26/10/06

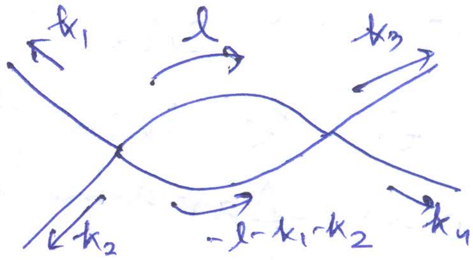
① Regularization

② Renormalization

(Instead of treating the original parameters as the input parameters, we take the renormalized parameters)

In ϕ^4 th., we have 2 parameters $\rightarrow \lambda$ & m

By adjusting finite no. of parameters we have to make infinite no. of quantities finite - remove ∞ no. of div. — But we need to make a limited class of div. diag. conv.



$$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{\{-l^2 - m^2 + i\epsilon\}\{-(k_1+k_2+l)^2 - m^2 + i\epsilon\}}$$

$\frac{\partial}{\partial k_1^{\mu}}$ act on this

we get

$$\frac{(k_1+k_2+l)_\mu}{\{-(k_1+k_2+l)^2 - m^2 + i\epsilon\}^2}$$

6 powers of l in denominator & 5 powers of l in num — so it isn't div.

this expression has the property that only the first term in the Taylor series expansion is divergent; all other terms are finite

Regularization

It's not that for every new k we have a new condn — once you make it finite for one choice of ext. mom., it will be finite for any other choice — so div. are not unrelated — so adjustment of a finite quantities is needed, though ~~apparently~~ naively we would have thought there were infinite no. of condns to be satisfied

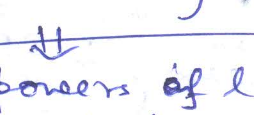
Make infinite integrals finite with the help of a cut-off ϵ.

As $\epsilon \rightarrow 0$, initially divergent integrals become divergent.

Consistency condition :- Initially finite integrals must approach their original values.

(otherwise you are changing the theory)

Adding more loops don't increase the div. — prop. of renormalizable th. ! — e.g.



But non-renormalizable th. — adding of more loops ↑ div.

Dimensional regularization

Work in $\underline{4-\epsilon}$ dimension

↓ by this we mean

Derive the formula by pretending that our QFT lives in 4 − ε dimension with integer ε & then treat ε as a continuous parameter.

Take $\epsilon \to 0$ limit $\xrightarrow[\text{us}]{\text{gives}}$ 4-dimensional theory

Working in $4+\epsilon$ doesn't help bcos reducing the no. of int. helps in reducing the div.

Working in continuous dim. has no meaning in QFT

Rules (formal) :-

$$\int \frac{d^4 k}{(2\pi)^4} \xrightarrow[\text{replaced by}]{\text{gets}} \int \frac{d^{4-\epsilon} k}{(2\pi)^{4-\epsilon}}$$

$$(2\pi)^4 \delta^{(4)}(k) \longrightarrow (2\pi)^{4-\epsilon} \delta^{(4-\epsilon)}(k)$$

$$\eta^{\mu\nu}\eta_{\mu\nu} = 4-\epsilon \qquad \boxed{D = 4-\epsilon}$$

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$$ (this doesn't depend on dim.)

$$\gamma_\mu \gamma^\mu = -(4-\epsilon) \longrightarrow \because (\gamma^0)^2 = 1 \;\&\; (\gamma^i)^2 = -1$$

$\gamma_\mu\gamma^\mu\gamma^\nu + \gamma_\mu\gamma^\nu\gamma^\mu = -2\eta^{\mu\nu}\gamma_\mu$

$\Rightarrow -(4-\epsilon)\gamma^\nu + 2\gamma^\nu + \gamma_\mu\gamma^\nu\gamma^\mu = 0$

$\Rightarrow \gamma_\mu\gamma^\nu\gamma^\mu = +(2-\epsilon)\gamma^\nu$

$$\gamma_\mu \gamma^\nu \gamma^\mu = (D-2)\gamma^\nu = (+2-\epsilon)\gamma^\nu \text{ etc.}$$

Dimension of γ-matrix in (4 − ε) dimension :-

Dim. of the γ-matrix = Dim. of the spinor repn.

~~SO(3,1)~~ SO(3−ε,1) is a subgroup of SO(3,1)

(4-dim. spinor is also a repn. of lower-dim. spinor)

One choice :- Take this to be 4

General " :- Take this to be $f(\epsilon)$

$f(0) = 4$

we can continue to use the higher-dim. repn., i.e., SO(3,1) repn. for SO(3−ε,1) also, though it can be a reducible repn. for this lower dim.

In this case, $\mathrm{Tr}(\gamma_\mu \gamma_\nu) = -f(\epsilon)\eta_{\mu\nu}$

Final result is independent of the choice of $f(\epsilon)$ as long as $f(0) = 4$ → one can show this

Integration rules :-

$$\boxed{\lim_{\epsilon\to 0} \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{1}{(k^2 + L - i\varepsilon)} = +i \frac{\Gamma(\epsilon/2) L^{1-\epsilon/2}}{(4\pi)^{2-\epsilon/2}(1-\epsilon/2)}}$$

In Minkowski space ; $k^2 = -(k^0)^2 + \vec{k}^2$

Note → ε has nothing to do with ϵ

$\varepsilon \xrightarrow[\text{to}]{\text{can go}} 0$

Scaling of L :-

Define $k = \sqrt{L}\, u$ (change variable)

∴ we have

scaling analysis

$$\int \frac{L^{\frac{4-\epsilon}{2}} d^{4-\epsilon}u}{(2\pi)^{4-\epsilon} L (u^2+1-i\varepsilon)} = L^{1-\epsilon/2} \int \frac{d^{4-\epsilon}u}{(2\pi)^{4-\epsilon}(u^2+1-i\varepsilon)}$$

(Whenever the integral is finite, it should agree with the RHS)

Differentiate this relation w.r.t. L n-times :-

LHS :-

$$\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{1}{(k^2+L-i\varepsilon)^{n+1}} (-1)(-2)\cdots(-n)$$

RHS :-

$$-i \frac{\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}(1-\epsilon/2)} (1-\epsilon/2)(-\epsilon/2)\cdots(1-\epsilon/2-(n-1)) \times L^{1-\epsilon/2-n}$$

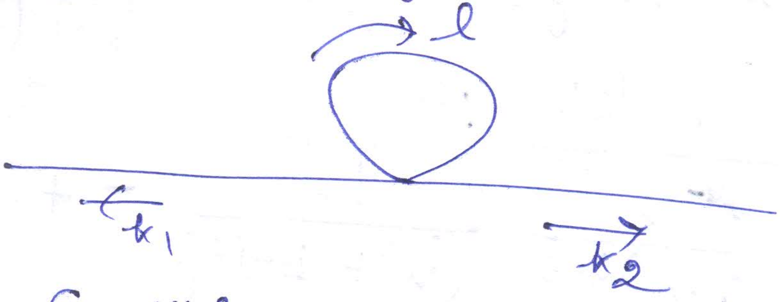
(These two must be equal)

Can be checked for $4-\epsilon < 2(n+1) \Rightarrow$ So that LHS is a finite int. → num. has more powers of k

or $\varepsilon > 4-2n-2 = 2-2n$

(Any part which isn't div., can be checked with $\epsilon \to 0$)

e.g. $\Rightarrow$ In ϕ^4 theory, consider the following graph:—



$$(\ldots) \int \frac{d^4 l}{(-l^2 - m^2 + i\varepsilon)}$$

"

$$-\int \frac{d^4 l}{l^2 + m^2 - i\varepsilon} \rightarrow -\int \frac{d^{4-\epsilon} l}{l^2 + m^2 - i\varepsilon}$$

$$= (2\pi)^{4-\epsilon} (i) \frac{\Gamma(\epsilon/2)\,(m^2)^{1-\epsilon/2}}{(4\pi)^{2-\epsilon/2}\,(1-\epsilon/2)}$$

these two can't be compared bcos the left one is div.

You don't get this kind of graphs if we used normal-ordered vertex — But this is an allowed diag. if we don't use normal-ordering — With renorm., we must get rid of all div.

Beyond the first 2 terms of the Taylor series expn., the left one isn't div.

Div. integrals have hidden in them finiteness

You regularize the div. & not the finite quantities

Infinite pieces are diff. in diff. reg. schemes — But we will see later that it won't matter

Sym. isn't lost in dim. reg. — you maintain Lor. inv. & gauge inv. throughout

Lor. Inv. & Gauge Inv. can hold by writing the th. appropriately

Renormalization

Valid for any scheme of regularization

Consider a field theory with fields $\phi_1, \phi_2, \ldots, \phi_N$ $\rightarrow$ scalars, vectors, fermions

$i \rightarrow$ contains all Lorentz, Dirac or vector indices

Parameters :- $g_1, \ldots, g_M \rightarrow$ masses, coupling constants etc.

Ansatz :- $g_s = F_s(\underbrace{g_{R1}, \ldots, g_{RM}}_{\vec{g}_R}; \epsilon)$

this fn. is actually infinite in $\epsilon \to$ limit

Our goal is s-matrix elements are finite in this limit

g_s need not be finite

$\vec{g}_R \downarrow$ set of physical parameters

limit $\epsilon \to 0$ is taken at fixed $\vec{g}_R$

$$\phi_i = \sum_{j=1}^{N} \left(\tilde{Z}^{1/2}_{(\vec{g}_R, \epsilon)}\right)_{ij} \phi_{jR}$$

ϕ_{jR} $\downarrow$ (Renormalized fields)

ϕ_i $\downarrow$ nobody tells us that they create normalized states out of vacuum

$S \xleftrightarrow[\text{to}]{\text{related}} \left\langle \prod_i \phi_i(x_i) \right\rangle \times \underbrace{(\ldots)}_{\text{factors}}$

$\times Z_i^{1/2}$ $\downarrow$ (wave-fn. renormalizing factors)

functions of $\vec{g} \rightarrow$ functions of $\vec{g}_R$

$\rightarrow Z_i^{1/2}(\vec{g}_R, \epsilon)$

Nobody tells us that $Z_i^{1/2}$ is finite but the prdt. of $\langle \prod_i \phi_i \rangle$ & $Z_i^{1/2}$ must be finite — New (renormalized) ϕ_i is should give finite corr. fns.

Div. piece of $Z_i^{1/2}$ must be absorbed into the redefn. of ϕ_i's

$$\phi_i = \sum_{j=1}^{N} \left(\tilde{Z}(\vec{g}_R, \epsilon)\right)^{1/2}_{ij} \phi_{jR}$$ such that $\left\langle \prod_i \phi_{iR}(x_i) \right\rangle$ is finite as $\epsilon \to 0$ at fixed $\vec{g}_R$.

Goal :- To show that we can find functions $F_s(\vec{g}_R, \epsilon)$ and $(\tilde{Z}^{1/2})_{ij}(\vec{g}_R, \epsilon)$ such that

$\langle \prod_i \phi_{iR}(x_i) \rangle$ is finite as $\epsilon \to 0$.

Treat $\phi_{iR}(x)$ as the basic fields & $\vec{g}_R$ as the basic parameters.

$$\langle \tilde{\Phi}_{iR}(p_1) \tilde{\Phi}_{jR}(p_2) \rangle = (2\pi)^4 \delta^{(4)}(p_1 + p_2) \left[\frac{Z^R_{ij}}{-p_1^2 - m^2_{ij} + i\epsilon} + \cdots \right]$$

finite by defn.

$Z^R_{ij} \to$ finite ($\because$ residue at the pole can't be ∞ for a finite th.)

finite (location of the pole is finite if the th. is finite)

$\to$ masses that come out of the th. are finite

($m_{ij} \to$ pole — mass of a possible particle)

S-matrix: $\longrightarrow \langle \prod_i \tilde{\Phi}_{iR}(p_i) \rangle \times Z^R$'s

$\to$ finite

Q) How do we find F_s & $(\tilde{Z})_{ij}$?

for all th. you can't find F_s & $\tilde{Z}^{1/2}$ $\longrightarrow$ non-renorm. theories

functional form of F_s depends on the choice of the set $\vec{g}_R$

$g_{R1} = \tilde{g}_{R1} + \lambda \tilde{g}^2_{R1}$

equally good choice

finite reparametrization of these parameters — this ambiguity is still there

Renormalization in ϕ^4 theory

$$S = \int \left(-\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{4!} \phi^4 \right) d^{4-\epsilon} x$$

correction must be $O(g_R^2)$

Ansatz
$$\begin{cases} g = F_1(g_R, m_R, \epsilon) = g_R\left[1 + g_R G_1(g_R, m_R, \epsilon)\right] \\ \phi = \tilde{Z}_\phi^{1/2}(g_R, m_R, \epsilon)\, \phi_R \\ m = F_2(g_R, m_R, \epsilon) \end{cases}$$

when coupling constant is small, tree level graphs are imp. which are non-div.

At tree level there is no need to renormalise ϕ

$$g = f_1(g_R, m_R, \epsilon) = g_R\left(1 + g_R G_1(g_R, m_R, \epsilon)\right)$$
$$\phi = \tilde{Z}_\phi^{1/2}(g_R, m_R, \epsilon)\, \phi_R = \phi_R\left(1 + g_R G_0(g_R, m_R, \epsilon)\right)$$
$$m = f_2(g_R, m_R, \epsilon) = m_R\left(1 + g_R G_2(g_R, m_R, \epsilon)\right)$$

We have used the fact that at the lowest order there is no UV divergence & hence no need to renormalize.

(We could have written this as) $\rightarrow$

$$g = g_R + O(g_2) = g_R + a g^2 + b g^3 + \cdots$$

lowest order in g means lowest order in g_R

(Solve iteratively) $\rightarrow$ $g = g_R$ to first order

Next iteration:–
$$g = g_R + a g_R^2$$

Next iteration:–
$$g = g_R + a\left(g_R + a g_R^2\right)^2 + b\, g_R^3$$
up to order g_R^3

etc.

In this expr., we are keeping ϵ fixed — kept a, b finite by keeping ϵ fixed

$$g = f_1(g_R, m_R, \epsilon) = g_R\left(1 + g_R\, G_1(g_R, m_R, \epsilon)\right) \equiv g_R\, Z_g$$

$$\phi = \tilde{Z}_\phi^{1/2}\, \phi = \phi_R\left(1 + g_R\, G_0(g_R, m_R, \epsilon)\right)$$

$$\equiv \phi_R\, \tilde{Z}_\phi^{1/2}$$

$$m = f_2 = m_R\left(1 + g_R\, G_2(g_R, m_R, \epsilon)\right) \equiv m_R\, Z_m$$

$$\mathcal{L} = \tfrac{1}{2}\tilde{Z}_\phi\left(-\partial_\mu \phi_R\, \partial^\mu \phi_R - Z_m^2\, m_R^2\, \phi_R^2\right)$$

$$- \tfrac{1}{4!}\, g_R\, \tilde{Z}_\phi^2\, Z_g\, \phi_R^4$$

~~everything~~ written in terms of renorm. couplings & renorm. fields

Goal :- Adjust $\tilde{Z}_\phi$, Z_m & Z_g in such a way that all correlation functions $\langle \prod_i \phi_{iR}(x_i) \rangle$ remain finite as $\epsilon \to 0$. with fixed g_R, m_R.

We can do it bcos choice of f_s is in our hands

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For 2-pt. fn., the lowest order (tree-level) diag. is

Then comes 

Consider a regularized theory with Lagrangian density $\mathcal{L}(\vec{\Phi}, \vec{g})$.

(fn. of the fields & parameters)

$\phi_1, \phi_2, \ldots, \phi_N = \{\phi_i\}$

$g_1, g_2, \ldots, g_M = \{g_s\}$

The Regulater is ϵ.

$\left\langle \prod_{i=1}^{k} \phi_i(x_i) \right\rangle$ is divergent as $\epsilon \to 0$ at fixed $\vec{g}$.

Introduce new parameter $\vec{g}_R$ & new field $\vec{\Phi}_R$ via the relation

$$g_s = F_s(\vec{g}_R, \epsilon)$$

$$\phi_i = \left(\tilde{Z}^{1/2}(\vec{g}_R, \epsilon)\right)_{ij} \phi_{jR}$$

<u>Goal</u> :- Find appropriate functions ~~F_s & $(\tilde{Z}^{1/2})_{ij}$~~ $F_s(\vec{g}_R, \epsilon)$ & $(\tilde{Z}^{1/2})_{ij}(\vec{g}_R, \epsilon)$ such that $\left\langle \prod_{i=1}^{k} \phi_{iR}(x_i) \right\rangle$ is finite as $\epsilon \to 0$ with fixed $\vec{g}_R$.

[we can find fns which satisfy this — what fn. we choice is acc. to our convenience]

Choos $\vec{g}_R$ such that

$\vec{g}_R = \vec{g}$ at the lowest order

$\phi_R = \phi$ " " "

If we can achieve this, $\vec{g}_R$ will be our renorm. parameters

For M g_R's look at results of M exp. & look at the S-matrix elements — then determine the M g_R's — $\vec{g}_R$ isn't sth we directly observe in exp.; the S-matrix elements are obsvd. in exp. — Now make predictions for the outcomes of other exp. with these values obtd.

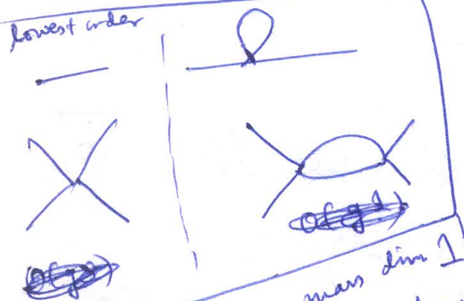
ϕ^4 field theory :-

$$\mathcal{L} = -\tfrac{1}{2}\,\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{g}{4!}\phi^4$$

Det. the finite no. of parameters by a finite no. of exp. — Everything else can be explained in terms of these finite no. of parameters once they are det.

Div. come at least at $O(g)$ — tree-level or leading order diag. don't have div.

We will focus only on the connected diag. bcos disconnected diag. are built out of connected diag.



Ansatz:

$$\begin{cases} \phi = \tilde{Z}^{1/2}(g_R, m_R, \epsilon)\,\phi_R, \quad \tilde{Z}^{1/2}(g_R, m_R, \epsilon) = 1 + O(g_R) \\ g = Z_g(g_R, m_R, \epsilon)\, g_R, \quad Z_g(g_R, m_R, \epsilon) = 1 + O(g_R) \\ m = Z_m(g_R, m_R, \epsilon)\, m_R, \quad Z_m(g_R, m_R, \epsilon) = 1 + O(g_R) \end{cases}$$

As long as these choices give the desired result, there is no reason why we can't make these choices for $\tilde{Z}^{1/2}$, Z_g & Z_m

In 4-D, ϕ has mass dim 1, g is dimensionless. In pert., it's useful to use a dimensionless parameter otherwise diff. order terms can't be compared easily

In $4-\epsilon$ dim, the action is

$$S = \int d^{4-\epsilon}x\, \mathcal{L} \rightarrow \text{dimensionless} \rightarrow$$ otherwise e^{iS} in path int. doesn't make sense

finding the mass dim. in $4-\epsilon$ dim $\rightarrow$

$$[\mathcal{L}] = 4-\epsilon$$

$$[\phi] = \frac{2-\epsilon}{2}$$

$$[m] = 1$$ $\rightarrow$ (which is good bcos m is the regular mass parameter)

$4-\epsilon = [g] + 4\times\frac{2-\epsilon}{2}$

$$\Rightarrow \quad [g] = \epsilon$$ $\rightarrow$ (coeff. of exp. of diff. powers of g will have diff. dim. — not good)

$\downarrow$

We will make g_R dimensionless

$\therefore \quad g = Z_g(g_R, m_R, \epsilon)\, \mu^\epsilon g_R$

where μ = arbitrary mass parameter

[Why don't we set m_R^ϵ? — μ gives an extra freedom - this new parameter is an useful thing to have - at the end one can set $\mu = m_R$ or ext. energy, or any other thing]

$$\therefore \mathcal{L} = -\tfrac{1}{2}\eta^{\mu\nu}\tilde{Z}(g_R, m_R, \epsilon)\,\partial_\mu\phi_R\,\partial_\nu\phi_R - \frac{1}{2}Z_m^2\tilde{Z}\,m_R^2\phi_R^2 - Z_g\tilde{Z}^2\frac{g_R\mu^\epsilon}{4!}\phi_R^4$$

We can rewrite $\mathcal{L}$ as:—

$$\mathcal{L} = \Big\{-\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_R\,\partial_\nu\phi_R - \tfrac{1}{2}m_R^2\phi_R^2 - \frac{1}{4!}g_R\mu^\epsilon\phi_R^4$$
$$-\tfrac{1}{2}\eta^{\mu\nu}(\tilde{Z}-1)\,\partial_\mu\phi_R\,\partial_\nu\phi_R - \tfrac{1}{2}\underbrace{(Z_m^2\tilde{Z}-1)}_{O(g_R)}m_R^2\phi_R^2 - \underbrace{(Z_g\tilde{Z}^2-1)}_{O(g_R)}\frac{g_R\mu^\epsilon}{4!}\phi_R^4$$

...can treat ϕ_R's as the fundamental fields - should give the same ans. ...n expressing ϕ's in ...terms of ϕ_R's at the end

...hich part of $\mathcal{L}$ we call ...e & which part int ...completely upto us as ...g as the free part is ...rable

this part looks like the original $\mathcal{L}$ with $\phi \to \phi_R$, $m \to m_R$ and $g \to g_R$

this coeff. is $O(g_R)$

$O(g_R)$

this part is called the <u>counter term Lagrangian</u>

(bcos the information of the adjustment of F's & Z's is completely contained here — coeff. should be adjusted in such a way that the div. coming from the other part cancel out)

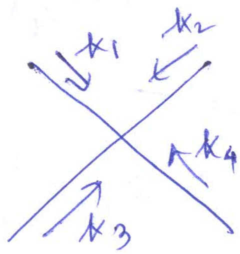
(We have already made the postulate that $\tilde{Z}$, $Z_m^2\tilde{Z}$ & $Z_g\tilde{Z}^2$ are $O(g_R)$ — of course we don't know what these are – we have set out to figure that out)

Counterterm $\longrightarrow$ Treat these as interaction terms

Feynman rules:–

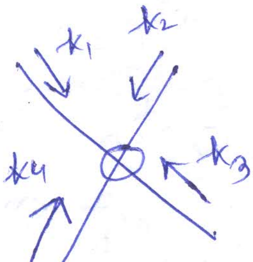
Propagator $\Rightarrow$

$$\frac{i}{-k_1^2 - m_R^2 + i\epsilon}(2\pi)^4\,\delta^{(4)}(k_1+k_2)$$

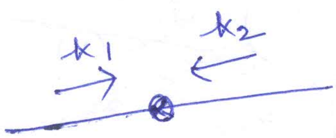


$$-i\,\frac{g_R}{4!}\,\mu^{\epsilon}\,(2\pi)^{4-\epsilon}\,\delta^{(4-\epsilon)}(k_1+k_2+k_3+k_4)$$

Counterterm vertices are denoted by a circle



$$-i\left(Z_g\tilde{Z}^2-1\right)\frac{g_R\,\mu^{\epsilon}}{4!}(2\pi)^{4-\epsilon}\,\delta^{(4-\epsilon)}(k_1+k_2+k_3+k_4)$$



$$\left[-\frac{i}{2}(\tilde{Z}-1)(k_1\cdot k_2)\,i^2-\frac{i}{2}(Z_m^2\tilde{Z}-1)m_R^2\right]$$
$$\times\,(2\pi)^{4-\epsilon}\,\delta^{(4-\epsilon)}(k_1+k_2)$$

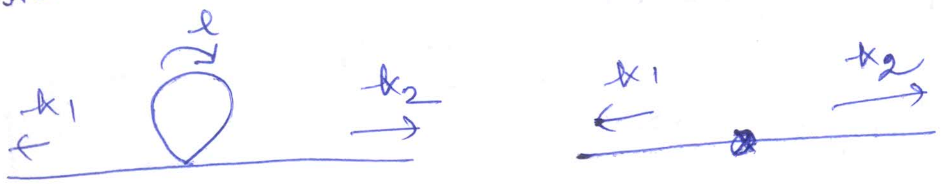
Adjust $\tilde{Z}$ & Z_m to remove div. – also we have committed to ourselves that
$\tilde{Z} = 1+\cdots$
$Z_m = 1+\cdots$

Two point function

(Renorm. procedure to 1 loop – 1 loop in the non-counterterm)

$\longrightarrow$ finite (at tree-level $O(g_R^0)$)

[Next contribution is of $O(g_R)$ — collect all terms of $O(g_R)$]



$$(2\pi)^{4-\epsilon}\,\delta^{(4-\epsilon)}(-k_1+k_2)\Bigg[\frac{i}{-k_1^2-m_R^2+i\varepsilon}\,\frac{i}{-k_2^2-m_R^2+i\varepsilon}\left(\frac{-i g_R\mu^\epsilon}{4!}\right)\int\frac{d^{4-\epsilon}l}{(2\pi)^{4-\epsilon}}\,\frac{i}{-l^2-m_R^2+i\varepsilon}\times\underbrace{4\times 3}_{\text{combinatorial factor}}$$

$$+\frac{i}{-k_1^2-m_R^2+i\varepsilon}\,\frac{i}{-k_2^2-m_R^2+i\varepsilon}\left\{-\frac{i}{2}(\tilde{Z}-1)(k_1^2)-\frac{i}{2}(Z_m^2\tilde{Z}-1)m_R^2\right\}\times\underbrace{2}_{\text{comb. factor}}\Bigg]$$

$$=(2\pi)^{4-\epsilon}\,\delta^{(4-\epsilon)}(k_1+k_2)\,\frac{i}{-k_1^2-m_R^2+i\varepsilon}\,\frac{i}{-k_2^2-m_R^2+i\varepsilon}$$

$$\Bigg[-i\,\frac{g_R\mu^\epsilon}{2}\times i\times i\times\frac{\Gamma(\epsilon/2)\,(m_R^2)^{1-\epsilon/2}}{(4\pi)^{2-\epsilon/2}(1-\epsilon/2)}-i(\tilde{Z}-1)k_1^2-i(Z_m^2\tilde{Z}-1)m_R^2\Bigg]$$

$\Gamma(x)\sim\frac{1}{x}$ for x small $\therefore\ \Gamma(\epsilon/2)\sim\frac{2}{\epsilon}$

$$=(2\pi)^{4-\epsilon}\,\delta^{(4-\epsilon)}(-k_1+k_2)\,\frac{i}{-k_1^2-m_R^2+i\varepsilon}\,\frac{i}{-k_2^2-m_R^2+i\varepsilon}$$

$$\times m_R^2\times\left[i\,\frac{g_R}{2}\,\frac{2}{\epsilon}\times\frac{1}{16\pi^2}+\text{finite}-i(\tilde{Z}-1)\,k_1^2/m_R^2-i(Z_m^2\tilde{Z}-1)\right]$$

finite $\rightarrow$ contains everything else

Note:– $\frac{1}{\epsilon}\times f(\epsilon)=\frac{f(0)}{\epsilon}+f'(0)+\epsilon/2\,f''(0)+\cdots$

need to expand also $\left(\mu/m_R\right)^\epsilon$

(In this case, we won't worry about the finite piece — but in higher order corrections they need to be handled carefully)

To this order we can choose

$$\tilde{Z} = 1 + K_\phi g_R + O(g_R^2)$$

$$Z_m^2 = 1 + \frac{1}{16\pi^2\epsilon} g_R + O(g_R^2)$$

$$\therefore Z_m = 1 + \frac{1}{32\pi^2\epsilon} g_R + K_m g_R + O(g_R^2)$$

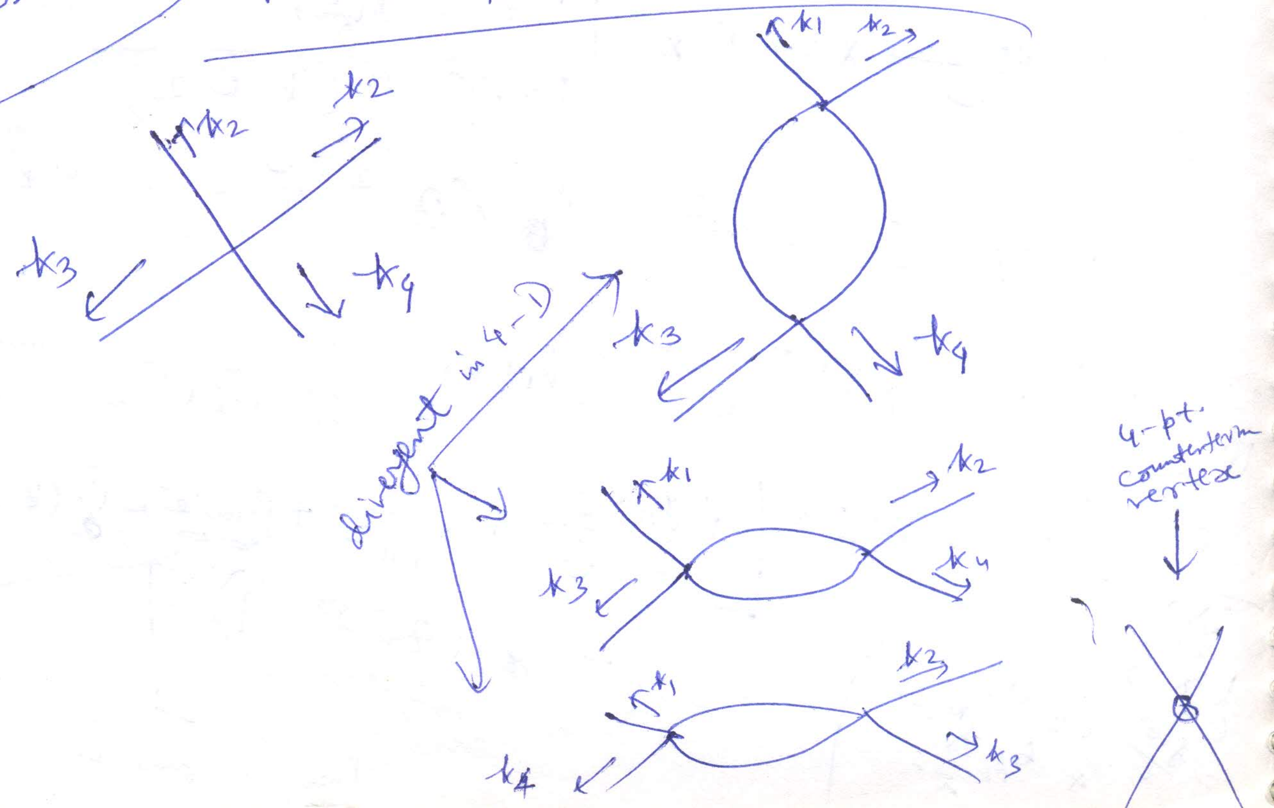
(where K_ϕ, K_m are finite nos.)

addn. of $K_\phi g_R$ or $K_m g_R$ won't give us infinities — any finite piece you can add

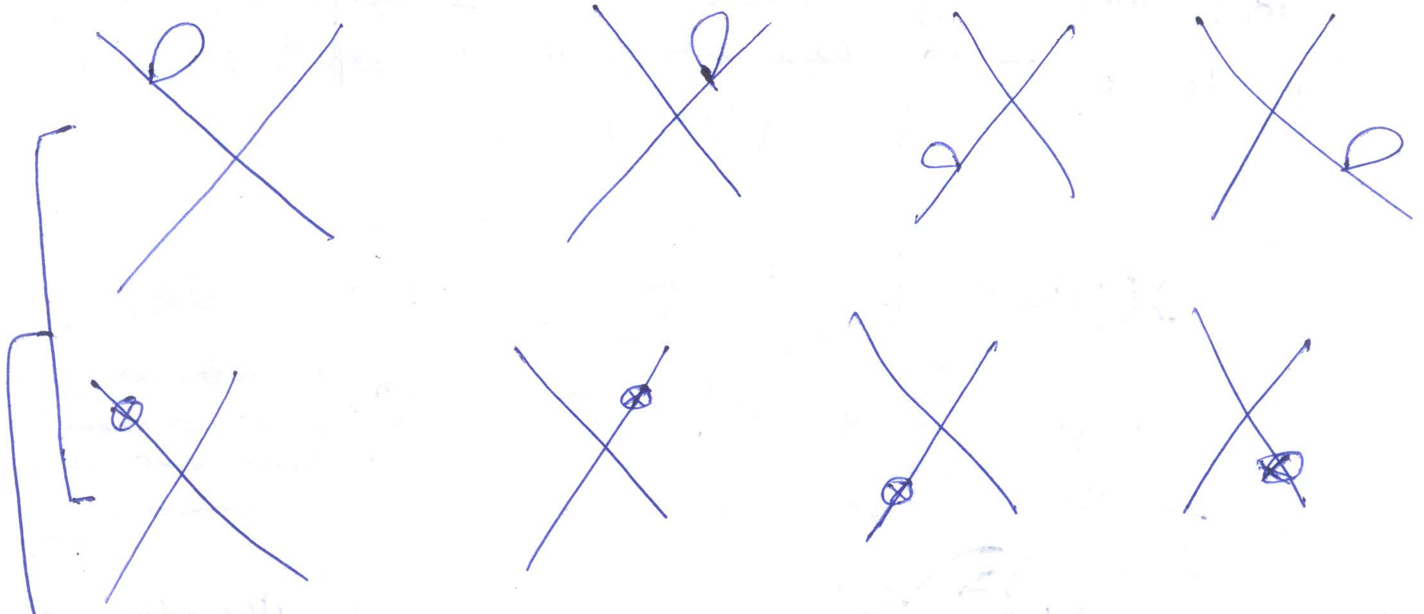
$\Downarrow$ not det. by renorm. procedure & this ambiguity can't be got rid of — nobody can prevent us from redefining our renorm. coupling constants

[Z_g appears in the coeff. of the 4-pt. fn. — so div. ~~for~~ 4-pt. fn. can be cancelled by adjusting Z_g]

Four point function



All are $O(g_R^2)$ terms



their sum is finite

[$\therefore$ these div. cancel & need not worry about — as long as we have made the subpieces finite, need not worry ~~about~~ about them — so consider only one-particle irreducible diag.]

$$\left(\prod_{i=1}^{4} \frac{i}{-k_i^2 - m_R^2 + i\epsilon}\right)\left[-\frac{i g_R}{\cancel{4!}}\,\mu^{\epsilon}\left(\tilde{Z}_\phi^2 Z_g - 1\right)\underbrace{\cancel{4\times 3\times 2}}_{\text{comb. factor}} + i\,\frac{3 g_R^2}{16\pi^2\epsilon} + \text{finite}\right]$$

from counterterm vertex

Ex. Check this.

$$\tilde{Z}_\phi^2 Z_g - 1 = \frac{3 g_R}{16\pi^2\epsilon} + K_g g_R + O(g_R^2)$$

$\mu^{\epsilon} \to 1$ as $\epsilon \to 0$

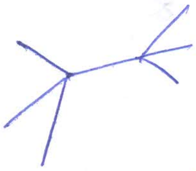
$$\Rightarrow Z_g = \left(1 + \frac{3 g_R}{16\pi^2\epsilon} + K_g g_R\right)\tilde{Z}^{-2}$$

$$= 1 + \frac{3 g_R}{16\pi^2\epsilon} + (K_g - 2K_\phi) g_R + O(g_R^2)$$

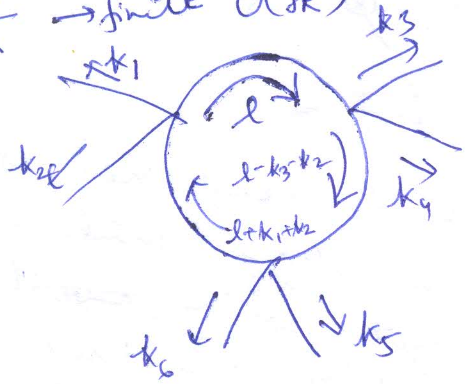
(But how do you ensure that 1-loop 6-pt. fn is finite ? — We have only made 1-loop 2-pt. & 4-pt. fns finite)

Higher point fn

Take 6-point fn → (it better be finite once we have chosen our renorm. parameters)



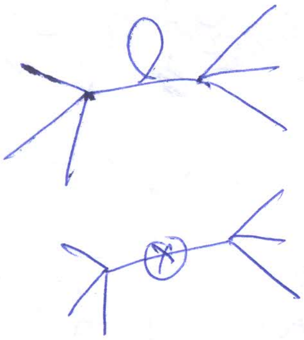
→ finite $O(g_R^2)$



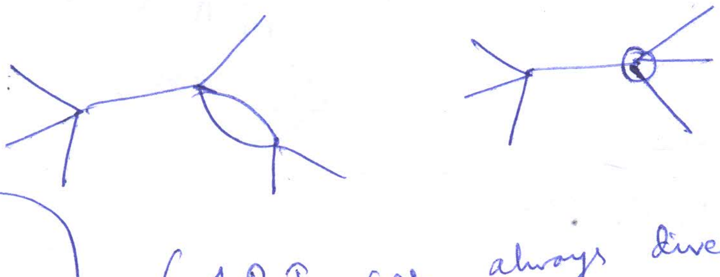
$$\int \frac{d^4 l}{l^6}$$

→ not divergent

The order of g_R is diff. — but as far as the loop counting is considered all these are 1-loop → so loop counting is more useful



} taken together they don't have any divergence



Together no divergence

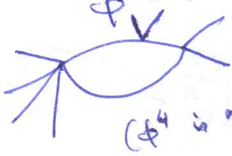
For ϕ^4 in 6-D,

is divergent

→ not renormalizable

(1PI are always divergenceless but 1P irreducible diag. div. are cancelled by the counterterm vertices)

ϕ^6 in 6-D isn't renorm: bcos :—



(ϕ^4 is also there)

$\Rightarrow$ 8-pt. fn. $\rightarrow \int \frac{d^6 l}{l^6} \Rightarrow$ divergent

→ Add ~~ϕ~~ 8-pt vertex, but then ~~ϕ~~ 10-pt will be divergent

(Eventually in the th. there will ∞ no. of parameters — can't predict anything — parameters can be fixed by a finite no. of exp.)

30/10/06

Renormalization of ϕ^4 theory

$$g = Z_g\, g_R\, \mu^\epsilon, \quad m = Z_m m_R, \quad \phi = \tilde{Z}^{1/2} \phi_R$$

fns of m_R, g_R, ϵ, μ

$S = \int d^{4-\epsilon}x \Big[\underbrace{\frac{1}{2}\left(-\partial_\mu \phi_R \partial^\mu \phi_R - m_R^2 \phi_R^2\right)}_{\text{free part}}$

$\underbrace{- \frac{g_R}{4!} \mu^\epsilon \phi_R^4}_{\text{Interaction}}$

counterterm $\begin{cases} -\frac{1}{2}(\tilde{Z}-1)\partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2}(\tilde{Z} Z_m^2 - 1) m_R^2 \phi_R^2 \\ - \frac{g_R}{4!}\mu^\epsilon (\tilde{Z}^2 Z_g - 1)\phi_R^4 \Big] \end{cases}$

Require

$\left\langle \prod_i \tilde{\phi}_R(k_i) \right\rangle$ is finite.

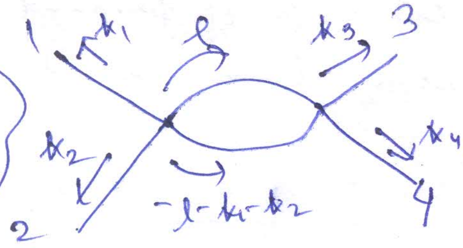
Finiteness of 2-pt. fn $\xrightarrow[\text{us}]{\text{gave}}$ $\tilde{Z} = 1 + K_\phi g_R$

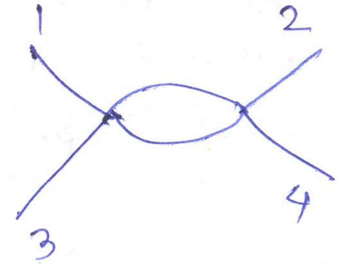
$$Z_m = 1 + \frac{g_R}{32\pi^2 \epsilon} + K_m g_R$$

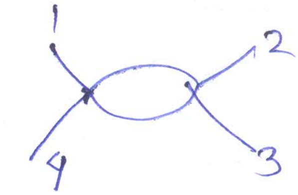
where K_ϕ & K_m are arbitrary constants.

To calculate Z_g we need to examine the 4-pt. fn.

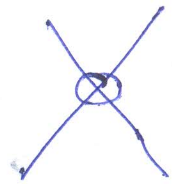
The basic 4-pt. fn. diags.







[The div. has to be cancelled by which is $O(g_R^2)$]



Other diag. need not be considered as there div. will be cancelled by

$$\prod_{i=1}^{4} \frac{i}{-k_i^2-m^2+i\varepsilon}(2\pi)^{4-\epsilon}\,\delta^{(4-\epsilon)}\Big(\sum k_i\Big)$$

$$\Bigg[\Big(-\frac{ig_R}{4!}\mu^{\epsilon}\Big)^2 \int \frac{d^{4-\epsilon}l}{(2\pi)^{4-\epsilon}} \frac{i}{-l^2-m_R^2+i\varepsilon}$$

$$\times \frac{i}{-(k_1+k_2+l)^2-m^2+i\varepsilon}$$

$$\times \frac{1}{2} \times \underbrace{8\times 3\times 4\times 3\times 2}_{\text{combinatorial factor}}$$

bcz it is 2nd order

+ Two other diagrams

(can be obtd. just by exchanging the momenta)

$$+\Big(-\frac{ig_R}{4!}\Big)\mu^{\epsilon}\,(\tilde{z}^2 z_g - 1)\times 4\times 3\times 2\Bigg]$$

$$\equiv I \text{ (say)}$$

there is an $O(g_R)$ term which we haven't written down as it is finite

Simple method (works only for one loop)

$$\int \frac{d^{4-\epsilon}l}{(2\pi)^{4-\epsilon}} \frac{i}{-l^2-m_R^2+i\varepsilon} \frac{i}{-(k_1+k_2+l)^2-m^2+i\varepsilon}$$

$$\longrightarrow \int \frac{d^{4-\epsilon}l}{(2\pi)^{4-\epsilon}} \frac{i^2}{(-l^2-m_R^2+i\varepsilon)^2}$$

$+$ finite

If you expand in a power series in terms of k_1 & k_2, only the first term is div. — the 2nd term involves a derivative & increase the power of l in denominator

Complete contribution :—

$$\frac{1}{2} g_R^2 \mu^{2\epsilon} \int \frac{d^{4-\epsilon}l}{(2\pi)^{4-\epsilon}} \frac{i^2}{(-l^2-m_R^2+i\varepsilon)^2} + \text{finite}$$

Claim

$$= \frac{1}{2} g_R^2 \mu^{2\epsilon}\, i \frac{(m_R^2)^{-\epsilon/2}\,\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}} + \text{finite}$$

Proof :—

Now,
$$\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{1}{k^2+L-i\varepsilon} = -i \frac{\Gamma(\epsilon/2)\, L^{1-\epsilon/2}}{(4\pi)^{2-\epsilon/2}(1-\epsilon/2)}$$

Taking $\frac{\partial}{\partial L}$ on both sides :—

$$-\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{1}{(k^2+L-i\varepsilon)^2} = -i \frac{\Gamma(\epsilon/2)\,\cancel{(1-\epsilon/2)}\, L^{-\epsilon/2}}{(4\pi)^{2-\epsilon/2}\,\cancel{(1-\epsilon/2)}}$$

$$\Rightarrow \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{1}{(k^2+m_R^2-i\varepsilon)^2} = \frac{i\,\Gamma(\epsilon/2)\,(m_R^2)^{-\epsilon/2}}{(4\pi)^{2-\epsilon/2}}.$$

In the $\epsilon \to 0$ limit, complete contribution is

$$\cancel{\frac{1}{2}} g_R^2\, i\, \frac{\cancel{2}}{\epsilon} \frac{1}{16\pi^2} + \text{finite}$$

(This could be done bcos we have a single pole — one loop calculations are simple bcos they involve single poles)

(You get $1/\epsilon$ only if there are 2 integrals)

Hence $$I = \prod_{i=1}^{4} \frac{i}{-k_i^2 - m_R^2 + i\varepsilon} (2\pi)^{4-\epsilon} \delta^{(4-\epsilon)}(\Sigma k_i)$$

$$\times \left[i \frac{g_R^2}{16\pi^2\epsilon} \times 3 - i g_R (\tilde{Z}^2 Z_g - 1) + \text{finite} \right]$$

→ from 3-diagrams

The div. part has no knowledge about the ext. mom. – so the div. piece of all the 3 diff. diag. must be the same

This gives us

$$\tilde{Z}^2 Z_g - 1 = \frac{3 g_R}{16\pi^2 \epsilon} + k_g g_R + O(g_R^2)$$

finite

(comes when into take into account higher loop div.)

there is a finite piece you can always add

$$Z_g = \tilde{Z}^{-2} \left(1 + \frac{3 g_R}{16\pi^2\epsilon} + k_g g_R + O(g_R^2) \right)$$

$$\tilde{Z} = 1 + K_\phi g_R + O(g_R^2)$$

$$Z_g = 1 + \frac{3 g_R}{16\pi^2\epsilon} + (K_g - 2K_\phi) g_R + O(g_R^2)$$

∴ 2 & 4-pt. fns can be made finite by this choice of Z_g, $\tilde{Z}$ & Z_m – higher loops can be made finite by adjusting terms having higher powers of g_R but we can't change lower order g_R terms

(If K_g, K_ϕ, K_m are there, are they new parameters of the theory?)

(If we change their values, will it give a new theory?)

Recall that $Z_m = 1 + \frac{g_R}{32\pi^2\epsilon} + K_m g_R + O(g_R^2)$

$$\therefore \quad m = Z_m m_R$$

$$= \left(1 + \frac{g_R}{32\pi^2\epsilon} + O(g_R^2)\right) \underbrace{\left(1 + K_m g_R + O(g_R^2)\right) m_R}_{m_R'}$$

$$= \left(1 + \frac{g_R}{32\pi^2\epsilon}\right) m_R'$$

$m_R' \to$ new parameters $\leftarrow g_R'$

Again, $$g = Z_g g_R = \left(1 + \frac{3 g_R}{16\pi^2\epsilon} + O(g_R^2)\right) \times \underbrace{\left(1 + (K_g - 2K_\phi) g_R\right) g_R}_{g_R'}$$

$$= \left(1 + \frac{3 g_R}{16\pi^2\epsilon}\right) g_R' = \left(1 + \frac{3 g_R'}{32\pi^2\epsilon}\right) g_R'$$

We can ignore $O(g_R g_R')$ if we ignore $O(g_R^2)$ terms

Also, $$\phi = \tilde{Z}^{1/2} \phi_R = \underbrace{\left(1 + \frac{1}{2} K_\phi g_R\right) \phi_R}_{\text{(call)} \to \phi_R'} = \phi_R'$$

Shift the constants to the next order for any given order

what we have achieved is

By postulating the relations:

$$\phi = \phi_R'$$

$$m = \left(1 + \frac{g_R'}{32\pi^2\epsilon}\right) m_R'$$

$$g = \left(1 + \frac{3 g_R'}{16\pi^2\epsilon}\right) g_R'$$

As long as g_R remains finite, g_R' is also finite; if g_R is fixed, g_R' is also fixed bcs no ϵ is involved even at $\epsilon \to 0$

The no. of parameters is still 2, namely, m_R' & g_R' → finite contribns getting dumped into m_R' & g_R'

then correln. fns of ϕ_R' are finite if we take $\epsilon \to 0$ at fixed m_R' & g_R'.

(If ϕ_R correln. fns are finite, ϕ_R' correln. fns are also finite to a given order)

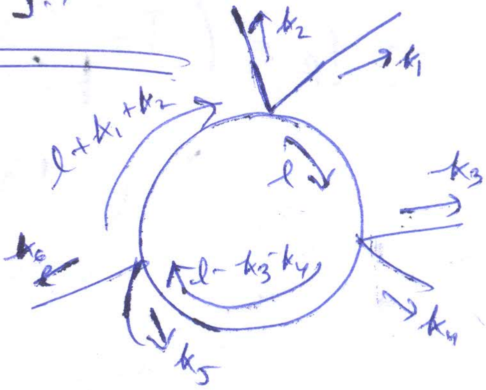
(The original th. is equiv. to the one with m_R', g_R' as well as to the one with m_R, g_R)

$$\left.\begin{aligned} \phi &= \phi_R' \\ m &= \left(1+\frac{g_R'}{32\pi^2\epsilon}\right)m_R' \\ g &= \left(1+\frac{3g_R'}{16\pi^2\epsilon}\right)g_R' \end{aligned}\right\}$$

Minimal subtraction scheme (Include only the pole part in the Z's)

Higher point fns

6-pt. fn

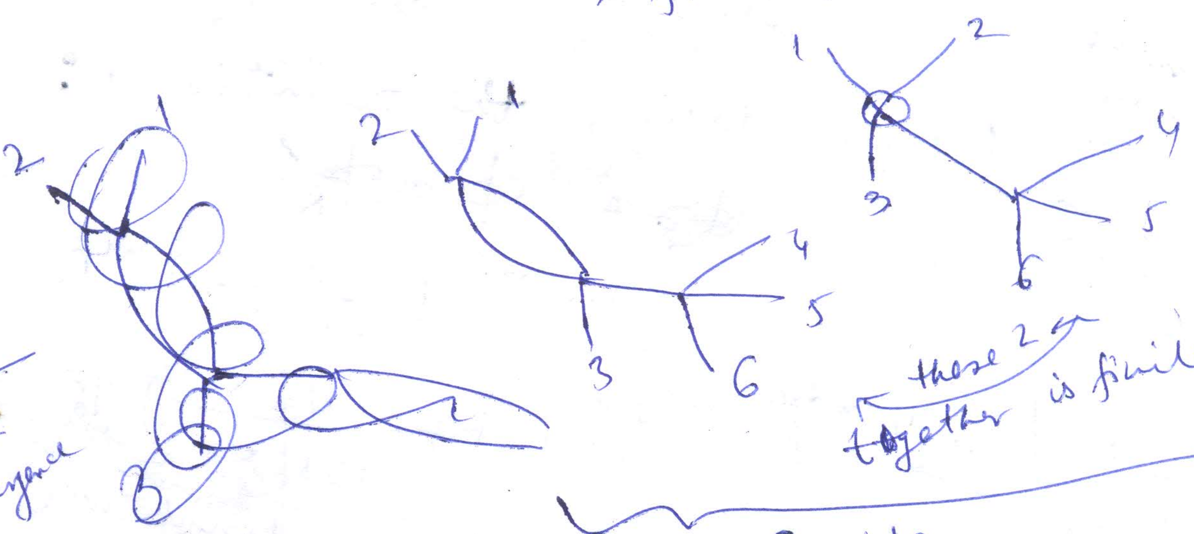


Possible 1-loop contribn to 6-pt fn → not div.

$$\int \frac{d^4 l}{\left[(-l^2-m_R^2+i\epsilon)\,(-(l-k_3-k_4)^2-m_R^2+i\epsilon) \times (-(l+k_1+k_2)^2 - m_R^2+i\epsilon)\right]}$$

→ finite

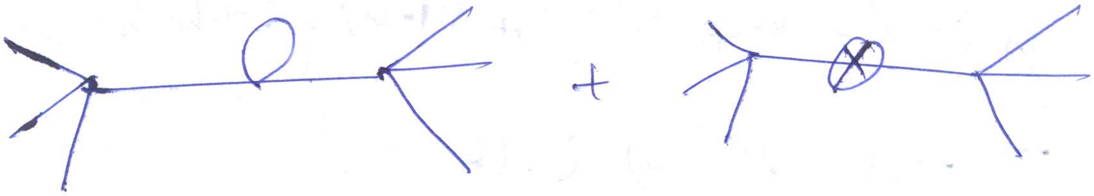
6 powers of l in den. & 4 " l in num → finite



not 1PI ← it is a sub-divergence

these 2 together is finite

finite

 finite

Conclusion :- Total 1-loop contribution to 6-pt. fn is finite.

This continues to hold for higher point fns.

(The basic 1PI particles are non-div. for higher pt. fns — we introduce more propagators

So for one loop it's very easy to see that higher pt. fns are finite)

Higher loops → additional complications

① Multiple denominators → technical complication

hence → we cannot apply dim. reg. formula directly.

e.g. → $\dfrac{1}{a\ b}$ (2 propagators) say, $a = (-l^2 - m_R^2 + i\varepsilon)$ & $b = (-\tilde{l}^2 - m_R^2 + i\varepsilon)$

$$\frac{1}{ab} = \int_0^1 du \frac{1}{\{au + b(1-u)\}^2} \cancel{= \int_0^1 \frac{1}{\{u(-l^2 - m_R^2 + i\varepsilon) + (1-u)(-\tilde{l}^2 - m_R^2 + i\varepsilon)\}^2}}$$

$$= \int_0^1 \frac{1}{\{u(-l^2 - m_R^2 + i\varepsilon) + (1-u)(-\tilde{l}^2 - m_R^2 + i\varepsilon)\}^2}$$

suppose $\tilde{l} = l + k_1 + k_2$

Then, $u(-l^2 - m_R^2 + i\epsilon) + (1-u)\{-(k_1+k_2+l)^2 - m_R^2 + i\epsilon\}$

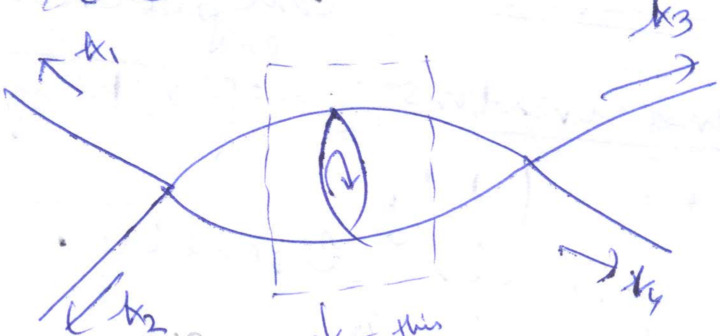
$$= -l^2 - 2(1-u)\, l\cdot(k_1+k_2) - \left(u m_R^2 + (1-u) m_R^2 + (1-u)(k_1+k_2)^2\right)$$

$$= -\underbrace{\left(l + (1-u)(k_1+k_2)\right)^2}_{l'} + (1-u)^2 (k_1+k_2)^2 - \left(m_R^2 + (1-u)(k_1+k_2)^2\right)$$

(→ replace int. over l by int. over l'; k_1 & k_2 are only constants as far as l-int. is concerned)

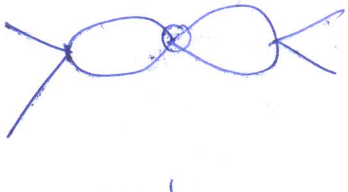
this was a technical issue
Real problem
→ now div. become nested

(2) Nested divergence (complicated problem)



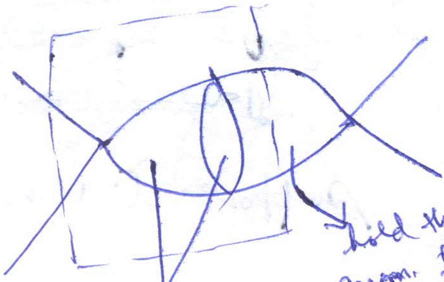
3 loop mom. are there

one might hope that it can be cancelled by



isolate this subdiagram → 2 prop. large ~~large~~ loop mom. (recall the 4-pt. fn.) (hold the other 2 mom. fixed)

which part of the div. is getting cancelled by corr. 2-loop counterterm is not clear



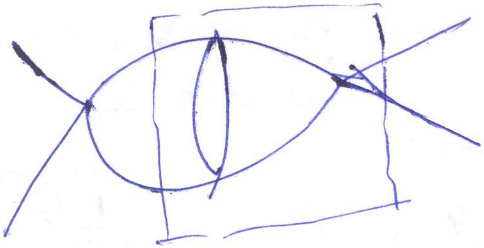
2 loops

hold this mom. fixed

8 integrals can be thought of as 2-loop 4-pt. fn

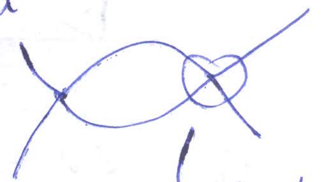
might hope that can be cancelled by

Two loop counterterm

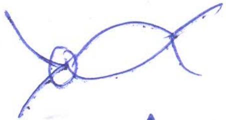
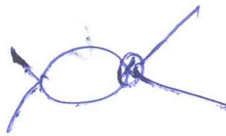


might hope that

cancelled by



Two loop counterterm

Problem → same diag. can be thought of as cancelled either by  or 

– can't say which part cancels with what

[Proof isn't simple but the fact is

To any given order in loop expn. it is possible to adjust Z's to cancel ~~adjust~~ the div. — have to show 2 pt., 4 pt. & all higher pt. fns are finite]

(So collect all such diag. & all cancelling counterterm diag. & then do the cancellation carefully without doing the cancellation separately)

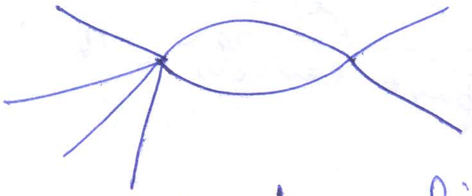
ϕ^6 in $D=4$:-

The Lagrangian includes

$$\frac{g}{4!}\phi^4 + \frac{\lambda}{6!}\phi^6$$

We have $\tilde{Z}_\phi, Z_m, Z_g, Z_\lambda$ → 4 renorm. constants to play with

6-pt. fn.

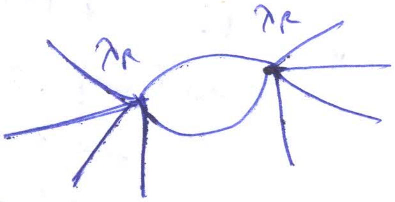


divergent

Cancelled by adjusting Z_λ

8-pt. fn.

divergent

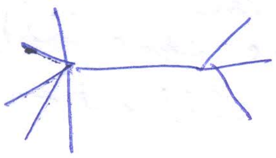


⇒ 1PI

There is no counterterm to ~~adj~~ cancel this divergence

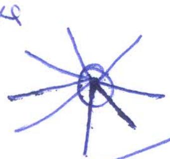
↳ This divergence cannot be cancelled by any available counterterm

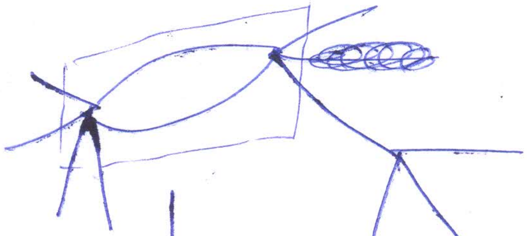
↳ (obtd. by redefining parameters in the Lag.)



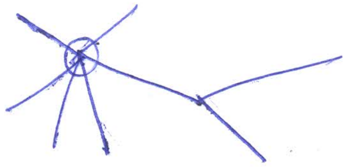
leading order

To cancel this we need a counterterm like





div. but cancelled by



31/10/06

Consider a scalar field theory in D-dimensions with ~~Lagrangian density~~ action:

$$S = \int d^D x \left[-\tfrac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \sum_{n=1}^{n_0} g^{(n)} \phi^n \right]$$

→ a general coupling of this form

(The results we get will be valid for theories with <u>other field</u> and <u>derivative interactions</u>).

Try to analyze under what condition this theory is renormalizable.

Considering the mass dimensions of ϕ & $g^{(n)}$'s

$$[\phi] = \frac{D-2}{2}$$

$$[g^n] + n \cdot \frac{D-2}{2} = D$$

$$\Rightarrow [g^n] = D - n \cdot \frac{D-2}{2} \equiv \lambda_n$$

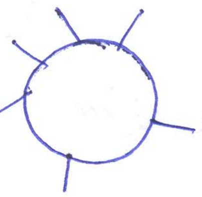
→ can be +ve or -ve depending on the value of n

For $n=2$, $[g^n] = 2$ → indep. of D. It just gives the mass term → $m^2\phi^2$

A specific graph in

$$\left\langle \prod_{i=1}^{N} \tilde{\phi}(k_i) \right\rangle = \prod_{i=1}^{N} \frac{i}{k_i^2 - m_k^2 + i\varepsilon} \times (2\pi)^D \delta^D(\sum k_i) \times \Gamma^{(N)}(k_1, \ldots, k_N)$$

Cutting off the external legs from the graph



(It contains prop. & vertices)

has the structure

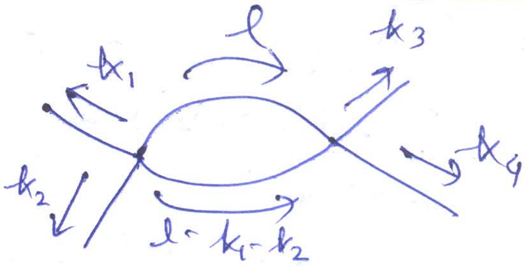
$$\prod_{n=3}^{n_0} \left(g^{(n)} \right)^{\alpha_n} \times \Gamma^{(N)}(k_1, \ldots k_N)$$

$$\int \prod_i \frac{d^D l_i}{(2\pi)^D} \times \text{propagators}$$

We will consider here only 1PI – once they are finite, the irreducible graphs are also finite

α_n = # of times $g^{(n)}$ vertex appears in the diagram

Example :-



$$\left\langle \prod_{i=1}^{4} \tilde{\Phi}(k_i) \right\rangle$$

$$= \prod_{i=1}^{4} \frac{i}{-k_i^2 - m_R^2 + i\varepsilon} \left(\frac{ig}{4!}\right)^2 \frac{1}{2} \times 8 \times 3 \times 2 \times 4 \times 3$$

$$\times (2\pi)^{(4-\epsilon)} \delta^{(4-\epsilon)}\left(\sum k_i\right) \times \underbrace{\int \frac{d^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} \frac{i}{-l^2 - m_R^2 + i\varepsilon} \frac{i}{-(l+k_1+k_2)^2 - m_R^2 + i\varepsilon}}_{\hat{\Gamma}(4)}$$

$\hat{\Gamma}(4) \rightarrow$ mom. int. with mom. factors

A +ve or zero dim of $\hat{\Gamma} \Rightarrow$ it is div. (zero dim. is log. div.)
A -ve dim of $\hat{\Gamma} \Rightarrow$ it is non-div.
$\Rightarrow$ So dim. of $\hat{\Gamma}$ is important

We have seen that

$$[\phi] = \frac{D-2}{2}$$

Now, $\tilde{\Phi}(k) = \int d^D x \, e^{-ik\cdot x} \phi(x)$

$$\therefore [\tilde{\Phi}(k)] = \frac{D-2}{2} - D = -\frac{D+2}{2}$$

$d^D x$ has dim. D

$$\therefore \left[\left\langle \prod_{i=1}^{N} \tilde{\Phi}(k_i) \right\rangle\right] = -N \frac{D+2}{2}$$

(Equate the dim. on both sides to get dim. of $\hat{\Gamma}$) $\rightarrow$

$$-N\frac{D+2}{2} = \underbrace{-2N}_{\text{dim. of } \prod_{i=1}^{N} \frac{i}{-k_i^2 - m_R^2 + i\varepsilon}} \underbrace{-D}_{\text{dim. of the } \delta^{(D)}(\sum k_i)} + [\Gamma^{(N)}]$$

$$\Rightarrow [\Gamma^{(N)}] = 2N + D - \frac{1}{2}ND - N = N + D - \frac{1}{2}ND$$
$$= D + N \cdot \frac{2-D}{2}$$

(Let us now find $[\hat{\Gamma}]$) →

$$D + N\frac{2-D}{2} = \sum_n \lambda_n \alpha_n + [\hat{\Gamma}^{(N)}]$$

$D + N\frac{2-D}{2}$ → dim of $\Gamma^{(N)}$

α_n → dim. of $g^{(n)}$

$$\Rightarrow [\hat{\Gamma}^{(N)}] = D + N\cdot\frac{2-D}{2} - \sum_n \lambda_n \alpha_n$$

= degree of divergence of $\hat{\Gamma}^{(N)}$

known as the degree of divergence of $\hat{\Gamma}^{(N)}$

$\hat{\Gamma}^{(N)}$ is divergent if

$$D + N\cdot\frac{2-D}{2} - \sum_n \lambda_n \alpha_n \geq 0$$

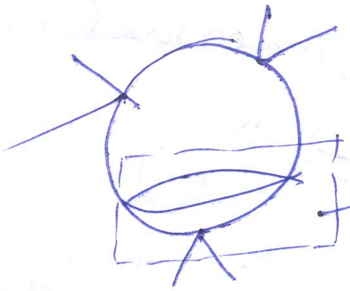
This is the div. which will occur if all the loop mom. become large – there can be subdivergence which is not accounted for here

This does not account for divergences of subdiagrams

e.g. →  in ϕ^4 theory in $D=4$

3 loops → 12 in num. & 14 in den.

→ nevertheless it has subdiv.



→ 2 loops
⇒ 8 integrals
& 4 internal propagators
→ 8 powers in den.

This is a div. in the 4-pt. fn. & has already been made finite

(So subdiv. we don't worry about, assuming it has been taken care of at a lower order)

While $\hat{\Gamma}^{(N)}$ div. means in order to cancel this you need a counterterm vertex – div. not coming from a lower order vertex.

Suppose for some ~~n, α_n~~ $n \leq n_0$, λ_n is negative.

α_n's are +ve (or zero) integers

↳ bcos some vertices may not appear

For any fixed N we can generate a divergent diagram by taking α_n corresponding to the negative λ_n sufficiently large, i.e., using that vertex many times.

If we can identify even a single diag. which is div., since there is no counterterm, the theory is non-renormalizable

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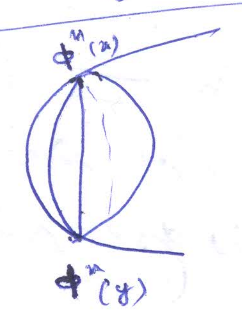
ϕ^n



It is possible to add as many ϕ^n vertices we like to generate a diag. for a fixed N

Contract lines on 2 ϕ^n vertices → eventually you have only 2 lines coming out

so it's easy to see that

Conclusion $\Longrightarrow$ A renormalizable field theory must have $\lambda_n \geq 0$ for all n.

$\because \lambda_n = D - n\frac{D-2}{2}$

$\xrightarrow[\text{means}]{\text{this}}$ $D - n\frac{D-2}{2} \geq 0$ for all n.

$\Longrightarrow$ (necessary but not yet sufficient).

e.g. → ① $D=4$ Then $4-n \geq 0 \Rightarrow n \leq 4$

② $D=6$ Then $6-2n \geq 0 \Rightarrow n \leq 3$ → so ϕ^4 in 6-dim isn't renormalizable

③ $D=2$ n can be any non-negative integer

(The above condn. isn't sufficient bcos →

Even if λ_n is +ve, we can choose N for which there is div.
What is max. possible div. can have?

$D+N\frac{2-D}{2}-\sum_n \lambda_n \alpha_n$

If all λ_n's ≥ 0, then an N-point function is definitely not divergent if

$$D+N\frac{2-D}{2}<0 \Rightarrow D-N\frac{D-2}{2}<0$$

We will only need counterterm for

$$D-N\frac{D-2}{2} \geq 0$$

We will need a counterterm vertex for N-pt. fn. for this inequality

Thus all N point vertices with $D-N\frac{D-2}{2} \geq 0$ must be present in the action.

this is in the worst possible case, i.e., there is no help from $\sum_n \lambda_n \alpha_n$

(there seem to be 2 conflicting condns →
① can't add too many vertices
② add as more vertices as you can)

But $D-n\frac{D-2}{2} \geq 0$ & $D-N\frac{D-2}{2} \geq 0$ condns are identical — otherwise we would have been in trouble

To get a renormalizable field theory (QFT) we must contain all terms in the action whose coefficients have dimension $\underline{\geq 0}$.

$\boxed{\text{N must be a subset of } n \rightarrow \text{now we see that } N = n}$

Such theories are called
<u>power counting renormalizable</u>.

This includes also terms containing derivatives.

e.g. $\rightarrow h^{(n)} \phi^n \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ → contributes to (n+2) point function

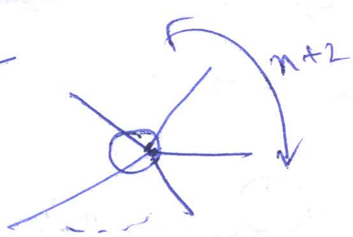
$\downarrow$

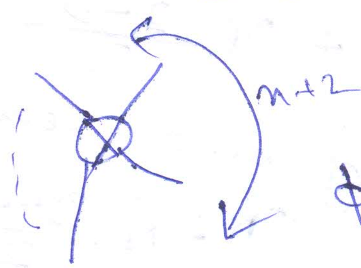
find $[h^{(n)}]$ (if it has −ve dim, then don't add it, otherwise add it to the Lagrangian)

(will give n+2 vertices — n+2 ϕ's)

Consider counterterms like →

$g^{(n)} \phi^{n+2}$





$\phi^n \partial_\mu \phi \partial^\mu \phi$

Suppose $h^{(n)} \phi^n \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ → has dim zero

$\Downarrow$ ϕ^n has dim zero

$g^{(n)} \phi^{n+2}$ → has dim 2