6/10/06

Actions with local $U(N)$ or $SU(N)$ invariance

Gauge fields: B^a_μ — a runs over generators T^a of $SU(N)$ or $U(N)$

(vector field — denoted by index μ)

→ $N \times N$ Hermitian matrices
traceless for $SU(N)$

$$\underset{\sim}{B}_\mu = B^a_\mu T^a$$

$$\underset{\sim}{G}_{\mu\nu} = \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu - i[\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu]$$

$$S_{gauge} = -\frac{1}{2g^2}\int d^4x \, \mathrm{Tr}(\underset{\sim}{G}_{\mu\nu}\underset{\sim}{G}^{\mu\nu})$$

invariant under $\underset{\sim}{B}_\mu \longrightarrow U \underset{\sim}{B}_\mu U^{-1} - i(\partial_\mu U)U^{-1}$

[If we consider $\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(\underset{\sim}{G}_{\mu\nu}\underset{\sim}{G}_{\rho\sigma})$ we can check that this is total derivative]

In pert. theory, this doesn't contribute – it won't contribute to Feyn. rules

this is also gauge inv. – why not add this term? – though it has non-pert. effects it doesn't contribute to pert.

In component form,

$$B^a_\mu \longrightarrow R_{ab} B^b_\mu - \alpha^a_\mu$$

$$U T^a U^{-1} = R_{ba} T^b \rightarrow \text{defines } R_{ab}$$

$$i(\partial_\mu U)U^{-1} = \alpha^a_\mu T^a \longrightarrow \text{defines } \alpha^a_\mu$$

For the boundary term from the total deriv. to vanish, you require fields to vanish – but to Feyn. rules it won't contribute

$$S_{fermion} = \int d^4x\, \bar{\Psi}(i\gamma^\mu \underset{\sim}{D}_\mu - m)\Psi$$

where $\underset{\sim}{D}_\mu = \partial_\mu - i\underset{\sim}{B}_\mu$

Invariant under

$\Psi \longrightarrow U\Psi$ together with

gauge trs. of B^a_μ.

Here we have considered Ψ to be N-comp. — but it need not be necessarily N-comp.

Suppose $R_G(U)$ represent U & $R_A(T^a)$ represent T^a.

Then $R_G(\mathbb{1} + i\epsilon^a T^a) = \mathbb{1} + i\epsilon^a R_A(T^a)$

$\llcorner$ defines $R_A(T^a)$

Suppose $\Psi \to R(U)\Psi$

$\underset{\sim}{D}_\mu\Psi = \partial_\mu - i B^a_\mu R_A(T^a)$

Define the repr. of the generators by this formula, given a repr. of the group

$\int d^4x\, \bar{\Psi}(i\gamma^\mu \underset{\sim}{D}_\mu - m)\Psi$ is gauge invariant.

Suppose a set of scalar fields $\phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\end{pmatrix}$ transform as $\phi \to R_G(U)\phi$

Define :— $D_\mu\phi = \left(\partial_\mu - iB^a_\mu R_A(T^a)\right)\phi$

Then, $(D_\mu\phi)^\dagger(D^\mu\phi)$ is gauge invariant.

NOTE $\underset{\sim}{D}_\mu\Psi \to R_G(U)\underset{\sim}{D}_\mu\Psi$ shows that the action is invariant.

$$(D_\mu \Phi) \longrightarrow R_a(U)(D_\mu \Phi)$$

$S_{gauge} = -\frac{1}{2g^2}\int d^4x \; \mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu} \underset{\sim}{G}^{\mu\nu}\right)$

where $\underset{\sim}{G}_{\mu\nu} = \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu - i[\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu]$

To be more general, we have put the const. factor $\frac{-1}{2g^2}$ by hand — But what is its role?

$B^a_\mu = g A^a_\mu \longrightarrow$ defines A^a_μ

$$\underset{\sim}{F}_{\mu\nu} = \partial_\mu \underset{\sim}{A}_\nu - \partial_\nu \underset{\sim}{A}_\mu - ig[\underset{\sim}{A}_\mu, \underset{\sim}{A}_\nu]$$

<u>Check</u> :- $\underset{\sim}{G}_{\mu\nu} = g \underset{\sim}{F}_{\mu\nu}$

Substituting this in Sgauge, we get,

$$S_{gauge} = -1/2 \int d^4x \; \mathrm{Tr}\left(\underset{\sim}{F}_{\mu\nu} \underset{\sim}{F}^{\mu\nu}\right)$$

(We will find that $\rightarrow$ the Quadratic term has no g, the Cubic term has a g, & the Quartic term has g^2.

['g' gives a strength of the coupling — Non-abelian gauge theory from the beginning is interacting, unlike the Maxwell theory]

Quadratic term is used to get the propagator — so it's good that it has no 'g' — don't want any arbitrary const. in the prop.

Cubic term gives a 3-pt. vertex

Quartic term gives a 4-pt. vertex

[Naively, we could have thought that cubic & quartic coupling could have diff. indep. strengths — but gauge inv. has fixed them to be g & g^2 ———]

$\cancel{\mathcal{D}_\mu \Psi}$

Now, for $\Psi \to R(U)\Psi$,

$$ {}_m\mathcal{D}_\mu \Psi \Rightarrow \partial_\mu - i B^a_\mu R_A(T^a) = \partial_\mu - ig A^a_\mu R_A(T^a) $$

$$ \therefore \int d^4x\, \bar{\Psi}(i\gamma^\mu\, {}_m\mathcal{D}_\mu - m)\Psi $$

$$ \longrightarrow \underline{\int d^4x\, \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi} $$

$$ + g\int d^4x\, \bar{\Psi}_i \gamma^\mu A^a_\mu(x) \{R_A(T^a)\}_{ij} \Psi_j $$

[corr. to free Dirac action for the multicomponent fermion field]

[this has a cubic interaction vertex & the coefficient of this term is given by 'g']

[$\therefore$ the same 'g' that controlled the couplings of the gauge fields, also controls that for the fermion fields]

[i, j → gauge indices
[Dirac indices are suppressed]

$(R_A(T^a))_{ij}$
↳ discrete choice depending on which repr. the fermion belongs to

For Maxwell field, suppose you couple to χ & Φ with strengths e & e' which can vary continuously in principle → nothing in the theory tells us that e, e' should be discrete

Here, the couplings can't be varied continuously, it must be some discrete choice — in SU(2) it is spin 1/2, 3/2 etc.

which repr. we use, or gauge grp. we use, will be fixed by expt.

Brief review of continuous groups & their unitary representations

Group G :- Collection of $N \times N$ unitary matrices U labelled by one or more continuous parameters such that if $U_1 \in G$, $U_2 \in G$, then $U_1 U_2 \in G$.

[If we choose all possible $N \times N$ unitary matrices, we are back to $U(N)$]

[$SU(2) \to$ all possible 2×2 matrices with det 1

But, we can also rep. $SU(2)$ by a

\# subset of the 3×3 matrices by constraining the same no. of indep. parameters
— set of all 3×3 orthogonal matrices
3×3 real unitary matrices (spin-1 repn.)

\# spin 3/2 repn. — subset of 4×4 matrices]

→ Defining representation

(For exceptional groups, you can't have a defining repn. (?) — or a repn. where you can put simple restrictions — like in $SU(N)$, you have $N \times N$ unitary matrices with det $= +1$)

Group elements close to identity

$$U = \mathbb{1} - i \sum_{a=1}^{k} \alpha^a T^a$$

↳ collection of $N \times N$ hermitian matrices

T^a → not all possible $N \times N$ Herm. matrices
— it depends on how ~~the grp we are considering~~ many we need in rep. an infinitesimal grp. element

where k :- dimension of the group/algebra

$\{T^a\}$:- Basis of a k-dimensional real vector space.

A generic element of the vector space :- $\sum_{a=1}^{k} \beta^a T^a$ where β^a : real numbers

[or you can think of the 'k' - parameters β^a as forming a k - dim. vector space which give a generic generator $\sum \beta^a T^a$]

If $T^a \in \mathcal{A}$, $T^b \in \mathcal{A}$, we can show that $-i[T^a, T^b] \in \mathcal{A}$.

i.e., $-i[T^a, T^b]$ is an element of the algebra

$$[T^a, T^b] = i \sum_c \underbrace{f^{abc}}_{\text{Real constants}} T^c$$

$-i$ is there bcos $[T^a, T^b]$ is anti-Herm. & can't be repntd as a linear combn of Herm. matrices

It follows from the following fact :—
If $U_1 \in G$, $U_2 \in G$, then $U_1 U_2 U_1^{-1} U_2^{-1} \in G$.

$$U_1 = \mathbb{1} - i\alpha_a T^a + O(\vec{\alpha}^2)$$

$$U_2 = \mathbb{1} - i\beta_a T^a + O(\vec{\beta}^2)$$

& calculate $U_1 U_2 U_1^{-1} U_2^{-1} \longrightarrow \mathbb{1} + (\text{constant})\alpha_a \beta_b [T^a, T^b]$ + higher order terms

[$\because$ $U_1 U_2 U_1^{-1} U_2^{-1}$ is a grp. element,
$[T^a, T^b] = i f^{abc} T^c$]

A generic element of G can be written as $U = \exp(i\alpha_a T^a)$, $\alpha_a \to$ real numbers

Claim :- ① $\forall \in G$ & $T^a \in A$,

$$U T^a U^{-1} \in A$$

belongs to the Algebra

Hence $U T^a U^{-1} = \sum_{b=1}^{K} T^b R_{ba}$ ↳ Real numbers

we proved an analogous result for SU(N), where we had Herm. matrices

Note :- This algebra doesn't contain all possible Herm. matrices

② If we take a G-valued function $U(x)$, then $i \partial_\mu U U^{-1} \in A$

$$\Rightarrow i \partial_\mu U U^{-1} = \sum_a \alpha_\mu^a T^a$$

↳ real numbers

Generators of SO(N) are antisym. Herm. matrices — these are not all possible Herm. matrices

Proof of ① :-

$$U \left(\mathbb{1} - i \epsilon_a T^a \right) U^{-1} \in G$$

$$= \mathbb{1} - i\epsilon_a U T^a U^{-1}$$

if $T^a \in A$ & $U \in G$

$$\xrightarrow[\text{implies}]{\text{this}} U T^a U^{-1} \in A$$

Proof of ② :-

Let ϵ be an infinitesimal parameter & n^μ be an arbitrary vector.

$$U(x) \in G$$

$$U(x + \epsilon n) \in G$$

$$U(x+\epsilon n)\, U(x)^{-1} \in G$$

$$\Downarrow$$

$$\left(U(x) + \epsilon n^\mu \partial_\mu U(x) + O(\epsilon^2) \right) U(x)^{-1} \in G$$

$$= \left(\mathbb{1} + \epsilon n^\mu \partial_\mu U(x) U^{-1} + O(\epsilon^2) \right) \in G$$

① is valid at a particular pt. — at every pt. it holds — But ② depends on 2 neighbouring pts.

Fair :- $\left(U(x) + \epsilon n^\mu \partial_\mu U(x) + O(\epsilon^2) \right) U(x)^{-1}$

$$= \left(\mathbb{1} + \epsilon \underbrace{n^\mu \partial_\mu U(x)\, U(x)^{-1}}_{-i\epsilon\, \alpha_\mu^a T^a} + O(\epsilon^2) \right) \in G$$

$\boxed{(A+B+C)D = AD+BD+CD}$

(∵ the whole thing is in the group & is infinitesimally close to identity)

$\xrightarrow[\text{that}]{\text{this shows}}$ $i\, \partial_\mu U\, U^{-1} = \alpha_\mu^a T^a$ for some real $\alpha_\mu^a(x)$.

Introduce K vector fields B_μ^a (where $a = 1, 2, \ldots, K$).

Gauge trs. law of B_μ^a under trs. $U(x)$ is $B_\mu^a(x) \longrightarrow R_{ab}(x)\, B_\mu^b(x) - \alpha_\mu^a(x)$

<u>Gauge invariant action</u> (is constructed as follows) →

Define : $\underset{\sim}{B}_\mu = \sum_{a=1}^{K} B_\mu^a T^a$

$$\underset{\sim}{G}_{\mu\nu} = \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu - i[\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu]$$

$$S_{gauge} = -\frac{1}{2g^2} \int d^4x\, \mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu} \underset{\sim}{G}^{\mu\nu} \right)$$

(Proof that S_{gauge} is gauge-inv.) →

<u>Proof</u> :- ① Show that

$$\underset{\sim}{B}_\mu \longrightarrow U \underset{\sim}{B}_\mu U^{-1} - i \partial_\mu U\, U^{-1}$$

(Multiply B_μ^a by T^a in $B_\mu^a \longrightarrow R_{ab} B_\mu^b - \alpha_\mu^a$)

② $G_{\mu\nu} \longrightarrow U G_{\mu\nu} U^{-1}$

Show that all terms where partial derivatives act on U, cancel

③ $\mathrm{Tr}\left(G_{\mu\nu} G^{\mu\nu}\right) \longrightarrow \mathrm{Tr}\left(U G_{\mu\nu} U^{-1} U G^{\mu\nu} U^{-1}\right)$

$= \mathrm{Tr}\left(G_{\mu\nu} G^{\mu\nu}\right)$

H/W

① $B_\mu^a T^a \xrightarrow{U(x)} R_{ab} T^a B_\mu^b - \alpha_\mu^a T^a$

$= U T^b U^{-1} B_\mu^b - i(\partial_\mu U) U^{-1}$

$= U B_\mu U^{-1} - i(\partial_\mu U) U^{-1} \equiv B'_\mu$

② $G_{\mu\nu} \to \partial_\mu B'_\nu - \partial_\nu B'_\mu - i[B'_\mu, B'_\nu]$

$= \partial_\mu [U B_\nu U^{-1} - i(\partial_\nu U) U^{-1}] - \partial_\nu [U B_\mu U^{-1} - i \partial_\mu U U^{-1}]$

$- i[U B_\mu U^{-1}, U B_\nu U^{-1}]$ + other terms of commutator

$= (\partial_\mu U) B_\nu U^{-1} + U(\partial_\mu B_\nu) U^{-1} + U B_\nu (\partial_\mu U^{-1})$

$- i\cancel{(\partial_\mu \partial_\nu U) U^{-1}} - i(\partial_\nu U)(\partial_\mu U^{-1})$

$- (\partial_\nu U) B_\mu U^{-1} - U(\partial_\nu B_\mu) U^{-1} - U B_\mu (\partial_\nu U^{-1})$

$+ i\cancel{(\partial_\nu \partial_\mu U) U^{-1}} - i(U B_\mu U^{-1} U B_\nu U^{-1}$

$+ i(\partial_\mu U)(\partial_\nu U^{-1})$ $- U B_\nu U^{-1} U B_\mu U^{-1})$

+ other terms of commutator

$= \cancel{(\partial_\mu U) B_\nu U^{-1}} - \cancel{(\partial_\nu U) B_\mu U^{-1}} + U B_\nu (\partial_\mu U^{-1}) - U B_\mu (\partial_\nu U^{-1}) + U[\partial_\mu B_\nu - \partial_\nu B_\mu] U^{-1}$

$- i(\partial_\nu U)(\partial_\mu U^{-1})$ $\quad - i U [B_\mu, B_\nu] U^{-1}$

$+ i(\partial_\mu U)(\partial_\nu U^{-1})$ $\quad - U B_\mu U^{-1}(\partial_\nu U) U^{-1} + \cancel{(\partial_\nu U) B_\mu U^{-1}} - \cancel{(\partial_\mu U) B_\nu U^{-1}}$

$+ U B_\nu U^{-1}(\partial_\mu U) U^{-1} + i(\partial_\mu U) U^{-1} \underbrace{(\partial_\nu U) U^{-1}}_{-U(\partial_\nu U^{-1})} - i(\partial_\nu U) U^{-1} \underbrace{(\partial_\mu U) U^{-1}}_{-U(\partial_\mu U^{-1})}$

$= U G_{\mu\nu} U^{-1} + \cancel{U B_\nu (\partial_\mu U^{-1})} - \cancel{U B_\mu (\partial_\nu U^{-1})} - \cancel{i(\partial_\nu U)(\partial_\mu U^{-1})}$

$+ \cancel{i(\partial_\mu U)(\partial_\nu U^{-1})} + \cancel{U B_\mu (\partial_\nu U^{-1})} - \cancel{U B_\nu (\partial_\mu U^{-1})}$

$- \cancel{i(\partial_\mu U)(\partial_\nu U^{-1})} + \cancel{i(\partial_\nu U)(\partial_\mu U^{-1})}$ $\quad = U G_{\mu\nu} U^{-1}$

Note $[B'_\mu, B'_\nu] = [U B_\mu U^{-1}, U B_\nu U^{-1} - i \partial_\nu U U^{-1}] - i[\partial_\mu U U^{-1}, U B_\nu U^{-1} - i \partial_\nu U U^{-1}]$

$= [B_\mu, B_\nu] - i[U B_\mu U^{-1}, \partial_\nu U U^{-1}] - i[\partial_\mu U U^{-1}, U B_\nu U^{-1}]$

$- [\partial_\mu U U^{-1}, \partial_\nu U U^{-1}]$

8/10/06

Matter fields (fermions, scalars) belong to certain representation.

Under a gauge transformation by $U(x)$

$$\begin{pmatrix}\psi_1\\ \vdots \\ \psi_M\end{pmatrix} \longrightarrow \underbrace{R_G(U)}_{\text{(repn. of the group element)}} \begin{pmatrix}\psi_1\\ \vdots \\ \psi_M\end{pmatrix}$$

$U \in G$, $R_G(U)$ is a representation of G.

We need to take a covariant derivative — ordinary deriv. won't work bcos of the presence of $R_G(U)$

<u>Review of group representation</u>

M-dimensional unitary representation.

For every $U \in G$, there is an $M \times M$ unitary matrix $R_G(U)$ such that

$$R_G(U_1)\, R_G(U_2) = R_G(U_1 U_2)$$

for $U_1 \in G$, $U_2 \in G$.

$$R_G(\mathbb{1}) = \mathbb{1}_{M\times M}$$

bcos suppose we choose $U_2 = \mathbb{1}$, then it follows that —

$$R_G(\mathbb{1} - i\underbrace{\epsilon^1}_{\text{small parameter}} T^1)$$

$\because R_G(\mathbb{1}) = \mathbb{1}_{M\times M}$, this expression follows

$$= \mathbb{1}_{M\times M} - i\epsilon^1 R_A(T^1) \rightarrow \text{defines } R_A(T^1)$$

Similarly, define $R_A(T^2), \ldots, R_A(T^K)$

Now, let us consider

$$R_G\left(1 - i\underbrace{\epsilon}_{\text{infinitesimal parameter}} \sum_{a=1}^{K} \underbrace{n_a}_{\text{some nos.}} T^a\right)$$

$$= R_G\left((1-i\epsilon n_1 T^1)(1-i\epsilon n_2 T^2)\cdots(1-i\epsilon n_k T^k)\right)$$

if we neglect $O(\epsilon^2)$ terms

$$= R_G(1-i\epsilon n_1 T^1)\, R_G(1-i\epsilon n_2 T^2)\cdots \quad \text{[By grp. prop.]}$$

$$= \left(1-i\epsilon n_1 R_A(T^1)\right)\left(1-i\epsilon n_2 R_A(T^2)\right)\cdots$$

$$= \left(1 - i\epsilon \sum_{a=1}^{K} n_a R_A(T^a)\right)$$

But, $R_G\left(1 - i\epsilon \sum_{a=1}^{K} n_a T^a\right) \rightsquigarrow 1 - i\epsilon R_A\left(\sum_{a=1}^{K} n_a T^a\right)$

Comparing these two, we get,

$$\boxed{R_A\left(\sum_a n_a T^a\right) = \sum_a n_a R_A(T^a)}$$

this is for the algebra

[This is the analog of $R_G(U_1)\,R_G(U_2) = R_G(U_1 U_2)$

(this is for the group)]

Recall $[T^a, T^b] = i f^{abc} T^c$ (sum over c)

(we can show) $\to$

$$[R_A(T^a), R_A(T^b)] = i f^{abc} R_A(T^c)$$

Proof :—

$$R_a(UVW) = R_a(U)\, R_a(V)\, R_G(W)$$

Take

group elements close to identity

$$U = (1 + i\epsilon\, \alpha_a T^a)$$

$$V = (1 + i\eta\, \beta_b T^b) \qquad \xi, \eta \text{ infinitesima[l]}$$

$$W = (1 - i\epsilon\, \alpha_a T^a)$$

ξ, η terms can be compared on both sides — but not $O(\xi^2)$ $O(\eta^2)$ ~~$O(\xi\eta)$~~ terms as we are neglecting them

We will get,

$$UVW = 1 - \epsilon\eta\, \alpha_a \beta_b [T^a, T^b] + i\eta\, \beta_b T^b$$

$$\Big[= (1 + i\epsilon\alpha_a T^a)(1 - i\epsilon\alpha_a T^a + i\eta\, \beta_b T^b + \epsilon\eta\, \beta_b T^b \alpha_a T^a)$$

$$= 1 - \cancel{i\epsilon\alpha_a T^a} + i\eta\beta_b T^b + \epsilon\eta\, \beta_b \alpha_a T^b T^a - \epsilon\eta\, \alpha_a \beta_b T^a T^b + \cancel{i\epsilon\alpha_a T^a} + O(\epsilon^2)\Big]$$

$$\therefore R_G(U)\, R_G(V)\, R_G(W)$$

$$= 1 - \epsilon\eta\, \alpha_a \beta_b [R_A(T^a), R_A(T^b)] + i\eta\, \beta_b R_A(T^b)$$

think of ϵ & η as indep. parameters & as if we have carried out Taylor series expn. retaining terms upto $O(\epsilon)$ & $O(\eta)$ for the fns.

Then compare each term of the Taylor series expn. on both sides

Note $U = 1 + i\epsilon\alpha_a T^a + O(\epsilon^2)$ $\downarrow$ no η-dependence ...

Also,

$$UVW = 1 + i\eta\, \beta_b T^b - i\epsilon\eta\, \alpha_a \beta_b f^{abc} T^c$$

$$\therefore R_G(UVW)$$

$$= 1 + i\eta\, R_A(\beta_b T^b - \epsilon\, \alpha_a \beta_b f^{abc} T^c)$$

$$= 1 + i\eta\, \beta_b R_A(T^b) - i\epsilon\eta\, \alpha_a \beta_b f^{abc} R_A(T^c)$$

Compare two sides

$$\Longrightarrow [R_A(T^a), R_A(T^b)] = i f^{abc} R_A(T^c)$$

(proved)

If $U T^a U^{-1} = T^b R_{ba}$,

then $R_G(U) R_A(T^a) (R_G(U))^{-1} = R_A(T^b) R_{ba}$

(To prove this, follow the same strategy by which we established it in the first place)

$\longrightarrow \quad U(1 - i\epsilon^a T^a) U^{-1} = 1 - i\epsilon^a T^b R_{ba}$,

Use

$$R_G\left(U(1 - i\epsilon^a T^a) U^{-1}\right)$$

$$= R_G(U)\, R_G(1 - i\epsilon^a T^a)\, (R_G(U))^{-1}$$

(Evaluate both sides & we will get the desired result)

$R_G(U)(1 - i\epsilon^a R_A(T^a))(R_G(U))^{-1} = 1 - i\epsilon^a R_A(T^b) R_{ba}$

$\Rightarrow 1 - i\epsilon^a R_G(U) R_A(T^a) (R_G(U))^{-1} = 1 - i\epsilon^a R_A(T^b) R_{ba}$

Suppose $U(x)$ is a group valued fn. of x.

$$i \partial_\mu U\, U^{-1} = \alpha^a_\mu(x)\, T^a$$

We defined $\alpha^a_\mu(x)$ by this relation

(Given this, we can show →)

Then, $i\partial_\mu (R_G(U)) (R_G(U))^{-1}$

$\because U(x)$ is a group valued fn., $\partial_\mu R_G(U)$ is defined

$$= \alpha^a_\mu(x)\, R_A(T^a)$$

(these coeffs. are indep. of which repn. we choose)

(To prove this, follow the steps we took to prove $i\partial_\mu U U^{-1} = \alpha^a_\mu(x) T^a$)

Proof :- Let n be a 4-vector.

$U(x + \epsilon n^\mu)\,(U(x))^{-1}$

(this was our starting point)

$$= (U(x) + \epsilon n^\mu \partial_\mu U)\, U(x)^{-1}$$

$$= \mathbb{1} + \epsilon n_\mu \partial_\mu U(x)\, (U(x))^{-1}$$

(Apply R_G on both sides) $\rightarrow$

$$R_G \left(U(x+\epsilon n^\mu) \, (U(x))^{-1} \right)$$
$$= R_G \left((U(x) + \epsilon n^\mu \partial_\mu U) \, U(x)^{-1} \right)$$
$$= R_G \left(\mathbb{1} + \epsilon n^\mu \partial_\mu U(x) \, (U(x))^{-1} \right)$$
$$= R_G \left(\mathbb{1} - i \epsilon n^\mu \alpha^a_\mu(x) \, T^a \right)$$

Now, $R_G \left(\mathbb{1} - i\epsilon n^\mu \alpha^a_\mu(x) T^a \right)$
$$= \mathbb{1} - i\epsilon n^\mu \alpha^a_\mu(x) R_A(T^a)$$

Again, $R_G \left(U(x+\epsilon n^\mu) \, (U(x))^{-1} \right)$
$$= R_G \left(U(x+\epsilon n) \right) R_G \left(U(x) \right)^{-1}$$
$$= \left(R_G(U(x)) + \epsilon n^\mu \partial_\mu R_G(U(x)) \right) R_G(U(x))^{-1}$$

$\boxed{\text{using the group property}}$

$$= \mathbb{1} + \epsilon n^\mu \partial_\mu R_G(U) \, R_G(U)^{-1}$$

Comparing, we get
$$\partial_\mu R_G(U) \, R_G(U)^{-1} = -i \alpha^a_\mu(x) R_A(T^a)$$

Note $\rightarrow$

$R_G(U^{-1}) = (R_G(U))^{-1}$

$$D'_\mu \psi' = \partial_\mu (R_G(U)\psi) - i R_G(U) B^a_\mu R_A(T^a) R_G(U^{-1}) R_G(U)\psi + i^2 \partial_\mu R_G(U) R_G(U)^{-1} R_G(U) \psi$$
$$= R_G(U) \partial_\mu \psi + \cancel{\partial_\mu R_G(U)\psi} - \cancel{\partial_\mu R_G(U)\psi} - i R_G(U) B^a_\mu R_A(T^a)\psi$$
$$= R_G(U) D_\mu \psi$$

Suppose ψ is M-component fermion such that

$$\psi \to R_G(U)\,\psi \text{ under gauge}$$

trs. by $U(x)$

actually it is 4M comp., if you count the Dirac indices

Here, we have suppress the Dirac indices

<u>Define</u> $\rightarrow$ $$\underset{\sim}{D}_\mu \psi = \partial_\mu \psi - i B_\mu^a(x) R_A(T^a)\psi$$

In the fundamental repn., the defn. was

$$\underset{\sim}{D}_\mu \psi = \partial_\mu \psi - i B_\mu^a(x)\, T^a \psi$$

in normal SU(N) gauge theory

<u>Ex.</u> Prove that

$$\underset{\sim}{D}_\mu \psi \to R_G(U)\, \underset{\sim}{D}_\mu \psi \text{ under}$$

gauge trs.

<u>HINT</u> :- $$B_\mu^a T^a \xrightarrow[\text{to}]{\text{transforms}} U B_\mu^a T^a U^{-1} - i\,\partial_\mu U\, U^{-1}$$

$$\stackrel{\text{call this}}{\equiv} \underbrace{B_\mu'^{a}}_{\text{(the transformed field)}} T^a$$

Show that

$$B_\mu'^a R_A(T^a) = R_G(U)\, B_\mu^a\, R_A(T^a)\, R_G(U)^{-1}$$
$$- i\,\partial_\mu R_G(U)\, R_G(U)^{-1}$$

Ex. Prove this

(Note this ~~so~~ repn. doesn't refer to the space-time spin, i.e., the spin of the particles)

Action (for the fermions) →

$\bar{\Psi}\,(i\gamma^\mu D_\mu - m)\,\Psi$ is gauge invariant

$m\bar{\Psi}\Psi$ is manifestly gauge inv. — $\bar{\Psi} D_\mu \Psi$ is gauge inv. bcos of the defns we have used

Use $R_G(U)^\dagger R_G(U) = \mathbb{1}$

(R_G's cancel by the unitarity of the repn.)

Suppose we have M complex scalars

$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_M \end{pmatrix}$ which transform as

$$\underbrace{\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_M \end{pmatrix}}_{\phi} \longrightarrow R_G(U) \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_M \end{pmatrix}$$

Define :- $D_\mu \phi = \partial_\mu \phi - i B_\mu^a(x)\, R_A(T^a)\, \phi$

$$-(D_\mu \phi)^\dagger D^\mu \phi - \tfrac{1}{2} m^2 \phi^\dagger \phi$$

is gauge invariant

In a theory with a given gauge group, one can have many different fermions & scalars in different representations.

Bcos we saw that the transfn. law for B_μ^a was fixed once & for all — By appropriate coupling to B_μ, we added fermions & scalars

Example $\Rightarrow$

Fermions Ψ in M dimensional representation R

Fermions χ in M' dimensional rep R'

Scalar ϕ in M'' " " R''

$$\# \; D_\mu \Psi = \partial_\mu \Psi - i B_\mu^a R_A(T^a) \Psi$$

$$\# \; D_\mu \chi = \partial_\mu \chi - i B_\mu^a R_A'(T^a) \chi$$

$\boxed{\text{same } B_\mu \text{ couples to all of these}}$

$$\# \; D_\mu \phi = \partial_\mu \phi - i B_\mu^a R_A''(T^a) \phi$$

$\boxed{\text{these are all gauge-inv. terms in the action}}$

$$\bar{\Psi}(i\gamma^\mu D_\mu - m_1)\Psi + \bar{\chi}(i\gamma^\mu D_\mu - m_2)\chi - (D_\mu \phi)^\dagger D_\mu \phi - m_3^2 \phi^\dagger \phi$$

(But these are not the only gauge inv. terms we can add to the action — we can also add) we can add

$$-\lambda(\phi^\dagger\phi)^2 + \alpha \bar{\Psi}_a \chi_b \phi_c C^{abc}$$

$a \to 1, 2, \ldots, M$; $b \to 1, 2, \ldots, M'$; $c \to 1, \ldots, M''$

($\because \phi^\dagger\phi$ is gauge inv.)

(You must be able to find coeff. C^{abc} to make the above gauge invariant)

Suppose, the gauge grp. $G = SU(2)$

Suppose ψ is in spin $2j_1+1$ repn

χ is in spin $2j_2+1$ repn

ϕ is in spin $2j_3+1$ repn

C^{abc} should be chosen such that the total spin is zero

$2j_1+1 \qquad 2j_2+1 \qquad 2j_3+1$

$|j_1-j_2|, |j_1-j_2|+1, \ldots, |j_1+j_2|$

We must have $|j_1-j_2| \leq j_3 \leq |j_1+j_2|$

(j_3 must be equal to one of these to get a spin-zero repn.)

We must be able to combine any of these with j_3 to get a singlet or spin-zero

$C^{abc} \longrightarrow$ Clebsh-Gordon coefficients

Given any 2 repns, in the prodt., we know what repn. we get — take a prodt. & see if we can get a singlet repn, which is gauge invariant $\longrightarrow$ Can be done for any gauge group

Product groups

Suppose G_1 is a group

G_2 is a group

Product group $G_1 \times G_2$ is defined as

set of all matrices U, V $\rightarrow \{(U, V)\}$ such that $U \in G_1, V \in G_2$.

(prdt. rule) $\rightarrow (U_1, V_1)(U_2, V_2) = (U_1 U_2, V_1 V_2)$

Gauge theories based on product group

Introduce gauge fields B_μ^a $(a = 1, \ldots, K_1)$

Dimension of the first group, i.e., G_1

C_μ^α $(\alpha = 1, 2, \ldots, K_2)$

$\hookrightarrow$ dim of the second group, i.e., G_2

Suppose $T^1, \ldots, T^{K_1}$ are the generators of G_1

& $L^1, \ldots, L^{K_2}$ " " " of G_2

Define $\underset{\sim}{B}_\mu = B_\mu^a T^a$, $\underset{\sim}{C}_\mu = C_\mu^\alpha L^\alpha$

Gauge trs. laws :-

Any generic grp. element is a pair

Under $(U(x), G(x)) \in G_1 G_2$

$$\underset{\sim}{B}_\mu \rightarrow U \underset{\sim}{B}_\mu U^{-1} - i \partial_\mu U \, U^{-1}$$

$$\underset{\sim}{C}_\mu \rightarrow V \underset{\sim}{C}_\mu V^{-1} - i \partial_\mu V \, V^{-1}$$

~~Gauge invariant action :-~~

Define :- $$\underset{\sim}{G}_{\mu\nu} = \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu - i [\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu]$$

$$\underset{\sim}{H}_{\mu\nu} = \partial_\mu \underset{\sim}{C}_\nu - \partial_\nu \underset{\sim}{C}_\mu - i [\underset{\sim}{C}_\mu, \underset{\sim}{C}_\nu]$$

These are just the field strengths — In general, these are matrices of diff. dim., bcos U & G can be matrices of diff. dim.

$\therefore$ the gauge invariant action :-

$$-\frac{1}{2g_1^2}\int d^4x \, \mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu} \underset{\sim}{G}^{\mu\nu}\right) - \frac{1}{2g_2^2}\int d^4x \, \mathrm{Tr}\left(\underset{\sim}{H}_{\mu\nu} \underset{\sim}{H}^{\mu\nu}\right)$$

where g_1, g_2 are arbitrary constants

(So, for prdt. group, you can take 2 indep. coupling constants for the 2 gauge grps)

Fields in one part has no coupling with fields in another — It is a ~~direct~~ sum of 2 indep. actions — But once we introduce matter fields, they can couple to both — Pure gauge fields don't have int. — just put the results together

Representation of product groups

(We must have a single matrix, not a pair of matrices)

Suppose R is an M-dimensional rep. of G_1 & S is an N-dimensional rep. of G_2.

$\therefore$ $\forall\ U \in G_1$, we have an $M \times M$ matrix $R_{G_1}(U)$

$$\left(R_{G_1}(U)\right)_{mn} \qquad m, n = 1, 2, \ldots, M$$

Similarly, $\forall\ V \in G_2$, we have an $N \times N$ matrix $S_{G_2}(V)$

$$\left(S_{G_2}(V)\right)_{\sigma\tau} \qquad \sigma, \tau = 1, 2, \ldots, N$$

Construct an $MN \times MN$ matrix

$$\left(R_{G_1 \times G_2}((U, V))\right)_{m\sigma,\, n\tau}$$

$n\tau$ → this pair can take MN values

$m\sigma$ ↓ this pair can take MN values

$$\overset{\text{defn.}}{=} \left(R_{G_1}(U)\right)_{mn} \left(S_{G_2}(V)\right)_{\sigma\tau}$$

Now Make sure that it forms a repr. →

<u>Need to prove :-</u>

$$\left(R_{G_1 \times G_2}\left((U_1, V_1)(U_2, V_2)\right)\right)_{m\sigma,\, n\tau}$$

$$= R_{G_1 \times G_2}(U_1, V_1)_{m\sigma,\, p\xi}\ R_{G_1 \times G_2}\left((U_2, V_2)\right)_{p\xi,\, n\tau}$$

Ex. Prove this (using the defns).

$$R((U,V)) = \begin{pmatrix} R(U) & 0 \\ 0 & S(V) \end{pmatrix}$$ → this itself isn't irreducible

$$R_{(G_1 \times G_2)}(U,V) = R_{G_1}(U)$$

$$R_{G_1 \times G_2}(U,V) = S_{G_2}(V)$$

where all the S's have been mapped to identity

Suppose we have a fermion

$\Psi_{m\sigma}$ $\quad m = 1, \ldots, M$
$\quad \sigma = 1, \ldots, N$

Need an MN component fermion

~~$(D_\mu \Psi)_{m\sigma} = \partial_\mu \Psi_{m\sigma} - i B_\mu^a R_{mn} (T^a) \Psi_{n\sigma}$~~

~~$- i C_\mu$~~

$$(D_\mu \Psi)_{m\sigma} = \partial_\mu \Psi_{m\sigma} - i B_\mu^a (R_{A_1}(T^a))_{mn} \Psi_{n\sigma}$$
$$- i C_\mu^\alpha S_{A_2}(L^\alpha)_{\sigma\tau} \Psi_{m\tau}$$

Ex. We can show that $(D_\mu \Psi)_{m\sigma} \longrightarrow (R_{G_1 \times G_2}(U,V))_{m\sigma, n\tau} (D_\mu \Psi)_{n\tau}$

Then $\bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi$ will be gauge-invariant.

So, B_μ & C_μ parts aren't decoupled any more.

12/10/06

Pure gauge theory based on gauge group G

Fields: B_μ^a $\quad a = 1, \ldots, K$

↳ runs over the generators of the gauge group

Define:- $\underset{\sim}{B}_\mu = B_\mu^a T^a$

↳ generators of G

$$\underset{\sim}{G}_{\mu\nu} = \partial_\mu \underset{\sim}{B}_\nu - \partial_\nu \underset{\sim}{B}_\mu - i\left[\underset{\sim}{B}_\mu, \underset{\sim}{B}_\nu\right] = G^c_{\mu\nu} T^c$$

$$G^c_{\mu\nu} = \partial_\mu B^c_\nu - \partial_\nu B^c_\mu + f^{abc} B^a_\mu B^b_\nu$$

Gauge field action

$$S_{gauge} = -\frac{1}{2g^2}\int \mathrm{Tr}\left(\underset{\sim}{G}_{\mu\nu}\underset{\sim}{G}^{\mu\nu}\right)$$

$$= -\frac{1}{2g^2}\mathrm{Tr}\left(T^a T^b\right)\int d^4x\, G^a_{\mu\nu} G^{b\mu\nu}$$

Convention :- $\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta_{ab}$ → Achieved by choice of T^a

By taking appropriate linear combns of the initial choice of generators

$$S_{gauge} = -\frac{1}{4g^2}\int G^a_{\mu\nu} G^{a\mu\nu} d^4x$$

$B^a_\mu = g A^a_\mu$ ⟶ defines A^a_μ

$\cancel{S_{gauge} = -1/4}$

$\cancel{F^a_{\mu\nu} = \frac{1}{g} G^a_{\mu\nu} = \partial_\mu A_\nu}$

$[T^a, T^b] = i f^{abc} T^c$

↓

depends on the particular choice of basis - on the particular T^a's we have chosen

$$F_{\mu\nu}^{c} = \frac{1}{g} G_{\mu\nu}^{c} = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g f^{abc} A_\mu^a A_\nu^b$$

$$\therefore \quad S_{gauge} = -\tfrac{1}{4}\int F_{\mu\nu}^{c} F^{c\mu\nu}\, d^4x$$

$$= S_{free} + S_{int}$$

S_{free} (contains quadratic piece)

S_{int} (contains cubic & higher order piece)

where,

$$S_{free} = -\tfrac{1}{4}\int d^4x \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\nu A^{a\mu} - \partial^\mu A^{a\nu}\right)$$

$$S_{int} = -g/2 \int d^4x \left(\partial_\mu A_\nu^c - \partial_\nu A_\mu^c\right) f^{abc} A^{a\mu} A^{b\nu}$$

$$-\tfrac{1}{4} g^2 f^{abc} f^{a'b'c} \int A_\mu^a A_\nu^b A^{a'\mu} A^{b'\nu}\, d^4x$$

<u>Gauge trs. law</u>

$$B_\mu^a \longrightarrow R_{ab}^{(x)} B_\mu^b(x) - \alpha_\mu^a(x)$$

where $R_{ab}(x)$ is defined through

$$U(x)\, T^a\, U^{-1}(x) = T^b R_{ba}(x)$$

& $\alpha_\mu^a(x)$ is defined through $i\partial_\mu U(x)\, U(x)^{-1} = \alpha_\mu^a(x) T^a$

If it is inv. under a finite gauge trs., it'll be inv. under an infinitesimal gauge trs.

Infinitesimal gauge trs:

$$U \doteq \mathbb{1} - i g \underbrace{f^a(x)}_{\text{infinitesimal fns}} T^a$$

(We have explicitly taken 'g' out — just a choice of normalization for convenience)

Ex: Show that

$\left(\mathbb{1} - ig\,\epsilon^{b} R_A(T^b)\right) R_A(T^a) \left(\mathbb{1} + ig\,\epsilon^{c} R_A(T^c)\right) = R(T^b) R_{ba}$

$\Rightarrow R(T^b) R_{ba} = \delta_{ba} R_A(T^b) + ig\epsilon^{c}\left[R_A(T^a), R_A(T^c)\right]$

$$R_{ca}(x) = \delta_{ca} + g f^{bac} \epsilon^{b}(x)$$

$$\alpha_\mu^{a}(x) = g\,\partial_\mu \epsilon^{a}(x)$$

$\alpha_\mu^{a} R_A(T^a) = i\partial_\mu\left(\mathbb{1} - ig\epsilon^{a} R_A(T^a)\right)\left(\mathbb{1} + ig\epsilon^{b} R_A(T^b)\right)$

$= g R_A(T^a)\,\partial_\mu \epsilon^{a}(\;''\;)$

$= g\,\partial_\mu \epsilon^{a} R_A(T^a)$

Ex: $B_\mu^{a} \longrightarrow B_\mu^{a} + \delta B_\mu^{a}$

where $\delta B_\mu^{a} = g f^{bca} \epsilon^{b} B_\mu^{c} + g\,\partial_\mu \epsilon^{a}$

Consequently, $A_\mu^{a} \longrightarrow A_\mu^{a} + \delta A_\mu^{a}$

where $\delta A_\mu^{a} = \frac{1}{g}\delta B_\mu^{a} = -\partial_\mu \epsilon^{a}(x) + g f^{abc} \epsilon^{b}(x) A_\mu^{c}(x)$

$\therefore$ Gauge invariance of S
$\Rightarrow \delta S = 0$ under this transformation

Naive path integral quantization

$$A_\mu^{a}(x) = \int \frac{d^4k}{(2\pi)^4}\, \tilde{A}_\mu^{a}(x)\, e^{ik\cdot x}$$

$\hookrightarrow$ (Go to Fourier space)

$$S_{free} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{A}_\mu^{a}(-k) \underbrace{\left(-k^2\eta^{\mu\nu} + k^\mu k^\nu\right)}_{M_{\mu\nu}(-k)} \tilde{A}_\mu^{a}(k)$$

S_{free} is the same as in QED except that we have a sum over 'k' — 'k' copies of Maxwell action $\longrightarrow$ so we will face the same problems as we faced in quantising Maxwell action

Propagator :-

$$\langle \tilde{A}_\mu^{a}(-k_1)\, \tilde{A}_2^{b}(-k_2)\rangle$$
$$= i\,\delta_{ab}\,(2\pi)^4\,\delta^{(4)}(-k_1+k_2)\left(M(-k_2)\right)^{-1}_{\mu\nu}$$

[We should think $M_{\mu\nu}(k)$ as $4k \times 4k$ matrix — we have $4k$ fields]
$\because a = 1, 2, \ldots, k$

But $\left(M(k_2)\right)^{-1}_{\mu\nu}$ does not exist because $M_{\mu\nu}(k)$ has a zero eigenvalue :-

$$(-k^2\eta^{\mu\nu} + k^\mu k^\nu)k_\nu = 0$$

(This 4×4 matrix has an e.value 0 with e.vector k_ν)

Physical origin of the zero eigenvalue

In the $g \to 0$ limit, gauge trs. law is

$$\delta A_\mu(x) = \partial_\mu \epsilon^a(x) \longrightarrow \int \frac{d^4k}{(2\pi)^4} e^{ik.x} \tilde{\epsilon}(k)$$

$$\| $$

$$\int \frac{d^4k}{(2\pi)^4} e^{ik.x} \delta \tilde{A}_\mu(k)$$

the problem comes in the free part — so we ~~stug~~ study the free part by taking $g \to 0$

$$\therefore \quad \delta \tilde{A}_\mu(k) = i k_\mu \tilde{\epsilon}(k)$$

(So the free theory, in the $g \to 0$ limit, comes with A-fixed, not with B-fixed)

$\delta S_{free} = 0$ under this

(The origin of the zero e.value is related to the gauge inv. of action)

Why does zero eigenvalue cause divergence?

→ Consider a finite dimensional integral

$$\langle f(x) \rangle = \frac{\int \prod_{i=1}^{n} dx_i \, e^{-A_{ij} x^i x^j} f(\vec{x})}{\int \prod_{i=1}^{n} dx_i \, e^{-A_{ij} x_i x_j}}$$

symmetric

Some Polynomial in $x_1, \ldots, x_n$

$A = U^T A_d U$ where $A_d = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

& $U^T U = \mathbb{1}$

orthogonal matrix

<u>Define</u>:- $\vec{y} = U\vec{x}$

$$\therefore \langle f(x) \rangle = \frac{\int \prod_{i=1}^{n} dy_i \, e^{-\sum_{i=1}^{n} \lambda_i y_i^2} f(U^{-1}\vec{y})}{\int \prod_{i=1}^{n} dy_i \, e^{-\sum_{i=1}^{n} \lambda_i y_i^2}}$$

(After change of variables)

[Using $e^{-x^T A x} = e^{-x^T U^T A_d U x} = e^{-y^T A_d y} = e^{-\sum_i \lambda_i y_i^2}$]

[In QFT also, we have to do this kind of ~~in~~ integrals except that you must do an infinite no. of integrals instead of a finite no.]

If $\lambda_i \neq 0$ & $\lambda_i > 0$, integral is well-defined.

If $\lambda_i < 0$ for some i, we can still define it by analytic continuation.

$$\frac{\int e^{-\lambda y^2} y^2 \, dy}{\int e^{-\lambda y^2} \, dy} \underset{\text{Check}}{=} (\text{constant}) \times \frac{1}{\lambda}$$

for $\lambda > 0$

for $\lambda < 0$, we can have $\frac{\int --}{\int --} = (\text{const}) \times \frac{1}{+\lambda}$ when $\lambda < 0$

But we can't make it work with $\lambda = 0$ $\longrightarrow$ divergence

Num. will be more div. than den. if f is y^1-dep. & is some polynomial of y^1 — no hope to get rid of div.

If $\lambda_1 = 0$, $A_{ij} x^i x^j$ has the same value for all y^1 at fixed $y^2, \ldots, y^n$ $\Rightarrow$ y^1 integral diverges

[If the action has some sym. dirn. — action remaining const. along a dirn. in the configuration space — ind. of action along that dirn. isn't damped]

[If the action has some sym. dirn. & the operator whose correlation fn. we are calculating, is unchanged in the sym. dirn., the div. cancels out — i.e., the operator should have the symmetries of the action

— In this case, the sym. is gauge inv. & the op. should then also be " "]

<u>Conclusion</u> $\longrightarrow$ In order to get sensible answer for correlation functions in a gauge theory, we must consider correlation functions of gauge invariant operators.

$\downarrow$

these are the only observables in the theory

[gauge inv. Hermitian op. are the only observables in gauge theories]

(This resolves the conceptual problem — what is the cause & we must do to resolve the problem (consider only gauge inv. obs.) — Practical diff. → how to develop our pert. theory to calculate ~~$\langle G\, G^{a\mu\nu}\rangle$ etc.~~ corr. fns of gauge inv. objects like $G^a_{\mu\nu}(x)\, G^{a\mu\nu}(x)$

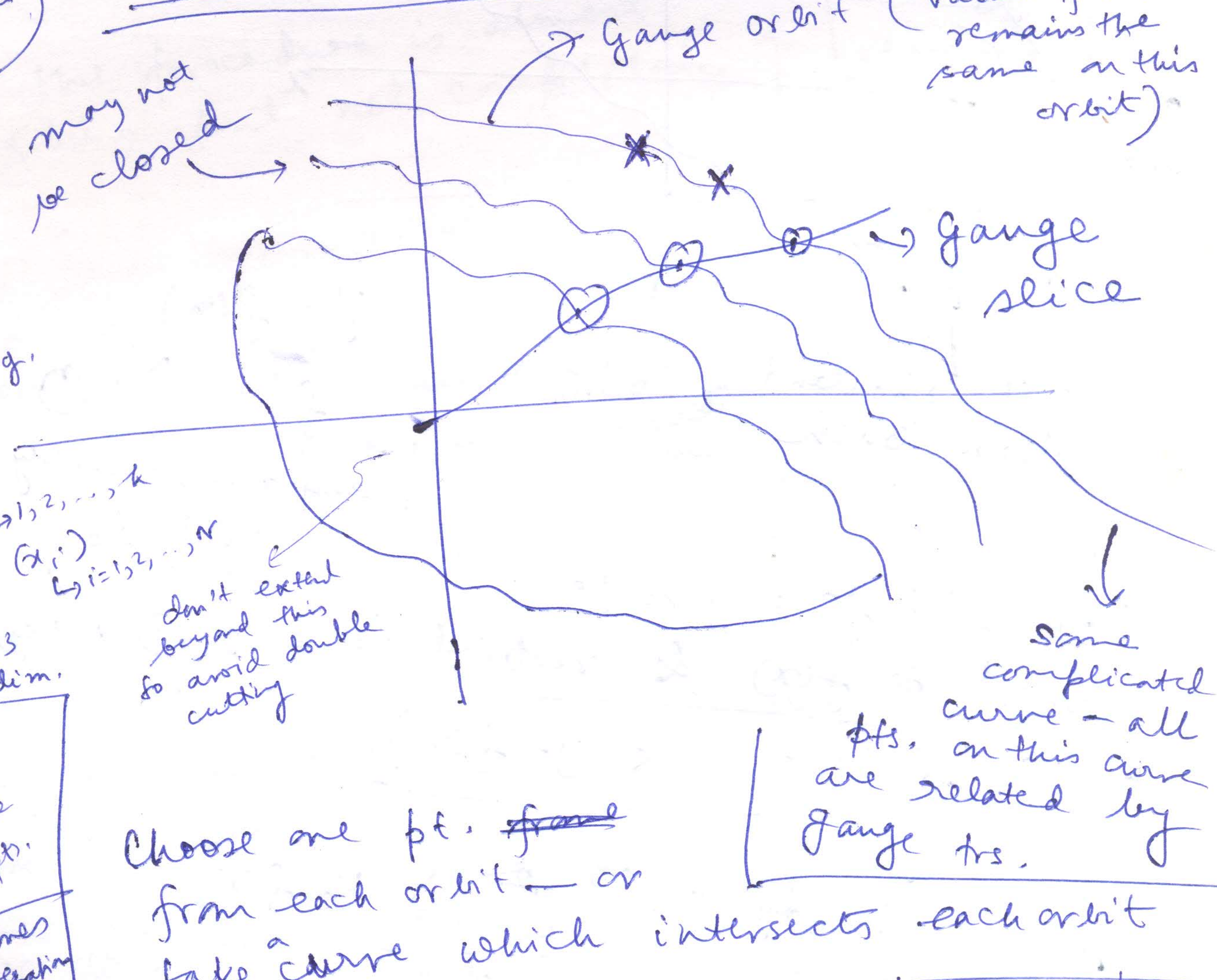
~~coor~~

2-pt. fn. like $\left\langle G^a_{\mu\nu}(x)\, G^{a\rho\sigma}(x)\; G^a_{\mu'\nu'}(y)\, G^{a\rho'\sigma'}(y) \right\rangle$ can be calculated

Lorentz indices should be same

Basic strategy

Configuration space



$N \times 4a$ variables if we take N lattice pts.

Problem comes from the integration along a orbit bcos action has the same value on an orbit

Choose one pt. ~~from~~ from each orbit — or take a curve which intersects each orbit only once

[curve should cut each orbit same no. of times — even if it cuts each orbit twice, it isn't a problem)

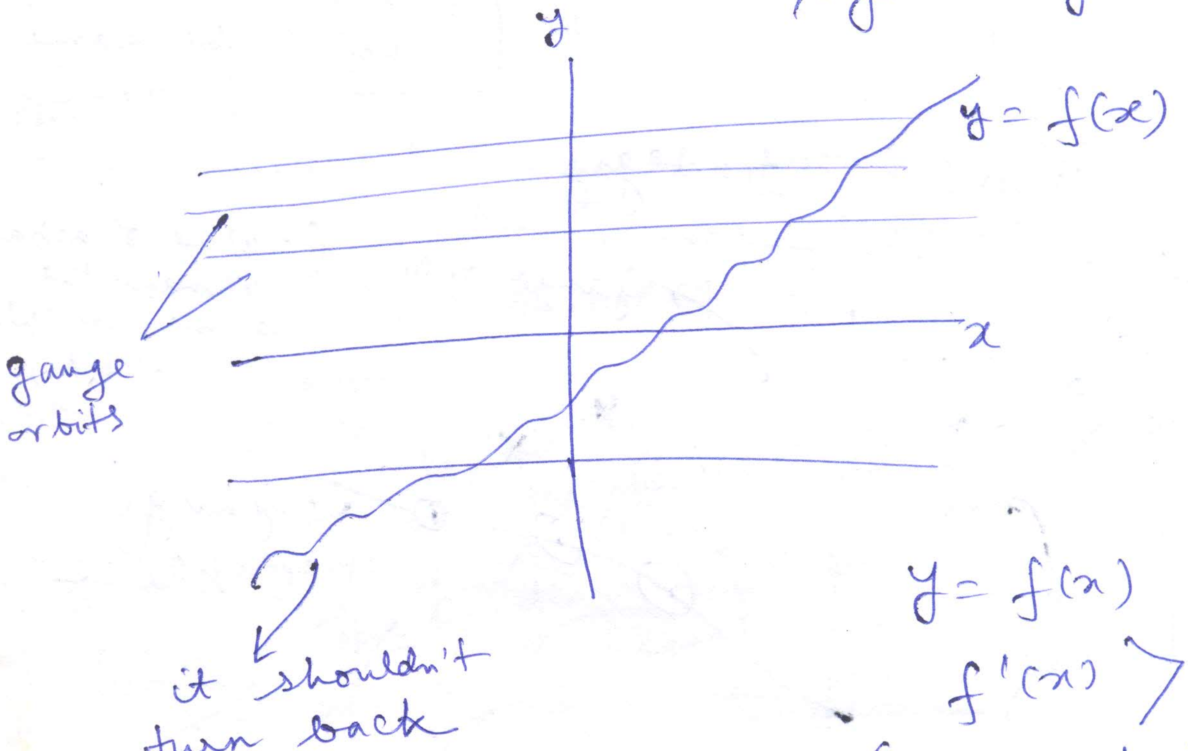
→ equiv. to not performing the y^1-integral

Take integral over the gauge slice

[Instead of a 2-D int. in this case, we are restricting to a 1-D int. along gauge-slice — avoiding the extra dim.]

e.g. $\to$ $$I = \frac{\int dx\, dy\, e^{-\lambda y^2} y^2}{\int dx\, dy\, e^{-\lambda y^2}}$$ (Both num. & den. are ill-defined)

symmetry $x \to x + a$



$y = f(x)$

$f'(x) > 0$

(any monotonically ↑ fn. will do)

(We will force y to be equal to $f(x)$ & restrict to a 1-D integral)

$\longrightarrow$ $\boxed{\text{Insert } \delta(y - f(x))}$

$$I = \frac{\int dx\, dy\, e^{-\lambda y^2} y^2\, \delta(y - f(x))}{\int dx\, dy\, e^{-\lambda y^2}\, \delta(y - f(x))}$$

$$= \frac{\int_{-\infty}^{\infty} dx\, e^{-\lambda (f(x))^2} (f(x))^2}{\int_{-\infty}^{\infty} dx\, e^{-\lambda (f(x))^2}}$$

(This has problem bcos the final ans. depends on the choice of $f(x)$ — although, ∵ $f(x)$ is mon. ↑, the final ans. is finite) → so $f(x)$ cuts each gauge orbit only once

Put $u = f(x)$, ~~$x(g(u))$~~ $x = g(u)$

$\therefore \; dx = g'(u)\, du$

$$\frac{\int_{-\infty}^{\infty} du\, g'(u)\, e^{-u^2} u^2}{\int_{-\infty}^{\infty} du\, g'(u)\, e^{-u^2}}$$

$g \rightarrow$ inverse fn. of f & so depends on the choice of 'f' & ∴ choice of gauge slice

[So the procedure is almost right, but not completely]

[The whole meaning of gauge inv. is that the ans. should not depend on which ~~pt~~ gauge slice we choice — gauge redundancy of course we ~~take~~ a gauge fix & calculate the ans. — but the ans. should be gauge choice indep.]

(By gauge-choice, we break manifest gauge inv. — I is no longer inv. under $x \rightarrow x+a$)

(Take $f(x) = x$ & $f(x) = x^3$ for another & see what happens)

We will show that the original integral can be manipulated to give

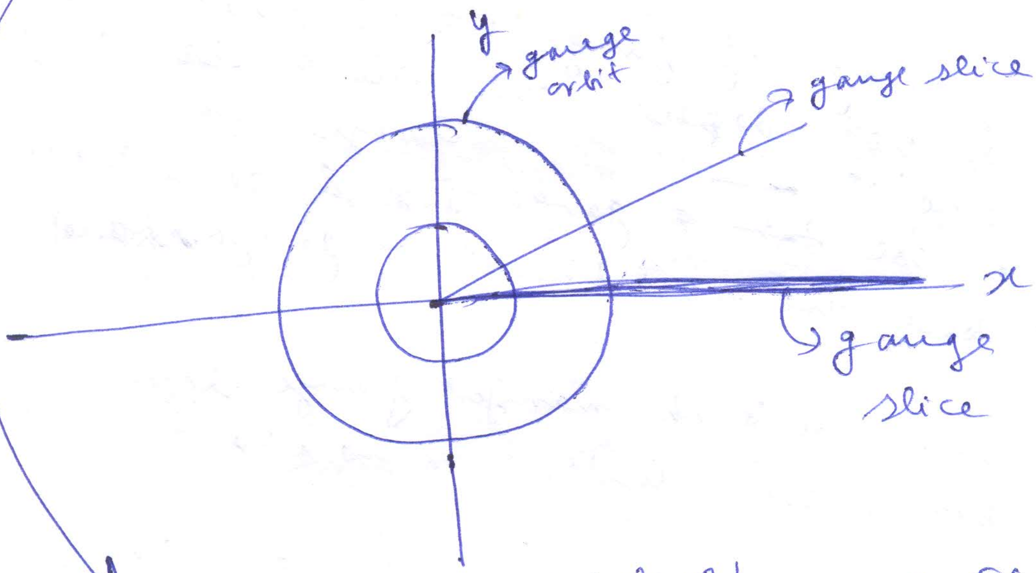
$$\frac{\int dx\, dy\, e^{-\lambda y^2}\, y^2 f'(x)\, \delta(y - f(x))}{\int dx\, dy\, e^{-\lambda y^2} f'(x)\, \delta(y - f(x))}$$

$$= \frac{\int dx\, e^{-\lambda (f(x))^2} (f(x))^2 f'(x)}{\int dx\, e^{-\lambda f(x)^2} f'(x)}$$

$$\Big\downarrow\ u = f(x) \Rightarrow du = \left(\frac{df}{dx}\right) dx$$

$$\frac{\int du\, e^{-\lambda u^2} u^2}{\int du\, e^{-\lambda u^2}}$$

$$\frac{\int dx\, dy\, e^{-\lambda(x^2+y^2)} (x^2+y^2)}{\int dx\, dy\, e^{-\lambda(x^2+y^2)}}$$



$$\frac{\int dx\, dy\, e^{-\lambda(x^2+y^2)} (x^2+y^2)\, \delta(y)}{\int dx\, dy\, e^{-\lambda(x^2+y^2)}\, \delta(y)}$$

corr. to choosing $f(x) = 0$ so that our gauge slice is $y = 0$

$$= \frac{\int dx\, e^{-\lambda x^2} x^2}{\int dx\, e^{-\lambda x^2}}$$

$$\text{Original integral} = \frac{\int r\, dr\, d\theta\, e^{-\lambda r^2} r^2}{\int r\, dr\, d\theta\, e^{-\lambda r^2}}$$

$$= \frac{\int r^3 e^{-\lambda r^2} dr}{\int r\, e^{-\lambda r^2} dr}$$

We haven't taken into account that
diff. gauge orbit has diff. vol.
— have to account for it to get the
correct answer