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Non-abelian gauge theories

Draw analogy with QED.

Recall :- QED action is

$$S = \int d^4x \, \mathcal{L}$$

where $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} \{ i\gamma^\mu (\partial_\mu - ieA_\mu) - m \} \Psi$

such that $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Dirac indices aren't written explicitly

Symmetry of $\mathcal{L}$

① Global symmetry :-

$$\Psi(x) \to e^{ie\lambda} \Psi(x), \quad A_\mu(x) \to A_\mu(x)$$

λ → constant

This is valid even in the absence of gauge fields A_μ & their coupling to Ψ.

② Local symmetry / gauge symmetry :-

$$\Psi(x) \to e^{ie\lambda(x)} \Psi(x), \quad A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x)$$

→ gauge fields are essential.

Here gauge field is essential to make $\mathcal{L}$ invariant

Not a symmetry of free fermion action.

But it is a symmetry of the

Maxwell action without fermions.

We will be led to postulating the presence of gauge fields if we start with free fermionic action & then take $\lambda(x)$ & demand it to be a sym. of $\mathcal{L}$

Define covariant derivative :-

$$D_\mu \psi = \partial_\mu \psi - ie A_\mu \psi$$

(Reason for defining this) → Under local gauge transformation

$$D_\mu \psi \longrightarrow e^{ie\lambda(x)} D_\mu \psi$$

Proof :-

$$D_\mu \psi \rightarrow \partial_\mu \left(e^{ie\lambda(x)} \psi\right) - ie\left(A_\mu + \partial_\mu \lambda\right)\left(e^{ie\lambda(x)}\psi\right)$$

$$= e^{ie\lambda(x)} \partial_\mu \psi + ie\, \partial_\mu \lambda(x) e^{ie\lambda(x)} \psi - ie A_\mu e^{ie\lambda(x)} \psi - ie\, \partial_\mu \lambda\, e^{ie\lambda(x)} \psi$$

$$= e^{ie\lambda(x)} \left(\partial_\mu \psi - ie A_\mu \psi\right)$$

Though $\partial_\mu \psi$ transforms in a complicated way, $D_\mu \psi$ transforms in a nice fashion — that's why it is called the covariant deriv.

Note → For $e^{ie\lambda}$ → ?
We have $(\partial_\nu - ieA_\mu) e^{ie\lambda(x)} (ie\partial_\nu \lambda - ieA_\nu)$
$= e^{ie\lambda(x)} (ie\partial_\mu \lambda - ieA_\mu)(ie\partial_\nu \lambda - ieA_\nu)$
$= (ie) e^{i\lambda} (\partial_\mu \partial_\nu \lambda - (\partial_\mu \lambda) A_\nu - A_\mu (\partial_\nu \lambda) + A_\mu A_\nu)$
⇓ sym. in μ, ν
$\therefore F_{\mu\nu} e^{ie\lambda} = 0$

$$[D_\mu, D_\nu]\psi = D_\mu D_\nu \psi - D_\nu D_\mu \psi$$

$$= \left(\partial_\mu - ie A_\mu(x)\right)\left(\partial_\nu - ie A_\nu(x)\right)\psi(x)$$

$$- \quad \nu \leftrightarrow \mu$$

$$= -ie\, F_{\mu\nu}(x)\, \psi(x)$$

$$[D_\mu, D_\nu]\psi \equiv D_\mu D_\nu \psi - D_\nu D_\mu \psi$$ → How does it transform?

$$\longrightarrow D'_\mu e^{ie\lambda(x)}(D_\nu \psi) - D'_\nu e^{ie\lambda(x)}(D_\mu \psi)$$

$$= e^{ie\lambda(x)} D_\mu D_\nu \psi - e^{ie\lambda(x)} D_\nu D_\mu \psi$$

$$= e^{ie\lambda(x)} [D_\mu, D_\nu]\psi$$

Suppose $F_{\mu\nu}(x) \longrightarrow F'_{\mu\nu}(x)$ under gauge transfn...

(The above 2 results together will tell us) →

$$-ie F'_{\mu\nu}(x)\, e^{ie\lambda(x)} \psi(x)$$

$$= e^{ie\lambda(x)} (-ie F_{\mu\nu}(x))\, \psi(x)$$

$$F'_{\mu\nu}(x) = F_{\mu\nu}(x)$$

Notice →
$(\partial_\mu - ie(A_\mu + \partial_\mu \lambda)$
$+ \partial_\nu - ie(A_\nu + \partial_\nu \lambda))e^{ie\lambda}$
$= e^{ie\lambda}(-ieA_\mu)(-ie)A_\nu$
$\therefore F'_{\mu\nu} e^{ie\lambda} = 0$
Hence
$F'_{\mu\nu} = F_{\mu\nu}$

This is a trivial exercise in ED but it provides a powerful tool to apply in Non-abelian gauge th. where we don't know beforehand how to construct $F_{\mu\nu}$

From the prop. that $[D_\mu, D_\nu]\psi = -ie F_{\mu\nu}(x)\,\psi(x)$, & it is guaranteed that $F'_{\mu\nu}(x) = F_{\mu\nu}(x)$ from here we can construct $F_{\mu\nu}$ for non-abelian gauge theories

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Consider a theory of N free Dirac fields, each with mass m.

$$S = \int d^4x\, \bar{\psi}^k (i\gamma^\mu \partial_\mu - m)\psi^k$$

where $k = 1, 2, \ldots, N$

repeated indices summed over

Dirac indices suppressed.

This action has a global symmetry :-

$$\psi^k \longrightarrow U_{kl}\,\psi^l, \Rightarrow \bar{\psi}^k = U^*_{kl}\,\bar{\psi}^l,$$

where $U^\dagger U = \mathbb{1}$

U: $N\times N$ unitary matrixe

$$S \xrightarrow[\text{transfn. goes to}]{\text{under this}} \int d^4x\; U^*_{kl}\,\bar{\psi}^l (i\gamma^\mu \partial_\mu - m)\,\psi^j\, U_{kj}$$

Now, $U^\dagger U = \mathbb{1} \Rightarrow U^*_{kl}\,U_{kj} = \delta_{jl}$

$$= \int d^4x\; \bar{\psi}^j (i\gamma^\mu \partial_\mu - m)\,\psi^j .$$

The set of $N\times N$ unitary matrices form a group.

$\downarrow$

$U(N)$ group

Special subset of $N\times N$ unitary matrices:

$$\left.\begin{array}{l} \det U = 1 \\ \longrightarrow SU(N) \text{ group} \end{array}\right\}$$ Examples of non-abelian group

[Prdt. are matrices & matrix prdt. doesn't commute]

this is also a group & is a subgroup of $U(N)$

Goal :- construct a theory where either the $SU(N)$ or the $U(N)$ symmetry becomes a local symmetry.

(This means that we want an action invariant under a transfn of the following kind)

$$S \longrightarrow S$$

under $\psi^k \longrightarrow U_{kl}(x)\,\psi^l(x)$

At every point x, $U_{kl}(x)$ describes a $U(N)$ or $SU(N)$ matrix.

different theories

ψ at every pt. x is multiplied by a diff. unitary matrix

We'll focus on the $SU(N)$ case for definiteness (but the result can be generalised for the $U(N)$ case).

We want an action invariant under

$$\psi^k(x) \longrightarrow U_{kl}(x)\,\psi^l(x) \xRightarrow{\text{consequently}} \bar{\psi}^k(x) \rightarrow U^*_{kl}(x)\,\bar{\psi}^l(x)$$

where $U_{kl}(x)$ is a special unitary matrix for every x.

We'll assume it is smooth — as $\psi(x)$ changes continuously, $U(x)$ also changes continuously

In the same spirit as abelian gauge theories, we will add terms to $\mathcal{L}$ to compensate for the change in original free fermionic action under this transfn.

$$S_{\text{free-fermion}} = \int d^4x \; \bar{\Psi}^k(x) \left(i\gamma^\mu \partial_\mu - m\right) \psi^k(x)$$

$$\xrightarrow[\text{to}]{\text{goes}} \int d^4x \; U^*_{kl}(x) \, \bar{\Psi}^l(x) \left(i\gamma^\mu \partial_\mu - m\right) \left(U_{kj}(x) \psi^j(x)\right)$$

$\boxed{\text{Now identify the problematic terms}}$

$$= \int d^4x \; \bar{\Psi}^l \left(i\gamma^\mu \partial_\mu - m\right) \psi^l(x)$$

$$+ \underbrace{\int d^4x \; \underset{``U^+_{lk}(x)}{U^*_{kl}(x)} \, \partial_\mu U_{kj}(x) \, \bar{\Psi}^l(x) \, i\gamma^\mu \psi^j(x)}_{\text{extra term}}$$

↓

(sth. you have to cancel by taking new terms)

Extra term in $\Delta S_{\text{free-fermion}}$

$$= \int d^4x \left(U^+(x) \partial_\mu U(x)\right)_{lj} \bar{\Psi}^l(x) \, i\gamma^\mu \psi^j(x)$$

Add new terms in the action :-

$$S_{int} = \bar{\Psi}^i \gamma^\mu \psi^k \underbrace{S_{\mu\, ik}(x)}_{\text{unknown combination of new fields}}$$

Suppose :-

$S_{\mu\, ik} \rightarrow S'_{\mu\, ik}$ under gauge trs.

$$S_{int} \longrightarrow \int d^4x \; U^*_{il}(x) \, \bar{\Psi}^l \gamma^\mu \psi^j \, U_{kj}(x) \, S'_{\mu\, ik}(x)$$

$$= \int d^4x \; \bar{\Psi}^l \gamma^\mu \psi^j \left(U^+ S'_\mu U\right)_{lj}$$

$$\overset{\text{want}}{=} \int d^4x\, \bar{\Psi}^l \gamma^\mu \psi^j\, S_{\mu\, lj}(x)$$

$$- \int d^4x \left(U^\dagger(x)\, \partial_\mu U(x)\right)_{lj} \bar{\Psi}^l i\gamma^\mu \psi^j$$

(just equate the coeff. of $\bar{\Psi}^l \gamma^\mu \psi^j$ on both sides) $\rightarrow$

$$\Rightarrow \quad U^\dagger S'_\mu U = S_\mu - i U^\dagger \partial_\mu U$$

$$\Rightarrow \quad \boxed{S'_\mu = U S_\mu U^\dagger - i\, \partial_\mu U\, U^{-1}}$$

$$S_{free} + S_{int} = S'_{free} + S'_{int}$$

$$\quad = S_{free} + \Delta S_{free} + S'_{int}$$

We haven't said so far that S_μ themselves are fundamental fields

$$\therefore \quad S'_{int} = S_{int} - \Delta S_{free}$$

Note :-

① $i\partial_\mu U\, U^{-1} = i\partial_\mu U\, U^\dagger$ is Hermitian if U is unitary.

$$\partial_\mu(\underbrace{UU^\dagger}_{= 1}) = 0$$

$$\Rightarrow \partial_\mu U\, U^\dagger = -U\partial_\mu U^\dagger$$

$$= -(\partial_\mu U\, U^\dagger)^\dagger \qquad [\because (AB)^\dagger = B^\dagger A^\dagger]$$

$$\underbrace{i\,\partial_\mu U\, U^\dagger}_{} = -i(\partial_\mu U\, U^\dagger)^\dagger = (i\partial_\mu U\, U^\dagger)^\dagger$$

($\therefore$ this is Hermian)

Ex, → 2 Show that

$i \partial_\mu U U^\dagger$ is traceless if $(\det U) = 1$

$$U(x+dx)U(x)^{-1} = (U(x) + \partial_\mu U\, dx^\mu)U(x)^{-1}$$
$$\Rightarrow \det U(x+dx) \det(U(x)^{-1}) = \det(\mathbb{1} + (\partial_\mu U)U^{-1} dx^\mu)$$
$$\Rightarrow 1 = 1 + \mathrm{Tr}((\partial_\mu U)U^{-1})\, dx^\mu$$
$$\Rightarrow \mathrm{Tr}((\partial_\mu U)U^{-1}) = 0$$

[We are trying to understand the nature of the matrix $i\partial_\mu U U^{-1}$]

How many linearly independent traceless Hermitian matrices are there?

→ L_{ik} : $N \times N$ traceless Hermitian matrise

$\therefore L_{ik} = L^*_{ki}$

(How many indep. coeff. are there to parametrize such matrise?)

L_{ik} for $i > k$ is determined in terms of L_{ik} with $i < k$.

$\frac{N(N-1)}{2}$ complex $\equiv N(N-1)$ real parameters.

On the diagonal $(N-1)$ real parameters. ↔ trace = 0 condition

$$N(N-1) + (N-1) = N^2 - 1$$

$$2 \times \frac{N^2 - N}{2} + N$$
$$= N^2$$
$N^2 - 1$ for traceless

$\therefore$ an $N \times N$ traceless Hermitian matrix can be expressed as :-

$$\sum_{a=1}^{N^2-1} \alpha^a T^a$$

$T^a \hookrightarrow$ fixed $N \times N$ traceless Hermitian matrices

$\boxed{\text{bcos there can be } \alpha^a \text{ indep. parameters}}$

<u>Example</u> :-

$N = 3$

$$\begin{pmatrix} a & c+id & e+if \\ c-id & b & g+ih \\ e-if & g-ih & -a-b \end{pmatrix} \longrightarrow$$ most general traceless 3×3 Hermitian matrix

$$= a\underbrace{\begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & -1 \end{pmatrix}}_{T_1} + b\underbrace{\begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & -1 \end{pmatrix}}_{T_2}$$

$$+ c\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{T_3} + d\underbrace{\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{T_4} + e\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{T_5}$$

$$+ f\underbrace{\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}}_{T_6} + g\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{T_7} + h\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}}_{T_8}$$

$$\therefore \; i\partial_\mu U \, U^{-1} = \sum_a \alpha_\mu^a(x)\, T^a$$

$\alpha_\mu^a(x) \hookrightarrow$ depend on $U(x)$

$\{T^a\}$: generators of $SU(N)$ algebra.

<u>Postulate</u> :- S_μ is an $N \times N$ traceless Hermitian matrix.

[This will be a consistent postulate only if $U S_\mu U^\dagger$ is also traceless Hermitian] →

Q) ① Is this Hermitian?

$$(U S_\mu U^\dagger)^\dagger = U S_\mu^\dagger U^\dagger = U S_\mu U^\dagger$$ ✓

② Is this traceless?

$$\mathrm{Tr}\,(U S_\mu U^\dagger) = \mathrm{Tr}\,(S_\mu \underbrace{U^\dagger U}_{\mathbb{1}}) = 0$$

So traceless as well as Herm —

We are trying a minimal set of fields which we add to make the theory work

$\Rightarrow$ S_μ' is also traceless & Hermitian

$$\Longrightarrow S_\mu(x) = \sum_{a=1}^{N^2-1} B_\mu^a(x)\, T^a$$

↓ some new fields.

[If we were working with $U(N)$, then $\partial_\mu U\, U^{-1}$ would have been Hermitian, but not traceless. Then we couldn't take S_μ to be traceless bcos, although $U S_\mu U^\dagger$ would have been traceless, $U S_\mu U^\dagger - i\partial_\mu U\, U^{-1}$ isn't]

Suppose $B^a_\mu \longrightarrow B'^a_\mu$ under the local gauge transfn:

$$S'_\mu = \sum_{a=1}^{N^2-1} B'^a_\mu T^a$$

$$\sum_{a=1}^{N^2-1} B'^a_\mu T^a = \sum_{a=1}^{N^2-1} B^a_\mu(x)\, U T^a U^+ - \sum_{a=1}^{N^2-1} \alpha^a_\mu T^a$$

$$\underbrace{U_{(x)} T^a U^{-1}_{(x)}}_{\text{also traceless \& Hermitian}} = U(x) T^a U^+(x) = \sum_{b=1}^{N^2-1} T^b R_{ba}(x)$$

bcs T^a is traceless & Hermitian

for some matrix $R_{ba}(x)$

$\Downarrow$

$(N^2-1)\times(N^2-1)$ matrix

$$\sum_{a=1}^{N^2-1} B'^a_\mu T^a = \underbrace{\sum_{a,b=1}^{N^2-1} B^a_\mu(x) R_{ba}(x) T^b}_{a \leftrightarrow b} - \sum_{a=1}^{N^2-1} \alpha^a_\mu(x) T^a$$

$$= \sum_{a,b=1}^{N^2-1} B^b_\mu(x) R_{ab}(x) T^a - \sum_{a=1}^{N^2-1} \alpha^a_\mu(x) T^a$$

$$\Rightarrow B'_\mu(x) = R_{ab}(x) B^b_\mu(x) - \alpha^a_\mu(x)$$

Postulate :-

B_μ^a's are the new fundamental fields.

We could have taken some other fundamental fields & expressed B_μ in terms of these — But this is the simplest choice

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Now, $S_{\text{free-fermion}} + S_{\text{int}}$

$$= \int d^4x \left[\bar{\Psi}^k(x) (i\gamma^\mu \partial_\mu - m) \psi^k(x) + \bar{\Psi}^l \gamma^\mu \psi^m(x) B_\mu^a(x) T^a_{lm} \right]$$

Transformation laws :-

$$\psi^k_{(x)} \longrightarrow U_{kl}(x) \psi^l(x) \,, \quad \bar{\Psi}^k(x) \to U^*_{kl}(x) \bar{\Psi}^l(x)$$

$$B_\mu^a(x) \to R_{ab}(x) B_\mu^b(x) - \alpha_\mu^a(x)$$

where $R_{ab}(x)$, $\alpha_\mu^a(x)$ are defined through :-

$$i \partial_\mu U U^\dagger = \sum_{a=1}^{N^2-1} \alpha_\mu^a(x) T^a \,,$$

$$U(x) T^a U^\dagger_{(x)} = \sum_{b=1}^{N^2-1} T^b R_{ba}(x)$$

[Gauge transfn is : generated by these $N \times N$ special unitary matrices $U(x)$]

[R_{ab}, $\alpha_\mu^a(x)$ are auxiliary objects constructed when $U(x)$ is given]

$U(1) \longrightarrow 1 \times 1$ Hermitian matrix
$\downarrow$
just a number

$\Downarrow$
case for QED

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Actions with local SU(N) symmetry

$$S = \int d^4x \left[\bar{\Psi}^k(x) (i\gamma^\mu \partial_\mu - m)\Psi(x) + \bar{\Psi}^l(x) \gamma^\mu \Psi^m(x) B^a_\mu(x) T'^a_{lm} \right]$$

T'^a : traceless, hermitian matrices

Invariant under:-

$$\Psi^k \longrightarrow U_{kl}(x)\, \Psi^l(x) \equiv \tilde{\Psi}^k$$

$$B^a_\mu(x) \longrightarrow R_{ab}(x)\, B^b_\mu(x) - \alpha^a_\mu(x) \equiv \tilde{B}^a_\mu$$

$U(x) \in SU(N)$ for every x.

$U^+ U = \mathbb{1}$, $\det U = 1$.

$U(x)\, T'^a\, U(x)^{-1} = R_{ba}(x)\, T'^b \rightarrow$ defines $R_{ba}(x)$

$i\partial_\mu U\, U^+ = \sum_a T'^a \alpha^a_\mu(x) \rightarrow$ defines $\alpha^a_\mu(x)$.

For $U(N)$ we drop the requirements:- $\det U = 1$, $\mathrm{Tr}(T^a) = 0$

We are missing the analog of $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ term.

We have introduced any kinetic for B_μ & so the Lag. can't give rise to propagator of B_μ — the quadratic term gives rise to the propagator

First we need the analog of $F_{\mu\nu}$.

Go back to electrodynamics.

$$D_\mu \psi = (\partial_\mu - ie A_\mu)\psi$$

$$[D_\mu, D_\nu] = -ie F_{\mu\nu}$$

Q) Is there an analogous construction here?

What is it that we are trying to achieve by this? In QED, $[D_\mu, D_\nu]$ was gauge inv. & that helped us to identify $F_{\mu\nu}$

[First step → construct the analog of D_μ]

We should be able to write

$$\left[\bar\psi^k(x)\,(i\gamma^\mu \partial_\mu - m)\,\psi^k(x) + \bar\psi^l(x)\gamma^\mu \psi^m(x)\, B^a_\mu(x)\, T^a_{lm}\right]$$

as $\bar\psi^k \left\{ i\gamma^\mu (D_\mu \psi)^k - m\psi^k \right\}$

$$\Rightarrow (D_\mu \psi)^k = \partial_\mu \psi^k - i B^a_\mu (T^a)_{km} \psi^m$$

\# D_μ is an operator [but it is a strange kind of operator b'cos it has ∂_μ but it is also a matrix]

$$(D_\mu)_{kl} = \delta_{kl}\, \partial_\mu - i B^a_\mu T^a_{lk}$$

$$(D_\mu)_{kl}\, \psi^l = \partial_\mu \psi^k - i \underbrace{B^a_\mu T^a_{kl}}_{(S_\mu)_{kl}} \psi^l$$

The whole thing can be written in this form →

$$\begin{pmatrix} \frac{\partial}{\partial x^\mu} & & 0 \\ & \partial/\partial x^\mu & \\ 0 & \cdots & \partial/\partial x^\mu \end{pmatrix} - i S_\mu$$

↳ also a sq. matrix

[Let us now see how $(\mathcal{D}_\mu \psi)$ transforms]:

$$\mathcal{D}_\mu \psi \xrightarrow[\text{transfn}]{\text{gauge}}$$

$$\partial_\mu \left(U_{kl}(x)\, \psi^l(x) \right) - i \left(R_{ab}(x)\, B_\mu^{\,b}(x) - \alpha_\mu^{\,a}(x) \right) T^a_{kl}\, U_{lm}(x)\, \psi^m(x)$$

Ex. S.T. $U_{kl}(x) \left(\mathcal{D}_\mu \psi \right)^l$ is the final result

$$(\tilde{\mathcal{D}}_\mu \tilde{\psi})^k = (\partial_\mu U_{kl})\psi^l + U_{kl}(\partial_\mu \psi^l) - i (U T^b U^{-1})_{kl} B_\mu^{\,b} U_{lm} \psi^m + i^2 ((\partial_\mu U) U^{-1})_{kl} U_{lm} \psi^m$$

Now, $U_{kl}(x) \left(\mathcal{D}_\mu \psi \right)^l \equiv \tilde{\mathcal{D}}_\mu(x)\, \tilde{\psi}(x)$ [By defn]

$$= U_{kl}(\partial_\mu \psi^l) + \cancel{(\partial_\mu U_{kl}) \psi^l} - i U_{k\alpha} (T^b)_{\alpha\beta} \delta_{\beta m} B_\mu^{\,b} \psi^m - \cancel{(\partial_\mu U)_{k\alpha} \delta_{\alpha m} \psi^m}$$

$$= U_{kl}(\partial_\mu \psi^l) - i U_{kl} (B_\mu^{\,b} T^b \psi)^l$$

$$= U_{kl} \left[\partial_\mu \psi^l - i B_\mu^{\,b} (T^b)_{lm} \psi^m \right]$$

$$= U_{kl} (\mathcal{D}_\mu \psi)^l \quad \text{(proved)}$$

$$\Rightarrow \tilde{\mathcal{D}}_\mu \tilde{\psi} = U \mathcal{D}_\mu \psi$$

But $\tilde{\psi} = U \psi$

$$\therefore \tilde{\mathcal{D}}_\mu U \psi = U \mathcal{D}_\mu \psi$$

ψ is an arbitrary vector

$$\Longrightarrow \tilde{\mathcal{D}}_\mu U = U \mathcal{D}_\mu$$

$$\Rightarrow \boxed{\tilde{\mathcal{D}}_\mu = U \mathcal{D}_\mu U^{-1}}$$

[This relation essentially encodes how the gauge fields B_μ transform — We could have gotten $\mathcal{D}_\mu$ transfn directly from B_μ transfn]

$$\tilde{\mathcal{D}}_\mu \tilde{\mathcal{D}}_\nu = U \mathcal{D}_\mu \mathcal{D}_\nu U^{-1}$$

$$\Rightarrow [\tilde{\mathcal{D}}_\mu, \tilde{\mathcal{D}}_\nu] = U [\mathcal{D}_\mu, \mathcal{D}_\nu] U^{-1}$$

[In QED, $[D_\mu, D_\nu]$ turned out to be the gauge field strength — Let's see what we get in this case] $\longrightarrow$

$$D_\mu D_\nu \psi^k = \partial_\mu \left(D_\nu \psi^k\right) - i B_\mu^a T^a_{kl} (D_\nu \psi)^l$$

$$= \partial_\mu \left(\partial_\nu \psi^k - i B_\nu^b T^b_{km} \psi^m\right)$$

$$- i B_\mu^a T^a_{kl} \left(\partial_\nu \psi^l - i B_\nu^b T^b_{lm} \psi^m\right)$$

$$= \partial_\mu \partial_\nu \psi^k - i \partial_\mu B_\nu^b T^b_{km} \psi^m$$

$$\underbrace{- i B_\nu^b T^b_{km} \partial_\mu \psi^m}_{\substack{b \to a \\ m \to l}} - i B_\mu^a T^a_{kl} \partial_\nu \psi^l$$

$$- B_\mu^a B_\nu^b T^a_{kl} T^b_{lm} \psi^m$$

$$= \partial_\mu \partial_\nu \psi^k - i \partial_\mu B_\nu^b T^b_{km} \psi^m - i B_\nu^a T^a_{kl} \partial_\mu \psi^l$$

$$- i B_\mu^a T^a_{kl} \partial_\nu \psi^l - B_\mu^a B_\nu^b T^a_{kl} T^b_{lm} \psi^m$$

Now, $[D_\mu, D_\nu] \psi = D_\mu D_\nu \psi - (\mu \longleftrightarrow \nu)$

$$= -i \overbrace{\left(\partial_\mu B_\nu^b - \partial_\nu B_\mu^b\right) T^b_{km}}^{b \to c} \psi^m$$

$$- B_\mu^a B_\nu^b \left(T^a T^b\right)_{km} \psi^m$$

$$\underbrace{+ B_\nu^a B_\mu^b \left(T^a T^b\right)_{km} \psi^m}_{a \longleftrightarrow b}$$

Need to consider only the 2nd & last terms — the others drop out in the commutator

$\rightarrow -i B_\nu^a T^a_{kl} \partial_\mu \psi^l$
$-i B_\mu^a T^a_{kl} \partial_\nu \psi^l$
is together sym. under $\mu \leftrightarrow \nu$ & drops out

$$= -i \left(\partial_\mu B_\nu^c - \partial_\nu B_\mu^c\right) T^c_{km} \psi^m$$

$$- B_\mu^a B_\nu^b \left(T^a T^b\right)_{km} \psi^m$$

$$+ B_\nu^b B_\mu^a \left(T^b T^a\right)_{km} \psi^m$$

$$\Rightarrow [D_\mu, D_\nu]\psi_k$$
$$= -i\left(\partial_\mu B_\nu^c - \partial_\nu B_\mu^c\right) T^c_{km}\psi^m$$
$$- B_\mu^a B_\nu^b \left[T^a, T^b\right]_{km}\psi^m$$

If T^a & T^b are hermitian, then $[T^a, T^b]$ is traceless & anti-hermitian.

__Proof__ :- $[T^a, T^b]^\dagger = \left(T^a T^b - T^b T^a\right)^\dagger$

$$= (T^b)^\dagger (T^a)^\dagger - (T^a)^\dagger (T^b)^\dagger$$
$$= T^b T^a - T^a T^b$$
$$= -[T^a, T^b]$$

Hence, $[T^a, T^b]$ is anti-hermitian

[[$i[T^a, T^b]$ is Hermitian]]

$$\mathrm{Tr}\left(T^a T^b - T^b T^a\right) = \mathrm{Tr}(T^a T^b) - \mathrm{Tr}(T^a T^b) = 0$$

[T^a constitutes a basis for traceless Herm. matrices — $\therefore$ iT^a constitutes a basis for traceless anti-Herm. matrices]

$$\Rightarrow [T^a, T^b] = i f^{abc} T^c$$ __for some constants f^{abc}__

[c is summed over]

$\therefore\ [\mathcal{D}_\mu, \mathcal{D}_\nu]\psi_k$

$$= -i\underbrace{\left[\partial_\mu B_\nu{}^c - \partial_\nu B_\mu{}^c + f^{abc} B_\mu{}^a B_\nu{}^b\right]}_{G^c_{\mu\nu}\ (\to\ \text{analog of } F_{\mu\nu})} T^c_{km}\psi^m$$

$$\left\{[\mathcal{D}_\mu, \mathcal{D}_\nu]\psi\right\}^k = -i\, G^c_{\mu\nu}\,(T^c)_{km}\psi^m$$

where $G^c_{\mu\nu} = \partial_\mu B^c_\nu - \partial_\nu B^c_\mu + f^{abc} B_\mu^a B_\nu^b$

[This is simply a defn. $G^c_{\mu\nu}$ — we haven't explored its properties yet & we don't know how it ~~will~~ will give rise to a gauge-inv. kinetic term]

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -i G^c_{\mu\nu} T^c \to \text{operator relation}$$

The commutator is a pure matrix — it's no longer a differential operator

↓

This is a crucial information bcos the analog of $F_{\mu\nu}$ should be a fn. & not an op.

Suppose under gauge ~~tr~~ trs.

$$G^c_{\mu\nu} \to \tilde{G}^c_{\mu\nu}$$

[[Take a short cut to find $\tilde{G}^c_{\mu\nu}$ using $[\tilde{\mathcal{D}}_\mu, \tilde{\mathcal{D}}_\nu] = U[\mathcal{D}_\mu, \mathcal{D}_\nu]U^{-1}$]]

We have,

$$[\tilde{\mathcal{D}}_\mu, \tilde{\mathcal{D}}_\nu] = -i\tilde{G}^c_{\mu\nu} T^c$$

this tells us that

$$\tilde{G}^c_{\mu\nu} T^c = U G^c_{\mu\nu} T^c U^{-1}$$

$$= G^c_{\mu\nu} R_{dc} \text{ (expd}$$

$$\Rightarrow \tilde{G}^{c}_{\mu\nu} T'^{c} = G^{d}_{\mu\nu} R_{cd}(x) T'^{c}$$

$$\Rightarrow \boxed{\tilde{G}^{c}_{\mu\nu} = R_{cd}(x)\, G^{d}_{\mu\nu}}$$

[We are almost there but not quite, bcos $\tilde{G}^{c}_{\mu\nu}$ is still not gauge-invariant]

Consider the following combination :-

$$Tr\left(G^{a}_{\mu\nu} T'^{a} G^{b\mu\nu} T'^{b}\right)$$

$$\rightarrow Tr\left(U G^{a}_{\mu\nu} T'^{a} \underbrace{U^{-1} U}_{\mathbb{1}} G^{b\mu\nu} T'^{b} U^{-1}\right)$$

$$= Tr\left(\underbrace{U^{-1} U}_{\mathbb{1}} G^{a}_{\mu\nu} T'^{a} G^{b\mu\nu} T'^{b}\right)$$

$$= Tr\left(G^{a}_{\mu\nu} T'^{a} G^{b\mu\nu} T'^{b}\right)$$

[This is an e.g. of a gauge-inv. action
→ it is the analog of Maxwell action]

$$Tr\left(T'^{a} T'^{b}\right) G^{a}_{\mu\nu} G^{b\mu\nu}$$

The full action :-

$$S = -\frac{1}{2g^2}\int d^4x\, Tr\left(T'^{a} T'^{b}\right) G^{a}_{\mu\nu} G^{b\mu\nu} + \int d^4x\, \bar{\psi}^{k}(x)\left\{ i\gamma^{\mu}(D_{\mu}\psi)^{k} - m\psi^{k}\right\}$$

$\boxed{g \text{ : constant}}$

[Given sth. is gauge-inv., multiply it by a const. & that will also be a gauge-inv. — for the second term you can't multiply by a constant bcos you can absorb it in ψ — for the first term you can't absorb it bcos of the term $f^{abc} B_\mu^a B_\nu^b$ in $G^c_{\mu\nu}$] → can't be absorbed in f^{abc} bcos changing f^{abc} gives a physically diff. theory – f^{abc} is det. by T^a's

[Could have multiplied by $g(x)$ within the integral — it will be gauge inv. but not translation-invariant]

Shorthand matrix notation :-

$$\underset{\sim}{B}_\mu \equiv B_\mu^a T^a$$

$$\underset{\sim}{G}_{\mu\nu} \equiv G^a_{\mu\nu} T^a$$

$$\underset{\sim}{G}_{\mu\nu} = \left[\partial_\mu B^c_\nu - \partial_\nu B_\mu^c + f^{abc} B_\mu^a B_\nu^b\right] T^c$$

$$= \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu - i\left[\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu\right]$$

(This is a more compact notation)

Similarly,

$$D_\mu \psi = \partial_\mu \psi - i \underset{\sim}{B}_\mu \psi$$

...ary ...ine → N×N matrix valued ...peration

Now, $S = \frac{-1}{2g^2}\int d^4x \; \mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu}\underset{\sim}{G}^{\mu\nu}\right)$

$+ \int d^4x \; \psi^* \left\{ i\gamma^\mu (D_\mu\psi)^* - m\psi^* \right\}$

Gauge transfn :-

$B_\mu^a \to R_{ab} B_\mu^b - \alpha_\mu^a \equiv \tilde{B}_\mu^a$

$\underset{\sim}{B}_\mu \to T^a R_{ab} B_\mu^b - \alpha_\mu^a T^a$

$\underset{\sim}{B}_\mu \to \underset{\sim}{\tilde{B}}_\mu = \tilde{B}_\mu^a T^a = T^a R_{ab} B_\mu^b - \alpha_\mu^a T^a$

$= U(x)\, T^b\, U^{-1}(x)\, B_\mu^b$

$- i\partial_\mu U \, U^{-1}$

$\Rightarrow \underset{\sim}{\tilde{B}}_\mu = U(x)\, \underset{\sim}{B}_\mu\, U(x)^{-1} - i\partial_\mu U\, U^{-1}$

$\underset{\sim}{\tilde{G}}_{\mu\nu} = \tilde{G}_{\mu\nu}^a T^a = U\, G_{\mu\nu}^c\, T^c\, U^{-1}$

$= U\, \underset{\sim}{G}_{\mu\nu}\, U^{-1}$

(∴ $\mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu}\underset{\sim}{G}^{\mu\nu}\right)$ is gauge-invariant)

Ex. Using the transformation law of $\underset{\sim}{B}_\mu$ & the expression for $\underset{\sim}{G}_{\mu\nu}$ in terms of $\underset{\sim}{B}_\mu$, find the gauge trs. law of $\underset{\sim}{G}_{\mu\nu}$.

14/9/06

Action with local SU(N) symmetry:-

$$S = S_{gauge} + S_{fermions}$$

$$S_{gauge} = -\frac{1}{2g^2}\int d^4x \, \mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu}\underset{\sim}{G}^{\mu\nu}\right)$$

$$S_{fermions} = \int d^4x \, \bar{\Psi}\left(i\gamma^\mu \underset{\sim}{D}_\mu - m\right)\Psi$$

where $\underset{\sim}{G}_{\mu\nu} = \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu - i\left[\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu\right]$

$\underset{\sim}{D}_\mu \Psi = \partial_\mu \Psi - i\underset{\sim}{B}_\mu \Psi$ | $\underset{\sim}{B}_\mu = B^a_\mu T^a$

S_{gauge} & $S_{fermions}$ are separately gauge-inv.
- like QED you can formulate an indep. theory of gauge fields, like the free Maxwell theory
- But unlike in QED, where $F_{\mu\nu}$ was linear in $\partial_\mu A_\nu$, you will get cubic & quartic coupling of gauge field & it is not a free theory (non-trivial theory & dynamics).

Gauge transformation

$$\underset{\sim}{B}_\mu \longrightarrow U\underset{\sim}{B}_\mu U^{-1} - i\,\partial_\mu U\,U^{-1}$$

$$\Psi \longrightarrow U\Psi$$

$U(x)$: A unitary matrix of det. 1 for every x

Consider N complese scalar fields
$\phi_1, \phi_2, \ldots, \phi_N$.

$$\phi_i = \frac{1}{\sqrt{2}}(\phi_{iR} + i\phi_{iI})$$

Q) How do we construct a gauge invariant action with local gauge transformation

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} \longrightarrow U(x)\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} \quad ?$$

Free action (for the ϕ_i-fields):-

$$\int d^4x \left(-\partial_\mu \phi_i^* \partial^\mu \phi_i - \sum_{i=1}^{N} m_i^2 \phi_i^* \phi_i\right)$$

[Even for global invariance, it won't be inv. if m_i's are diff. — so m_i's should be the same]

$$\phi_i \longrightarrow U_{ij}(x)\,\phi_j(x)$$ (The action isn't inv. under a local gauge transfn)

$$\cancel{D_\mu \phi_i = \partial_\mu \phi_i}$$

$$D_\mu \phi = \partial_\mu \phi - i B_\mu \phi$$

$$(D_\mu \phi)_i = \partial_\mu \phi_i - i B_\mu^a T^a_{ik} \phi_k$$

$$D_\mu \phi \longrightarrow U D_\mu \phi \text{ under } \phi \to U\phi$$

ϕ & ψ (fermion) transform similarly - the fact that ψ carries a Dirac index doesn't matter

$$S_{scalar} = \int d^4x \left\{ -(D_\mu \phi)^\dagger (D_\mu \phi) - m^2 \phi^\dagger \phi \right\}$$

where $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}$ & $\phi^\dagger = (\phi_1^* \ldots \phi_N^*)$

Note → The gauge kinetic must always be there

(Representation means) →

for every $U \in G$, there is a $k \times k$ matrix $R(U)$ such that

$$R(U_1 U_2) = R(U_1) R(U_2)$$

⇒ (this gives a) → k-dimensional representation

If $R_G(U)$ is a unitary matrix for every $U \in G$, then it is a unitary reprentation.

[This can be done for any $SU(N)$]

(You have to choose the repn properly so that you can explain ~~the~~ a particular experiment properly — theoretically, you are free to choose any repn)

[For $SU(N)$, N-dim. repn. is called the fundamental repn. — no reason why all fermions & scalars will transform acc. to the fundamental repn.] [Transfn. of gauge fields can't ~~transform~~ depend on which repn. we choose]

Q) How do we construct a gauge invariant action for fermions or scalar fields which do not transform in the fundamental representation of $SU(N)$?

(We can have a fn. where some ϕ transform in repn. & some in other)

Postpone the answer for a while

Infinitesimal gauge transformation:-

$\xrightarrow[\text{are}]{\text{these}}$ gauge trs. for which the transformed fields are infinitesimally close to their original values.

$$U = \mathbb{1} \underbrace{-}_{\text{convention}} i \underbrace{\epsilon}_{\text{small number}} \Omega \rightarrow N \times N \text{ matrix}$$

$\boxed{\epsilon \text{ is fixed}}$

$\boxed{U(x) = \mathbb{1} - i\epsilon\Omega(x)}$

Here we working with fundamental repr. ↓ so U is $N \times N$

$$U^{+}U = \mathbb{1} \quad (\text{bcos } U \text{ is unitary})$$

$$\Rightarrow (\mathbb{1} + i\epsilon\Omega^{+})(\mathbb{1} - i\epsilon\Omega) = \mathbb{1}$$

$$\Rightarrow i\epsilon(\Omega^{+} - \Omega) + O(\epsilon^2) = 0 \Rightarrow \boxed{\Omega^{+} = \Omega}$$

$$\det U = 1$$

$$\det(\mathbb{1} - i\epsilon\Omega) = 1 - i\epsilon \, \mathrm{Tr}(\Omega) + O(\epsilon^2)$$

$$\Rightarrow \boxed{\mathrm{Tr}(\Omega) = 0}$$

$$\det \begin{bmatrix} 1 - i\epsilon\Omega_{11} & -i\epsilon\Omega_{12} & \cdots \\ -i\epsilon\Omega_{21} & 1 - i\epsilon\Omega_{22} & \cdots \\ & & \ddots \end{bmatrix}$$

$$= (1 - i\epsilon\Omega_{11}) \cdots (1 - i\epsilon\Omega_{NN}) + O(\epsilon^2)$$

$$= 1 - i\epsilon(\Omega_{11} + \Omega_{22} + \cdots + \Omega_{NN}) + O(\epsilon^2)$$

$$\epsilon\Omega = \sum_a \underbrace{\epsilon^a}_{\text{real infinitesimal parameters}} T^a \rightarrow \text{complete set of linearly independent traceless hermitian matrices}$$

An infinitesimal gauge trs. is generated by

$$U(x) = \mathbb{1} - i\epsilon^a(x) T^a$$

[Finite gauge trs.

$$U(x) = \exp(-i\lambda_a(x) T^a)$$

$\hookrightarrow$ arbitrary fns of x]

exp^n can't be always done as it doesn't always give a faithful repn.

A convenient normalization :-

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta_{ab}$$

~~Exp~~ Under this infinitesimal gauge transfm,

$$\psi^* \longrightarrow (U\psi)^* = \{(\mathbb{1} - i\epsilon^a(x) T^a)\psi\}^*$$

$$= \psi^* - i\epsilon^a(x)(T^a\psi)^*$$

$$\delta\psi^* = -i\epsilon^a(x)(T^a\psi)^*$$

$\because T$ is Herm.

Taking the C.C. $\longrightarrow$

$$\delta\bar{\psi}^k = i\epsilon^a(x) T^a_{lk}\bar{\psi}^l$$

(∴ there is no change in the above result at all in the fundamental repn) if we consider scalar fields Φ

$$B^a_\mu \longrightarrow B^a_\mu + \delta B^a_\mu$$

$$= R_{ab} B^b_\mu(x) - \alpha^a_\mu(x)$$

$U = \mathbb{1} - i\epsilon^a T^a$

$U^{-1} = \mathbb{1} + i\epsilon^a T^a$

$$U T^a U^{-1} = R_{ba} T^b$$

$$\left(\mathbb{1} - i\epsilon^b(x) T^b\right) T^a \left(1 + i\epsilon^c(x) T^c\right)$$

$$= \underline{\mathbb{1}} - i\epsilon^b(x) T^b T^a + i\epsilon^c(x) T^a T^c + \mathcal{O}(\epsilon^2)$$

$$= \underline{\mathbb{1}} - i\epsilon^b(x) T^b T^a + i\epsilon^b(x) T^a T^b + \mathcal{O}(\epsilon^2)$$

$$\Rightarrow \quad U T^a U^{-1} = T^a + i\epsilon^b(x)\,\underbrace{[T^a, T^b]}_{i f^{abc} T^c}$$

$$= T^a - f^{abc}\,\epsilon^b(x)\,T^c$$

But $U T^a U^{-1} = R_{ba} T^b = R_{ca} T^c$

$$\therefore \quad R_{ca} = \delta_{ca} - f^{abc}\,\epsilon^b(x)$$

[Comparing the two]

$$\alpha_\mu^a(x)\, T^a = i\partial_\mu U\, U^{-1}$$

$$= \partial_\mu \epsilon^a(x)\, T^a \left(1 + i\epsilon^b(x) T^b + O(\epsilon^2)\right)$$

$$= \partial_\mu \epsilon^a(x)\, T^a + O(\epsilon^2)$$

Hence $\Rightarrow \quad \alpha_\mu^a(x) = \partial_\mu \epsilon^a(x) + O(\epsilon^2)$

$$B_\mu^a + \delta B_\mu^a = \left(\delta_{ca} - f^{cba}\epsilon^b(x)\right) B_\mu^c - \partial_\mu \epsilon^a(x)$$

$$= B_\mu^a - f^{cba}\epsilon^b(x) B_\mu^c - \partial_\mu \epsilon^a(x)$$

$$\Rightarrow \quad \delta B_\mu^a(x) = -\partial_\mu \epsilon^a(x) - f^{cba}\epsilon^b(x)\, B_\mu^c(x)$$

Gauge invariance

$\delta S = 0$ to order ϵ under the transformations on B_μ^a, $\psi^\star$ & $\phi^\star$.

Ex: check ~~that~~ this directly.

We will need :-

$$f^{abc} f^{cde} + f^{cae} f^{bdc} + f^{dac} f^{cbe} = 0$$

This follows from

$$[[T^a, T^b], T^d] + [[T^b, T^d], T^a] + [[T^d, T^a], T^b] = 0$$

Arbitrary representation (unitary)

- $R_a(U)$: a $k \times k$ matrix representing $U \in SU(N)$

$R_a(U_1 U_2) = R_a(U_1) R_a(U_2)$ etc.

$(R_a(U))^\dagger R_a(U) = \mathbb{1}$ (consequence of unitarity of the representation)

Consider $U = \mathbb{1} - i\epsilon^a T^a$

$\boxed{U \text{ is } N \times N \text{ but } R_a(U) \text{ is } k \times k}$

$$\Rightarrow U = (\mathbb{1} - i\epsilon^1 T^1)(\mathbb{1} - i\epsilon^2 T^2) \cdots (1 - i\epsilon^{N^2-1} T^{N^2-1}) + O(\epsilon^2)$$

$$R_a(U) = R_a(\mathbb{1} - i\epsilon^1 T^1) R_a(\mathbb{1} - i\epsilon^2 T^2) \cdots R_a(\mathbb{1} - i\epsilon^{N^2-1} T^{N^2-1}) + O(\epsilon^2)$$

Now,

$R_a(\mathbb{1} - i\epsilon^1 T^1)$ [Note $\rightarrow R_a(\mathbb{1}) = \mathbb{1}$]

$= \mathbb{1}_{k\times k} - i\epsilon^1 R_A(T^1)$

$\longrightarrow$ defines $R_A(T^1)$

$\boxed{\text{For every generator } T^a, \text{ there will be } \cancel{\text{a matrix}} \text{ a repr. matrix } R_A(T^a)}$

$$R_a(U) = (\mathbb{1} - i\epsilon^1 R_A(T^1))(\mathbb{1} - i\epsilon^2 R_A(T^2)) \cdots$$

$$= \mathbb{1} - i\epsilon^a R_A(T^a) + O(\epsilon^2)$$

Unitarity of $R_a(U) \Longrightarrow R_A(T^a)$ is Hermitian

$\boxed{R_A(T^a) \text{ isn't in the group sense}}$

Ex: $[R_A(T^a), R_A(T^b)] = i f^{abc} R_A(T^c)$

↳ This follows from
$$R_a(\mathbb{1} - i\epsilon^a T^a)\, R_a(\mathbb{1} - i\eta^b T^b)\, R_a(\mathbb{1} + i\epsilon^c T^c)$$
$$= R_a\big((\mathbb{1} - i\epsilon^a T^a)(\mathbb{1} - i\eta^b T^b)(\mathbb{1} + i\epsilon^c T^c)\big)$$
Compare the order $\eta^a \epsilon^b$ term on both sides.

Suppose we have a ~~b~~ A fermions which transform as
$$\cancel{\psi^k \to (R_a(U_x))\ \psi^l}$$
$$\psi^k \to (R_a U(x))_{kl} \psi^l$$
under an SU(N) gauge trs. $U(x)$.

Claim :- Define
$$(\underset{\sim}{D}_\mu \psi)^m = \partial_\mu \psi^m - i B_\mu^a (R_A(T^a))_{mn} \psi^n$$

Then $S_{fermion} = \int \bar{\psi}^m \left(i\gamma^\mu (\underset{\sim}{D}_\mu \psi)^m - m\psi^m\right) d^4x$

is gauge invariant.

Ex. check this for infinitesimal gauge trs.

$$R_G\left(1-i\sum_a \epsilon_a T^a\right) = \mathbb{1} - i\sum_a \epsilon_a R_A(T^a)$$

$$\parallel$$

$$\mathbb{1} - iR_A\left(\sum_a \epsilon^a T^a\right)$$

Comparing, we get

$$R_A\left(\sum_a \epsilon_a T^a\right) = \sum_a \epsilon^a R_A(T^a)$$

$$R_G(U)\ R_A(T^a)\ R_G(U^{-1})$$

$$= R_A(U T^a U^{-1}) \longrightarrow$$ have to prove & use this prove the inv. of S_{f}

for finite gauge trans.

$$D_\mu\psi \longrightarrow R_G(U)\, D_\mu \psi$$

$$\phi^\dagger \begin{pmatrix} m_1^2 & & & \\ & m_2^2 & & \\ & & \ddots & \\ & & & m_N^2 \end{pmatrix} \phi$$

$$\phi^\dagger U^\dagger \begin{pmatrix} m_1^2 & & & \\ & m_2^2 & & \\ & & \ddots & \\ & & & m_N^2 \end{pmatrix} U\phi$$

$$R_G(\alpha) = \exp\left(-i\chi^a(x)\, R_A(T^a)\right)$$

For $U = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $U = \mathbb{1}$ & so we won't get a faithful repn.

$$R_G\left(1 - i\sum_a \epsilon_a T^a\right) = \mathbb{1} - i\sum_a \epsilon_a R_A(T^a)$$

$$\parallel$$

$$\mathbb{1} - i R_A\left(\sum_a \epsilon^a T^a\right)$$

comparing, we get

$$R_A\left(\sum_a \epsilon_a T^a\right) = \sum_a \epsilon^a R_A(T^a)$$

$$R_G(U)\, R_A(T^a)\, R_G(U^{-1})$$

$$= R_A(U T^a U^{-1}) \longrightarrow$$ have to prove & use this prove the inv. of S_{ferm}

for finite gauge trans.

$$D_\mu \psi \longrightarrow R_G(U)\, D_\mu \psi$$

$$\phi^\dagger \begin{pmatrix} m_1^2 & & & \\ & m_2^2 & & \\ & & \ddots & \\ & & & m_N^2 \end{pmatrix} \phi$$

$$\phi^\dagger U^\dagger \begin{pmatrix} m_1^2 & m_2^2 & \cdots & m_N^2 \end{pmatrix} U\phi$$

$$R_G(U) = \exp\left(-i\lambda^a(x)\, R_A(T^a)\right)$$

For $U = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $U = \mathbb{1}$ &) we won't get a faithful repn.

Pfaffian (Definition)

The Pfaffian is an analog of the determinant that is defined only for a $2n \times 2n$ antisymmetric matrix. It is a polynomial of degree n in elements of the matrix, such that its square is equal to the determinant of the matrix.

The Pfaffian is applied in the generalized Gauss-Bonnet theorem.

Examples

$$\mathrm{Pf}\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a,$$

$$\mathrm{Pf}\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + dc.$$

Standard definition

Let

$$A = \begin{bmatrix} 0 & a_{1,2} & .. & a_{1,2n} \\ -a_{1,2} & 0 & .. & a_{2,2n} \\ .. & .. & .. & .. \\ -a_{2n,1} & -a_{2n,2} & .. & 0 \end{bmatrix}.$$

Let Π be the set of all partition of $\{1, 2, .., 2n\}$ into pairs of elements $\alpha \in \Pi$, can be represented as

$$\alpha = \{(i_1, j_1), (i_2, j_2), .., (i_n, j_n)\}$$

with $i_k < j_k$ and $i_1 < i_2 < \cdots < i_n$, let

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & .. & 2n \\ i_1 & j_1 & i_2 & j_2 & .. & j_n \end{bmatrix}$$

be a corresponding permutation and let us define $sgn(\alpha)$ to be the signature of π; clearly it depends only on the partition α and not on the particular choice of π. Given a partition α as above let us set $a_\alpha = a_{i_1,j_1} a_{i_2,j_2} .. a_{i_n,j_n}$, then we can define the Pfaffian of A as

$$\mathrm{Pf}(A) = \sum_{\alpha \in \Pi} sgn(\alpha) a_\alpha.$$

Alternative definition

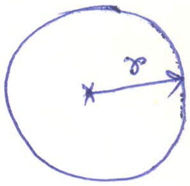
One can asociate to any antysimmetric $2n \times 2n$ matrix $A = \{a_{ij}\}$ a bivector : $\omega = \sum_{i<j} a_{ij} e_i \wedge e_j$ in a basis $\{e_1, e_2, ..., e_{2n}\}$ of $\mathbb{R}^{2n}$, then

$$\omega^n = n! \mathrm{Pf}(A) e_1 \wedge e_2 \wedge \ldots \wedge e_{2n},$$

P.T. For n dim.,

$$dV = r^{n-1}\, dr\, d\Omega_n$$

In n-D, on a sphere, $x_1^2 + x_2^2 + \cdots + x_n^2 = r^2$ — (1)



$V \propto r^n$

$A \propto r^{n-1}$

$$\begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_2}{\partial r} & \cdots & \frac{\partial x_n}{\partial r} \\ \frac{\partial x_1}{\partial \theta_1} & \frac{\partial x_2}{\partial \theta_1} & \cdots & \frac{\partial x_n}{\partial \theta_1} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_1}{\partial \theta_{n-1}} & \frac{\partial x_2}{\partial \theta_{n-1}} & \cdots & \frac{\partial x_n}{\partial \theta_{n-1}} \end{vmatrix}$$

$$J = \left| \quad \right|$$

$$dx_1\, dx_2 \cdots dx_n = J\, dr\, d\Omega_n$$

$$\boxed{\frac{\partial r}{\partial x_1} = \frac{x_1}{r}}$$

$$x_1 \frac{\partial x_1}{\partial r} = r \Rightarrow \boxed{\frac{\partial x_1}{\partial r} = r/x_1}$$

from (1),

$$x_1 = r\, f_1(\theta_1, \ldots, \theta_{n-1})$$

$$\cdots$$

$$x_n = r\, f_n(\theta_1, \ldots, \theta_{n-1})$$

Such that $\sum f_i^2(\theta_1, \theta_2, \ldots, \theta_{n-1}) = 1$

$\therefore$ J is a sum of terms $\propto \frac{r}{x_i} r^{n-1} \propto \underline{\underline{r^{n-1}}}$

'Connected' → fully connected, i.e., with no vacuum bubbles, & all external legs connected to each other.

Amputation → Starting from the tip of each ext. leg, find the last pt. at which the diag. can be cut by removing a single propagator, such that this operation separates the leg from the rest of the diag.

1PI diag → Any diag. that can't be split into 2 by removing a single line.

Problem Set 1: Date Due: September 15, 2006

$i\mathcal{M}$ = sum of all connected, amputated diag.

1. Consider the action of a free scalar field theory:

$$S = -\frac{1}{2}\int d^4x \left[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right]$$

where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$. Now introduce a new field χ through the relation:

$$\phi = \chi + \frac{1}{2}\lambda\chi^2$$

(a) Express the action in terms of χ and its derivatives.

(b) Now forget about the description in terms of the ϕ field altogether, and treat this as a new field theory of the field χ. Derive the momentum space Feynman rules for calculating correlation functions for the χ field.

(c) Calculate the three and four point correlation functions:

$$\langle\tilde{\chi}(k_1)\tilde{\chi}(k_2)\tilde{\chi}(k_3)\rangle_c$$

and

$$\langle\tilde{\chi}(k_1)\tilde{\chi}(k_2)\tilde{\chi}(k_3)\tilde{\chi}(k_4)\rangle_c$$

to order λ and λ^2 respectively. Here

$$\langle\prod_{i=1}^{n}\tilde{\chi}(k_i)\rangle \equiv \int[\mathcal{D}\tilde{\chi}(k)]e^{iS}\prod_{i=1}^{n}\tilde{\chi}(k_i)\Big/\int[\mathcal{D}\tilde{\chi}(k)]e^{iS} \qquad (1)$$

and $\langle\ \rangle_c$ denotes sum over only the connected Feynman diagrams.

Answer for S-matrix elements shouldn't depend on what field variables you are choosing — the original action being non-int., it's no surprise that you get zero answer

(d) Calculate the S-matrix element for the scattering of two χ particles with momenta k_1 and k_2 to go into two χ particles of momenta p_1 and p_2 to order λ^2. In this calculation you can ignore the forward scattering amplitude (*i.e.* the contribution from disconnected Feynman diagrams).

You can simplify the answer by expressing the final result using the Mandelstam variables:

$$s = -(k_1+k_2)^2, \qquad t = -(k_1-p_1)^2, \qquad u = -(k_1-p_2)^2$$

Note that only two of these variables are independent since we have a relation:

$$s + t + u = 4m^2$$

which you should be able to prove. Thus the final result should be expressed as a function of only two of these three variables (say of s and t).

2. Consider an action describing a vector field A_μ and a scalar field ϕ:

$$S = \int d^4x \Big[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + c\eta^{\mu\nu}A_\mu\partial_\nu\phi \Big]$$

where $F_{\mu\nu} \equiv (\partial_\mu A_\nu - \partial_\nu A_\mu)$ and c is a constant. Calculate the two point correlation functions

(a) $\langle \widetilde{A}_\mu(k_1)\widetilde{A}_\nu(k_2)\rangle$

(b) $\langle \widetilde{A}_\mu(k_1)\widetilde{\phi}(k_2)\rangle$

(c) $\langle \widetilde{\phi}(k_1)\widetilde{\phi}(k_2)\rangle = 0$ becos the cofactor is zero (as it has a zero e.value)

by defining these correlation functions through the usual path integral formulation as given *e.g.* in eq.(1) of problem 1.

Hint: In solving this problem you may need to invert a 5×5 matrix. ~~You can try to use Lorentz covariance to simplify the analysis.~~

Quantum Field Theory

1. Consider an n component real scalar field $(\phi_1, \ldots \phi_n)$ with action

$$S = \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu} \sum_{\alpha=1}^{n} \partial_\mu\phi_\alpha\partial_\nu\phi_\alpha - V(\phi_1, \ldots \phi_n) \right]$$

where

$$V(\phi_1, \ldots \phi_n) = -\mu^2 \sum_{\alpha=1}^{k} \phi_\alpha\phi_\alpha + \lambda \left(\sum_{\alpha=1}^{n} \phi_\alpha\phi_\alpha \right)^2, \qquad k < n$$

for some real, positive constants μ and λ. We shall work at tree level and hence will not worry about renormalizability issues.

   (a) Find the full symmetry group as well as the unbroken symmetry group of the theory.

   (b) Find the masses of different scalar particles in the theory and show that the results are consistent with Goldstone's theorem.

   (c) If we couple the system to gauge fields so as to make all the global symmetries into gauge symmetries, how many massless and how many massive gauge fields shall we get?

2. Suppose A^a_μ are a set of non-abelian gauge fields in three dimensions with gauge group G. Let T^a's denote the generators of the group, and define $\mathbf{A}_\mu = \sum_a A^a_\mu T^a$. We also define $\epsilon^{\mu\nu\rho}$ to be the totally antisymmetric tensor in three dimensions, with $\epsilon^{012} = 1$.

   Consider the action

$$S = \int d^3x \, \epsilon^{\mu\nu\rho} \, Tr \left[\mathbf{A}_\mu \partial_\nu \mathbf{A}_\rho + \lambda \, \mathbf{A}_\mu \mathbf{A}_\nu \mathbf{A}_\rho \right]$$

   where Tr denotes trace and λ is a constant. Find the value of λ for which the action is gauge invariant after throwing away total derivative terms.

P.T.O.

3. Consider a four dimensional field theory with a complex scalar field ϕ and action

$$S = \int d^4x \left[-\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 \right]$$

(a) Calculate the renormalization constants $\widetilde{Z}_\phi$, Z_m and Z_λ to first order in the renormalized coupling constant λ_R.

(b) Calculate the β-function for λ_R to order λ_R^2.