13/10/06

Consider a set of variables $u_1, u_2, \ldots, u_n$, which are the integration variables.

We want to calculate

$$\langle O \rangle = \frac{\int \left(\prod_i du_i\right) e^{iS(\vec{u})} \mathcal{O}(\vec{u})}{\int \left(\prod_i du_i\right) e^{iS(\vec{u})}}$$

This in principle, can be a prdt. of several fields

$n = 4NK$ for pure gauge theory

in 3+1 dim. spacetime

K → dim. of G

N → # of lattice points

Symmetries :- $u_i \xrightarrow[\text{transform}]{\text{gauge}} F_i(\vec{u}, \vec{\theta})$

$\vec{\theta}$ — gauge trs. parameters

a set of sym. transfns. labelled by θ_α's

At each lattice pt. we will have K gauge trs. parameters — so total = NK

$\theta_\alpha : \alpha = 1, 2, \ldots, NK$

$\mathcal{G}$:- group of symmetries generated by $\vec{\theta}$

dimension NK

G → gauge group

$$\mathcal{G} = G \otimes G \otimes \cdots \otimes G \quad (N \text{ times})$$

$S(\vec{u}) = S(\vec{F}(\vec{u}, \vec{\theta}))$ → action is invariant

$\mathcal{O}(\vec{u}) = \mathcal{O}(\vec{F}(\vec{u}, \vec{\theta}))$ → $\mathcal{O}$ is a gauge invariant operator

assumptions

If $v_i = F_i(\vec{u}, \vec{\theta})$, then $\prod_{i=1}^{n} dv_i = \prod_{i=1}^{n} du_i$

i.e., $\det\left(\frac{\partial F_i(\vec{u}, \vec{\theta})}{\partial u_j}\right) = 1$

If these are not satisfied, then there is no hope to get gauge-inv. result becos the integration measure on change of variables shouldn't change, in addn. to $O(\vec{u}') = O(\vec{F}(\vec{u}', \vec{\theta}'))$

Ex. Check that $\prod_{\mu=0}^{3} \prod_{a=1}^{K} \mathcal{D} A_\mu^a$ is gauge invariant.

(i.e.) check that the int. measure doesn't change under gauge trs. in the above case)

[We can separately take the $\prod du_i$ & the integral to be non-gauge inv. but the 2 together to be gauge inv. But to make things simpler, the above method is better.) Here, the whole analysis will be carried out assuming that there is a symmetry grp. $\mathcal{G}$ — it's another matter that $\mathcal{G} = G \otimes G \otimes \cdots G$]

$$\langle O \rangle = \frac{\int \left(\prod_i du_i\right) e^{iS(\vec{u})} \, O(\vec{u})}{\int \left(\prod_i du_i\right) e^{iS(\vec{u})}} = \frac{N}{D}$$

Suppose we have some representation of $\mathcal{G}$

not necessarily the direct prodt. matrix repn.

$\mathcal{G} = G \otimes G \otimes \cdots \otimes G$

grp. element $\to (U_1, U_2, \ldots, U_{NK}) \to V$

$v_{i_1 i_2 \cdots i_{NK}}$ (d^{NK} dim rep.)

$\downarrow$

d dim rep.

$\Rightarrow (U_1)_{i_1 i_1'} (U_2)_{i_2 i_2'} \cdots (U_{NK})_{i_{NK} i_{NK}'}$

$\downarrow$

one possible repn.

$\Rightarrow v_{i_1' \cdots i_{NK}'}$

Generators of $\mathcal{G}$ are T^A $[A = 1, 2, \ldots, NK]$

Now,

$$U(\vec{\theta}' + \delta\vec{\theta}) \, U(\vec{\theta}')^{-1} = \mathbb{1} + i T^A \delta_W \theta^A \to \text{define } \delta_W \theta^A$$

This relation is indep. of the rep. (or parametrisation) we choose — the relation bet. $\delta_W \theta^A$ & $\delta\vec{\theta}$ will of course depend on the choice of repn.

A physical field can't ~~do~~ ~~be~~ be rep. by diff. rep. at diff. pts. — but mathematically, there is no reason why $i_1, i_2, \ldots i_{NK}$ will belong to the same rep.

dep. on the choice of repn.

$$\therefore \delta_W O^A = \sum_B S_{AB}(\vec{\theta})\,\delta O^B + O(\delta O^2)$$

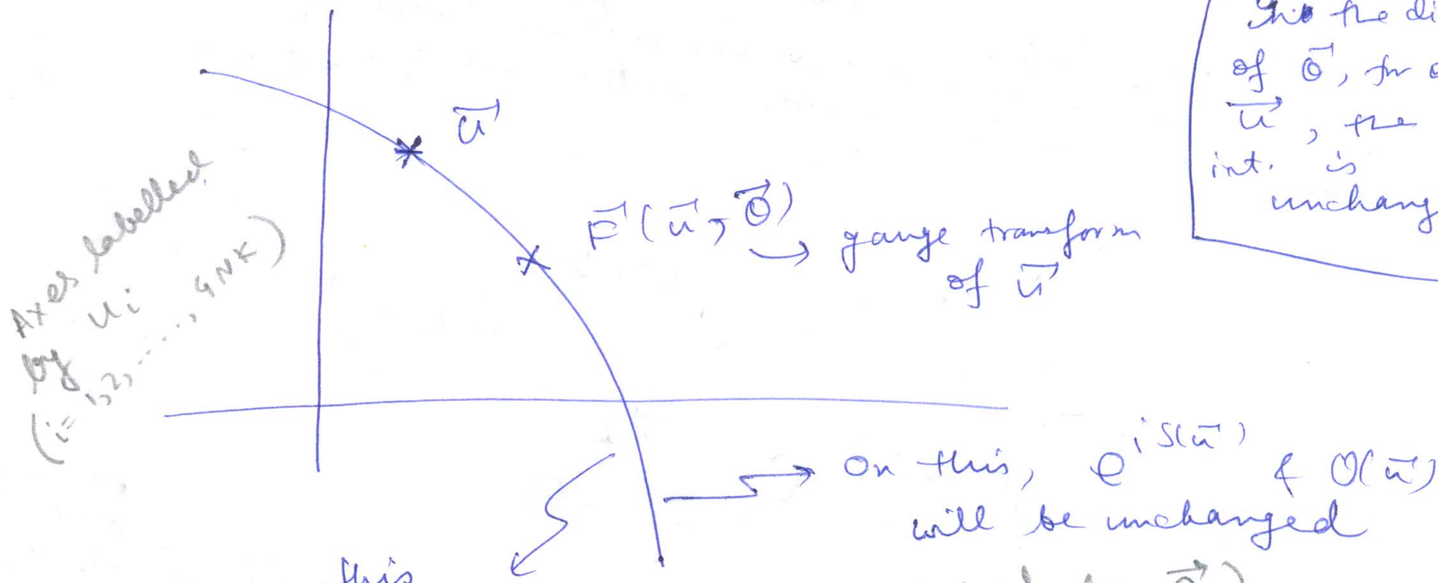
→ general str. of $\delta_W O^A$, bcos for $\delta\vec{O} = 0$, $\delta_W O^A = 0$

We haven't made any commitment about the parametrisation of $U(\vec{\theta})$

Now,

$$N = \int \prod_i du_i\, e^{iS(\vec{u})}\, O(\vec{u})$$

In the dirn. of $\vec{\theta}$, for every $\vec{u}$, the int. is unchanged



The whole space is $4Nk$ dimensional.

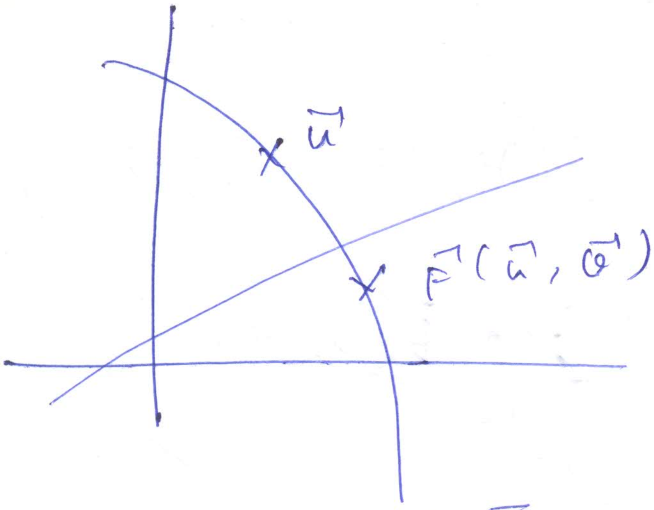
For closed orbit, then int. around it won't give ∞

Total volume $= (V_a)^N$ $a \to k$-dim. group

[For each lattice pt., we have a finite vol. V_a (for a finite group) — but there are N such pts. — & in the $N \to \infty$ limit, we get ∞]

[In lattice gauge theory, we don't gauge fix bcos we take a finite grp. & a finite no. of lattice pts. — $(V_a)^N$ cancels in N & D]

— Div. comes in the continuum ($N \to \infty$) limit.



Dim. of the gauge slice $= 4NK - NK = 3NK$

We need to have NK eqns to have 3NK-dim gauge slice in 4KN-dim space

Gauge fixing condition :-

$\mathcal{H}_A(\vec{u}) = \underbrace{B_A}_{\text{constants}}$

functions

For any $\vec{u}$, $\mathcal{H}_A(\vec{F}(\vec{u},\vec{\theta})) = B_A$ is true for a single $\vec{\theta}$. → orbit is intersected only once

Naive guess :- Insert $\prod_A \delta(\mathcal{H}_A(\vec{u}) - B_A)$ in the integrand

But we don't recover the integral we began with by this procedure, as we have seen already

Begin with the identity!

$$\int \left(\prod_A d\theta_A\right) \prod_A \delta\left(\mathcal{H}_A(\vec{F}(\vec{u},\vec{\theta})) - B_A\right) \det\left(\frac{\partial \mathcal{H}_A(\vec{F}(\vec{u},\theta))}{\partial \theta_B}\right) = 1$$

think of $\vec{u}$'s as fixed objects here

Now, $$\int \prod_i dx^i \prod_j \delta(f_j(\vec{x})) = \left.\frac{1}{\det\left(\frac{\partial f_j}{\partial x^i}\right)}\right|_{f_j(\vec{x})=0}$$

Sum over all $f_i(\vec{x}) = 0$ for multiple zeroes

$\Updownarrow$

In other words,

$$\int \prod_i dx^i \prod_j \delta(f_j(\vec{x})) \det\left(\frac{\partial f_j(\vec{x})}{\partial x^i}\right) = 1$$

$$\therefore \quad N = \int \prod_i du_i \, e^{iS(\vec{u})} \, O(\vec{u})$$

$$\times \int \left(\prod_A d\theta^A \right) \prod_A \delta\left(\mathcal{H}_A(\vec{F}(\vec{u}, \vec{\theta})) - B_A \right)$$

$$\det \left(\frac{\partial H_A(\vec{F}(\vec{u}, \theta))}{\partial \theta_B} \right)$$

$\boxed{\text{We have increased the no. of int. variables by inserting the delta functions}}$

(We will first do the u_i-int. for a fixed $\vec{\theta}$ & then the θ^A-int.) → $\boxed{\text{Change the int. order}}$

Define $v_i = F_i(\vec{u}, \vec{\theta})$

and change variable from $\vec{u}$ to $\vec{v}$.

Using → $S(\vec{u}) = S(\vec{v})$ → $\boxed{\because \text{ action is gauge inv.}}$

& $O(\vec{u}) = O(\vec{v})$ → $\boxed{\because \text{ Op. is gauge inv.}}$

& $\prod_i du_i = \prod_i dv_i$ → $\boxed{\because \text{ the measure is gauge inv.}}$

Then,

$$N = \int \left(\prod_A d\theta^A \right) \int \left(\prod_i dv_i \right) e^{iS(\vec{v})} \, O(\vec{v})$$

$$\prod_A \delta(\mathcal{H}_A(\vec{v}) - B_A) \det \left(\frac{\partial H_A(\vec{F}(\vec{u}, \vec{\theta}))}{\partial \theta_B} \right)$$

(there is still some θ-dependence in this part of the ~~second~~ integral - so θ-dep. part is still not decoupled)

$\boxed{\text{this we don't yet know how to handle}}$

Now,

$$\frac{\partial H_A(\vec{F}(\vec{u},\vec{\theta}))}{\partial \theta_B} = \frac{\partial H_A(\vec{v})}{\partial v_i}\,\frac{\partial F_i(\vec{u},\vec{\theta})}{\partial \theta_B}$$

by Chain rule

$\vec{v}$ is now an indep. variable

Defn of $\frac{\partial F_i(\vec{u},\vec{\theta})}{\partial \theta_B}$ is

$$F_i(\vec{u}, \vec{\theta}+\delta\vec{\theta}) - F_i(\vec{u},\vec{\theta}) = \frac{\partial F_B(\vec{u},\vec{\theta})}{\partial \theta_B}\,\delta\theta_B$$

Use the group property

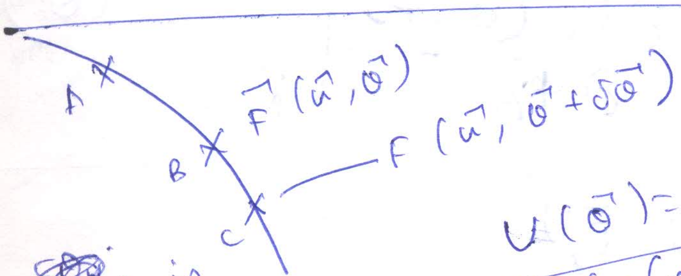
$$U(\vec{\theta}+\delta\vec{\theta})\,U(\vec{\theta})^{-1} = 1 + i\,\delta_W \theta^A T^A$$

$$\Rightarrow \quad U(\vec{\theta}+\delta\vec{\theta}) = (1 + i\,\delta_W \theta^A T^A)\,U(\vec{\theta}) = U(\delta_W\theta^A)\,U(\vec{\theta})$$

(This physically means) →

$\Longrightarrow$ Trs. by $\vec{\theta}+\delta\vec{\theta}$ = trs. by $\vec{\theta}$ followed by a trs. by $\delta_W\theta^A$

$$\therefore \quad F_i(\vec{u}, \vec{\theta}+\delta\vec{\theta}) = F_i(\vec{F}(\vec{u},\vec{\theta}), \delta_W\vec{\theta}) = F_i(\vec{v}, \delta_W\vec{\theta})$$



($\delta_W\vec{\theta}$ is causing the change from B to C)

$$U(\vec{\theta}) = \exp(i\theta_A T^A)$$

$$\therefore \quad \exp(i(\theta_A + \delta\theta_A)T^A)\,\exp(-i\theta_A T^A) = 1 + i\,\delta_W \theta^A T^A$$

The infinitesimal transfn. that connects these two isn't $\delta\vec{\theta}$ — it is $\delta_W\vec{\theta} \neq \delta\vec{\theta}$ in general if T^A's don't commute

If the generators don't commute, $\delta_W\theta^A \neq \delta\theta_A$

$$\therefore F_i(\vec{u}, \vec{\theta} + \delta\vec{\theta}) = F_i(\vec{v}, 0) + \left.\frac{\partial F_i(\vec{v}, \vec{\Phi})}{\partial \Phi_B}\right|_{\vec{\Phi}=0} \delta_W \theta_B + O(\delta_W \theta_B^2)$$

(By Taylor expn)

~~$\partial\theta$~~ we have seen that, $\vec{u} \to \vec{F}(\vec{u}, \vec{\theta})$

If $\vec{\theta} = 0$, $\vec{u} \to \vec{u}$

$$= v_i + \left.\frac{\partial F_i(\vec{v}, \vec{\Phi})}{\partial \Phi_B}\right|_{\vec{\Phi}=0} \delta_W \theta_B$$

$$F_i(\vec{u}, \vec{\theta}) = v_i$$

$$\Rightarrow F_i(\vec{u}, \vec{\theta} + \delta\vec{\theta}) - F_i(\vec{u}, \vec{\theta}) = \left.\frac{\partial F_i(\vec{v}, \vec{\Phi})}{\partial \Phi_B}\right|_{\vec{\Phi}=0} \times \delta_W \theta_B$$

But $\left.\dfrac{\partial F_i(\vec{v}, \vec{\Phi})}{\partial \Phi_B}\right|_{\vec{\Phi}=0} \delta_W \theta_B$

$\qquad \qquad \qquad \qquad \qquad \qquad \shortparallel \; S_{BC}(\theta)\, \delta\theta^C$

$$= \left.\frac{\partial F_i(\vec{v}, \vec{\Phi})}{\partial \Phi_B}\right|_{\vec{\Phi}=0} S_{BC}(\theta)\, \delta\theta^C$$

Compare this with $\dfrac{\partial F_i(\vec{u}, \vec{\theta})}{\partial \theta_C} \delta\theta_C$

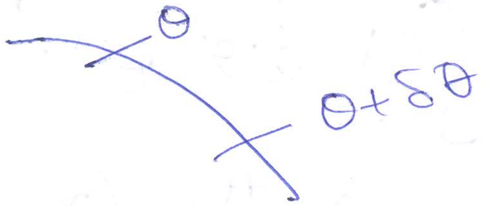
$$\Rightarrow \frac{\partial F_i(\vec{u}, \vec{\theta})}{\partial \theta_C} = \left.\frac{\partial F_i(\vec{v}, \vec{\Phi})}{\partial \Phi_B}\right|_{\vec{\Phi}=0} S_{BC}(\vec{\theta}).$$

Hence, $\dfrac{\partial H_A(\vec{v})}{\partial v_i} \dfrac{\partial F_i(\vec{u}, \vec{\theta})}{\partial \theta_B}$

$$= \frac{\partial H_A(\vec{v})}{\partial v_i} \left.\frac{\partial F_i(\vec{v}, \vec{\Phi})}{\partial \Phi_D}\right|_{\vec{\Phi}=0} S_{DB}(\vec{\theta})$$

We have eliminated $\vec{u}$ in terms of $\vec{v}$

→ these are not gauge-inv. quantities & $\therefore$ can't replace $\vec{u}$ by $\vec{v}$ for them - so look at the relation betn. $\vec{u}$ & $\vec{v}$ & ~~cat~~ derive this final result

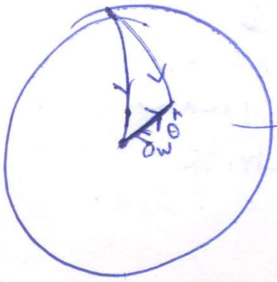


The gauge trs. that takes us from Θ to $\Theta + \delta\Theta$ is

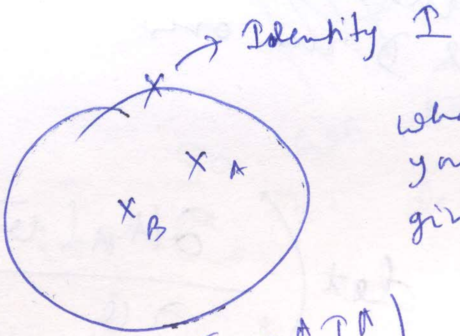
$\delta_W \Theta$ → completely a prop. of the grp.

For any Abelian grp.,

$$\delta_W \Theta^A = \delta \Theta_A$$

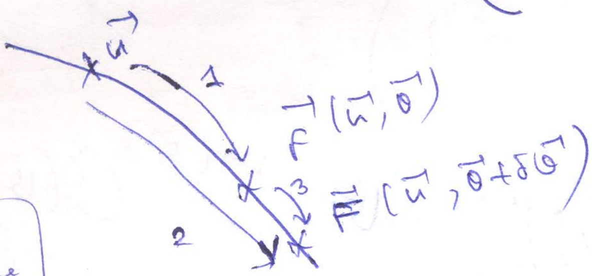


→ If the grp. space has curvature



What grp. element should you multiply with A to give you the grp. element B? ⟶ $U(\delta_W \Theta^A)$

$$\left(\mathbb{1} + i\delta_W \Theta^A T^A\right)$$



These trs. follow grp. prop.

So 2nd element = prdt. of first & 3rd elements

R_a has unique ~~uni~~group prop. (a uni grp?)

$\Big[\exp(-i\Theta_A T^A)$ ⟶ Rotates the generators in the sense that it doesn't ~~com~~ commute with T^A

→ can't add the arguments of $\exp(-i(\Theta_A + d\Theta_A)T^A)$ & $\exp(-i\Theta_A T^A)\Big]$

$$\mathcal{N} = \int \left(\prod_A d\theta^A\right) \int \prod_i d\vartheta_i \, e^{iS(\vec{\vartheta})} \, \mathcal{O}(\vec{\vartheta})$$

$$\times \prod_A \delta\left(H_A(\vec{\vartheta}) - B_A\right)$$

$$\times \det\left(\frac{\partial H_A(\vec{F}(\vec{u},\vec{\theta}))}{\partial \theta_B}\right)$$

this isn't a square matrix

$$\because \left\{ \frac{\partial H_A(\vec{\vartheta})}{\partial \vartheta_i} \frac{\partial F_i(\vec{\vartheta}, \vec{\phi})}{\partial \phi_D}\Big|_{\phi=0} \right\} S_{DB}(\vec{\theta})$$

this is a square matrix bcs both A & D run over N_k values

another square matrix

$$\therefore \det\left(\frac{\partial H_A(\vec{F}(\vec{u},\vec{\theta}))}{\partial \theta_B}\right) = \det\left(\frac{\partial H_A(\vec{\vartheta})}{\partial \vartheta_i} \frac{\partial F_i(\vec{\vartheta}, \vec{\phi})}{\partial \phi_D}\Big|_{\vec{\phi}=0}\right) \times \left(\det S(\vec{\theta})\right)$$

Hence,

$$\mathcal{N} = \left\{ \int \left(\prod_A d\theta^A\right) \det S(\vec{\theta}) \right\} \int \prod_i d\vartheta_i \, e^{iS(\vec{\theta})} \, \mathcal{O}(\vec{\vartheta})$$

$$\prod_A \delta\left(H_A(\vec{\vartheta}) - B_A\right) \det\left(\frac{\partial H_A(\vec{\vartheta})}{\partial \vartheta_i} \frac{\partial F_i(\vec{\vartheta},\vec{\phi})}{\partial \phi_D}\right)\Big|_{\vec{\phi}=0}$$

factors out & cancels with the denominater

this is the standard measure of the grp. elements

this measure is indep. of the parametrisation

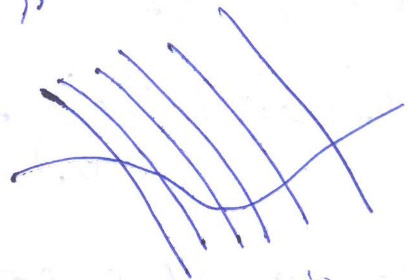
measure $\prod_A d\theta^A$ is dep. on parametrisation

Haar measure (of the grp.)

sampling more in this region than in this region first case (making a small

[~~not~~ grazing in the slope) & large slopes in the 2nd case]

Slope det. how it samples



So we ~~is~~ ~~have~~ are forced to include the det $\left(\frac{\partial H_A(\vec{v})}{\partial v_i} \frac{\partial f_i(\vec{v},\vec{\phi})}{\partial \phi_D}\right)\Big|_{\phi=0}$

to compensate for the fact that the sampling of pts. is dep. on slope.

[Int. $\longrightarrow$ limit of summation – so this is necessary that sampling is uniform]

(We have factored out ~~the~~ & cancelled vol. of gauge grp. element) — (But the factor for the config. space is still remaining)