

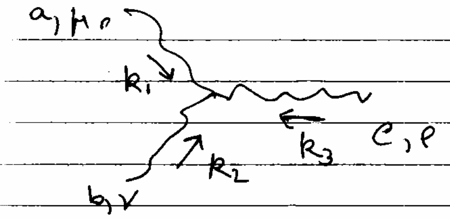
$$\tilde{A}^a_\mu(k_1)\tilde{A}^b_\nu(k_2)\tilde{A}^d_\rho(k_3)\tilde{A}^e_\sigma(k_4)\ \{\,f^{abc}f^{dec}\,\eta^{\mu\rho}\eta^{\nu\sigma}$$

$$+\ 23\ \text{more terms}\,\}\times\frac{1}{24}$$

$$+\,g\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\quad\tilde{\bar{c}}^a(k_1)\ \tilde{A}^c_\mu(k_2)\ \tilde{c}^b(k_3)$$

$$(2\pi)^4\quad\delta^{(4)}(k_1+k_2+k_3)\quad f^{bca}\,(i k_1^\mu)$$

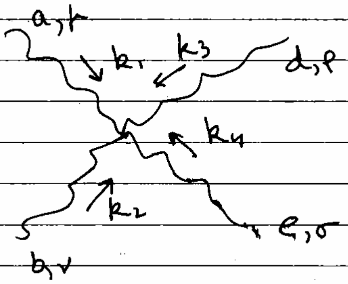
Vertices:



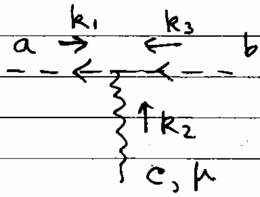
$$-\frac{ig}{2} \times (2\pi)^4 \delta^{(4)}(k_1+k_2+k_3)$$

$$\times \frac{1}{3}\{ f^{abc}(\eta^{\nu\rho} i k_3^{\mu} - \eta^{\mu\rho} i k_3^{\nu})$$

+ 2 more terms}



$$-i\frac{g^2}{4} \times \cdots$$



$$ig(2\pi)^4 \delta^{(4)}(k_1+k_2+k_3)$$

$$f^{bca}(ik_1^{\mu})$$

It is easy to incorportate fermions and scalars.

In either cases the gauge fixing procedure we have used goes through.

Only subtlety: Recall that we changed variables from $\vec{u}$ to $\vec{v} = \vec{f}(\vec{u}, \vec{\theta})$.

gauge transformation.

This change of variables must act on the additional fields also.

$S(\vec{u}) = S(\vec{v})$ only if all fields transform according to the gauge trs. law.

Rest of the analysis remains the same, including renaming $\vec{v}$ as $\vec{u}$ at the end.

Fermions in representation R_A:

$$S_{fermion} = \bar{\Psi} (i \gamma^\mu D_\mu - m) \Psi$$

$$(D_\mu \Psi)^k = \partial_\mu \Psi^k - i g A_\mu^a (R_A(T^a))_{kl} \Psi^l$$

$$\Rightarrow S_{fermion} = \bar{\Psi}^k (i \gamma^\mu \partial_\mu - m) \Psi^k$$

$$+ g \bar{\Psi}^k \gamma^\mu (R_A(T^a))_{kl} \Psi^l A_\mu^a$$

(Note: For fundamental representation $R_A(T^a) = T^a$).

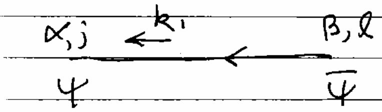
Momentum space:

$$S_{fermion} = \int \frac{d^4k}{(2\pi)^4} \bar{\Psi}^j(-k) (\not{k} - m) \Psi^j(k)$$

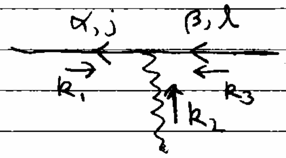
$$+ \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \bar{\Psi}_\alpha^j(k_1) A_\mu^a(k_2)$$

$$\Psi_\beta^l(k_3) (2\pi)^4 \delta^{(4)}(k_1+k_2+k_3) \gamma^\mu_{\alpha\beta} (R_A(T^a))_{jl}$$

Feynman rules.



$$\delta_{jl} (2\pi)^4 \delta^{(4)}(k_1 + k_2) \left(\frac{i}{\not{k}_1 - m}\right)_{\alpha\beta}.$$



$$i \cdot (2\pi)^4 \delta^{(4)}(k_1 + k_2 + k_3)$$
$$g (\gamma^\mu)_{\alpha\beta} \; (R(T^a))_{jl}$$

The sign associated with a given Feynman diagram has to be determined in terms of # of crossings.

Note: Like the γ matrices, the matrices $R(T^a)$ are also multiplied in the order opposite the arrow from left to right.

Coupling scalars to gauge fields.

$$S_{scalar} = -(D_\mu \Phi)^\dagger (D^\mu \Phi) - m^2 \Phi^\dagger \Phi$$

$$D_\mu \phi = \partial_\mu \Phi - i g A_\mu^a \; R_A(T^a) \, \phi$$

$$(D_\mu \phi)^\dagger = \partial_\mu \Phi^\dagger + i g A_\mu^a \; \cancel{R_A(T^a)} \; \phi^\dagger (R_A(T^a))^\dagger$$

$$= \partial_\mu \Phi^\dagger + i g A_\mu^a \; \phi^\dagger R_A(T^a)$$

(If the representation is unitary then $R_A(T^a)$ is hermitian).

$$R_a(U)^\dagger R_a(U) = \mathbb{1} \Rightarrow R_A(T^a)^\dagger = R_A(T^a))$$

$$-(D_\mu \Phi)^\dagger D^\mu \Phi - m^2 \Phi^\dagger \Phi$$

$$= -\partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi$$

$$- i g A_\mu^a \; \Phi^\dagger R_A(T^a) \partial_\mu \Phi + i g A_\mu^a \partial_\mu \Phi^\dagger R_A(T^a) \Phi$$

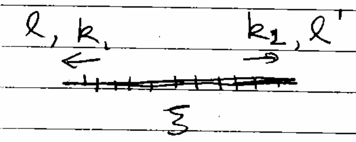
$$- g^2 A_\mu^a A^{\mu b} \; \Phi^\dagger R_A(T^a) \; R_A(T^b) \; \Phi$$

Free action:

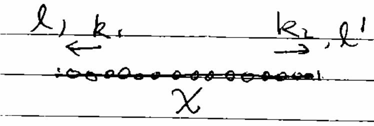
$$- \partial_\mu \Phi_k^* \partial^\mu \Phi_k - m^2 \Phi_k^* \Phi_k$$

$$\Phi_k = \frac{\xi_k + i \chi_k}{\sqrt{2}}, \quad \Phi_k^* = \frac{\xi_k - i \chi_k}{\sqrt{2}}$$

$$\Rightarrow S_{free} = -\frac{1}{2} [\partial_\mu \xi_k \partial^\mu \xi_k + m^2 \xi_k \xi_k + \partial_\mu \chi_k \partial^\mu \chi_k + m^2 \chi_k \chi_k]$$



$$(2\pi)^4 \delta^{(4)}(k_1 + k_2) \quad \frac{i}{-k^2 - m^2} \quad \delta_{\ell\ell'}$$



$$(2\pi)^4 \delta^{(4)}(k_1 + k_2) \quad \frac{i}{-k^2 - m^2} \delta_{\ell\ell'}$$

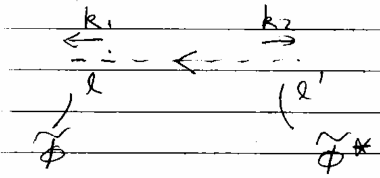
$$\Rightarrow \langle \tilde{\phi}_\ell(k_1) \; \tilde{\phi}^*_{\ell'}(k_2) \rangle$$

$$= (2\pi)^4 \delta^{(4)}(k_1 + k_2) \quad \frac{i}{-k_1^2 - m^2} \quad \delta_{\ell\ell'}$$

$$\langle \tilde{\phi}_\ell(k_1) \; \tilde{\phi}_{\ell'}(k_2) \rangle = \langle \tilde{\phi}^*_\ell(k_1) \; \tilde{\phi}_{\ell'}(k_2) \rangle = 0.$$

This allows us to introduce

complex scalar propagator:



$$(2\pi)^4 \, \delta^{(4)}(k_1+k_2) \, \frac{i}{-k_1^2-m^2} \, \delta_{\ell\ell'}$$

~~Vertex~~

$$S_{int} = -ig \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4}$$

$$\tilde{A}^a_\mu(k_1) \; \tilde{\phi}^*_j(k_2) \; \tilde{\phi}_\ell(k_3) \; (2\pi)^4 \delta^{(4)}(k_1+k_1+k_3)$$

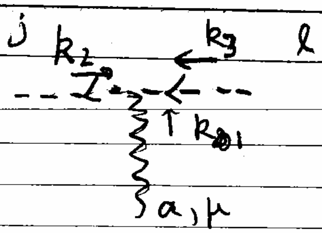
$$\left\{ R(T^a)_{j\ell} \, (ik_3 - ik_2) \right\}$$

$$-g^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4}$$

$$\tilde{A}^a_\mu(k_1) \; \tilde{A}^b_\nu(k_2) \; \tilde{\phi}^*_j(k_3) \; \tilde{\phi}_\ell(k_4) (2\pi)^4 \delta^{(4)}(k_1+\cdots k_4)$$

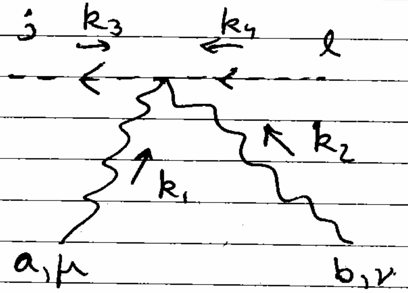
$$\left(R(T^a)\,R(T^b)\right)_{j\ell} \times \eta^{\mu\nu}$$

Note $\tilde{\phi}_j^*(k) = \left(\tilde{\phi}_j(-k)\right)^*$ (as for fermion



$$(2\pi)^4\,\delta^{(4)}(k_1+k_2+k_3)$$

$$g\,\big(R(T^a)\big)_{jl}\;i(k_3-k_2)$$



$$(2\pi)^4\,\delta^{(4)}(k_1+k_2+k_3+k_4)$$

$$(-ig^2)\,\big(R(T^a)\,R(T^b)\big)_{jl}$$

$$\eta^{\mu\nu}$$

A note on U(N) gauge groups:

$\underset{\sim}{B}_\mu = B_\mu^a T^a$.

↳ linearly independent N × N hermitian matrices.

special generator : $T^{N^2} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \times \frac{1}{\sqrt{2N}}$

T^{N^2} commutes with all other T^a's.

Gauge field action:

$$-\frac{1}{4g^2}\,\mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu}\,\underset{\sim}{G}^{\mu\nu}\right)$$

$$\underset{\sim}{G}_{\mu\nu} = \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu + [\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu]$$

We could add:

$$-\frac{1}{4g'^2}\,\mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu}\right)\,\mathrm{Tr}\left(\underset{\sim}{G}^{\mu\nu}\right)$$

→ Involves only $B_\mu^{N^2}$ or $G_{\mu\nu}^{N^2}$ since $\mathrm{Tr}(T^{N^2}) \neq 0$. $\mathrm{Tr}(T^a) = 0$ for other a.

g' : an independent coupling constant.

Alternatively we could ~~take~~ introduce gauge fields B_μ^a for $a = 1, \ldots N^2-1$ and $C_\mu = B_\mu^{N^2}$ as separate sets.

$$\mathcal{L} = -\frac{1}{4g^2}\,\mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu}\,\underset{\sim}{G}^{\mu\nu}\right) - \frac{1}{(g'')^2}\, H_{\mu\nu} H^{\mu\nu}$$

$$\downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\sum_{a=1}^{N^2-1} G^a_{\mu\nu}\, \underset{\text{traceless}}{T^a} \qquad\qquad \partial_\mu C_\nu - \partial_\nu C_\mu .$$

Physically this reflects the fact that $U(N)$ is (locally) isomorphic to

$SU(N) \times U(1)$

Standard model action

Gauge group:

$$\underset{\substack{\downarrow \\ \text{strong}}}{SU(3)} \times \underbrace{SU(2) \times U(1)}_{\text{electroweak}}$$

Gauge fields:

$$\underset{SU(3)}{\underset{\sim}{B}_\mu = B_\mu^a T^a}, \qquad \underset{SU(2)}{\underset{\sim}{C}_\mu = C_\mu^A T^A} \qquad \underset{\substack{\downarrow \\ U(1)}}{S_\mu}$$

Action:

$$-\frac{1}{4g_1^2} \mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu} \underset{\sim}{G}^{\mu\nu}\right) - \frac{1}{4g_2^2} \mathrm{Tr}\left(\underset{\sim}{H}_{\mu\nu} \underset{\sim}{H}^{\mu\nu}\right)$$

$$- \frac{1}{4g_3^2} K_{\mu\nu} K^{\mu\nu}$$

$$\underset{\sim}{G}_{\mu\nu} = \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu + [\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu]$$

$$\underset{\sim}{H}_{\mu\nu} = \partial_\mu \underset{\sim}{C}_\nu - \partial_\nu \underset{\sim}{C}_\mu + [\underset{\sim}{C}_\mu, \underset{\sim}{C}_\nu]$$

$$K_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu$$

We shall denote the U(1) charge by Y & Call it hyper charge.

~~A~~ For a field ψ of hypercharge Y,

$$D_\mu \psi = \left(\partial_\mu - i Y S_\mu + \cdots\right) \psi$$

$$\psi \to e^{i Y \Lambda(x)} \psi , \quad S_\mu \to S_\mu + \partial_\mu \Lambda$$

under U(1) gauge transformation.

Other fields (Fermions):

Quarks:

Left handed fermions in fundamental of SU(23), fundamental of SU(3) and hyper charge 1/3.

$$U_{\underset{SU(3)}{i}\,\underset{SU(2)}{\alpha}\,L}$$

$$i \bar{U}_L \gamma^\mu D_\mu U_L$$

$$= i \bar{U}_L \gamma^\mu \left(\partial_\mu - i B^a_\mu T^a - i C^A_\mu T^A - \frac{i}{3} S_\mu \right) U_L$$

$$= i \bar{U}_{i\alpha} \gamma^\mu \left(\delta_{ij} \delta_{\alpha\beta} \partial_\mu - i B^a_\mu (T^a)_{ij} \delta_{\alpha\beta} \right.$$

$$\left. - i C^A_\mu (T^A)_{\alpha\beta} \delta_{ij} - \frac{i}{3} S_\mu \delta_{ij} \delta_{\alpha\beta} \right) U_L$$

There are two more fields $C_{i\alpha L}$ and $T_{i\alpha L}$ with same property.

⇒ Similar action with $U \to C, T$.

Physical interpretation:

$$U_{i\underset{\downarrow}{\alpha} L}$$

Colour index
(Strong interaction)

Each quark comes in 3- colours.

U_{i1L} ⇒ Left-handed component of up quark

U_{i2L} ⇒ Left handed component of down quark

C_{i1} → charm quark

C_{i2} → strange quark

T_{i1} → top quark

T_{i2} → bottom quark

$\Big\}$ All left-handed.

What about the right-handed components?

Fundamental of SU(3), neutral under SU(2) (singlet) ~~hypercharge~~

$$\left.\begin{array}{l} u_{iR} \\ c_{iR} \\ t_{iR} \end{array}\right\} \rightarrow \text{hypercharge } \frac{4}{3}$$

d_{iR}, s_{iR}, b_{iR} → hypercharge $-\frac{2}{3}$.

$$i\,\bar{u}_R \gamma^\mu D_\mu u_R$$

$$= i\,\bar{u}_{iR} \gamma^\mu \left(\delta_{ij}\, \partial_\mu - B^a_\mu (T^a)_{ij} - \frac{4}{3}\, i\, S_\mu \right) u_{iR}$$

→ Similarly for t_{iR}, c_{iR}.

$$i\bar{d}_{R}\gamma^\mu D_\mu d_R = i\bar{d}_R\gamma^\mu(\partial_\mu - iB^a_\mu T^a + \frac{2i}{3}S_\mu)d_R$$

$$= i\bar{d}_{iR}\gamma^\mu(\delta_{ij}\partial_\mu - iB^a_\mu(T^a)_{ij} + \frac{i}{3}S_\mu \delta_{ij})d_{jR}$$

Similarly for s_{iR}, b_{iR}.

Can we write down any mass term?

$\bar{u}_L u_R$ $\quad \bar{U}_{iL} u_{iR}$.

→ not gauge invariant.

No gauge invariant mass term is possible.

Leptons:

Left-handed: Fundamental of SU(2), singlet of SU(3), hypercharge -1.

$E^{(1)}_{\alpha L}$, $E^{(2)}_{\alpha L}$, $E^{(3)}_{\alpha L}$

$$i\bar{E}^{(1)}_L\gamma^\mu D_\mu E^{(1)}_L = i\bar{E}^{(1)}_L\gamma^\mu(\partial_\mu - iC^A_\mu T^A + iS_\mu)E^{(1)}_L$$

Right handed leptons:

e_R, μ_R, τ_R

→ Singlet of SU(3) & SU(2), hypercharge -2.

$$i \bar{e}_R \gamma^\mu D_\mu e_R = i \bar{e}_R \gamma^\mu (\partial_\mu + 2 i S_\mu) e_R.$$

Similarly for μ_R, τ_R.

Interpretation:

$$E^{(1)}_{1L}, E^{(2)}_{1L}, E^{(2)}_{3R} \Leftrightarrow e_L, \mu_L, \tau_L$$

→ left handed components of

~~[illegible]~~ electron, muon, τ.

$$E^{(1)}_{2L}, E^{(2)}_{2L}, E^{(3)}_{2L} \Leftrightarrow \nu_{eL}, \nu_{\mu L}, \nu_{\tau L}.$$

Left-handed components of ν_e, ν_μ, ν_τ.

Note: No right handed ν_e, ν_μ, ν_τ in the standard model.

Physical electric charge:

$$Q = \underset{\downarrow}{T_3} + \frac{Y}{2}$$

3-rd generator of SU(2): $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

Problem with the model as it stands:

No particle has any mass.

① Gauge fields are massless.

② No gauge invariant mass term possible for quarks, electrons, etc.

$\rightsquigarrow$ Contradiction to experiment.

To solve these problems we need to introduce a set of complex scalar fields called the Higgs field H.

Higgs scalar:

Singlet of SU(3), fundamental of SU(2), hypercharge $Y = 1$

$$\mathcal{L} = -(\mathcal{D}_\mu \phi)^\dagger (\mathcal{D}^\mu \phi)$$

$$= -(\mathcal{D}_\mu \phi)^*_\alpha (\mathcal{D}^\mu \phi)_\alpha$$

$$(\mathcal{D}_\mu \phi)_\alpha = \Big(\partial_\mu \phi_\alpha - i C^A_\mu (T^A)_{\alpha\beta} \phi_\beta - i S_\mu \phi_\alpha\Big)$$

What other terms can we add?

$$-\mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \dots$$

What about scalar-fermion-fermion coupling?

$$\lambda_d \underset{\substack{\downarrow \\ Y=-\frac{1}{3}}}{\bar{U}_{i\alpha L}} \underset{\substack{\searrow \\ Y=1}}{\phi_\alpha} \underset{\substack{\searrow \\ Y=-\frac{2}{3}}}{d_{iR}} + h.c. \rightarrow \text{possible.}$$

Similarly for c, t.

Another possible term:

$$\left\{ \lambda_u \bar{U}_{i\alpha L} \, u_{iR} \, \epsilon_{\alpha\beta} \, \phi^*_\beta + h.c. \right\}$$

$$Y = -\tfrac{1}{3} \qquad Y = \tfrac{4}{3} \qquad Y = -1$$

The Higgs field gives mass to the fermions and SU(2) gauge fields via Higgs mechanism.

Finally we also have

$$\lambda_e \, \bar{E}^{(1)}_{\alpha L} \, \phi_\alpha \, e_R$$

$$Y = 1 \qquad Y = 1 \qquad Y = -2.$$

Spontaneous symmetry breaking

Consider a scalar field theory.

$$S = \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi) \right]$$

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$$

Invariant under $\phi \to -\phi$.

$\Rightarrow$ All correlation functions have this symmetry.

$$\left\langle \prod_{i=1}^{n} \phi(x_i) \right\rangle = \left\langle \prod_{i=1}^{n} \left(-\phi(x_i)\right) \right\rangle$$

Consequence: It vanishes if n ~~even~~ = odd.

Formal proof:

$$\int [\mathcal{D}\phi]\, e^{iS[\phi]} \prod_{i=1}^{n} \phi(x_i) \Big/ \int \mathcal{D}\phi\, e^{iS[\phi]}$$

$$\underset{\phi=-\chi}{=} \int [\mathcal{D}\chi]\, e^{iS[-\chi]} \prod_{i=1}^{n} (-1)^n \chi(x_i) \Big/ \int \mathcal{D}\chi\, e^{iS[-\chi]}$$

$$= \int \mathcal{D}\chi\, e^{iS[\chi]} \prod_{i=1}^{n} \left(-\chi(x_i)\right) \Big/ \int \mathcal{D}\chi\, e^{iS[\chi]}$$

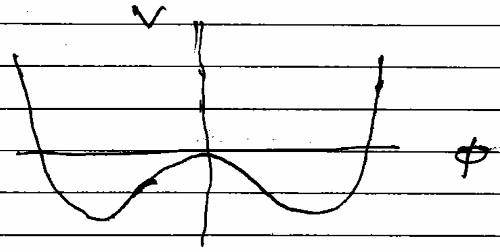
$$= \int \mathcal{D}\phi\, e^{iS[\phi]} \prod_{i=1}^{n} \left(\phi(x_i)\right) \Big/ \int \mathcal{D}\phi\, e^{iS[\phi]}$$

Now consider:

$$S = \int d^4x \left[-\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi) \right]$$

$$V(\phi) = -\frac{1}{2}\mu^2 \phi^2 + \phi \frac{\lambda}{4} \phi^4$$

Naively : $m^2 = -\mu^2$ in lowest order



group velocity

$$E^2 = \vec{p}^2 - \mu^2 \Rightarrow \vec{v} = \frac{dE}{dp} = \frac{\vec{p}}{\sqrt{\vec{p}^2 - \mu^2}}$$

$|\vec{v}| > 1 \Rightarrow$ tachyon.

Note: This conclusion is reached by doing perturbation theory where $O(\phi^4)$ term is neglected w.r.t. ϕ^2 term. Is this justified?

Consider linearized eq. of motion:

$$-\Box\phi = \mu^2\phi$$

If $\vec{\nabla}\phi = 0 \Rightarrow \partial_0^2\phi = \mu^2\phi$

$$\phi = Ae^{\mu x^0} + Be^{-\mu x^0}$$

⇝ ϕ grows with time for generic initial condition.

(Note: For $\mu = 2m$, ϕ will be oscillatory & hence small perturbation remains small).

⇒ Perturbation theory breaks down.

Remedy: Expand $V(\phi)$ around the minimum and carry out perturbation theory.

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4$$

$$V'(\phi) = 0 \Rightarrow -\mu^2\phi + \lambda\phi^3 = 0. \Rightarrow \phi = \underset{\text{max}}{0}, \underset{\text{min}}{\pm\sqrt{\frac{\mu^2}{\lambda}}}$$

Define $\chi = \phi - \sqrt{\frac{\mu^2}{\lambda}}$

$$V(\phi) = V\left(\chi + \sqrt{\frac{\mu^2}{\lambda}}\right) = -\frac{\lambda}{4}\left(\frac{\mu^2}{\lambda}\right)^2 + \mu^2\chi^2 + \lambda\sqrt{\frac{\mu^2}{\lambda}}\,\chi^3 + \frac{\lambda}{4}\chi^4.$$

Note: χ^2 coefficient is +ve.

Interpretation: We have a particle of mass. $\sqrt{2}\,\mu$.

Perturbation theory: Represent χ^3 and χ^4 terms as interaction terms.

The correlation functions do not have any obvious symmetry.

Why?

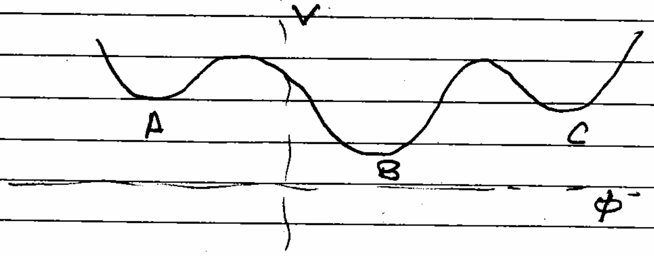
Canonical formalism: The vacuum $|\Omega\rangle$ is not invariant under the symmetry.

Path integral formulation: Boundary condition at ∞ ($\alpha \to 0$ as $r \to \infty$) breaks the symmetry).

$\Rightarrow$ Spontaneous symmetry breaking.

~~Physics around $\phi = \sqrt{\frac{\mu^2}{\lambda}}$ & $\phi = -\sqrt{\frac{\mu^2}{\lambda}}$ equivalent~~

An aside: Consider a $V(\phi)$ of the form

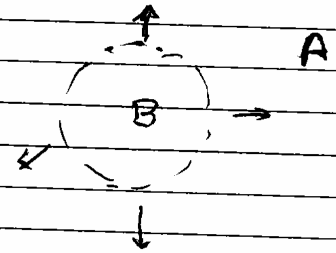


We could do sensible perturbation theory around A, B or C.

If ~~A, B~~ they are not related by symmetry, then the 'theories' are inequivalent.

A, C: metastable vacua, B: stable vacua.

Non-perturbatively one finds that metastable vacua are unstable → decays into stable vacuum.



The life-time may be large.
Our universe is most likely a metastable phase with large life-time.

Spontaneous breaking of continuous symmetry:

Consider a complex scalar field with

$$S = \int d^4x \left[-\eta^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - V(\phi) \right]$$

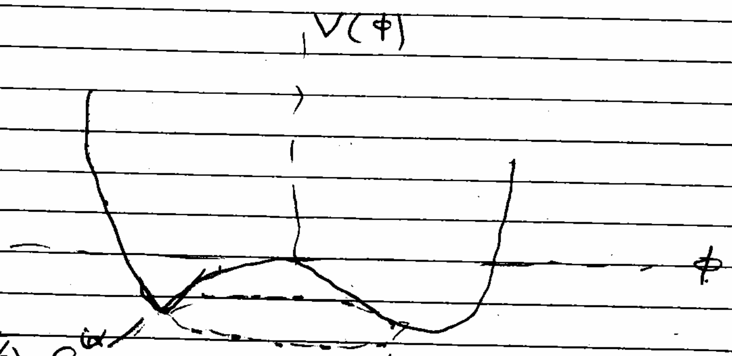
$$V[\phi] = -\mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2$$

symmetry: $\phi \to e^{i\theta} \phi$

θ arbitrary constant.

If $\phi = (\phi_1 + i \phi_2) \frac{1}{\sqrt{2}}$

$$V(\phi) = -\frac{1}{2} \mu^2 (\phi_1^2 + \phi_2^2) + \frac{\lambda}{4} (\phi_1^2 + \phi_2^2)^2$$



$\phi = \sqrt{\mu^2/2\lambda}\, e^{i\alpha}$ α = arbitrary.

Expand around any point.

$\alpha = 0.$

$$\phi = \sqrt{\frac{\mu^2}{2\lambda}} + \underset{\frac{1}{\sqrt{2}}\left(\chi_1 + i\chi_2\right)}{\underset{\shortparallel}{\chi}}.$$

$$S = \int d^4x \Big[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu \chi_1 \partial_\nu \chi_1 - \frac{1}{2}\eta^{\mu\nu}\partial_\mu \chi_2 \partial_\nu \chi_2 - \tilde{V}(\chi_1, \chi_2) \Big]$$

$$\tilde{V}(\chi_1, \chi_2) = -\frac{\lambda}{4}\left(\frac{\mu^2}{\lambda}\right)^2 + \mu^2 \chi_1^2 + \text{Cubic} + \text{Quartic in } \chi_1 \chi_2$$

Note: χ_2 is massless.

$m_{\chi_1}^2 = 2\mu^2.$

Interpretation: χ_2 denotes the flat direction of the potential.

$\to$ Goldstone boson.

One can argue on general grounds that this particle remains massless even after quantum corrections.

(Pole at $k^2=0$ is not shifted).

General result: Suppose we have a continuous group G and a scalar field ϕ in representation R.

$$\phi \to R(g)\,\phi \quad \text{for } g \in G.$$

Suppose we have a potential $V(\phi)$ with a minimum at ϕ_0.

In general $R(g)\,\phi_0 \neq \phi_0$.

$\Rightarrow g$ is not a symmetry of the perturbative vacuum around ϕ_0.

g is "spontaneously broken".

Suppose $h \in G$ such that

$$R(h)\ \phi_0 = \phi_0$$

Then h is a symmetry of perturbation vacuum.

Note: If $R(h_1)\ \phi_0 = \phi_0$, $R(h_2)\ \phi_0 = \phi_0$ then

$$R(h_1 h_2)\ \phi_0 = R(h_1)\ R(h_2)\ \phi_0 = \phi_0$$

$\ni$ The subset of the elements of G, describing symmetries of the vacuum, forms a group.

"Unbroken subgroup" H of G.

Suppose T^a: generators of G
$a = 1, \cdots n_G$

Infinitesimally close to identity element of G:

$$1 + i \sum_{a=1}^{n_G} \epsilon_a T^a$$

Generators of H : T^{α} : $\alpha = 1, \cdots n_H$.

Infinitesimally close to identity element of H : $1 + i \sum_{\alpha=1}^{n_H} \epsilon_{\alpha} T^{\alpha}$.

Generators of G which are not generators of H :

$$T^{k} : k = n_H + 1, \cdots n_G.$$

$$\left(1 + i \sum_{\alpha=1}^{n_H} \epsilon_{\alpha} R(T^{\alpha})\right) \phi_0 = \phi_0$$

$$\Rightarrow \quad R(T^{\alpha}) \phi_0 = 0.$$

$$R(T^{k}) \phi_0 \neq 0.$$

Now $V(\phi_0) = V\left(\left(1 + i \sum_{k=n_H+1}^{n_G} \epsilon_k R(T^k)\right) \phi_0\right)$

$$= V\left(\phi_0 + i \sum_{k=n_{H+1}}^{n_G} \epsilon_k R(T^k) \phi_0\right)$$

$n_G - n_H$ flat directions

$\Rightarrow$ $n_G - n_H$ massless Goldstone bosons.

U(1) gauge field coupled to ~~matter~~ Complex Scalar.

$$S = \int d^4x \left[- (D_\mu \phi)^* D_\nu \phi \, \eta^{\mu\nu} - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$

$$V(\phi) = -\mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2$$

$$D_\mu \phi = (\partial_\mu - i g A_\mu) \phi \, , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Gauge trs. $\phi \to e^{i\theta(x)} \phi \, , \quad A_\mu \to A_\mu + \frac{1}{g} \partial_\mu \theta$

The potential has minimum at

$$\phi = \sqrt{\frac{\mu^2}{2\lambda}} \, e^{i\alpha}$$

Define $\chi = \frac{1}{\sqrt{2}} (\chi_1 + i \chi_2)$

$$\phi = \sqrt{\frac{\mu^2}{2\lambda}} + \frac{1}{\sqrt{2}} (\chi_1 + i \chi_2)$$

Substitute into the action.

$$S = \int d^4x \left[- \partial_\mu \phi^* \partial^\mu \phi + i g \, \partial_\mu \phi^* A_\mu \phi \right.$$
$$- i g \, \phi^* \partial_\mu \phi A_\mu - g^2 A_\mu A^\mu \phi^* \phi$$
$$\left. - \mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$

Substitute ϕ in terms of χ:

$$S = \int d^4x \Big[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\chi_1\,\partial_\nu\chi_2 - \frac{1}{2}\eta^{\mu\nu}\partial_\mu\chi_2\,\partial_\nu\chi_2$$

$$+\, g\sqrt{\frac{\mu^2}{\lambda}}\,\eta^{\mu\nu}A_\mu\,\partial_\nu\chi_2 - \frac{g^2}{2}\frac{\mu^2}{\lambda}A_\mu A^\mu$$

$$+\, \frac{\lambda}{4}\left(\frac{\mu^2}{\lambda}\right)^2 - \mu^2\chi_1^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \cdots\Big]$$

Cubic + quartic.

Note χ-term between A_μ & χ_2

→ need to diagonalize.

Define $B_\mu = \left(A_\mu - \sqrt{\frac{\lambda}{\mu^2}}\,\frac{1}{g}\,\partial_\mu\chi_2\right)$.

$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$.

$$S = \int d^4x \Big[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\chi_1\,\partial_\nu\chi_2 - \frac{1}{2}g^2\frac{\mu^2}{\lambda}\eta^{\mu\nu}B_\mu B_\nu$$

$$+\, \frac{\lambda}{4}\left(\frac{\mu^2}{\lambda}\right)^2 - \mu^2\chi_1^2 - \frac{1}{4}\eta^{\mu\mu'}\eta^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'}$$

$$+ \cdots\Big].$$

~~Note:~~

Note: ① χ_2 has disappear completely from the quadratic action.

② Gauge trs. laws:

$$A_\mu \to A_\mu + \partial_\mu \theta(x) \qquad \phi \to e^{ig\theta(x)}\phi$$

$$= (\phi + ig\,\theta(x)\,\phi + \cdots)$$

$$\frac{\delta\chi_1}{\sqrt{2}} + \frac{i\delta\chi_2}{\sqrt{2}} = ig\,\theta(x)\left[\sqrt{\frac{\mu^2}{2\lambda}} + \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2)\right]$$

$$\delta\chi_1 = -g\,\theta(x)\,\chi_2$$

$$\delta\chi_2 = g\sqrt{\frac{\mu^2}{\lambda}}\,\theta(x) + g\,\cancel{\sqrt{\frac{\mu^2}{2\lambda}}}\,\theta(x)\,\chi_1$$

$$\delta B_\mu = \partial_\mu\theta - \sqrt{\frac{\lambda}{\mu^2}}\,\frac{1}{g}\,\partial_\mu\left[g\sqrt{\frac{\mu^2}{\lambda}}\,\theta(x)\right.$$

$$\left. + g\,\theta(x)\,\chi_1\right]$$

$$= 0 - \sqrt{\frac{\lambda}{\mu^2}}\,\partial_\mu(\theta(x)\,\chi_1)$$

$\Rightarrow$ No field independent term.

~~For free action $\delta B_\mu = 0$~~

$\Rightarrow$ B_μ kinetic operator is invertible even before gauge fixing!

Nevertheless we need to gauge fix since the original theory had gauge invariance.

Unitary gauge: $\mathcal{H}(x) = \chi_2(x)$

$$\prod_x \delta(\mathcal{H}(x)) = \prod_x \delta(\chi_2(x))$$

$\Rightarrow$ sets χ_2 to 0 in the path integral.

No need to exponentiate.

Ghost action:

$$\delta \mathcal{H}(x) = \delta\chi_2(x) = \left(\sqrt{\frac{\mu^2}{2\lambda}} + \chi_1(x)\right) \epsilon(x)$$

$$\epsilon(x) = \frac{1}{g}\,\theta(x).$$

$$\left.\frac{\delta \mathcal{H}_\epsilon(x)}{\delta\epsilon(y)}\right|_{\epsilon=0} = \left(\sqrt{\frac{\mu^2}{2\lambda}} + \chi_1(x)\right) \delta^{(4)}(x-y)$$

Ghost action:

$$\int d^4x\; b(x)\left(\sqrt{\frac{\mu^2}{\lambda}} + \chi_1(x)\right) c(y)\; \delta^{(4)}(y-x)$$

$$= \int d^4x\; b(x)\left(\sqrt{\frac{\mu^2}{\lambda}} + \chi_1(x)\right) c(x).$$

b - c propagator: $i\sqrt{\frac{\lambda}{\mu^2}}\,(2\pi)^4\,\delta^{(4)}(k_1+k_2)$

No $1/k^2$.

Scalar field action:

$$\int d^4x\left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\chi_1\partial_\nu\chi_1 - \mu^2\chi_1^2\right]$$

$\Rightarrow$ a particle of mass $2\mu^2$.

Gauge field action:

$$\int d^4x\left[-\frac{1}{2}\eta^{\mu\mu'}\eta^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'} - \frac{1}{2}\overset{g^2\mu^2/\lambda}{\overset{\shortparallel}{M^2}}\eta^{\mu\nu}B_\mu B_\nu\right]$$

$$= \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,\tilde{B}_\mu(-k)\overbrace{\left\{\eta^{\mu\nu}(-k^2-M^2)+k^\mu k^\nu\right\}}^{M^{\mu\nu}(k)}\tilde{B}_\nu(k)$$

Gauge field propagator:

$$i\,(2\pi)^4\,\delta^{(4)}(k_1+k_2)\,\big(M^{-1}(k)\big)_{\mu\nu}.$$

Ex. $$\big(M^{-1}(k)\big)_{\mu\nu} = \frac{1}{-k^2-M^2}\left(\eta_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right)$$

$\Rightarrow$ a particle of mass $M^2 = \mu^2 g^2/\lambda$.

How many states?

① ~~Need to~~ Could quantize the theory in the canonical formalism and check.

② Or we could check how many eigenvalues of $\eta_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}$ actually has pole.

(Note: Since at the free level B_μ is gauge invariant all poles are physical).

Easiest: go to rest frame: $k \simeq (M,0,0,0) + O(k^2+M^2)$

00 component vanishes. $\Rightarrow$ 3 poles.

$\Rightarrow$ 3 states. spin 1.

Find spectrum:

A massive vector of mass $g\mu/\sqrt{\lambda}$

A scalar of mass $\sqrt{2}\,\mu$ → Higgs

Note: # of d.o.f. (states) is the same in broken and unbroken phase.

Broken phase: 1 vector → 3 states } 4
1 - Scalar → 1 state }

Unbroken phase: 1 massless vector - 2 states

2 massive scalars → 2 states

→ 4.

Note: The gauge we used for analyzing the spectrum is not suitable for S-matrix calculation.

→ Not renormalizable.

→ requires another gauge (like Feynman gauge)

General Case:

Gauge group G broken to H.

Generator of G: T^a : $a=1, \cdots n_G$.

Generator of H: T^α : $\alpha=1, \cdots n_H$.

Generator of G not in H: T^K : $K=n_H+1, \cdots n_G$

Gauge fields A^α remain massless.

Gauge fields A^K: "absorb the goldstone bosons" and become massive.

Revisiting the standard model:

Gauge group $SU(3) \times SU(2) \times U(1)$

Gauge fields:

$$\begin{tikzcd} B_\mu^a \arrow[d] & C_\mu^A \arrow[d] & S_\mu \arrow[d] \\ SU(3) & SU(2) & U(1) \arrow[d] \\ 1 \leq a \leq 8 & 1 \leq A \leq 3 & \text{hypercharge} \end{tikzcd}$$

Fermions:

① Left handed fermions in fundamental of $SU(3)$, fundamental of $SU(2)$, hypercharge $y = \frac{1}{3}$

$$\begin{tikzcd} & U_{i\alpha L} \arrow[dl] \arrow[dr] & \\ SU(3) \text{ Color} & & SU(2) \end{tikzcd}$$

$U_{i1} \rightarrow$ up quark

$U_{i2} \rightarrow$ down quark.

+2 more generations

② Right handed fermions in fundamental of $SU(3)$, singlet of $SU(2)$

$u_{iR} \rightarrow$ hypercharge $y = \frac{4}{3}$

$d_{iR} \rightarrow$ hypercharge $y = -\frac{2}{3}$

+2 more generations.

③ Left handed ~~leptons~~ fermions in fundamental of SU(2), singlet of SU(3) and hypercharge $Y = -1$.

$E_{\alpha L}$ $\quad E_{1L}$: neutrino | + 2 more

SU(2) $\quad E_{2L}$: electron. | generation

④ Right handed fermions in singlet of SU(3), singlet of SU(2), hypercharge $Y = -2$

e_R

+ 2 more generation.

⑤ Higgs field : ~~a~~ complex scalar in the ~~doublet~~ fundamental of SU(2), singlet of SU(3), hypercharge $Y = 1$.

Coupling of gauge fields among themselves and to scalars and fermions

→ Completely fixed by gauge invariance in terms of g_1, g_2, g_3.

Only freedom: Add gauge invariant terms involving fermions and ~~gauge~~ ~~fields~~ scalars.

ⓐ No mass term for fermions.

ⓑ Yukawa coupling terms:

$$\underset{}{\lambda_d}\ \underset{Y=-\frac{1}{3}}{\bar{U}_{i\alpha L}}\ \underset{Y=1}{\phi_\alpha}\ \underset{Y=-\frac{2}{3}}{d_{iR}} + \lambda_u\ \underset{Y=-\frac{1}{3}}{\bar{U}_{i\alpha L}}\ \underset{Y=\frac{4}{3}}{u_{iR}}\ \underset{Y=-1}{\epsilon_{\alpha\beta}\phi^*_\beta}$$

$$\text{~~other~~} + \lambda_e\ \underset{Y=1}{\bar{E}_{\alpha L}}\ \underset{Y=1}{\phi_\alpha}\ \underset{Y=-2}{e_R}$$

$$+\ h.c.$$

ⓒ Higgs mass and self-coupling:

$$-m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \equiv -V(\phi)$$

Replace m^2 by $-\mu^2$

$$V(\phi) = -\mu^2\,\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$$

$$= \lambda\left(\phi^\dagger\phi - v^2\right)^2 \text{ ~~+ constant~~ } - \lambda v^4$$

$$v = \sqrt{\frac{\mu^2}{2\lambda}}$$

As a result, $V(\phi)$ has a minimum at $\phi^\dagger\phi = v^2$

Any ϕ satisfying this will give equivalent physics.

Take $\phi = \begin{pmatrix} 0 \\ v \end{pmatrix}$.

Gauge group is broken.

$SU(3) \to$ intact.

$SU(2) \times U(1)$ action:

$$e^{i\alpha}\, U \begin{pmatrix} 0 \\ v \end{pmatrix}$$

Infinitesimal: $\alpha = \epsilon$, $U = 1 + i\epsilon_a T^a$

$$\begin{pmatrix} 0 \\ v \end{pmatrix} + \delta\phi = \left(1 + i\overset{\epsilon Y_\phi}{\cancel{\alpha}}\right)\left(1 + i\epsilon_a T^a\right)\begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$= \left(1 + i\epsilon Y_\phi + i\epsilon_a T^a\right)\begin{pmatrix} 0 \\ v \end{pmatrix}$$

Unbroken symmetry generators:

$$\delta\phi = 0.$$

$$\begin{pmatrix} \epsilon + \frac{\epsilon_3}{2} & \epsilon_1 - i\epsilon_2 \\ \epsilon_1 + i\epsilon_2 & \epsilon - \frac{\epsilon_3}{2} \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = 0.$$

$$\epsilon - \frac{\epsilon_3}{2} = 0 \qquad \epsilon_1 - i\epsilon_2 = 0.$$

$$\Downarrow \qquad \overset{\shortparallel}{\epsilon_1} = \epsilon_2 = 0$$

$$\epsilon = \frac{\epsilon_3}{2}$$

~~$\epsilon_1 = \epsilon_2 = 0$~~

If we choose $\epsilon = \frac{\epsilon_3}{2}$, then the corresponding symmetry is unbroken.

What is this symmetry?

In general the symmetry generator is

$$\epsilon Y + \epsilon_a T^a$$

$$\to \epsilon_3 \left(\frac{Y}{2} + T^3 \right)$$

$$\Downarrow$$

Unbroken U(1) generator

→ electric charge.

Check:

		$T_3 + Y/2$	
Q :	U_{i1L}	$\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$	u_L
	U_{i2L}	$-\frac{1}{2} + \frac{1}{6} = -1/3$	d_L
	u_{iR}	$0 + \frac{2}{3} = 2/3$	u_R
	d_{iR}	$0 - 1/3 = -1/3$	d_R
	E_{1L}	$1/2 - 1/2 = 0$	ν_L
	E_{2L}	$-1/2 - 1/2 = -1$	e_L
	e_R	$0 - 1 = -1$	e_R

Mass terms for fermions:

For $\phi = \begin{pmatrix} 0 \\ v \end{pmatrix} + \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ φ_1 — complex, φ_2 — complex.

$$\lambda_d \bar{U}_{i\alpha L} \phi_\alpha d_{iR} = v \lambda_d \bar{U}_{i1L} d_{iR} + \text{Cubic}$$

$$= \underbrace{\lambda_d v}_{m_d} \bar{d}_L d_R + \text{Cubic}$$

$$\lambda_u \bar{U}_{i\alpha L} u_{iR} \epsilon_{\alpha\beta} \phi^*_\beta = \underbrace{\lambda_u v}_{m_u} \bar{u}_{iL} u_{iR} + \text{Cubic}$$

Note: Neutrinoes are massless.

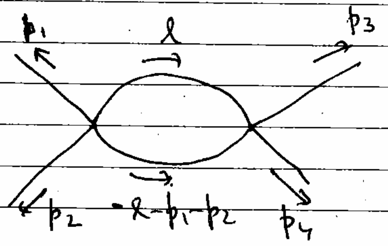
SU(2) gauge fields C^1_μ, C^2_μ $\overset{W^\pm}{\text{massive}}$

A linear combination of C^3_μ, S_μ $\underset{\text{photon}}{\text{massless}}$

the other is massive → Z

Ultraviolet divergences and renormalization:

Take a typical loop diagram in ϕ^4 field theory with loops:



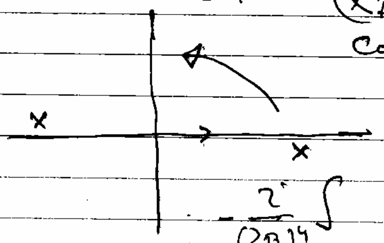
$$\prod_{k=1}^{4} \frac{i}{-p_k^2-m^2} \quad (2\pi)^4 \quad \delta^{(4)}(p_1+p_2+p_3+p_4)$$

$$\int \frac{d^4 l}{(2\pi)^4} \quad \frac{i}{-l^2-m^2+i\varepsilon} \quad \frac{i}{-(l+p_1+p_2)^2-m^2+i\varepsilon}$$

Divergences:

① $l^2+m^2=0 \to$ controlled by $i\varepsilon$.

② Large l: $-\frac{1}{(2\pi)^4}\int \frac{d^4 l}{(l^2+m^2)^2} \to$ log divergent.



Complex l_0 plane

$l_0 = i\, l_4$

$$-\frac{i}{(2\pi)^4}\int \frac{dl_1 \cdots dl_4}{(l_1^2+\cdots l_4^2+m^2)^2}$$

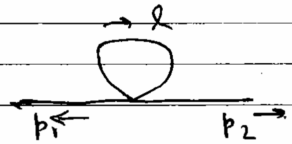
$$= \frac{i}{(2\pi)^4} \int \frac{r^3\, dr\, \Omega_3}{(r^2+m^2)^2} \qquad r = \sqrt{l_1^2 + \cdots l_4^2}$$

Ω_3 : volume of unit 3-sphere.

$$\underset{r \to \infty}{\sim} \int^{\infty} \frac{dr}{r}$$ → logarithmically divergent from $r \to \infty$.

Note: No divergence at $r = 0$.

Similar divergences occur in other theories and diagrams.



$$\sim (\cdots) \int \frac{d^4 l}{(l^2+m^2)}$$

→ quadratic divergence.

etc.

Q. How do we make sense of this?

→ Renormalization.

Two steps.

① Regularization: Make the theory finite by using a cut-off parametrized by some small parameter ϵ (not ε of $i\varepsilon$).

Examples:

a) Momentum cut-off: Restrict integration over l^μ s.t. $|l^\mu| < M/\epsilon$ for some mass M.

→ not Lorentz invariant.

→ not a serious problem if at the end we can recover Lorentz invariance.

b) Dimensional regularization: Work in $4-\epsilon$ dimension instead of 4.

~~This~~ ~~makes~~ ~~the~~ ~~to~~ By choosing ϵ to be sufficiently large we can make the integral finite.

Then analytically continue to small ϵ. We shall discuss this in detail later.

Step. 2: Make the ~~coupling constants~~ parameters of the theory e.g. m^2 and λ in ϕ^4 theory ϵ dependent and divergent in $\epsilon \to 0$ limit such that in the $\epsilon \to 0$ limit these divergences cancel the divergences from loop integral.

Reason: Parameters in the action are not directly measured.

As long as the observables, i.e. the S-matrix elements are all finite, the theory is consistent.

Non-trivial constraint: There are only finite no. of parameters but ∞ no. of S-matrix elements.

Thus we need to ensure that by adjusting the ϵ dependence of these finite no. of parameters we can make all the observables finite.

Only special class of theories have this property

$\sim$ Renormalizable theories.

Another viewpoint: Since there are finite no. (say N) of parameters we can eliminate them to express all S-matrix elements in terms of N-independent S-matrix elements.

These relations must be finite as $\epsilon \to 0$.

Dimensional regularization:

$$\int \frac{d^D k}{(2\pi)^D} \quad \frac{1}{(k^2+L)^\alpha}$$

$$k^2 = k_1^2 + \cdots k_D^2$$

How to make sense of this for ~~fractional~~ non-integer D and for $D \geq \alpha$?

Scaling : $k = \sqrt{L}\, l$.

$$L^{D/2-\alpha} \underbrace{\int \frac{d^D l}{(2\pi)^D} \quad \frac{1}{(l^2+1)^\alpha}}_{A}$$

Our goal is to compute A

Differentiate both sides n-times w.r.t. L.

$$(-\alpha)(-\alpha-1)\cdots(-\alpha-n+1) \int \frac{d^D k}{(2\pi)^D} \quad \frac{1}{(k^2+L)^{\alpha+n}}$$

$$= \left(\frac{D}{2}-\alpha\right)\left(\frac{D}{2}-\alpha-1\right)\cdots\left(\frac{D}{2}-\alpha-n+1\right) A \; L^{\frac{D}{2}-\alpha-n}$$

~~At~~ Thus

$$A = \frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-n+1)}{\left(\frac{D}{2}-\alpha\right)\left(\frac{D}{2}-\alpha-1\right)\cdots\left(\frac{D}{2}-\alpha-n+1\right)} \int \frac{d^D k}{(2\pi)^D} \frac{L^{\alpha+n-\frac{D}{2}}}{(k^2+L)^{\alpha+n}}$$

For $\alpha+n > \frac{D}{2}$ this integral is convergent.

But it does not help for even integer D since one of the denominator factors vanish.

Strategy: Try to define r.h.s. for D a real no.

$$d^D k = r^{D-1} \Omega_{D-1}\, dr. \qquad r = \sqrt{k^2}$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{L^{\alpha+n-\frac{D}{2}}}{(k^2+L)^{\alpha+n}} = \int \frac{r^{D-1} \Omega_{D-1}\, dr}{(2\pi)^D (r^2+1)^{\alpha+n}}$$

$$\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

Valid for integer D but makes sense for all real D.

(21.8)

$$\int_0^\infty r^{D-1}\left(r^2+1\right)^{-\alpha-n}\,dr$$

$$= \frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(\alpha+n-\frac{D}{2}\right)}{2\Gamma(\alpha+n)}$$

$$A = \frac{1}{2}\,\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{\Gamma(\alpha+n)}\,\frac{\Gamma\left(\alpha+n-\frac{D}{2}\right)}{\left(\alpha+n-\frac{D}{2}-1\right)\cdots\left(\alpha-\frac{D}{2}\right)}$$

$$\times\ \frac{2\,\pi^{D/2}}{(2\pi)^D}$$

$$= \frac{1}{(4\pi)^{D/2}}\,\frac{\Gamma\left(\alpha-\frac{D}{2}\right)}{\Gamma(\alpha)}$$

Dealing with more denominators:

$$\frac{1}{\left\{(l+k_1)^2+m^2\right\}\cdots\left\{(l+k_n)^2+m^2\right\}}$$

$$= \int_0^1 dx_1 \cdots \int_0^1 dx_n\ \delta(x_1+\cdots x_n-1)\ \frac{(n-1)!}{\left(x_1\left\{(l+k_1)^2+m^2\right\}+\cdots x_n\left\{(l+k_n)^2+m^2\right\}\right)^n}$$

$$\Big(l^2 \underset{\overset{\shortparallel}{1}}{(x_1+\cdots x_n)} + 2l\cdot(x_1k_1+\cdots x_nk_n) + \sum x_i\cdot k_i^2 + m^2\Big)$$

→ Complete the square & define

$$\tilde{l} = l + x_1k_1+\cdots x_nk_m$$

More loop integrals:

$$d^D l_1 \quad d^D l_2 \cdots .$$

Combine all denominators first.

$$\frac{1}{\big(A_{ij}\, l_i\cdot l_j + 2l_i\cdot B_i + C\big)^n}$$

functions of external momenta and the x_i's.

Diagonalize A by $l_i = V_{ij}\tilde{l}_j$

Den: $\big(y_i\, \tilde{l}_i^{\,2} + 2\tilde{l}_i\cdot\tilde{B}_i + \tilde{C}\big)^n$

21.10

$$\sqrt{y_i}\left(\tilde{l}_i + \frac{\tilde{B}_i}{y_i}\right) = \hat{l}_i$$

Den. $\left(\sum_i \hat{l}_i^2 + \hat{C}\right)$

Now treat $(\hat{l}_1, \ldots \hat{l}_k)$ as a k D. dimensional vector and apply dimensional regularization formula.

Numerator factors:

$$\int \frac{d^D k}{(k^2 + L)^\alpha} k_\mu = 0.$$

$$\int \frac{d^D k}{(k^2+L)^\alpha} k_\mu k_\nu = A \, \delta_{\mu\nu}.$$

$$A \underbrace{\delta^{\mu\nu} \delta_{\mu\nu}}_{D} = \int \frac{d^D k}{(k^2+L)^\alpha} (k^2 + L - L)$$

$\Rightarrow$ A in terms of known formula.

General replacement rules:

$$\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}}$$

$$(2\pi)^4 \delta^{(4)}(k) \to (2\pi)^{4-\epsilon} \delta^{(4-\epsilon)}(k)$$

$$\eta^{\mu\nu}\eta_{\mu\nu} = 4-\epsilon, \quad \{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$$

$$\gamma_\mu \gamma^\mu = 4-\epsilon \qquad \gamma_\mu \gamma^\nu \gamma^\mu = (2-D)\gamma^\nu$$

etc.

Note: # of γ-matrices $= 4-\epsilon$.

Dimension of γ-matrices in $4-\epsilon$ dim:

$= f(\epsilon)$

$f(4) = 4$

$$\mathrm{Tr}\,[\gamma_\mu \gamma_\nu] = f(\epsilon)\,\eta_{\mu\nu}$$

The final result is independent of the choice of $f(\epsilon)$ but we shall choose $f(\epsilon) = 4$.

Renormalization: General procedure

Consider a field theory with fields

$\phi_i \quad (i = 1, \cdots N)$

(scalars, vectors, fermions, ...)

Parameters $g_s \quad (s = 1, \cdots M)$

~ includes masses, coupling constants, gauge fixing parameter α etc.

$$\vec{g} = (g_1, \cdots g_M)$$

Introduce new fields ϕ_{iR} and new parameters g_{sR} related to the old ones as

$$\left.\begin{aligned} g_s &= F_s(\vec{g}_R, \epsilon) \\ \phi_i &= K_{ij}(\vec{g}_R, \epsilon)\, \phi_{jR} \end{aligned}\right\} \text{ϵ-dependent redefinition of fields + parameters}$$

ϵ : UV regulator

F_s, K_{ij} : Some function of their arguments.

$$\left\langle \prod_{l=1}^{n} \phi_{i_l R}(x_l) \right\rangle$$

$$= \sum'_{j_1, \cdots j_n} \prod_{l=1}^{n} K^{-1}_{i_l j_l}(\vec{g}_R, \epsilon) \left\langle \prod_{l=1}^{n} \phi_{j_l}(x_l) \right\rangle$$

Question: Can we choose $K_{ij}(\vec{g}_R, \epsilon)$ and $F_s(\vec{g}_R, \epsilon)$ such that

$$\left\langle \prod_{l=1}^{n} \phi_{i_l R}(x_l) \right\rangle$$

is finite for all Greens functions, as $\epsilon \to 0$.

ϕ_{iR} : Renormalized field

g_{sR} : Renormalized parameters.

If the answer is yes then the theory is renormalizable.

Note: To lowest order in perturbation theory there is no divergence.

⇒ We can take

$$g_s = F_s(\vec{g}_R, \epsilon) = g_{sR} + G_s(\vec{g}_R, \epsilon)$$

$$K_{ij}(\vec{g}_R, \epsilon) = \delta_{ij} + L_{ij}(\vec{g}_R, \epsilon)$$

G_s and L_{ij} contains at least one order of the perturbation parameter.

$$\mathcal{L}(\{\phi_i\}, \{g_s\}) = \mathcal{L}(\{\phi_{iR} + L_{ij}(\vec{g}_R, \epsilon)\phi_{jR}\}, \{g_{sR} + G_s(\vec{g}_R, \epsilon)\})$$

$$= \mathcal{L}(\{\phi_{iR}\}, \{g_{sR}\}) + \overset{\text{Counterterm}}{\mathcal{L}_{ct}}$$

$\downarrow$

contains at least one power of perturbation parameter.

Treat all terms in $\mathcal{L}_{ct}$ as perturbation (including quadratic terms) since they contain at least one order in the expansion parameter.

To each order try to adjust $\mathcal{L}_{ct}$ in such a way that the divergences in the Feynman loop integral cancel against explicit ϵ-dependent divergent terms in $G_s(\vec{g}_R, \epsilon)$ and $L_{ij}(\vec{g}_R, \epsilon)$.

Note: This does not fix the finite parts of G_s and L_{ij}.

$\rightsquigarrow$ an ambiguity related to redefinition of the parameters g_{sR} and fields ϕ_{iR}

$\rightsquigarrow$ Does not affect physical results.

Example: Renormalization of ϕ^4 theory.

$$S = \int d^{4-\epsilon}x \left[-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4 \right]$$

Dimensions: $2[\phi] + 2 - (4-\epsilon) = 0$

$$[\phi] = 1 - \frac{\epsilon}{2}.$$

$$[m] = 1 \qquad [g] + 4\left(1-\frac{\epsilon}{2}\right) - (4-\epsilon) = 0$$

$$\Rightarrow [g] = \epsilon$$

Introduce:

$$\phi = \tilde{Z}_\phi^{1/2}(g_R, m_R, \epsilon)\,\phi_R$$

$$m^2 = Z_m^2(g_R, m_R, \epsilon)\,m_R^2$$

$$g = Z_g(g_R, m_R, \epsilon)\,g_R\,\underbrace{\mu^\epsilon}_{\text{some mass scale.}}$$

g_R is dimensionless.

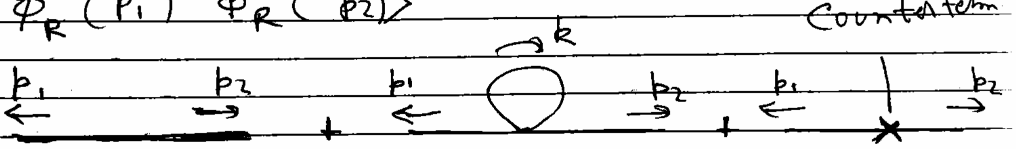
→ a good expansion parameter.

$$\mathcal{L} = \frac{1}{2}\left(- \tilde{Z}_\phi \partial_\mu \phi_R \partial_\mu \phi_R - \tilde{Z}_\phi Z_m^2 m_R^2 \phi_R^2 \right)$$
$$- \frac{1}{4!} \mu^\epsilon g_R \tilde{Z}_\phi^2 Z_g \phi_R^4$$
$$= -\frac{1}{2} \partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2} m_R^2 \phi_R^2 - \frac{g_R \mu^\epsilon}{4!} \phi_R^4$$
$$- \frac{1}{2}(\tilde{Z}_\phi - 1) \partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2}(\tilde{Z}_\phi Z_m^2 - 1) m_R^2 \phi_R^2$$
$$- \frac{1}{4!} \mu^\epsilon g_R (\tilde{Z}_\phi^2 Z_g - 1) \phi_R^4$$

We need to fix $\tilde{Z}_\phi$, Z_m, Z_g by requiring finiteness of the Greens function

Two point function:

$\langle \tilde{\phi}_R(p_1) \tilde{\phi}_R(p_2) \rangle$



$$(2\pi)^{4-\epsilon} \delta^{(4-\epsilon)}(p_1+p_2) \left[\frac{i}{-p_1^2 - m_R^2 + i\epsilon} \right.$$
$$+ \frac{i}{-p_1^2 - m_R^2 + i\epsilon} \frac{i}{-p_1^2 - m_R^2 + i\epsilon} \left\{ -\frac{i g_R \mu^\epsilon}{4!} \times 4 \times 3 \times \int \frac{d^{4-\epsilon}k}{(2\pi)^\epsilon (-k^2 - m_R^2 + i\epsilon)} \right.$$
$$\left. - i (\tilde{Z}_\phi Z_m^2 - 1) m_R^2 - i(\tilde{Z}_\phi - 1) p_1^2 \right\}$$

$\{ \quad \}$

$$= - \frac{i g_R \mu^\epsilon}{4!} \times 4 \times 3 \times i \times (-2) \frac{\Gamma(-1+\frac{\epsilon}{2}) (m_R^2)^{1-\frac{\epsilon}{2}}}{(4\pi)^{2-\epsilon}}$$

$$- i (\tilde{Z}_\phi Z_m^2 - 1) m_R^2 - i (\tilde{Z}_\phi - 1) p_1^2$$

$$= (-i) \Big\{ (\tilde{Z}_\phi Z_m^2 - 1) m_R^2 + (\tilde{Z}_\phi - 1) p_1^2$$

$$- \frac{g_R}{2} \times \frac{1}{16\pi^2} \times \frac{2}{\epsilon} m_R^2 + \text{finite} \Big\}$$

Strategy : Choose:

$$\tilde{Z}_\phi = 1, \quad \tilde{Z}_\phi Z_m^2 = 1 + \frac{m_R^2}{(4\pi)^2} \frac{1}{\epsilon} g_R$$

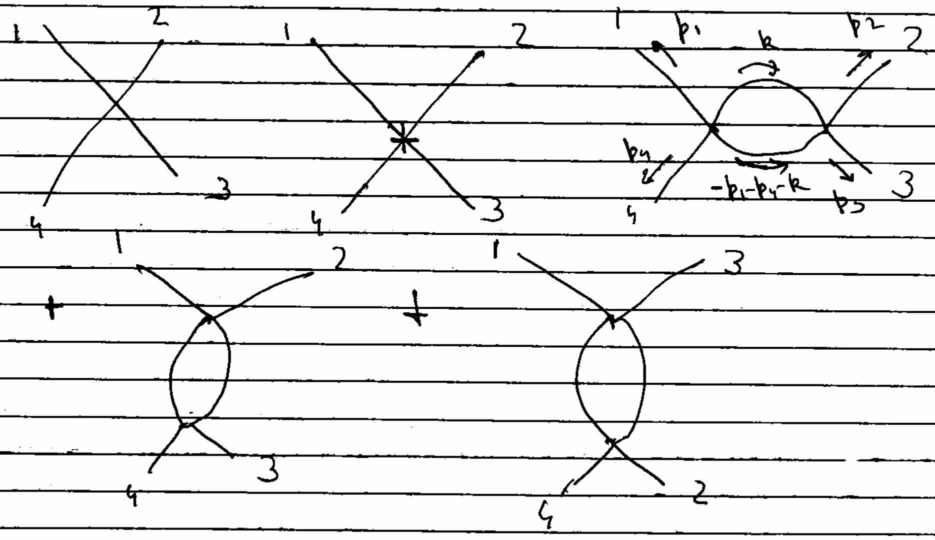
to this order.

$$Z_m^2 = 1 + \frac{m_R^2}{16\pi^2 \epsilon} g_R + O(g_R^2)$$

$$Z_m = 1 + \frac{m_R^2}{32\pi^2 \epsilon} g_R + O(g_R^2)$$

$$Z_\phi = 1 + O(g_R^2).$$

Four point function:



$$-\frac{i g_R}{4!}\mu^\epsilon(\tilde{Z}_\phi^2 Z_g - 1) \times 4 \times 3 \times 2$$

$$+ \; i \frac{3 g_R^2}{16\pi^2\epsilon} + \text{Finite.}$$

$$\Rightarrow (\tilde{Z}_\phi^2 Z_g - 1) = \frac{3 g_R}{16\pi^2\epsilon} + \text{finite.}$$

$$\tilde{Z}_\phi = 1 + O(g_R^2)$$

$$\Rightarrow Z_g = 1 + \frac{3 g_R}{16\pi^2\epsilon} + O(g_R^2)$$

Evaluation of the divergent diagram:

$$(2\pi)^{4-\epsilon}\,\delta^{(4-\epsilon)}\left(\sum p_i\right)\frac{1}{2}\left(-\frac{2 g_R}{4!}\right)^2 \times 8 \times 3 \times 4 \times 3 \times 2$$

$$\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}}\;\frac{i}{-k^2-m_R^2+i\varepsilon}\;\frac{i}{-(p_1+p_4+k)^2-m_R^2+i\varepsilon}$$

$$\prod_{j=1}^{4}\frac{i}{-p_j^2-m_R^2+i\varepsilon}$$

$$= (2\pi)^{4-\epsilon}\,\delta^{(4-\epsilon)}\left(\sum p_i\right)\left(+\frac{g_R^2}{2}\right)\left(\prod_{j=1}^{4}\frac{i}{-p_j^2-m_R^2+i\varepsilon}\right)$$

$$2\int\frac{d^{4-\epsilon}k_E}{(2\pi)^{4-\epsilon}}\underbrace{\frac{1}{k_E^2+m_R^2}\;\frac{1}{(k_E+p_1+p_4)^2+m_R^2}}$$

$$\Downarrow$$

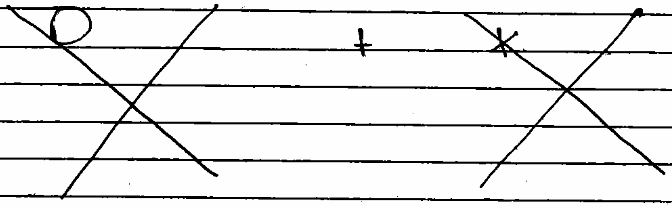
$$\frac{1}{(k_E^2+m_R^2)^2}+\text{finite}$$

$$= (2\pi)^{4-\epsilon}\,\delta^{(4-\epsilon)}\left(\sum p_i\right)\prod_{j=1}^{4}\frac{i}{-p_j^2-m_R^2+i\varepsilon}\left(+\frac{2 g_R^2}{2}\right)$$

$$\times\;\underbrace{\frac{1}{(4\pi)^{2-\frac{\epsilon}{2}}}\;\frac{\Gamma\left(\frac{\epsilon}{2}\right)}{\Gamma(2)}}\to\left(\frac{1}{16\pi^2}\times\frac{2}{\epsilon}+\text{finite}\right)$$

$$= (2\pi)^{4-\epsilon}\,\delta^{(4-\epsilon)}\left(\sum p_i\right)\left(\prod_{j=1}^{4}\frac{i}{-p_j^2-m_R^2+i\varepsilon}\right)\left\{i\, g_R^2\,\frac{1}{16\pi^2\epsilon}+\text{finite}\right\}$$

Note:



Cancel.

etc.

⇒ ⊙ For checking finiteness we only need to analyze one particle irreducible (1PI) diagrams.

↓

diagrams which ~~do~~ ~~not~~ cannot be separated into two disconnected parts by cutting a single line.

Summary: Could choose

$$Z_m = 1 + \frac{g_R}{32\pi^2\epsilon} \qquad Z_m = 1 + \frac{g_R}{32\pi^2\epsilon} + K_m g_R$$

$$\tilde{Z}_\phi = 1 \qquad \tilde{Z}_\phi = 1 + K_\phi g_R$$

$$Z_g = 1 + \frac{3 g_R}{16\pi^2\epsilon} \qquad Z_g = 1 + \frac{3 g_R}{16\pi^2\epsilon} + K_g g_R$$

What is the interpretation of these undetermined constants?

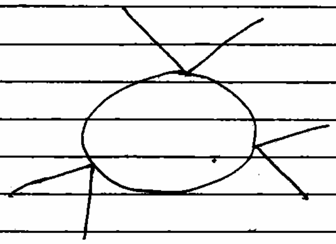
$$m_0 = Z_m m_R = \left(1 + \frac{g_R}{32\pi^2\epsilon} + K_m g_R\right) m_R$$

$$= \left(1 + \frac{g_R}{32\pi^2\epsilon}\right) \underset{\overset{\parallel}{(1 + K_m g_R) m_R}}{m_R'}$$

Similarly

K_ϕ can be absorbed into a definition of ϕ_R & Z_g can be absorbed into a redefinition of g_R.

What about 6-point function?



$$\sim \int \frac{d^{4-\epsilon} l}{(l^2 - m^2 + i\epsilon)^3}$$

→ finite

~~Note: If we had bo~~ ⇒ The theory is renormalizable to one loop.

Note: If we had been in six dimensions this would diverge.

There are no Z's to cancel it.

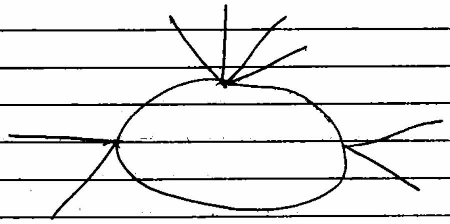
Possible remedy: Add a ϕ^6 coupling:

$$\lambda \phi^6$$

$$\lambda = Z_\lambda \lambda_R$$

↳ Can be adjusted to cancel divergence in 6-point fn.

However now the eight point function is divergent.



$$\sim \int \frac{d^6 l}{(l^2)^3}$$

We need ∞ no. of coupling constants, one of each 2n-point vertex. (& more for vertices with derivatives).

$\Rightarrow$ The theory is non-renormalizable.

Under what conditions is a theory renormalizable?

Consider a theory in D-dimensions:

$$\frac{1}{2}\int d^D x \; \partial_\mu \phi \, \partial^\mu \phi + \cdots$$

$$M^{-D+2+2d(\phi)} \sim M^0 \Rightarrow d(\phi) = \frac{D-2}{2}$$

$$\tilde{\phi}(k) = \int \frac{d^Dx}{(2\pi)^D}\, \phi(x)\, e^{ik.x}$$

$$\Rightarrow d(\tilde{\phi}) = \frac{D-2}{2} - D = -\frac{D+2}{2}$$

Consider a coupling

$$g^{(n)} \int d^Dx\, \phi^n \qquad\qquad D - n\,\frac{D-2}{2}$$

$$d[g^{(n)}] = -n\, d(\phi) + D \overset{=}{\cancel{\text{[illegible]}}} \equiv \lambda_n$$

(say).

If the vertex has derivatives then we have to take this into account.

For fermions $d[\psi] = \frac{D-1}{2}$ etc.

Now consider:

$$\left\langle \prod_{i=1}^{N} \tilde{\phi}(p_i) \right\rangle = \left(\prod_{i=1}^{N} \frac{i}{-p_i^2 - m^2 + i\epsilon} \right) (2\pi)^D \delta^D(\Sigma p_i)$$

$$\Gamma^{(N)}(p_1, \cdots p_N)$$

$$d[\Gamma^{(N)}] = N.\left(-\frac{D+2}{2}\right) + 2N + D$$

$$= N\,\frac{2-D}{2} + D$$

$\Gamma^{(N)}$ Contains powers of coupling Constant.

Suppose the total mass dimension of all the coupling constants in $\Gamma^{(N)}$ is α i.e.

If $\Gamma^{(N)} = \prod_n \left(g^{(n)}\right)^{\alpha_n} \hat{\Gamma}^{(N)}$

then $\alpha = \sum \alpha_n \lambda_n$

$\hat{\Gamma}^{(N)}$ has dimension $N\frac{2-D}{2} + D - \alpha$

$\downarrow$

Maximum degree of divergence.

(Could be less due to powers of external momenta or masses in the numerator).

⇒ There can be UV divergence if

$$N\frac{2-D}{2}+D-\alpha \geq 0.$$

$$\alpha = \sum_n \underset{\text{integers} > 0.}{\underbrace{\alpha_n}} \lambda_n$$

If any $\lambda_n < 0$ then for any given set of external lines we can make $-\alpha$ as large as we like by taking α_n to be large for that n.

⇒ All the λ_n's must be positive or 0. $\Rightarrow D - n\frac{D-2}{2} \geq 0 \Rightarrow n \leq \frac{2D}{D-2}$

In this case we need

$$N \geq \frac{2D}{D-2}$$ for divergence.

Thus the divergences are present precisely for amplitudes for which there is already an elementary vertex.

→ We can adjust its renormalization constant to cancel divergence.

Ex Extend this to include

① Interactions with derivatives

② fermions.

Note: In order that a theory is renormalizable we have to include all possible terms in the action for which the coefficient has dimension >0.

The operator has dimension $< D$.

~~Then~~ otherwise we shall not have enough constants to remove all divergences.

Results: A QFT is power counting renormalizable if all coupling constants have dimension ≥ 0, and all such couplings are present

$\Rightarrow$ Operators multiplying the coupling constant have dimension $\leq D$.

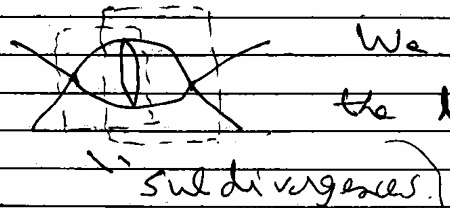
Note: Hidden assumption: Propagators do not give $1/M$ terms

e.g. bosons $\sim \frac{1}{k^2}$, fermions $1/k$ for large k.

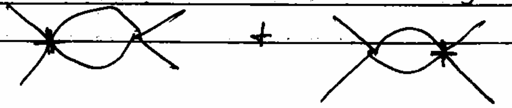
We still need to do some work to prove renormalizability.

e.g. overlapping divergence.

In ϕ^4 theory; consider



We need to ensure that the lower order counterterms cancel these subdivergences.



→ This can be shown.

Role of symmetry: Some time we can relax the constraint of all couplings being present if the theory has a symmetry.

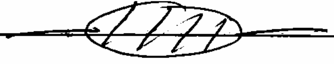
→ All couplings consistent with a given symmetry must be present.

e.g. ϕ^4 theory:

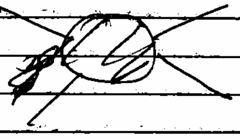
$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\lambda\phi^4$$

$$\phi = Z_\phi^{1/2}\phi_R, \quad m = Z_m m_R, \quad \lambda = Z_\lambda \lambda_R \mu^\epsilon$$

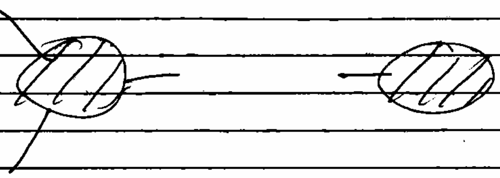
By adjusting Z_ϕ, Z_m we cancel divergences in



By ~~[illegible]~~ adjusting Z_λ we cancel divergences in



By power counting



are also divergent, but these vanish by $\phi \to -\phi$ symmetry.

A more non-trivial example: Complex scalar field with global U(1) symmetry.

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2$$

~ all possible terms of dimension ≤ 4 with $\phi \to e^{i\alpha} \phi$ symmetry.

(Excludes $(\phi^* \phi^* \phi + c.c.)$ $(\phi^* \phi^* \phi^* \phi + c.c.)$ etc.)

Work with real variables:

$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2) \Rightarrow$$

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2$$

$$-\frac{1}{2}m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 - \frac{\lambda}{\text{[illegible]}}\left(\phi_1^2+\phi_2^2+2\phi_1\phi_2\right)^2$$

~~Many terms are miss~~

Symmetry: $\phi_1 \to \cos\alpha\,\phi_1 + \sin\alpha\,\phi_2$

$\phi_2 \to -\sin\alpha\,\phi_1 + \cos\alpha\,\phi_2$

Many terms are missing, e.g.

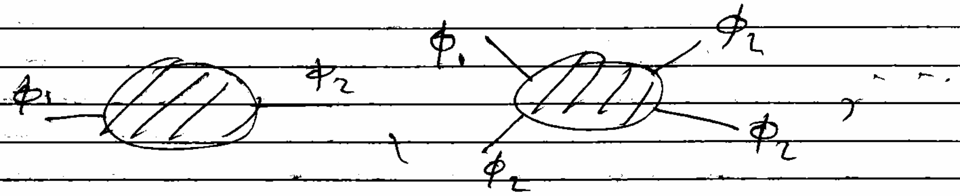
$\phi_1^2\phi_2$, $\phi_1\phi_2^3$, $\phi_1^3\phi_2$ etc.

~~Many~~ no could be ruled out using

$\phi_1 \to -\phi_1$ & $\phi_2 \to -\phi_2$ symmetry.

~~Seen~~ Absence of those Green fns

can be seen trivially in Feynman diagram

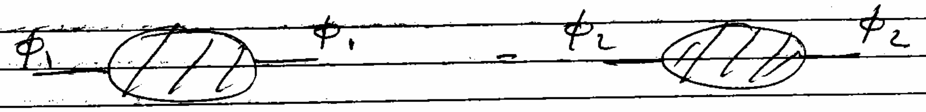


all vanish.

More unfortantly we see that many coefficients are related.

ϕ_1^2 & ϕ_2^2 coeff., ϕ_1^4, ϕ_2^4 and $\phi_1^2 \phi_2^2$ coeff.

Some of these can be understood as consequences of $\phi_1 \Leftrightarrow \phi_2$ symmetry, e.g.



⇒ If we define:

$\Phi_1 = Z_\phi^{1/2} \phi_1$, $\Phi_2 = Z_\phi^{1/2} \phi_2$, $m = Z_m m_R$

the the counterterm:

$$-\frac{1}{2}(Z_\phi - 1)\, \partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2}(Z_m^2 Z_\phi - 1) m_R^2 \phi_R^2$$

can be adjusted to cancel the divergences in both graphs.

However some relations are non-trivial.

Consider the four point counterterm.

$$\lambda = \lambda_R \mu^\epsilon Z_\lambda .$$

$\Rightarrow$ Counterterm:

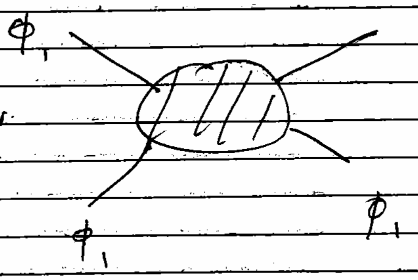
$$-\left(Z_\lambda Z_\phi^2 - 1\right) \lambda_R \mu^\epsilon \left(\phi_{1R}^4 + \phi_{2R}^4 + 2\, \phi_{1R}^2 \phi_{2R}^2 \right)$$

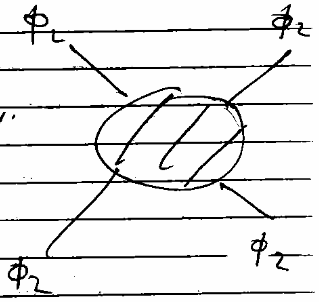
~~Con~~ Only Z_λ is undetermined.

For this to cancel all divergences we need.

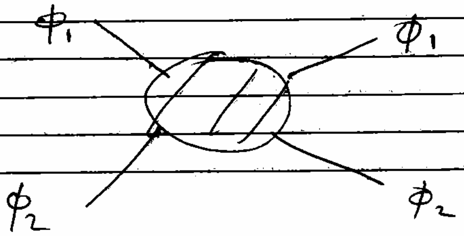
Consequence of $\phi \leftrightarrow \phi_2$

$\longleftrightarrow$

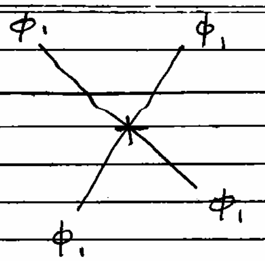
Div 

$\frac{1}{4!}$ $= \frac{1}{4!}$ Div 

$\updownarrow$?

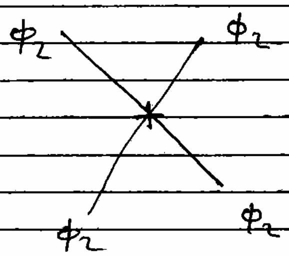
$= \frac{1}{2 \times 2! \times 2!}$ Div. 

24.7



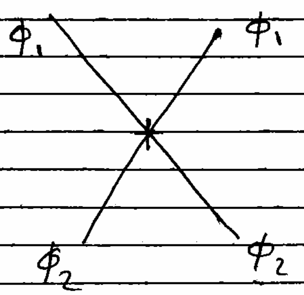
$$-i(Z_\lambda Z_\phi^2 - 1)\,\lambda_R\,\mu^\epsilon \times 4!$$

$\times$ Common factors



$$-i\,(Z_\lambda Z_\phi^2 - 1)\,\lambda_R\,\mu^\epsilon \times 4!$$

$\times$ Common factors



$$-i\,(Z_\lambda Z_\phi^2 - 1)\,\lambda_R\,\mu^\epsilon$$

$$\times\, 2 \times 2! \times 2!$$

$\times$ Common factor

Seems non-trivial but is nevertheless true as a consequence of symmetry.

Infinitesimal symmetry trs:

$$\phi_1 \to \phi_1 + \theta\phi_2 \qquad \phi_2 \to \phi_2 - \theta\phi_1$$

$$0 = \langle \tilde\phi_1(p_1)\,\tilde\phi_1(p_2)\,\tilde\phi_1(p_3)\,\tilde\phi_2(p_4)\rangle$$

$$= \langle(\tilde\phi_1 + \theta\tilde\phi_2)(\tilde\phi_1 + \theta\tilde\phi_2)(\tilde\phi_1 + \theta\tilde\phi_2)(\tilde\phi_2 - \theta\tilde\phi_1)\rangle$$

$$\overset{0_{\text{''}}}{=} \langle\tilde\phi_1\tilde\phi_1\tilde\phi_1\tilde\phi_2\rangle + \theta\{-\langle\tilde\phi_1\tilde\phi_1\tilde\phi_1\tilde\phi_1\rangle$$

$$+ \langle\tilde\phi_1\tilde\phi_2\tilde\phi_1\tilde\phi_2\rangle + \langle\tilde\phi_1\tilde\phi_1\tilde\phi_2\tilde\phi_2\rangle$$

$$+ \langle\tilde\phi_2\tilde\phi_1\tilde\phi_1\tilde\phi_2\rangle\}$$

$$\Rightarrow \text{Div.}\langle\tilde\phi_1\tilde\phi_1\tilde\phi_1\tilde\phi_1\rangle$$

$$= 3\ \text{Div.}\langle\tilde\phi_1\tilde\phi_1\tilde\phi_2\tilde\phi_2\rangle$$

→ as required to cancel divergences.

We now turn to renormalization of gauge theories.

$$S_{tot} = S_{gauge} + S_{gf} + S_{ghost}$$

Example: Choose the gauge fixing function to be $\partial_\mu A_\mu^a(x)$.

$$\Rightarrow S_{tot} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}^a(x)\, F^{a\mu\nu}(x)$$
$$-\frac{1}{2\alpha}\int d^4x\, \partial^\mu A_\mu^a(x)\, \partial^\nu A_\nu^a(x)$$
$$+\int d^4x\, b_a(x)\, \Box\, C_a(x)$$
$$+ g f^{bca}\int d^4x\, \partial^\mu b_a(x)\, A_\mu^c(x)\, C_b(x)$$

$$F_{\mu\nu}^c(x) = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g f^{abc} A_\mu^a A_\nu^b$$

- $d(A_\mu^c) = 1$ from $\int d^4x\, (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$

$d(b_a(x)) = 1, \quad d(C^a(x)) = 1$

(Could choose $d(b_a(x)) = \lambda$, $d(C^a(x)) = 2 - \lambda$ but final result does not depend on λ)

$$\frac{1}{\alpha}\int d^4x\, \partial^\mu A_\mu^a\, \partial^\nu A_\nu^a \Rightarrow d[\alpha]=0.$$

$$g\, f^{abc}\int d^4x\, \partial_\mu A_\nu^c\, f^{abc} A_\mu^a A_\nu^b$$

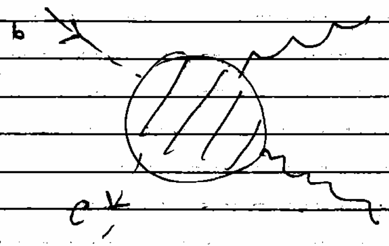
$\Rightarrow g$ has dimension 0.

The theory satisfies the first condition.

However there are many more terms of dimension ≤ 4 which are not present.

e.g. $b_a c_a A_\mu^b A^{\mu b}$.

What cancels possible divergence is



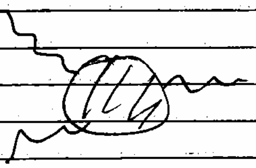
Many couplings are related, e.g.

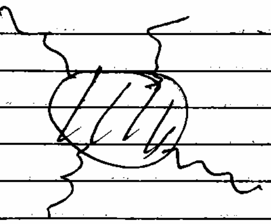
$$-\frac{1}{2} g f^{abc} A_\mu^a A_\nu^b (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a]$$

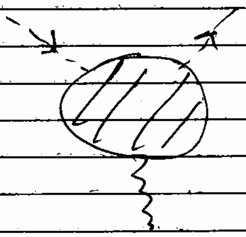
$$-\frac{1}{4} g^2 f^{abc} f^{ade} A_\mu^a A_\nu^b A^{d\mu} A^{e\nu}$$

$$+ g f^{bca} \partial^\mu b_a(x) A_\mu^c c_b(x)$$

Same Z_g will have to cancel the divergences in:







What gaurantees this?

Naive answer: gauge invariance.

Problem: Due to the gauge fixing term the Feynman rules are not gauge invariant

Q. Does the gauge fixed action have any symmetry such that only the terms we have in the gauge fixed action are invariant under this symmetry?

There is such a symmetry.

→ BRST symmetry.

Symmetry transformation parameter: a grassman variable ζ

not a fn of x.

For ordinary fields (gauge fields, scalars, fermions) the infinitesimal BRST transformation = infinitesimal gauge tr. with parameter $\zeta c^a(x)$.

$$\delta A_\mu^a(x) = -\partial_\mu(\xi C^a(x)) - g f^{bca}\,\xi C^c(x)\, A_\mu^b(x)$$

$$\delta \psi_s(x) = -ig\,\xi C^a(x)\,(R_f(T^a))_{st}\,\psi^t(x)$$

$$\delta \phi_k(x) = -ig\,\xi C^a(x)\,(R_s(T^a))_{kl}\,\phi^l(x)$$

etc.

$$\delta C^a(x) = -\frac{1}{2} g f^{bca}(\xi C^c(x))\, C^b(x)$$

$$\delta b^a(x) = -\frac{1}{\alpha} F_a(x)\,\xi$$

if the gauge fixing term is

$$-\frac{1}{2\alpha}\int d^4x\, F^a(x)\, F^a(x)$$

Claim $S = S_{gauge} + S_{g.f.} + S_{ghost} + S_{matter}$

$\Uparrow$

gauge invariant action of fermions & scalars

is invariant under BRST trs.

Note: Invariance of $S_{gauge} + S_{matter}$ follows from gauge invariance.

If true this symmetry can be used to prove renormalizability of the theory if the theory is power counting renormalizable.

S_{gauge} has operators of dimension ≤ 4

$S_{g.f.}$ and S_{ghost} have operators of dimension ≤ 4 if the gauge fixing function is choosen correctly.

e.g. $S_{gf} = -\frac{1}{2\alpha}\int d^4x \left(\partial_\mu A^{a}_{\mu} + A^{a}_{\mu} A^{b}_{\nu} A^{b\nu}\right)^2$

will violate this.

What about S_{matter}?

$$-\frac{1}{2}\int d^4x \left(\partial_\mu \Phi - i g A^a_\mu R_S(T^a)\Phi\right)^\dagger \left(\partial^\mu \Phi - i g A^{b\mu} R_S(T^b)\Phi\right)$$

$$+\int d^4x\, \bar{\Psi}\left(i\gamma^\mu \partial_\mu - m + g\gamma^\mu A^a_\mu R_S(t^a)\right)\Psi$$

$d[\phi] = 1$, $d[\Psi] = \frac{3}{2}$ $\Rightarrow$ All terms have dimension ≤ 4.

Can we add other terms of dimension ≤ 4.

$\Phi \Phi \Phi \to$ dimension 3

$\Phi \Phi \Phi \Phi \to$ dimension 4

$\bar{\Psi} \Psi \Phi \to$ dimension 4.

General structure:

$$\left(C^{(1)}_{mnp} \Phi_m \Phi_n \Phi_p + C^{(2)}_{mnp} \Phi^{\dagger}_m \Phi_n \Phi_p + c.c. \right)$$

$$+ \left(D_{k\ell m} \bar{\Psi}^k \Psi^\ell \Phi_m + c.c. \right) \text{ etc.}$$

$\to$ Possible only if we can find $C^{(1)}_{mnp}$, etc. to construct a gauge invariant term.

$\to$ determined by group theory.

e.g. if $G = SU(2)$ and $\Phi_m \in$ spin 2 representation then:

Clebsch Gordon. $\to$ $C^{2\,2\,2}_{m_1\, m_2\, m_3} \Phi^{\dagger}_{m_1} \Phi_{m_2} \Phi_{m_3}$ is invariant.

Proof of BRST invariance

$$S_{gf} = -\frac{1}{2\alpha}\int d^4x\, F^a(x)\, F^a(x)$$

$$\delta S_{gf} = -\frac{1}{\alpha}\int d^4x\, F^a(x)\, \delta F^a(x)$$

$$= -\frac{1}{\alpha}\int d^4x\, F^a(x) \int \left.\frac{\delta F^a_\theta(x)}{\delta\theta^b(y)}\right|_{\theta=0} \zeta c^b(y)\, d^4y$$

$$S_{ghost} = -\int d^4x\, d^4y\, b^a(x) \left.\frac{\delta F^a_\theta(\cancel{\theta}x)}{\delta\theta^b(y)}\right|_{\theta=0} c^b(y)$$

$$\delta S_{ghost} = -\int d^4x\, d^4y \left(-\frac{1}{\alpha} F_a(x)\, \zeta\right) \left.\frac{\delta F^a_\theta(x)}{\delta\theta^b}\right|_{\theta=0} c^b(y)$$

$$-\frac{1}{\alpha}\int d^4x\, d^4y\, b^a(x)\, \delta\left[\left.\frac{\delta F^a_\theta(x)}{\delta\theta^b(y)}\right|_{\theta=0}\right] c^b(y)$$

$$-\frac{1}{\alpha}\int d^4x\, d^4y\, b^a(x)\, \frac{\delta F^a_\theta(x)}{\delta\theta^b(y)} \left(-\frac{1}{2} g\right) f^{bca}$$

$$\zeta\, c^c(y)\, c^b(y)$$

$\delta S_{gf} + \delta S_{ghost}$

$$= -\frac{1}{\alpha}\int d^4x\, d^4y\, d^4z\; b^a(x) \left[\frac{\delta}{\delta \phi_{c}(z)} \left\{ \frac{\delta F^a_\theta(x)}{\delta \theta^b(y)}\Big|_{\theta=0} \right\}_\phi \right]_{\phi=0}$$

$$\delta\, c^c(z)\, c^b(y)$$

$$+\frac{1}{2\alpha} g \int d^4x\, d^4y\; b^a(x)\; \frac{\delta F^a_\theta(x)}{\delta \theta^b(y)}\Big|_{\theta=0}\; f^{bca}$$

$$\delta\, c^c(y)\, c^b(y)$$

$$\left[\frac{\delta}{\delta \phi_c(z)} \left\{ \frac{\delta F^a_\theta(x)}{\delta \theta^b(y)}\Big|_{\theta=0} \right\}_\phi \right]_{\phi=0}$$

$$-\left[\frac{\delta}{\delta \phi_b(y)} \left\{ \frac{\delta F^a_\theta(x)}{\delta \theta^c(z)}\Big|_{\theta=0} \right\}_\phi \right]_{\phi=0}$$

= ~~Some~~ Gauge trs. of $F^a_\theta(x)$ with some gauge trs. parameter (y)

~~$= \frac{\delta F^a_\chi(x)}{\delta \chi^d(z)}\Big|_{\chi=0}$~~ The parameter χ_d depends on b, c, y, z but not on the functions F^a.

X^d depends on b, c but is independent of the choice of F^a.

Thus we can try to evaluate X^d by acting the same operation on a different function.

Take a field ψ^s transforming in representation R

$$\psi^s_\theta(x) = \psi^s(x) - ig\left(R_s(T^b)\right)_{st} \theta^b(x)\, \psi^t(x)$$

$$\left.\frac{\delta \psi^s_\theta(x)}{\delta \theta^b(y)}\right|_{\theta=0} = -ig\left(R_s(T^b)\right)_{st} \psi^t(x)\, \delta^{(4)}(x-y)$$

$$\left[\frac{\delta}{\delta\phi^c(z)}\left\{\left.\frac{\delta \psi^s_\theta}{\delta \theta^b(y)}\right|_{\theta=0} \phi\right\}\right]_{\phi=0}$$

$$= -ig\left(R_s(T^b)\right)_{st}\left(-ig\, R_s(T^c)\right)_{tu} \psi^u(x)\, \delta^{(4)}(x-y)\, \delta^{(4)}(x-z).$$

$$= -g^2 \left(R_s(T^b)\, R_s(T^c)\right)_{su} \psi^u(x)$$

$$\delta^{(4)}(x-y)\,\delta^{(4)}(x-z)$$

$$\left(\left[\frac{\delta}{\delta\phi^c(z)}\left\{\left(\frac{\delta\psi^s(x)}{\delta\theta^b(y)}\right|_{\theta=0}\right\}_{\phi}\right]_{\phi=0}\right.$$

$$\left. - \left(b \leftrightarrow c,\; y \leftrightarrow z\right)\right) \psi_s(x)$$

$$= -g^2\, \delta^{(4)}(x-y)\,\delta^{(4)}(x-z)$$

$$\left(R_s(T^b)\, R_s(T^c) - R_s(T^c)\, R_s(T^b)\right)_{su} \psi_u(x)$$

$$\parallel$$

$$i f^{bcd} R_s(T^d).$$

Thus the function with which we gauge transform is

$$\chi^d(x) = i f^{bcd} (-g^2)\, \delta^{(4)}(x-y)\, \delta^{(4)}(x-z)$$

$$\Rightarrow \left(\left[\frac{\delta}{\delta \phi^c(z)} \left\{ \frac{\delta F^a_\theta(x)}{\delta \theta^b(y)} \Big|_{\theta=0} \right\}_\phi \right]_{\phi=0} \right.$$

$$\left. - \left(b \leftrightarrow c, \quad y \leftrightarrow z \right) \right)$$

$$= \int d^4 w \, \chi^d(w) \left(\frac{\delta F^a_\chi(x)}{\delta \chi^d(w)} \Big|_{\chi=0} \right)$$

$$= \int d^4 w \, . \, (g) \, f^{bcd} \, \delta^{(4)}(w-y) \, \delta^{(4)}(w-z)$$

$$\frac{\delta F^a_\chi(x)}{\delta \chi^d(w)} \Big|_{\chi=0}$$

$$- \frac{1}{\alpha} \int d^4x \, d^4y \, d^4z \; b^a(x) \left[\frac{\delta}{\delta \phi_c(z)} \left\{ \frac{\delta F^a_\theta(x)}{\delta \theta^b(y)} \Big|_{\theta=0} \right\}_\phi \right]_{\phi=0}$$

$$\varsigma c^c(z) \; c^b(y)$$

$$= -\frac{1}{2\alpha} \, (g) \, f^{bcd} \int d^4x \, d^4y \, d^4z \, d^4w$$

$$\delta^{(4)}(w-y) \; \delta^{(4)}(w-z) \quad \frac{\delta F^a(x)}{\delta \chi^d(w)} \Big|_{w=0} \; \varsigma c^c(z) \, c^b(y)$$

$$= -\frac{g}{2\alpha} f^{bcd} \int d^4x \, d^4y \quad \frac{\delta F^a_\chi(x)}{\delta \chi^d(y)} \Big|_{\chi=0} \quad \varsigma c^c(y) \, c^b(y)$$

This establishes BRST invariance and renormalizability of the gauge fixed action.

Explicit one loop renormalization:

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})$$

$$-\frac{g}{2}(\partial_\mu A_\nu^c - \partial_\nu A_\mu^c) f^{abc} A^{a\mu} A^{b\nu}$$

$$-\frac{1}{4} g^2 f^{abc} f^{a'b'c} A_\mu^a A_\nu^b A^{a'\mu} A^{b'\nu}$$

$$-\frac{1}{2\alpha} \partial^\mu A_\mu^c \partial^\nu A_\nu^c$$

$$+ b_a(x)\, \partial^\mu \partial_\mu c_a(x) + g f^{bca} \partial^\mu b_a(x) A_\mu^c(x) c^b(x)$$

$$+ \bar{\psi}^k (i\gamma^\mu \partial_\mu - m)\psi + g \bar{\psi}_k \gamma^\mu (T^a)_{k\ell} \psi_\ell A_\mu^a$$

$$A_\nu^a = \tilde{Z}_A^{1/2} A_{\nu R}^a, \quad b_a = \tilde{Z}_{ghost}^{1/2} b_{aR},$$

$$c_a = \tilde{Z}_{ghost}^{1/2} c_{aR}, \quad \psi = \tilde{Z}_\psi^{1/2} \psi_R$$

$$g = Z_g g_R \mu^{\epsilon/2}, \quad \alpha = Z_\alpha \alpha_R \quad m = Z_m m_R$$

Ex. check that

$$\tilde{Z}_A = 1 - \frac{g_R^2}{8\pi^2\epsilon}\left(\frac{4}{3}T_R - \frac{1}{2}C_G\left(\frac{13}{3} - \overset{\alpha}{\cancel{\phi}}_R\right)\right)$$

$$\mathrm{Tr}\left(R(T^a)\,R(T^b)\right) = T_R\,\delta_{ab}$$

$$f^{acd} f^{bcd} = C_G\,\delta_{ab}$$

$$\tilde{Z}_\alpha = \tilde{Z}_A$$

$$\tilde{Z}_{ghost} = 1 + \frac{g_R^2}{32\pi^2\epsilon} C_G (3 - \alpha_R)$$

$$\tilde{Z}_\psi \cdot Z_m = 1 - \frac{g_R^2}{8\pi^2\epsilon} C_R (3 + \alpha_R)$$

$$\sum_a T^a T^a = C_R\, I$$

$$Z_m = 1 - \frac{3 g_R^2}{8\pi^2\epsilon} C_R$$

$$Z_g = 1 - \frac{g_R^2}{8\pi^2\epsilon}\,\frac{11 C_G - 4 T_R}{6}.$$

Renormalization group

~~a~~ illustrated in the context of gauge theory coupling to fermions in irreducible representation R.

$$g = Z_g \, g_R \, \mu^{\epsilon/2}, \quad \alpha = Z_\alpha \alpha_R, \quad m = Z_m m_R$$

Z_g, Z_α, Z_m: functions of $g_R(\mu, \epsilon)$

We began with 3 unrenormalized parameters α, g, m

We end up with four parameters

$$g_R, \alpha_R, m_R, \mu$$

($\epsilon \to 0$ limit ∋ ϵ is not a parameter).

How can the no. of parameters increase?

Note: As long as ϵ is kept finite, the full action depends only on g, α, m and all quantities are finite.

$\Rightarrow$ For fixed ϵ, if we change g_R, m_R, α_R, μ s.t. g, m, α are fixed then the [illegible] physical results should not change.

Q. Under a change of μ to $\mu + \delta\mu$, how should we change g_R, m_R, α_R to achieve this?

$$\frac{d}{d\mu}(g) = 0 \quad \Rightarrow \quad \frac{d}{d\mu}\left(Z_g \, g_R \, \mu^{\epsilon/2} \right) = 0.$$

$$\frac{d}{d\mu}(\alpha) = 0 \quad \Rightarrow \quad \frac{d}{d\mu}\left(Z_\alpha \, \alpha_R \right) = 0.$$

$$\frac{d}{d\mu}(m) = 0 \quad \Rightarrow \quad \frac{d}{d\mu}\left(Z_m \, m_R \right) = 0.$$

Minimal subtraction scheme:

$$Z_g = 1 + \frac{A_g g_{(R)}^2}{\epsilon} \quad Z_m = 1 + \frac{A_m g_R^2}{\epsilon}, \quad Z_\alpha = 1 + \frac{A_\alpha}{\epsilon} g_R^2$$

for some constants A_g, A_m, A_α (α_R independent)

$$\mu \frac{d}{d\mu}\left(g_R \left(1 + \frac{g_R^2 A_g}{\epsilon}\right) \mu^\epsilon \right) = 0.$$

$$\Rightarrow \mu \frac{dg_R}{d\mu} \mu^\epsilon + 3 g_R^2 \mu \frac{dg_R}{d\mu} \frac{A_g}{\epsilon} \mu^\epsilon$$

$$+ g_R \left(1 + \frac{A_g g_R^2}{\epsilon}\right) \mu^\epsilon \cdot \frac{\epsilon}{2} = 0.$$

$$\mu \frac{dg_R}{d\mu} \left(1 + 3 g_R^2 \frac{A_g}{\epsilon}\right)$$

$$= - g_R \frac{\epsilon}{2}\left(1 + \frac{A_g g_R^2}{\epsilon}\right)$$

$$\mu \frac{dg_R}{d\mu} = -\left(1 + 3 g_R^2 \frac{A_g}{\epsilon}\right)^{-1} g_R \frac{\epsilon}{2}\left(1 + \frac{A_g g_R^2}{\epsilon}\right)$$

$$= -\frac{\epsilon}{2} g_R \left(1 - 3 g_R^2 \frac{A_g}{\epsilon} + g_R^2 \frac{A_g}{\epsilon}\right)$$

$$= -\frac{\epsilon}{2} g_R + g_R^3 A_g$$

As $\epsilon \to 0$

$$\mu \frac{dg_R}{d\mu} = \underbrace{A_g\, g_R^3}_{\beta_g(g_R)}$$

β- function for the coupling g_R.

Note: $\epsilon \to 0$ limit will have to be taken at the very end.

$$\mu \frac{d}{d\mu}(\alpha) = 0 \Rightarrow \mu \frac{d}{d\mu}\left(Z_\alpha \overset{\alpha_R}{\cancel{g_R}} \right) = 0.$$

$$\Rightarrow \mu \frac{d}{d\mu}\left(\alpha_R \left(1 + \frac{A_{\cancel{g}\alpha}}{\epsilon} g_R^2 \right) \right) = 0$$

$$\Rightarrow \mu \frac{d\alpha_R}{d\mu} \left(1 + \frac{A_\alpha}{\epsilon} g_R^2 \right)$$

$$+ \alpha_R \frac{A_\alpha}{\epsilon} \times 2 g_R \mu \frac{dg_R}{d\mu} = 0$$

$$\Rightarrow \mu \frac{d\alpha_R}{d\mu} {}_{2}= \left(1 + \frac{A_\alpha}{\epsilon} g_R^2 \right)^{-1} \frac{A_{\cancel{\alpha}}}{\epsilon} \left(+ 2 \alpha_R g_R \mu \frac{dg_R}{d\mu} \right)$$

$$= - 2 \alpha_R g_R \left(1 - \frac{A_\alpha}{\epsilon} g_R^2 \right) \frac{A_{\cancel{\alpha}}}{\epsilon} \left(- \frac{\epsilon}{2} g_R + g_R^3 A_g \right)$$

$$= \alpha_R A_{\cancel{g}\alpha}\, g_R^2$$

$$\Rightarrow \quad \mu \frac{d\alpha_R}{d\mu} = \beta_\alpha(g_R)$$

$$\parallel$$

$$\alpha_R \, g_R^2 \, A_\alpha$$

Similarly

$$\mu \frac{dm_R}{d\mu} = m_R \, g_R^2 \, A_m$$

$$\parallel\!|$$

$$\beta_m . (g_R).$$

This way we can get β functions for all parameters.

Implication for renormalized Greens functions:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g_R} + \beta_\alpha \frac{\partial}{\partial \alpha_R} + \beta_m \frac{\partial}{\partial m_R}\right)$$

$$\left\langle \prod_i \phi_i(x_i) \right\rangle = 0$$

$$\Downarrow$$

$$\left\langle \prod_i \left(Z_i^{1/2} \phi_{iR}(x_i) \right) \right\rangle$$

$$\tilde{Z}_i^{1/2} = \left(1 + \frac{A_i(\alpha_R)}{\epsilon}\, g_R^2\right)$$

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_g\frac{\partial}{\partial g_R} + \beta_\alpha\frac{\partial}{\partial\alpha_R} + \beta_m\frac{\partial}{\partial m_R}\right)\tilde{Z}_i^{1/2}$$

$$= \frac{1}{2}\tilde{Z}_i^{1/2}\left(\mu\frac{\partial}{\partial\mu} + \beta_g\frac{\partial}{\partial g_R} + \beta_\alpha\frac{\partial}{\partial\alpha_R} + \beta_m\frac{\partial}{\partial m_R}\right) \ln\tilde{Z}_i$$

$$/\!/$$

$$\frac{A_i(\alpha_R)}{\epsilon}\, g_R^2$$

$$= \frac{1}{2\epsilon}\tilde{Z}_i^{1/2}$$ ~~$\left\{\frac{A_i(\alpha_R)}{\epsilon}\,\mu\,\partial A_i(\alpha_R)\right.$~~

$$\times\left\{\left(\mu\frac{\partial}{\partial\mu} + \beta_g\frac{\partial}{\partial g_R} + \beta_\alpha\frac{\partial}{\partial\alpha_R}\right) A_i(\alpha_R) \times g_R^2\right.$$

$$= \gamma_i\,\tilde{Z}_i^{1/2}$$

Ex. Calculate γ_{A_μ}, γ_ψ, γ_{ghost}

$$\frac{1}{2\epsilon}\, 2 Z_i^{1/2}\, g_R\left\{-\frac{\epsilon}{2}\, g_R\right\} A_i = -\frac{1}{2}A_i\, g_R^2$$

$$\left\{\mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g_R} + \beta_\alpha \frac{\partial}{\partial \alpha_R} + \beta_m \frac{\partial}{\partial m_R} + \sum_{j=1}^{n} \gamma_j \right\}$$

$$\left\langle \prod_{i=1}^{n} \phi_{iR}(x_i) \right\rangle = 0.$$

→ determines the μ-dependence of renormalized Greens function in terms of the dependence on other parameters.

(28.1)

Renormalization group equation: general form:

Fields $\{\phi_i\}$. Parameters $\{g_\alpha\}$

$$\phi_i = \left(Z_{ij}(\vec{g}_R, \mu)\right)^{1/2} \phi_{jR} \qquad i = 1, \cdots N.$$

$$g_\alpha = f_\alpha(\vec{g}_R, \epsilon, \mu) \qquad \alpha = 1, \cdots M$$

β_α : Soln to the equation:

$$\underbrace{\mu \frac{\partial}{\partial \mu} f_\alpha}_{\text{calculable}} + \beta_\gamma \underbrace{\frac{\partial f_\alpha}{\partial g_{\gamma R}}}_{\text{calculable.}} = 0.$$

~~If~~ ~~there~~

If there are M parameters this gives M linear eq. in M unknowns $\beta_1, \cdots \beta_M$.

$\Rightarrow$ Can be solved to find $\beta_1, \cdots \beta_M$.

γ_{ij} : Obtained by solving:

$$\underbrace{\mu \frac{\partial}{\partial \mu} \left(\tilde{Z}^{1/2}\right)_{ij}}_{\text{calculable}} = \underbrace{\left(\tilde{Z}^{1/2}\right)_{ik}}_{\text{calculable}} \gamma_{kj}$$

$\Rightarrow$ Can be solved to find γ_{kj}.

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_\alpha\frac{\partial}{\partial g_{\alpha R}}\right)\left\langle\prod_{i=1}^{n}\phi_{s_i}(x_i)\right\rangle = 0$$

$s_1, \cdots s_n$: labels which fields we calculate correlation fn. of.

$1 \le s_i \le N$.

$$\phi_{s_i}(x_i) = \sum_{r_i} \tilde{Z}^{1/2}_{s_i r_i}\, \phi_{r_i R}(x_i)$$

Substitute.

$$\Rightarrow \left(\mu\frac{\partial}{\partial\mu}+\sum_\alpha\beta_\alpha\frac{\partial}{\partial g_{\alpha R}}\right)\left\langle\prod_{i=1}^{n}\phi_{s_i}(x_i)\right\rangle$$

$$+\sum_{k=1}^{n}\sum_{r_k}\gamma_{s_k r_k}\left\langle\left(\prod_{\substack{i=1\\ i\neq k}}^{n}\phi_{s_i}(x_i)\right)\phi_{r_k}(x_k)\right\rangle$$

For simplicity we shall take γ_{ij} to be diagonal: $\gamma_{ij} = \gamma_i \delta_{ij}$

$$\Rightarrow \left(\mu\frac{\partial}{\partial\mu}+\sum_\alpha\beta_\alpha\frac{\partial}{\partial g_{\alpha R}}+\sum_k\gamma_{s_k}\right)\left\langle\prod_{i=1}^{n}\phi_{s_i}(x_i)\right\rangle$$

$$= 0.$$

Use of RG:

Consider the correlation fn.

$$\langle \prod_i \tilde{\Phi}_{\Delta_i}(k_i) \rangle \equiv F(\{k_i\}, \{g_\alpha\}, \mu)$$

$\rightsquigarrow$ satisfies the same RG equation.

If $\prod_i \tilde{\Phi}_{\Delta_i}(k_i)$ has dimension d then

we have $F(\{k_i\}, \{g_{\alpha R}\}, \mu)$

$$~~\langle \prod_i \tilde{\Phi}_{\Delta_i}(k_i) \rangle~~ = (k_1^0)^d$$

$$\times \, f\left(\frac{k_i}{k_1^0}, \frac{g_{\alpha R}}{(k_1^0)^{d_\alpha}}, \frac{\mu}{k_1^0} \right)$$

$$F(\{\lambda k_i\}, \{g_{\alpha R}\}, \mu)$$

$$= \lambda^d (k_1^0)^d \, f\left(\frac{k_i}{k_1^0}, \frac{g_{\alpha R}\lambda^{-d_\alpha}}{(k_1^0)^{d_\alpha}}, \frac{\mu\lambda^{-1}}{k_1^0} \right)$$

$$= \lambda^d F(\{k_i\}, \{g_{\alpha R}\lambda^{-d_\alpha}\}, (\mu\lambda^{-1}))$$

Now consider the limit $\lambda \to \infty$ (UV)
or $\lambda \to 0$ (IR).

In this limit $\lambda^{-d_\alpha} g_\alpha \to 0$ (UV) or ∞ (IR)
for $d_\alpha > 0$.
$\lambda^{-d_\alpha} g_\alpha$ remains fixed for $d_\alpha = 0$..

This can be used to simplify the theory.

e.g. for massive ϕ^4 theory in the UV
$m_R \to 0$, g_R remains fixed.

The theory becomes massless.

In the $\lambda \to 0$ (IR) limit $m_R \to \infty$,
g_R remains fixed.

$\Rightarrow$ The particle aquires ∞ mass.

Since finite energy processes will not be affected by this, the theory becomes trivial

In gauge theory coupled to massive fermions:

In UV ($\lambda \to \infty$) we have ~~a~~ ~~the~~ gauge theory coupled to massless fermions

In IR ($\lambda \to 0$) we have pure gauge theory.

However these simple minded results are destroyed by μ.

See As $\lambda \to \infty$, $\mu\lambda^{-1} \to 0$.

As $\lambda \to 0$, $\mu\lambda^{-1} \to \infty$.

We have to study how this affects the theory.

Idea: Use RG equation to find dependence on μ.

$$F(\{\lambda k_i\}, \{g_{\alpha R}\}, \mu)$$

$$= \lambda^{d} F(\{k_i\}, g_{\alpha R}\lambda^{-d_\alpha}, \mu\lambda^{-1})$$

~~Define: $F_\alpha(\{k_i\}, \{g_{\alpha R}\}, \mu) = \frac{\partial}{\partial g_{\alpha R}} F(\{k_i\}, \{g_{\alpha R}\}, \mu)$~~

~~$F_\mu(\{k_i\}, \{g_{\alpha R}\}, \mu)$~~

~~$= \mu \frac{\partial}{\partial \mu} F(\{k_i\}, \{g_{\alpha R}\}, \mu)$~~

$$\Rightarrow \lambda \frac{\partial}{\partial \lambda} F(\{\lambda k_i\}, \{g_{\alpha R}\}, \mu)$$

$$= \lambda \frac{\partial}{\partial \lambda}\left[\lambda^{d} F(\{k_i\}, g_{\alpha R}\lambda^{-d_\alpha}, \mu\lambda^{-1})\right]$$

$$= d\, F(\{\lambda k_i\}, g_{\alpha R}, \mu)$$

$$- \sum_\alpha d_\alpha g_{\alpha R} \frac{\partial}{\partial g_{\alpha R}} F(\{\lambda k_i\}, g_{\alpha R}, \mu)$$

$$- \mu \frac{\partial}{\partial \mu} F(\{\lambda k_i\}, g_{\alpha R}, \mu)$$

Now use:

$$\left(\mu\frac{\partial}{\partial\mu} + \sum_\alpha \beta_\alpha \frac{\partial}{\partial g_{\alpha R}} + \sum_{i=1}^{n} \gamma_{A_i}\right) F(\{\lambda k_i\},\{g_{\alpha R}\},\mu) = 0$$

$$\Rightarrow \lambda\frac{\partial}{\partial\lambda} F(\{\lambda k_i\},\{g_{\alpha R}\},\mu)$$

$$= \underbrace{\left(d + \sum_{i=1}^{n} \gamma_{A_i}\right)}_{} F(\{\lambda k_i\},\{g_{\alpha R}\},\mu)$$

$$+ \sum_\alpha \left(\underbrace{\beta_\alpha - g_{\alpha R}\, d_{\alpha R}}_{\tilde{\beta}_\alpha(g_R)}\right) \frac{\partial}{\partial g_{\alpha R}} F(\{\lambda k_i\},\{g_{\alpha R}\},\mu)$$

anomalous dimension

Boundary condition: At $\lambda = 1$

$$F(\{\lambda k_i\},\{g_{\alpha R}\},\mu) = F(\{k_i\},\{g_{\alpha R}\},\mu)$$

Claim: The solution to this eq. is given by:

$$F(\{\lambda k_i\}, \{g_{\alpha R}\}, \mu)$$

$$= F(\{k_i\}, \{\bar{g}_{\alpha R}(\lambda, \vec{g}_R)\}, \mu)$$

$$\times \exp\left(\sum_{i=1}^{n} \tilde{d}_{s_i} \ln \lambda\right)$$

$$\exp\left(\sum_{i=1}^{n} \int_1^{\lambda} \frac{d\lambda'}{\lambda'} \gamma_{s_i}(\{\bar{g}_\alpha(\lambda', \vec{g}_R)\}\right)$$

$\bar{g}_\alpha(\lambda, \vec{g}_R)$ is the solution to:

$$\lambda \frac{\partial \bar{g}_\alpha(\lambda, \vec{g}_R)}{\partial \lambda} = \tilde{\beta}_\alpha (\vec{\bar{g}}_R(\lambda))$$

$$\bar{g}_{\alpha R}(\lambda = 1, \vec{g}_R) = g_{\alpha R}$$

check that it gives us back the old result in absence of renormalization, e.g. for

$$\beta_\alpha = 0 , \qquad \gamma_i = 0. \quad \& \text{ no } \mu \text{ dependence}$$

$$\tilde{\beta}_\alpha = - d_\alpha \, g_{\alpha R}$$

$$\Rightarrow \lambda \frac{\partial \bar{g}_\alpha (\lambda, \vec{g}_R)}{\partial \lambda} = - d_\alpha \, \bar{g}_\alpha (\vec{\lambda}, \vec{g}_R)$$

$$\Rightarrow \bar{g}_\alpha = \lambda^{-d_\alpha} g_{\alpha R}.$$

$$F(\{\lambda k_i\}, \{g_{\alpha R}\}, \mu)$$

$$= \underbrace{\exp\left(\sum_i d_{s_i} \ln \lambda\right)}_{\lambda^{\sum_i d_{s_i}}} F(\{k_i\}, \{\lambda^{-d_\alpha} g_{\alpha R}\}, \mu)$$

→ Original result in absence of renormalization.

Renormalization group equation + dimensional analysis

$$\Rightarrow \lambda \frac{\partial}{\partial \lambda} F(\{\lambda k_i\}, \{g_{\alpha R}\})$$

$$= \left(d + \sum_{i=1}^{n} \gamma_{s_i}(\{g_{\alpha R}\})\right) F(\{\lambda k_i\}, \{g_{\alpha R}\})$$

$$+ \sum_{\alpha} \underbrace{\left(\beta_\alpha(\{g_{\gamma R}\}) - d_{\alpha R}\, g_{\alpha R}\right)}_{\tilde{\beta}_\alpha} \frac{\partial}{\partial g_{\alpha R}} F(\{\lambda k_i\}, \{g_{\alpha R}\})$$

$$F(\{k_i\}, \{g_{\alpha R}\}) = \left\langle \prod_{i=1}^{n} \tilde{\Phi}_{s_i}(k_i) \right\rangle$$

$$d = \sum_i \tilde{d}_{s_i} \rightarrow \text{dimension of } \tilde{\Phi}_{s_i}$$

$\tilde{d}_\alpha$: dimension of g_α

Note: We have ~~suppressed~~ omitted displaying μ-dependence since the equation does not contain $\frac{\partial}{\partial \mu}$ any more.

Use short hand notation: $\vec{g}_R = \{g_{\alpha R}\}$.

Solution: First define $\bar{g}_{\alpha R}(\lambda, \vec{g}_R)$ as solutions to

$$\lambda \frac{\partial \bar{g}_{\alpha R}(\lambda, \vec{g}_R)}{\partial \lambda} = \tilde{\beta}_\alpha(\vec{\bar{g}}_R(\lambda, \vec{g}_R))$$

$$\bar{g}_{\alpha R}(\lambda = 1, \vec{g}_R) = g_{\alpha R}$$

Thus $g_{\alpha R}$ enters the argument of $\bar{g}_{\alpha R}$ through boundary conditions.

Claim: The solution is

$$F(\{\lambda k_i\}, \{g_{\alpha R}\})$$
$$= \lambda^d \exp\left(\sum_{i=1}^{n} \int_1^\lambda \frac{d\lambda'}{\lambda'} \gamma_{\psi_i}(\vec{\bar{g}}(\lambda', \vec{g}_R))\right)$$
$$F(\{k_i\}, \vec{\bar{g}}_R(\lambda, \vec{g}_R))$$

→ relates large / small momentum behaviors of F to F at fixed momentum but different couplings $\bar{g}_R(\lambda, \vec{g}_R)$

Note: If $\gamma_{\lambda i} = 0$, $\beta_\alpha = 0$ i.e. there is no μ dependence introduced through renormalization, then

$$\bar\beta_\alpha = -d_\alpha g_{\alpha R} \quad (\text{no sum over } \alpha)$$

$$\Rightarrow \quad \bar g_{\alpha R}(\lambda, \vec g_R) = g_{\alpha R}\, \lambda^{-d_\alpha}$$

& we get:

$$F(\{\lambda k_i\}, \{g_{\alpha R}\})$$

$$= \lambda^{d} F(\{k_i\}, \{g_{\alpha R}\lambda^{-d_\alpha}\})$$

$\sim$ earlier relation without rescaling of $\mu \to \mu\lambda^{-1}$.

Thus the R.G. ~~generalizes the result of~~ effectively takes into account the effect of $\mu \to \mu\lambda^{-1}$ scaling by additional changes in effective coupling and dimensions.

Proof: We shall make use of the relation

$$\beta_{\gamma}(\vec{g}_R)\,\frac{\partial \bar{g}_{\alpha R}(\lambda, \vec{g}_R)}{\partial g_{\gamma R}} = \beta_\alpha(\vec{\bar{g}}_R(\lambda, \vec{g}_R))$$

This will be proved later but for now we proceed assuming this to be correct.

We need to check that the r.h.s. satisfies the differential equation.

$$\lambda \frac{\partial}{\partial\lambda}(\text{r.h.s.})$$

$$= \Big(d + \sum_{i=1}^{n} \gamma_{s_i}(\vec{\bar{g}}_R(\lambda, \vec{g}_R))\Big)\ (\text{r.h.s.})$$

$$+\ \lambda^{d} \exp\Big(\sum_{i=1}^{n} \int_1^{\lambda} \frac{d\lambda'}{\lambda'}\, \gamma_{s_i}(\vec{\bar{g}}_R(\lambda', \vec{g}_R))\Big)$$

$$\times\ \frac{\partial \bar{g}_{\alpha R}}{d\lambda}\,\frac{\partial}{\partial \bar{g}_{\alpha R}}\ F(\{k_i\}, \vec{\bar{g}}_R(\lambda, \vec{g}_R))$$

$$\beta_\alpha(\vec{\bar{g}}_R(\lambda, \vec{g}_R)) = \beta_\gamma(\vec{g}_R)\,\frac{\partial \bar{g}_{\alpha R}(\lambda, \vec{g}_R)}{\partial g_{\gamma R}}$$

~~$+ \left(d + \sum_{i=1}^{n} \gamma_{s_i}\left(\vec{g}_R(\lambda, \vec{g}_R)\right)\right) \Big] \text{(r.h.s.)}$~~

~~$+ \lambda^{d} \exp\left(\sum_{i=1}^{n} \int_1^{\lambda} \frac{d\lambda'}{\lambda'} \gamma_{s_i}\left(\vec{g}_R(\lambda', \vec{g}_R)\right)\right)$~~

Write the differential eq. as

$$\lambda \frac{\partial}{\partial \lambda} \ln F^{(\{\lambda k_i\}, \vec{g}_R)}_{\Lambda} - \sum_{\alpha} \tilde{\beta}_\alpha(\vec{g}_R) \frac{\partial}{\partial g_{\alpha R}} \ln F(\{\lambda k_i\}, \vec{g}_R)$$

$$- \left(d + \sum_{i=1}^{n} \gamma_{s_i}(\vec{g}_R)\right) = 0$$

Proposed soln.

$$\ln F^{(\{\lambda k_i\}, \vec{g}_R)}_{\Lambda} = d \ln \lambda + \sum_{i=1}^{n} \int_1^{\lambda} \frac{d\lambda'}{\lambda'} \gamma_{s_i}\left(\vec{g}_R(\lambda', \vec{g}_R)\right)$$

$$+ \ln F\left(\{k_i\}, \vec{g}_R(\lambda, \vec{g}_R)\right)$$

~~check~~ to check that r.h.s. satisfies the differential eq.

$$\lambda \frac{\partial}{\partial\lambda}(\text{r.h.s.}) = d + \sum_{i=1}^{n} \gamma_{\Delta_i}\left(\vec{\bar{g}}_R(\lambda, \vec{g}_R)\right)$$

$$+ \underbrace{\frac{\partial \bar{g}_{\alpha R}(\lambda, \vec{g}_R)}{\partial \lambda}}_{\tilde{\beta}_\alpha\left(\vec{\bar{g}}_R(\lambda, \vec{g}_R)\right)} \frac{\partial}{\partial \bar{g}_{\alpha R}} \ln F\left(\{k_i\}, \vec{\bar{g}}_R(\lambda, \vec{g}_R\right)$$

$$\Downarrow$$

$$\beta_\gamma(\vec{g}_R) \frac{\partial \bar{g}_{\alpha R}(\lambda, \vec{g}_R)}{\partial g_{\gamma R}}$$

$$\lambda \frac{\partial}{\partial\lambda}(\text{r.h.s.}) = d + \sum_{i=1}^{n} \gamma_{\Delta_i}\left(\vec{\bar{g}}_R(\lambda, \vec{g}_R)\right)$$

$$+ \tilde{\beta}_\gamma(\vec{g}_R) \frac{\partial}{\partial g_{\gamma R}} \ln F\left(\{k_i\}, \vec{\bar{g}}_R(\lambda, \vec{g}_R)\right)$$

$$- \sum_\alpha \tilde{\beta}_\alpha(\vec{g}_R) \frac{\partial}{\partial g_{\alpha R}} (\text{r.h.s.})$$

$$= - \sum_\alpha \int_1^\lambda \frac{d\lambda'}{\lambda'} \sum_i \tilde{\beta}_\alpha(\vec{g}_R) \frac{\partial \gamma_{\Delta_i}\left(\vec{\bar{g}}_R(\lambda', \vec{g}_R)\right)}{\partial \bar{g}_{\gamma R}}$$

$$\times \frac{\partial \bar{g}_{\gamma R}(\lambda', \vec{g}_R)}{\partial g_{\alpha R}}$$

$$- \sum_\alpha \tilde{\beta}_\alpha(\vec{g}_R) \frac{\partial}{\partial g_{\alpha R}} \ln F\left(\{k_i\}, \vec{\bar{g}}_R(\lambda, \vec{g}_R)\right)$$

$$\left(\lambda \frac{\partial}{\partial \lambda} - \sum_{\alpha} \tilde{\beta}_{\alpha}(\vec{g}_R) \frac{\partial}{\partial g_R}\right)(\text{r.h.s.})$$

$$= d + \sum_{i=1}^{n} \gamma_{s_i}(\vec{g}_R(\lambda, \vec{g}_R))$$

$$- \sum_{i} \int_{1}^{\lambda} \frac{d\lambda'}{\lambda'} \sum_{\alpha} \tilde{\beta}_{r}(\vec{g}_R(\lambda', \vec{g}_R))$$

$$\times \frac{\partial \gamma_{s_i}(\vec{g}_R(\lambda', \vec{g}_R))}{\partial \bar{g}_{rR}}$$

$$= d + \sum_{i=1}^{n} \gamma_{s_i}(\vec{g}_R(\lambda, \vec{g}_R))$$

$$- \sum_{i=1}^{n} \int_{1}^{\lambda} \frac{d\lambda'}{\lambda'} \left[\lambda' \frac{\partial}{\partial \lambda'} g_{rR}(\lambda', \vec{g}_R)\right.$$

$$\left. \times \frac{\partial \gamma_{s_i}(\vec{g}_R(\lambda', \vec{g}_R))}{\partial \bar{g}_{rR}}\right]$$

$$\Downarrow$$

$$\lambda' \frac{\partial}{\partial \lambda'} \gamma_{s_i}(\vec{g}_R(\lambda', \vec{g}_R))$$

$$= d + \sum_{i=1}^{n} \gamma_{s_i}(\vec{g}_R(\lambda, \vec{g}_R))$$

$$- \sum_{i=1}^{n} \left(\gamma_{s_i}(\vec{g}_R(\lambda, \vec{g}_R)) - \gamma_{s_i}(\underset{\substack{\Downarrow \\ \vec{g}_R}}{\vec{g}_R(1, \vec{g}_R)})\right)$$

$$= \left(d + \sum_{i=1}^{n} \gamma_{A_i}(\vec{g}_R)\right).$$

Thus

$$\left(\lambda \frac{\partial}{\partial \lambda} - \sum_{\alpha} \tilde{\beta}_{\alpha}(\vec{g}_R) \frac{\partial}{\partial g_R} - d - \sum_{i=1}^{n} \gamma_{A_i}(\vec{g}_R)\right)$$

$$\text{r.h.s.} = 0$$

$\Rightarrow$ l.h.s. = r.h.s.

(Note: they satisfy same b.c. at $\lambda = 1$)

Proof of:

$$\frac{\underbrace{\partial \bar{g}_{\alpha R}(\lambda, \vec{g}_R)}_{}}{\partial g_{\gamma R}}\, \beta_\gamma(\vec{g}_R) = \beta_\alpha(\vec{\bar{g}}_R(\lambda, \vec{g}_R))$$

Note: At $\lambda = 1$, $\bar{g}_{\alpha R}(\lambda, \vec{g}_R) = g_{\alpha R}$.

$$\Rightarrow \quad \text{l.h.s.} = \delta_{\alpha\gamma}\, \beta_\gamma(\vec{g}_R) = \beta_\alpha(\vec{g}_R)$$

$$\text{r.h.s.} = \beta_\alpha(\vec{\bar{g}}_R(\lambda = 1, \vec{g}_R)) = \beta_\alpha(\vec{g}_R)$$

Thus it is enough to show that

$$\frac{\partial}{\partial \lambda}(\text{lhs}) = \frac{\partial}{\partial \lambda}(\text{rhs})$$

Call l.h.s. − r.h.s. = $f_\alpha(\lambda, \vec{g}_R)$

$$\lambda \frac{\partial}{\partial \lambda}(\text{l.h.s.}) = \frac{\partial}{\partial g_{\gamma R}}\left(\lambda \frac{\partial \bar{g}_{\alpha R}(\lambda, \vec{g}_R)}{\partial \lambda}\right) \beta_\gamma(\vec{g}_R)$$

$$= \frac{\partial}{\partial g_{\gamma R}}\, \beta_\alpha(\vec{\bar{g}}_R(\lambda, \vec{g}_R))\, \beta_\gamma(\vec{g}_R)$$

$$= \frac{\underbrace{\partial \beta_\alpha(\vec{\bar{g}}_R}_{}(\lambda, \vec{g}_R))}{\partial \bar{g}_{\delta R}}\, \underbrace{\frac{\partial \bar{g}_{\delta R}(\lambda, \vec{g}_R)}{\partial g_{\gamma R}}\, \beta_\gamma(\vec{g}_R)}_{\longleftrightarrow \; (\beta_{\cancel{\alpha}\delta}(\vec{\bar{g}}_R(\lambda, \vec{g}_R)) + f_\delta(\lambda, \vec{g}_R))}$$

$$\lambda \frac{\partial}{\partial\lambda}(\text{r.h.s.}) = \frac{\partial\beta_\alpha(\vec{\bar{g}}_R(\lambda,\vec{g}_R))}{\partial\bar{g}_{\delta R}}\,\beta_\delta(\vec{\bar{g}}_R(\lambda,\vec{g}_R))$$

$$\lambda\frac{\partial}{\partial\lambda} f_\alpha(\lambda,\vec{g}_R) = \lambda\frac{\partial}{\partial\lambda}\left(\text{l.h.s.} - \text{r.h.s.}\right)$$

$$= \frac{\partial\beta_\alpha(\vec{\bar{g}}_R(\lambda,\vec{g}_R))}{\partial\bar{g}_{\delta R}}\, f_\delta(\lambda,\vec{g}_R)$$

$\rightsquigarrow$ First order differential eq. for f_α.

Solution is unique once we specify $f_\alpha(\lambda=1, \vec{g}_R)$.

$$f_\alpha(\lambda=1,\vec{g}_R) = 0$$

$$\Rightarrow\quad f_\alpha(\lambda,\vec{g}_R) = 0$$

$$\Rightarrow\quad \text{l.h.s.} = \text{r.h.s.}$$

Renormalization group equation:

$$\left(\mu\frac{\partial}{\partial\mu} + \sum_{\alpha}\beta_\alpha\frac{\partial}{\partial g_{\alpha R}} + \sum_{k}\gamma_{s_k}\right)\underset{F(\{k_i\},\{g_{\alpha R}\},\mu)}{\underset{\Vert}{\left\langle\prod_{i=1}^{n}\tilde{\phi}_{s_i}(k_i)\right\rangle}} = 0.$$

Dimensional analysis:

$$F(\{\lambda k_i\},\{g_{\alpha R}\},\mu) = \lambda^{\overset{\sum_i \tilde{d}_{s_i}}{\overset{\shortparallel}{d}}} F(\{k_i\},\{g_{\alpha R}\lambda^{-d_\alpha}\},\mu\lambda^{-1})$$

Combining the two.

$$\Rightarrow \left\{\lambda\frac{\partial}{\partial\lambda} - \sum_{i}(d_{s_i}+\gamma_{s_i}) - \sum_{\alpha}\overset{\tilde{\beta}_\alpha}{\overleftrightarrow{(\beta_\alpha - g_{\alpha R}d_{\alpha R})}}\frac{\partial}{\partial g_{\alpha R}}\right\}$$

$$F(\{\lambda k_i\},\{g_{\alpha R}\},\mu) = 0$$

Solution:

$$F(\{\lambda k_i\},\{g_{\alpha R}\},\mu)$$

$$= F(\{k_i\},\{\bar{g}_{\alpha R}(\lambda,\vec{g}_R\},\mu)$$

$$\exp\left(\sum_{i=1}^{n} d_{s_i}\ln\lambda + \sum_{i=1}^{n}\int_1^{\lambda}\frac{d\lambda'}{\lambda'}\gamma_{s_i}(\{\bar{g}_\alpha(\lambda',\vec{g}_R)\}\right)$$

$$\lambda\frac{d\bar{g}_{\alpha R}}{\partial\lambda} = \beta_\alpha(\vec{\bar{g}}_R(\lambda)), \quad \bar{g}_{\alpha R}(\lambda=1,\vec{g}_R) = g_{\alpha R}$$

High energy behaviour

$$\lambda \frac{\partial \bar{g}_{\alpha R}}{\partial \lambda} = \tilde{\beta}_\alpha(\{\bar{g}_{\alpha R}\}) \qquad \bar{g}_{\alpha R}(\lambda=1) = g_{\alpha R}$$

$$\tilde{\beta}_\alpha = \beta_\alpha - d_\alpha g_{\alpha R}$$

If $\tilde{\beta}_\alpha < 0$ then $\bar{g}_{\alpha R}$ decreases for large λ.

For parameters with $d_\alpha > 0$, e.g. masses, the dominant term at weak coupling is $-d_\alpha g_{\alpha R}$

$$\lambda \frac{\partial \bar{g}_{\alpha R}}{\partial \lambda} \simeq -d_\alpha \bar{g}_{\alpha R}$$

$$\Rightarrow \bar{g}_{\alpha R} \simeq g_{\alpha R} \exp(-d_\alpha \lambda)$$

$$\to 0 \text{ as } \lambda \to \infty.$$

Example: For gauge theory coupled to massive fermions

$$\beta_m = \underset{\text{Constant}}{A_m} m_R g_R^2 \Rightarrow \tilde{\beta}_m = A_m m_R g_R^2 - m_R$$

$\simeq m_R$ for small g_R.

Thus if the coupling is small then all the effective mass parameters → 0 as $\lambda \to \infty$.

However we now need to find conditions under which the dimensionless couplings, i.e. the perturbation expansion parameters, remain small.

For dimensionless parameters

$$\lambda \frac{\partial g_{\alpha R}}{\partial \lambda} = \beta_{\alpha}(\vec{g}_{\alpha})$$

If all β_α are negative at weak coupling then all $g_{\alpha R} \to 0$ as $\lambda \to \infty$.

⇒ The ~~theory~~ correlation fn. at large λ ~~are given by~~ can be computed in perturbation theory with weak coupling.

Example: Consider gauge theory coupled to fermions in representation R.

$$g = Z_g \, g_R \, \mu^{\epsilon/2}$$

$\swarrow$ ↳ renormalized gauge coupling

$$1 + \frac{A}{\epsilon} g \, g_R^2 + O(g_R^4)$$

$$A_g = - \frac{g_R^2}{8\pi^2} (11 C_G - 4 T_R)$$

$$f^{acd} f^{bcd} = C_G \delta_{ab}, \quad \mathrm{Tr}(R(T^a) \, R(T^b)) = T_R \delta_{ab}$$

$$\beta_{\phi g} = A g_R^3 + O(g_R^5)$$

Thus $\beta_g < 0$ requires $A < 0$.

$$11 C_G > 4 T_R$$

$C_G > 0$. Thus gauge theory coupled to fermions has negative β_g for small g_R for certain choices of R.

Strong interaction : gauge group SU(3).

Fermions in fundamental (3) and anti-fundamental representation of SU(3).
conjugate

Total: 6 Dirac fermions in fundamental
(+ complex conjugate)

$C_a = 3$, $T_{\text{fundamental}} = \frac{1}{2}$.

$\Rightarrow 11\, C_a - 4\, T_R$

$= 11 \times 3 - 4 \times \frac{1}{2} \times n$ ← number of fundamental

$= (33 - 2n). > 0$ for $n = 3$.

$\Rightarrow$ QCD at large momentum is weakly coupled.

Behaviour of $\bar{g}_R(\lambda)$ for large λ:

$$\lambda\frac{\partial}{\partial\lambda}\bar{g}_R = \underset{\substack{\parallel \\ -B.}}{A}\,\bar{g}_R^3 + O(\bar{g}_R^5)$$

$$-\frac{1}{2}\cancel{\partial}\,\frac{\partial}{\partial\ln\lambda}\left(\frac{1}{\bar{g}_R^2}\right) = -B.$$

$$\bar{g}_R^{-2} = 2B\ln\lambda + g_R^{-2}$$

$$\bar{g}_R^2 = \frac{1}{g_R^2 + 2B\ln\lambda}$$

Thus for large λ, $\bar{g}_R^2 \sim \frac{1}{2B\ln\lambda}$

An amplitude ~~which~~ whose leading connected contribution goes as $\bar{g}_R^{2K}$ will go as:

$$\lambda^{\sum d_{s_i}}\left(\frac{1}{\ln\lambda}\right)^K \times \text{constant} + \cdots$$

$\searrow$ scaling violation.

Discovery of SU(3) as theory of strong interactions.

↔ followed the experimental observation that strongly interacting particles seem to be made of weakly interacting constituents at high energy.

→ Look for a theory with negative β_g.

Non-abelian gauge theories are the only candidates.

A caveat to our analysis:

$$F(\{\lambda k_i\}, \{g_{\alpha R}\}) = F(\{k_i\}, \{\bar{g}_{\alpha R}\})$$
$$\times \exp\left(\quad\right)$$

$\bar{m}_R \to 0$ as $\lambda \to \infty$.

If in this limit F develops singularities, e.g. if it has terms like $\ln m_R$, then we cannot use our argument.

$\rightsquigarrow$ Need to carefully analyze the divergent parts as $\overline{m}_R \to 0$ and analyze the λ-dependence coming from there.

Trick: Look for quantities which are finite as $m_R \to 0$.

Only those quantities have simple behaviour at large λ.

e.g. $\sigma(e^+ e^- \to$ all strongly interacting particles$)$

or deep inelastic scattering.

What about small λ.?

for Massive fermions $\bar{m}_R \to \infty$ (for small $\bar{g}_R$)

However now $\bar{g}_R(\lambda)$ becomes large for small λ.

→ We can no longer justify keeping $O(\bar{g}_R^3)$ contribution to $\beta_0(\bar{g}_R)$

$O(\bar{g}_R^5)$ etc. terms become important.

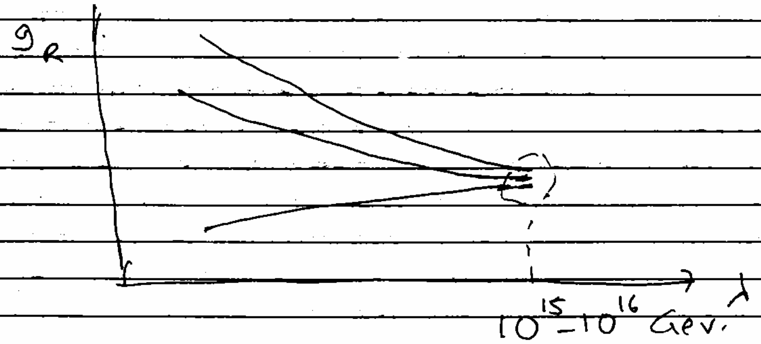
⇒ Perturbation theory becomes useless.

→ Need non-perturbative methods. (Lattice gauge theory)

Standard model:

Since the particle spectrum is known we can calculate the $\bar{g}_R(\lambda)$ at $\lambda \to \infty$. for U(1), SU(2) & SU(3).

Result:



They almost meet.

→ suggests existence of a ~~[illegible]~~ unified group (e.g. SU(5)) ~~[illegible]~~ broken to SU(3) x SU(2) x U(1) at $10^{15}-10^{16}$ GeV.

More accurate analysis: Does not quite work for SM but works for SUSY SM.

Some properties & uses of BRST symmetry

δ(original fields) = gauge trs. with parameter $\theta^a(x) = J c^a(x)$

$$\delta c^a(x) = -\frac{1}{2} g f^{bca} J c^c(x) c^b(x)$$

$$\delta b^a(x) = -\frac{1}{\alpha} F_a(x) . J$$

for gauge fixing term $-\frac{1}{2\alpha}\int d^4x\, F^a(x) F^a(x)$.

A slight generalization: Introduce a new field $\chi^a(x)$ and write the gauge fixing term as

$$\int d^4x \left[\frac{\alpha}{2} \chi^a \chi^a + \chi^a(x) F^a(x) \right]$$

After ~~eliminating~~ integrating over χ^a we get back $-\frac{1}{2\alpha}\int d^4x\, F^a F^a$.

$$\delta b^a = \chi^a(x) J$$

$$\delta \chi^a = 0.$$

We can treat the gauge fixed action as fundamental and go to canonical formulation.

⇒ Conserved charge Q_B

$$[\zeta Q_B, \text{field}] = i\,\delta(\text{field}).$$

Ex. check that

$$[\xi Q_B, [\zeta Q_B, \text{field}]] = 0 \quad \text{for every field}$$

e.g.

$$[\xi Q_B, [\zeta Q_B, b^a(x)]] = [\xi Q_B, \chi^a(x)\,\zeta]$$

$$= [\xi Q_B, \chi^a(x)]\,\zeta = 0.$$

Consequence:

$$\xi Q_B (\zeta Q_B A - A \zeta Q_B) - (\zeta Q_B A - A \zeta Q_B)\, \xi Q_B \overset{}{=} 0$$

$$\Rightarrow -\xi \zeta Q_B^2 A - \zeta Q_B A \zeta Q_B - \zeta Q_B A \xi Q_B$$

$$+ A \zeta Q_B \xi Q_B = 0$$

$$\Rightarrow \zeta \xi [Q_B^2, A] = 0$$

Q_B^2 commutes with all operators.

One finds $Q_B^2 = 0$ (nilpotent)

Define $[Q_B, A\} = [Q_B, A]$ for bosonic A

$= \{Q_B, A\}$ for fermionic A.

$$[\xi Q_B, A] = \xi [Q_B, A\}$$

~~Now in the canonical formulation we~~ ~~do then as a cons~~ Now in the canonical formulation we declare that an operator A is physical if

$$[Q_B, A\} = 0. \quad \text{i.e. } [\xi Q_B, A] = 0.$$

~~Now~~ All gauge invariant operators are automatically physical.

Now take the operator $[Q_B, B\}$ arbitrary.

$$[\xi Q_B, [\xi Q_B, B]] = 0$$

$$\Rightarrow [\xi Q_B, \xi [Q_B, B\}] = 0$$

$$\Rightarrow \xi [\xi Q_B, [Q_B, B\}] = 0 \Rightarrow [Q_B, B\} \text{ is physical.}$$

However all correlation functions involving $[Q_B, B\}$ and other physical operators vanish.

$$\langle \prod_{i=1}^{n} [Q_B, B(x)\} \prod_{i} A_i(x_i) \rangle$$

$$= \langle \Omega | T\left(\prod_{i=1}^{n} A_i(x_i) \; [Q_B, B(x_i)\} \right| \Omega \rangle$$

$$\int \langle [Q_B, B(x)) \prod_{i} A_i(x_i) \rangle$$

$$= \int \langle \Omega | T\Big([Q_B, B(x_0)\} \prod_{i=1}^{m} A_i(x_i) | \Omega \rangle$$

$$= \langle \Omega | T\left([\int Q_B, B(x_0)] \prod_{i=1}^{n} A_i(x_i) \right) | \Omega \rangle$$

$$= \langle \Omega | \left[\int Q_B, T\left(B(x_0) \prod_{i=1}^{n} A_N(x_i) \right] | \Omega \right\rangle$$

$$= 0 \quad \text{if} \quad Q_B |\Omega\rangle = \langle \Omega | Q_B = 0$$

Note: $\int Q_B$ is conserved & hence time independent.

$$\Rightarrow \langle \Omega\, T\Big([Q_B, B(x_0)\} \prod_{i=1}^{n} A_i(x_i) \Big) | \Omega \rangle = 0$$

Conclusion: Even though $\{Q_B, B(x)\}$ is physical this is trivial.

(All its correlation functions vanish).

We declare that two physical operators A_1 and A_2 are equivalent if

$$A_1 - A_2 = \{Q_B, B\} \quad \text{for some operator } B.$$

Equivalence classes $\Rightarrow$ complete set of physical operators.

This gives a complete description of physical operators in the gauge fixed theory without going back to the original gauge invariant formulation.

Proof of independence of gauge fixing terms. (α and $F^a(x)$).

$S_{gf} + S_{ghost}$

$$= \int d^4x \left[\frac{\alpha}{2} \chi^a(x) \chi^a(x) + \chi^a(x) F^a \right]$$

$$- \int d^4x \; d^4y \; b^a(x) \left. \frac{\delta F_\theta^a(x)}{\delta \theta^b(y)} \right|_{\theta=0} c^b(y)$$

Change $\alpha \to \alpha + \delta\alpha$.

$$\zeta \, \delta_\alpha (S_{gf} + S_{ghost}) = \frac{1}{2} \underset{\tilde{\zeta}}{\underleftarrow{\zeta \, \delta\alpha}} \int d^4x \, \chi^a(x) \chi^a(x)$$

$$= \frac{1}{2} \left[\tilde{\zeta} Q_B, \int d^4x \, b^a(x) \, \chi^a(x) \right]$$

Thus $\zeta \, \delta_\alpha \left\langle \prod_i A_i(x_i) \right\rangle$

$$= \frac{1}{2} \langle \Omega | T \Big(i \left[\tilde{\zeta} Q_B, \int d^4x \, b^a(x) \chi^a(x) \right]$$

$$\prod_i A_i(x_i) \Big) | \Omega \rangle = 0.$$

for physical $A_i(x_i)$.

Change $F^a \to F^a + \Delta F^a$

$$\zeta\, \delta_F \left(S_{gf} + S_{ghost} \right)$$

$$= \int d^4x \; \zeta\, \chi^a(x)\, \Delta F^a(x)$$

$$+ \int d^4x \, d^4y \; b^a(x) \; \zeta \, \frac{\delta(\Delta F^a_\theta(x))}{\delta\theta^b(y)}\bigg|_{\theta=0} c^b(y)$$

$$= \left[\zeta\, Q_B , \int d^4x \; b^a(x)\, \Delta F^a(x) \right]$$

The same argument as before now shows that

$$\zeta\, \delta_F \left\langle \prod_i A_i(x_i) \right\rangle = 0$$

for physical $A_i(x_i)$.