# Statistical Mechanics Exam: Part I

1. Consider a cylinder of radius $a$, kept vertically, containing $N$ molecules of a monatomic gas. $m$ is the mass of each atom, $g$ is the acceleration due to gravity, and $T$ is the temperature of the gas. The upper end of the cylinder is closed with a piston of weight $w$. In the equilibrium situation the piston is at a height $h$ above the lower end of the cylinder. The height $h$ is large so that the effect of the gravitational pull on the atoms cannot be ignored. The temperature is sufficiently large so that we can use classical statistical mechanics.

   a) Find an expression for the free energy $F$ of the system as a function of $a$, $N$, $m$, $g$, $h$.

   b) Using the result of part a), and the equation $dF = -dW - SdT$ where $dW$ is the work done by the system, and $S$ is the entropy of the system, calculate the net force exerted by the gas on the piston, i.e. calculate the weight $w$ in terms of $a$, $N$, $m$, $g$, $h$.

   c) Calculate the equilibrium density of atoms at a height $x$ above the bottom of the cylinder.

   d) Using the result of part c), calculate the pressure at the top of the cylinder and the force exerted by the gas on the piston. Compare with the result of part a).

2. Consider a two dimensional ideal monatomic Bose gas containing $N$ atoms each of mass $m$, confined in an area $A$ with the geometry of a square, and periodic boundary condition in both directions. Find the temperature $T$ at which the fraction $f$ ($f$ is small compared to 1 but $fN$ is large compared to 1) of the total number of particles will be in the ground state. $T$ should be determined in terms of $N$, $A$ and $m$. (Note that you need to do this calculation for large but finite $A$.)

3. Consider a lattice containing two rows of sites as follows

$$\begin{array}{ccccccccccccccccc}
x & x & x & x & x & x & x & x & \cdots & x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x & \cdots & x & x & x & x & x & x & x & x
\end{array}$$

   Each row contains $2N$ sites, with the $(2N+1)$th site identified with the first site. To the $i$th site in the upper row we associate a variable

$s_i$, and to the $i$th site in the lower row we associate a variable $\sigma_i$. $s_i$ and $\sigma_i$ take values $\pm 1$. The total energy of the system for a given configuration of spins is given by:

$$E(\{s_i\}, \{\sigma_i\}) = -J \sum_{i=1}^{2N} (\sigma_i s_{i+1} + s_i \sigma_{i+1})$$

Find an expression for the free energy and the specific heat of this system.

1/2/06

# Fermions with spins

$\longrightarrow$ $(2s+1)$ degenerate states for every $\vec{p}$.

[in absence of mag. field]

$$\ln \mathcal{Q} = (2s+1)\frac{V}{h^3}\int d^3p \ln\left(1+ze^{-\beta e(\vec{p})}\right)$$

$$N = (2s+1)\frac{V}{h^3}\int d^3p \frac{ze^{-\beta e(\vec{p})}}{1+ze^{-\beta e(\vec{p})}}$$

$$\frac{PV}{NkT} = V/N \frac{\partial}{\partial V}(\ln \mathcal{Q}) = \frac{f_1(z)}{f_2(z)}$$ same functions (as before)

$$\longrightarrow \frac{N}{2s+1} = \frac{V}{h^3}\int d^3p \frac{ze^{-\beta e(\vec{p})}}{1+ze^{-\beta e(\vec{p})}}$$

$\longrightarrow$ same eqn. as in the earlier case with $N$ replaced by $\frac{N}{2s+1}$.

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## Ideal Bose gas

$$\ln \mathcal{Q} = -\sum_{\vec{p}} \ln\left(1-ze^{-\beta e(\vec{p})}\right)$$

$$\bar{N} = \sum_{\vec{p}} \frac{ze^{-\beta e(\vec{p})}}{1-ze^{-\beta e(\vec{p})}}, \quad \overline{n(\vec{p})} = \frac{ze^{-\beta e(\vec{p})}}{1-ze^{-\beta e(\vec{p})}}$$

$\overline{n(\vec{p})} \geq 0$ for every $\vec{p}$

$\xrightarrow[\text{us to have}]{\text{force}} z \leq 1$

[why is this happening? $\rightarrow$ Bcos $\sum_k \{ze^{-\beta e(\vec{p})}\}^k = \frac{1}{1-ze^{-\beta e(\vec{p})}}$ only when $z \leq 1$]

$$\bar{N} = \sum_{\vec{p}} \frac{z\, e^{-\beta e(\vec{p})}}{1 - z e^{-\beta e(\vec{p})}}$$

used to solve for $z$ in terms of $\bar{N}$

(Prop. of the R.H.S. !-)

$$\sum_{\vec{p}} \frac{z e^{-\beta e(\vec{p})}}{1 - z e^{-\beta e(\vec{p})}} = \sum_{\vec{p}} \frac{1}{z^{-1} e^{\beta e(\vec{p})} - 1}$$

increases as $z$ increases

As $z \to 0$ $\quad \sum_{\vec{p}} \frac{z e^{-\beta e(\vec{p})}}{1 - z e^{-\beta e(\vec{p})}} \longrightarrow 0$

for small $z$, we make the approxmn

$$z \underbrace{\sum_{\vec{p}} e^{-\beta e(\vec{p})}}$$

this thing is finite (bcos it is sufficiently convergent for large $\vec{p}$)

so neglecting the den. is justified

we aren't making any thing conv. by $\vec{p}\,\theta$ div. this approxmn

As $z \to 1$

$$\sum_{\vec{p}} \frac{z e^{-\beta e(\vec{p})}}{1 - z e^{-\beta e(\vec{p})}} = \frac{z}{1-z} + \sum_{\vec{p} \neq 0} \frac{z e^{-\beta e(\vec{p})}}{1 - z e^{-\beta e(\vec{p})}}$$

$\to \infty$ as $z \to 1$

free bosons, & so $e(\vec{p}) = \vec{p}^2/2m$

$$\Rightarrow \bar{N} = \sum_{\vec{p}} \frac{z e^{-\beta e(\vec{p})}}{1 - z e^{-\beta e(\vec{p})}}$$

has a unique soln. in the range $0 < z < 1$ for any $0 < \bar{N} < \infty$

An overall shift in energy can be absorbed in the redefn. of $z$

bcos $\bar{N}$ is a monotonically $\uparrow$ fn. of $z$.

Take the $V \to \infty$ limit & replace the sum by integral.

$$\ln \mathcal{Q} = -\frac{V}{h^3}\int d^3p \ln\left(1 - z e^{-\beta \vec{p}^2/2m}\right)$$

$$= -\frac{V}{h^3} 4\pi \int p^2 dp \ln\left(1 - z e^{-\beta p^2/2m}\right)$$

$$p = |\vec{p}|$$

$$\ln \mathcal{Q} = -\frac{4\pi V}{h^3}(2mkT)^{3/2}\int_0^\infty u^2 du \ln\left(1 - z e^{-u^2}\right)$$

$$p = \sqrt{\frac{2m}{\beta}} \cdot u$$

$$= \frac{4\pi V}{h^3}(2mkT)^{3/2} f_1(z)$$

$$\hookrightarrow -\int_0^\infty u^2 du \ln\left(1 - z e^{-u^2}\right)$$

$$\bar{N} = \frac{V}{h^3}\int d^3p \frac{z e^{-\beta \vec{p}^2/2m}}{1 - z e^{-\beta \vec{p}^2/2m}}$$

$$= \frac{4\pi V}{h^3}(2mkT)^{3/2} F_2(z)$$

where $$F_2(z) = \int_0^\infty u^2 du \frac{z e^{-\cancel{\beta} u^2}}{1 - z e^{-\cancel{\beta} u^2}}$$

Need to solve

$$\boxed{\bar{N} = \frac{4\pi V}{h^3}(2mkT)^{3/2} F_2(z)}$$

$$F_2(z) = \int_0^\infty u^2 du \frac{1}{z^{-1}e^{u^2} - 1}$$ increases as $z$ increases

In order to get a good stat. ensemble for bosons, we have to restrict $z$ to $0 < z < 1$ & for fermions $0 < z < \infty$

$e^{\beta\mu N} \to$ weight factor
Allowed wt. factor as long as the sum is convergent — not physically ... is non-conv.

$\mu$ is an auxiliary variable — $\mu$ is a trick we have introduced

$z$ is an intermediate contraction to solve $\bar{N}$

~~For bosons,~~ Grand canonical ensemble won't work when you find the sum is non-conv. → maybe use some other ensemble

But here we see we can use grand can. for $\bar{N}$ ($0 < \bar{N} < \infty$

It was seen in the discrete version — so no surprise that we see the same behavior in the continuum version

As $z \to 0$, $F_2(z) \simeq \int_0^\infty u^2 \, du \, z e^{-u^2}$

$\simeq z \times$ finite integral

$\to 0$ as $z \to 0$

As $z \to 1$

$F_2(z) \longrightarrow \int_0^\infty \frac{u^2 \, du}{e^{u^2} - 1}$

2 things to check:- whether it is div. for large u region & small u region

for large u, $\int du \, u^2 e^{-u^2} \to$ finite

for small u,

$\int du \, u^2 \frac{1}{1+u^2-1} \sim \int du \to$ finite

$\int_0^\infty = \int_0^1 + \int_1^\infty$

finite finite

$\int_0^1 + \int_1^\infty$

$\int_0^1 \frac{u^2 \, du}{e^{u^2}-1}$

$= \int_0^1 \frac{u^2 \, du}{u^2 + u^4/2 + \cdots}$

$= \int_0^1 \cancel{u^2} \, du \frac{1}{\cancel{u^2}} \{1 + u^2/2 + \cdots\}^{-1}$

As $u \to 0$, there is a pole of the integrand

$\frac{1}{e^{u^2}-1} \sim \frac{1}{u^2}$ as $u \to 0$

Even if the integrand is div., the integral may or may not diverge

e.g. $\to \int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_0^1 = 2$

Suppose $F_2(z) \to K$ as $z \to 1$

calculable constant

$K = \int u^2 \, du \frac{e^{-u^2}}{1 - e^{-u^2}}$

$0 < F_2(z) < K$ if $0 < z < 1$

Without taking a radial cond. (i.e., taking $u^2 du$), we could have used $\frac{dx \, dy \, dz}{e^{(x^2+y^2+z^2)} - 1}$

this'll give the same result

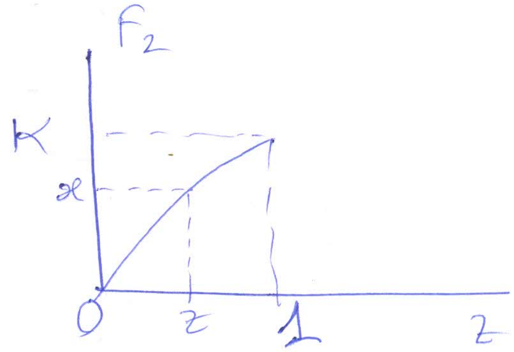
$$\bar N = \frac{4\pi V}{h^3}(2mkT)^{3/2} F_2(z)$$

If $\bar N < \frac{4\pi V}{h^3}(2mkT)^{3/2} K$

then this eqn has a unique solution in the range $0 < z < 1$

$$x = \frac{h^3}{\bar N}\frac{1}{4\pi V}(2mkT)^{3/2}$$

$$x = F_2(z)$$




If $x > K$ $\overset{\text{same}}{\underset{\text{as}}{\Longleftrightarrow}}$ $\bar N > \frac{4\pi V}{h^3}(2mkT)^{3/2} K$

then there is no soln to $\bar N = \frac{4\pi V}{h^3}(2mkT)^{3/2} F_2(z)$

in the range $0 < z \leq 1$.

Go back to the discrete version

$$\bar N = \sum_{\vec p} \frac{z\, e^{-\beta e(\vec p)}}{1 - z\, e^{-\beta e(\vec p)}}$$

$\downarrow$ ~ requires that $\frac{z\, e^{-\beta e(\vec p)}}{1 - z\, e^{-\beta e(\vec p)}}$ is smooth

$$\frac{V}{h^3}\int d^3 p$$

If $z < 1$ then $\frac{z\, e^{-\beta e(\vec p)}}{1 - z\, e^{-\beta e(\vec p)}}$ is finite

We can go from one pt. in the lattice to another, if it doesn't jump by a large amt.

Sth has gone wrong in going from discrete to continuum version $\because$ in discrete version we had a soln for every value of $\bar N$, but in continuum version we find this is not true

If $z \simeq 1$ then the summand is not ~~so~~ smooth at $e(\vec{p}) \simeq 0$.

Lesson:– As $z \to 1$ we cannot use the naive thermodynamic limit ~~the sum~~ of replacing

$$\sum_{\vec{p}} \xrightarrow{\text{by}} \frac{V}{h^3}\int d^3p$$

Take $\frac{1}{f}$

Suppose you are starting from $f=0$ case.
Even for $f=\epsilon$, $f=2\epsilon$
$\frac{1}{\epsilon}$, $\frac{1}{2\epsilon}$ differ by large amt.
↓
can't write this as an integral

$\frac{1}{f=0}$ & $\frac{1}{\epsilon}$ is enormous

$$e(\vec{p}=0,0,1) = \frac{1}{2m}\left(\frac{2\pi n_1 \hbar}{L}\right)^2$$

~~at~~ also $\to 0$ in the thermodynamic limit — so you can't say that $e(\vec{p}=0)$ is separate & treat the rest as integral)

Problem is not only at $e=0$, the problem is at small $e$, in principle

If $\bar{N} < \frac{4\pi V}{h^3}(2mkT)^{3/2}K$

then $\bar{N} = \frac{4\pi V}{h^3}(2mkT)^{3/2}F_2(z)$ has a unique solution for $0 < z < 1$.

↓

thermodynamic limit has no subtlety

$z=1$ has a problem, so we don't take $\bar{N} \leq \cdots$

self-consistent way of det. $z$
⇓
assume the thermodyn. limit, find $z$ — & then ~~check~~ see that the thermodyn. limit is valid

$$\bar{N} < \frac{4\pi V}{h^3}(2mkT)^{3/2}K$$

$$\Rightarrow T > \left(\frac{\bar{N}}{4\pi VK}\right)^{2/3}\frac{h^2}{2mk}$$

$\equiv T_c$
↓
critical temperature

We can use this to study high temperature limit.

$\beta \to 0 \; (T \to \infty)$ at fixed $\bar{N}/V$.

$$\frac{\bar{N}}{V} = \frac{4\pi}{h^3}(2mkT)^{3/2} \, F_2(z)$$

If $T \to \infty$ at fixed $\bar{N}/V$, then

$$F_2(z) \longrightarrow 0$$
$$\Downarrow$$
$$z \to 0$$

$$F_1(z) = -\int_0^\infty u^2 \, du \, \ln(1 - z e^{-u^2})$$

$$\simeq -\int_0^\infty u^2 \, du \left\{ - z e^{-u^2} - \frac{1}{2} z^2 e^{-2u^2} - \cdots \right\}$$

$$\simeq \frac{\sqrt{\pi}}{4}\left(z + \frac{1}{4\sqrt{2}} z^2 + \cdots\right)$$

same integral as for fermions except for the '$-$' sign before $ze^{-u^2}$

$$F_2(z) = \int_0^\infty u^2 \, du \, \frac{z e^{-u^2}}{1 - z e^{-u^2}}$$

$$= \int_0^\infty u^2 \, du \left( z e^{-u^2} + z^2 e^{-2u^2} + \cdots \right)$$

$$\simeq \frac{\sqrt{\pi}}{4}\left(z + \frac{1}{2\sqrt{2}} z^2 + \cdots\right)$$

$$\frac{PV}{\bar{N}kT} = \frac{F_1(z)}{F_2(z)} = \left(\frac{1 + \frac{1}{4\sqrt{2}} z + \cdots}{1 + \frac{1}{2\sqrt{2}} z + \cdots}\right)$$

$$\simeq \left(1 - \frac{1}{4\sqrt{2}} z + \cdots\right)$$

$$\frac{\bar{N}}{V} = \frac{4\pi}{h^3}(2mkT)^{3/2}\frac{\sqrt{\pi}}{4}z\left(1+\frac{z}{2\sqrt{2}}+\cdots\right)$$

Leading soln:-

$$z = \frac{\bar{N}}{V}\frac{h^3}{\pi\sqrt{\pi}}(2mkT)^{-3/2} + \cdots$$

$\therefore$ we get

$$\frac{pV}{\bar{N}kT} = \left\{1 - \frac{1}{4\sqrt{2}}\frac{\bar{N}}{V}\frac{h^3}{(\pi)^{3/2}}\frac{1}{(2mkT)^{3/2}} + \cdots\right\}$$

$<$ classical ideal gas answer

again illustrates the fact that classical stat. is somewhere in bet. Bose & Fermi

Bose $\rightarrow$ more states available than for classical

Case for $T < T_c$ :-

$$\bar{N} = \sum_{\vec{p}} \frac{z e^{-\beta e(\vec{p})}}{1 - z e^{-\beta e(\vec{p})}}$$

When the thermodyn. limit breaks down

We need to take $z \to 1$ as $V \to \infty$.

($z$ close to 1 as you are taking the thermodyn. limit)

$$\bar{N} = \underbrace{\sum_{\substack{\vec{p} \\ e(\vec{p}) < \eta}} \frac{z e^{-\beta e(\vec{p})}}{1 - z e^{-\beta e(\vec{p})}}}_{I_1} + \underbrace{\sum_{\substack{\vec{p} \\ e(\vec{p}) \geq \eta}} \frac{z e^{-\beta e(\vec{p})}}{1 - z e^{-\beta e(\vec{p})}}}_{I_2}$$

$\eta$ : A fixed number

Now take $V \to \infty$ limit.

(In this case we know that somehow) $z \longrightarrow 1$

(In this limit) 2nd term

$$I_2 = \frac{V}{h^3}\int_{\vec{p}^2 \geq 2m\eta} d^3p \, \frac{z e^{-\beta \vec{p}^2/2m}}{1 - z e^{-\beta \vec{p}^2/2m}}$$

$$= \underbrace{G(\eta, \beta, z)}_{\text{some function}} \rightarrow \text{\# of particles with energy} \geq \eta$$

First we have taken thermodyn. limit — this is imp.

η-dep. of G is the imp. part

$$I_2 \xrightarrow[\eta \to 0]{} \underbrace{\frac{4\pi V}{h^3}(2mkT)^{3/2} K}_{\text{\# of particles with energy} \geq \eta \text{ as } \eta \to 0}$$

If we take $\eta \to 0$ first & then the thermodyn. limit then we are in trouble

& η must be much much larger than the energy spacing betw. 2 levels

$$I_1 = \sum_{\substack{\vec{p} \\ e(\vec{p}) < \eta}} \frac{z e^{-\beta e(\vec{p})}}{1 - z e^{-\beta e(\vec{p})}} = \text{\# of particles with energy} < \eta$$

$$= \bar{N} - I_2$$

$$\underset{\eta \to 0}{=} \bar{N} \underbrace{\left\{ 1 - \frac{4\pi V}{\bar{N} h^3}(2mkT)^{3/2} K \right\}}_{\text{some fraction} > 0}$$

$$\bar{N} > \frac{4\pi V}{h^3}(2mkT)^{3/2} K$$

# of particles with energy η in the $\eta \to 0$ limit is a finite fraction of the total no. of particles.

the fact that we cannot account for the particles with energy close to zero shows that thermodyn. breaks down for them

5/1/06

# Ideal Bose gas

$$\bar{N} = \sum_{\vec{p}} \frac{z e^{-\beta e(\vec{p})}}{1 - z e^{-\beta e(\vec{p})}} = \sum_{\vec{p}} \underbrace{\frac{1}{z^{-1} e^{\beta e(\vec{p})} - 1}}_{n(\vec{p})} \equiv G(z)$$

1) $n(\vec{p})$ for each $\vec{p}$ is a monotone increasing function of $z$ for $0 < z < 1$

$\Rightarrow G(z)$ is a monotone increasing function of $z$.

2) $G(z) \to 0$ as $z \to 0$

3) $G(z) \to \infty$ as $z \to 1$.

4) Replacing (the summation by an integration for large $V$)

$\sum_{\vec{p}}$ by $\frac{V}{h^3} \int d^3p$ for large $V$ breaks down when $z \simeq 1$, $e(\vec{p}) \simeq 0$.

$$\bar{N} = \sum_{\substack{\vec{p} \\ e(\vec{p}) < \epsilon}} \frac{1}{z^{-1} e^{\beta e(\vec{p})} - 1} + \sum_{\substack{\vec{p} \\ e(\vec{p}) \geq \epsilon}} \frac{1}{z^{-1} e^{\beta e(\vec{p})} - 1}$$

$\epsilon$ : fixed energy.

$u^2 \geq \beta\epsilon$

$V \to \infty$

$$\frac{V}{h^3} \int d^3p \, \frac{1}{z^{-1} e^{\beta e(\vec{p})} - 1}$$

$$= \frac{4\pi V}{h^3} (2m k T)^{3/2} \, H_2(z, \epsilon)$$

$$H_2(z,\epsilon) = \int_{\sqrt{\beta\epsilon}}^{\infty} u^2 \, du \, \frac{1}{z^{-1} e^{u^2} - 1}$$

$$\lim_{\epsilon \to 0} H_2(z, \epsilon) = F_2(z)$$

$$H_2(z, \epsilon) \leq F_2(z) = \int_0^\infty u^2 du \, \frac{1}{z^{-1} e^{u^2} - 1}$$

$F_2(z) \to 0$ as $z \to 0$

$\to K_0$ as $z \to 1$

$$\int_0^\infty u^2 du \, \frac{1}{e^{u^2} - 1} = \text{finite constant}$$

$$\frac{\bar{N}}{V} = \frac{1}{V} \sum_{\substack{\vec{p} \\ e(\vec{p}) < \epsilon}} \frac{1}{z^{-1} e^{\beta e(\vec{p})} - 1}$$

$$+ \frac{4\pi}{h^3} (2mkT)^{3/2} H_2(z, \epsilon)$$

$$\lim_{\epsilon \to 0} \lim_{V \to \infty} \frac{\bar{N}}{V} = \lim_{\epsilon \to 0} \lim_{V \to \infty} \frac{1}{V} \sum_{\substack{\vec{p} \\ e(\vec{p}) < \epsilon}} \frac{1}{z^{-1} e^{\beta e(\vec{p})} - 1}$$

$$+ \frac{4\pi}{h^3} (2mkT)^{3/2} F_2(z)$$

$\frac{\bar{N}}{V}$ is a quantity that we'll give

Define: $\rho = \bar{N}/V$

Define $T_c$ through

$$\rho = \frac{4\pi}{h^3} (2mkT_c)^{3/2} K_0$$ → defn of $T_c$

for a given $\rho$, we define $T_c$ by this eqn for a given $\rho$

$T_c$ isn't an intrinsic prop. of the material — it depends upon $\rho$

$$\frac{4\pi}{h^3} (2mkT_c)^{3/2} K_0 - \frac{4\pi}{h^3} (2mkT)^{3/2} F_2(z)$$

$$= \lim_{\epsilon \to 0} \lim_{V \to \infty} \frac{1}{V} \sum_{\substack{\vec{p} \\ e(\vec{p}) < \epsilon}} \frac{1}{z^{-1} e^{\beta e(\vec{p})} - 1}$$

If $T < T_c$, then L.H.S. is a positive number since $F_2(z) \le K_0$.

Hence ⟹ right hand side $\underset{\text{cannot be}}{\neq} 0$ → this can happen only if $z \to 1$ as $V \to \infty$

$\Rightarrow F_2(z) \longrightarrow K_0$

Bcos if $z < 1$, we can replace $\Sigma$ by $\int$ & the upper limit being $\epsilon$, it goes to zero as $\epsilon \to 0$

$$\frac{4\pi}{h^3} k_0 (2m k)^{3/2} \left\{ T_c^{3/2} - T^{3/2} \right\}$$

$$= \lim_{\epsilon \to 0} \lim_{V \to \infty} \frac{1}{V} \sum_{\substack{\vec{p} \\ e(\vec{p}) < \epsilon}} \frac{1}{z^{-1} e^{\beta e(\vec{p})} - 1}$$

R.H.S. $= \frac{1}{V} \times$ Total no. of particles with energy $< \epsilon$

for $T < T_c$, the total number of particles with energy $< \epsilon$

$= V \times$ some fixed no.

$= \frac{N}{\rho} \times$ some fixed no.

$\rho = N/V$ = fixed

Fraction of particles with energy $< \epsilon$ = a fixed number even as $\epsilon \to 0$

We can set $z = 1$ in the LHS, not in the RHS bcos $z \to 1$ only as $V \to \infty$

In the LHS the $V$-dependence has dropped out & so we could put $z = 1$ there

We can set $z = 1$ only in those terms where $V \to \infty$ limit is smooth

This is not what we expect - we expect that # of particles having energy $< \epsilon$ is a fn. of $\epsilon$ & that goes to zero as $\epsilon \to 0$

Q) How are these particles distributed among the energy levels $< \epsilon$?

First $V \to \infty$ has been taken & then $\epsilon \to 0$; so $\epsilon$ is much greater than energy spacings; we have already made the spacings approach zero

Consider a box of lengths $(L_1, L_2, L_3)$

$$e = \frac{4\pi^2 \hbar^2}{2m} \left( \frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2} \right)$$

$$= \frac{2\pi^2 \hbar^2}{m L^2} \left( n_1^2 + n_2^2 + n_3^2 \right)$$

$L_1 = L_2 = L_3 = L$ for simplicity

$$\text{R.H.S.} = \frac{1}{V} \sum_{\substack{\vec{p} \\ e(\vec{p}) < \epsilon}} \frac{1}{z^{-1} e^{\beta e(\vec{p})} - 1}$$

$$= \underbrace{\frac{1}{V} \frac{1}{z^{-1} - 1}}_{= I_1} + \underbrace{\sum_{\substack{\vec{p} \neq 0 \\ e(\vec{p}) < \epsilon}} \frac{1}{z^{-1} e^{\beta e(\vec{p})} - 1}}_{= I_2}$$

We will show

$$\lim_{\epsilon\to 0} \lim_{V\to\infty} I_2 = 0$$

$\Rightarrow$ All the contribution must come from $I_1$

$\Rightarrow$ A finite fraction of the particles will be in $\vec{p}=0$ state.

Each term in $I_2$ is positive.

$$I_2 \leq \frac{1}{V} \sum_{\substack{\vec{p}\neq 0 \\ e(\vec{p})\leq\epsilon}} \frac{1}{e^{\beta e(\vec{p})}-1}$$

( $\because$ it was a monotone $\uparrow$ fn. of $z$ & if we put $z=1$, that is maximum)

$$e^{\beta e} > 1+\beta e$$

$$\therefore I_2 \leq \frac{1}{V} \sum_{\vec{p}\neq 0} \frac{1}{1+\beta e(\vec{p})-1}$$

$$= \frac{1}{\beta V} \sum_{\substack{\vec{n}\neq 0 \\ e<\epsilon}} \frac{1}{\frac{4\pi^2\hbar^2}{2m L^2}\underbrace{(n_1^2+n_2^2+n_3^2)}_{\vec{n}^2}}$$

$$\therefore I_2 \leq \frac{1}{\beta V} \frac{mL^2}{2\pi^2\hbar^2} \sum_{\substack{\vec{n}\neq 0 \\ \vec{n}^2 < \frac{mL^2\epsilon}{2\pi^2\hbar^2}}} \frac{1}{n_1^2+n_2^2+n_3^2}$$

$$\frac{2\pi^2\hbar^2}{mL^2}\vec{n}^2 < \epsilon$$

$$\Rightarrow \vec{n}^2 < \frac{mL^2\epsilon}{2\pi^2\hbar^2}$$

$V = L^3$

$\therefore$ Individual terms go to zero in the $L\to\infty$ limit

When $e \sim \frac{1}{L^2}$

$$\frac{1}{e^{\beta e}-1} \quad \text{Vs} \quad \frac{1}{z^{-1}e^{\beta e}-1}$$

$$\sim \frac{1}{\beta e} = \frac{1}{\beta\cdot\frac{1}{L^2}} \sim L^2/\beta$$

these 2 terms can differ appreciably

$$\frac{z}{1+\beta e - z}$$

$$= \frac{z}{\frac{\beta}{L^2}+1-z}$$

we know $1-z$ is small but how small is it?

It may be $\sim \frac{1}{L}$

then we will ~~do~~ be wrong in setting $z=1$

~~So~~ we can't put $z=1$ in the 2nd term of the last page viz., $I_2$

The limit is subtle — $e(\vec{p})\to 0$ we don't know which is going faster

The no. of terms in the sum is unbounded as $L\to\infty$ so it isn't enough to show that individual terms go to zero

$$\sum_{\substack{\vec{n} \neq 0 \\ \vec{n}^2 < \frac{mL^2\epsilon}{2\pi^2\hbar^2}}} \frac{1}{n_1^2+n_2^2+n_3^2}$$

has vanishing contribution as $L\to\infty$ due to $\frac{1}{L}$ factor

$$= \underbrace{\sum_{\substack{\vec{n} \neq 0 \\ \vec{n}^2 < M}} \frac{1}{n_1^2+n_2^2+n_3^2}}_{\text{finite}}$$

$$\boxed{M = \text{A fixed large number}}$$

$$+ \sum_{M \le \vec{n}^2 \le \frac{mL^2\epsilon}{2\pi^2\hbar^2}} \frac{1}{n_1^2+n_2^2+n_3^2}$$

∵ for large M, $\frac{1}{n_1^2+n_2^2+n_3^2}$ does not change appreciably on changing $n_i$ by 1

$$\int_{\vec{n}^2 = M}^{\vec{n}^2 = \frac{mL^2\epsilon}{2\pi^2\hbar^2}} d^3n \, \frac{1}{\vec{n}^2}$$

$$= 4\pi \int_{\sqrt{M}}^{\sqrt{m\epsilon}\frac{L}{\sqrt{2}\pi\hbar}} \cancel{n^2}\, dn \cdot \frac{1}{\cancel{n^2}}$$

$$= 4\pi \boxed{\frac{L}{\pi\hbar}\sqrt{\frac{m\epsilon}{2}} - \sqrt{M}}$$

(same as $\lim V\to\infty$)

$$\lim_{\epsilon\to 0}\lim_{L\to\infty} I_2$$

$$= \lim_{\epsilon\to 0}\lim_{L\to\infty} \frac{m}{2\pi^2\hbar^2\beta}\cdot\frac{1}{L}\left\{4\pi\left(\frac{L}{\pi\hbar}\sqrt{\frac{m\epsilon}{2}} - \sqrt{M}\right) + \text{finite}\right\}$$

(contribution which came from the sum $\sum_{\vec{n}^2 < M}$)

$$= \lim_{\epsilon\to 0} \frac{m}{2\pi^2\hbar^2\beta}\frac{4}{\hbar}\sqrt{\frac{m\epsilon}{2}} = 0$$

All the contribution must come from $I_1$

($\vec{p} = 0$ term)

$$\lim_{V\to\infty} \frac{1}{V}\frac{1}{z^{-1}-1} = \frac{4\pi}{h^3} k_0 (2mk)^{3/2}\left\{T_c^{3/2} - T^{3/2}\right\}$$

$$\left( z^{-1} \simeq 1 + c/V \quad \lim_{V\to\infty}\frac{1}{V}\frac{1}{1+c/V-1} = 1/c \right)$$

The ε limit isn't ready here

where $\frac{1}{c} = \frac{4\pi}{h^3} K_0 (2mk)^{3/2} \left\{ T_c^{3/2} - T^{3/2} \right\}$

fraction of particles at $\vec{p} = 0$ level

$$\frac{1}{N} \frac{1}{z^{-1}-1} \simeq \frac{V}{N} \cdot \frac{1}{V} \frac{1}{z^{-1}-1}$$

$$= \underbrace{\frac{1}{\rho} \frac{4\pi}{h^3} K_0 (2mk)^{3/2} \left\{ T_c^{3/2} - T^{3/2} \right\}}$$

finite as $V \to \infty$, $N \to \infty$

$\rho = N/V$ fixed

$\sim N_A$ (macroscopic no.) sit at $\vec{p} = 0$

$\to$ Bose - Einstein condensation

$N_0$ = no. of particles in ground state $= \frac{z}{1-z}$

$\Rightarrow z = \frac{N_0}{N_0 + 1}$ $\rightsquigarrow$ Pathria

## Thermodynamics below $T_c$

$$z^{-1} \simeq 1 + c/V \qquad z \simeq 1 - c/V$$

$$\ln Q = - \sum_{\vec{p}} \ln \left(1 - z e^{-\beta e(\vec{p})}\right)$$

$$= -\ln(1-z) - \sum_{\vec{p} \neq 0} \ln \left(1 - z e^{-\beta e(\vec{p})}\right)$$

$\to -V/h^3 \int d^3p \ln \left(1 - z e^{-\beta p^2/2m}\right)$

$= \frac{4\pi V}{h^3} (2mkT)^{3/2} F_1(z)$

$-\ln(c/V) \sim \ln V$

$\ln V$ becomes large in the large thermodyn. limit but doesn't grow fast enough to affect the thermodynamics

$$\frac{1}{N} \ln Q = \frac{1}{N} \ln V + \cdots$$

$\to 0$ in thermodynamic limit

$$F_1(z) = - \int_0^\infty u^2 du \ln \left(1 - z e^{-u^2}\right)$$

So no special treatment reqd. for the $\vec{p} = 0$ term in this case

The same treatment works for both $T < T_c$ & $T > T_c$ for $\ln Q$ - no term contributes appreciably so that this is reqd.

this is due to the presence of log

$$P = kT \frac{\partial}{\partial V} \ln Q$$

$$= kT \cdot \frac{4\pi}{h^3} (2mkT)^{3/2} f_1(z)$$

$P$, as it stands, doesn't depend on $V$ — $V$ term has dropped out

Usually $z$ is a fn of $\frac{N}{V}$ & $T$ through the $f_2$ equation.
(this generates a dependence of $P$ on $N/V$ & $T$)

For $T < T_c$, $z = 1 - c/N$

$$f_1(z) \simeq f_1(1) + O\left(\frac{1}{N}\right)$$

$$P = kT \frac{4\pi}{h^3} (2mkT)^{3/2} f_1(1)$$

for $T < T_c$.

$f_1(1) = -\int_0^\infty u^2 \ln(1 - e^{-u^2})$ is a finite integral

$\longrightarrow$ independent of density

(this time it is truly indep. of density)

strange system bcos as you change the vol. keeping the $N$ & $T$ fixed, $P$ doesn't change

This is not sth. unusual, when we have a mixture of 2 phases (say, liq. $H_2O$ & vapour) — now reduce the vol. — pressure won't change, only some vapour will condense into liquid

$$\frac{1}{1 - z e^{-\beta e(\vec{p})}}$$

$\frac{1}{1 - z e^{-\beta e(\vec{p})}}$ changes as $z$ changes & so $P$ changes

For $z = 1$, the distribution is fixed. As you are compressing, only more & more particles are accumulating in the zero-energy state which don't exert pressure (density $\uparrow$)

distribn. of particles in the $e \neq 0$ states remain the same

$$\ln Q = \frac{4\pi V}{h^3} \left(\frac{2m}{\beta}\right)^{3/2} f_1(z)$$

$$E = \left(-\frac{\partial}{\partial \beta} \ln Q\right)_{z, V}$$

$$= \frac{4\pi V}{h^3} \frac{3}{2\beta} \cdot \left(\frac{2m}{\beta}\right)^{3/2} f_1(z)$$

is a fn of $(N/V, \beta)$ for $T > T_c$

$= 1$ for $T < T_c$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{N, V}$$

$C_V$ & all higher derivatives will have different behaviour for $T < T_c$ & $T > T_c$

Ex. Check that for $T > T_c$

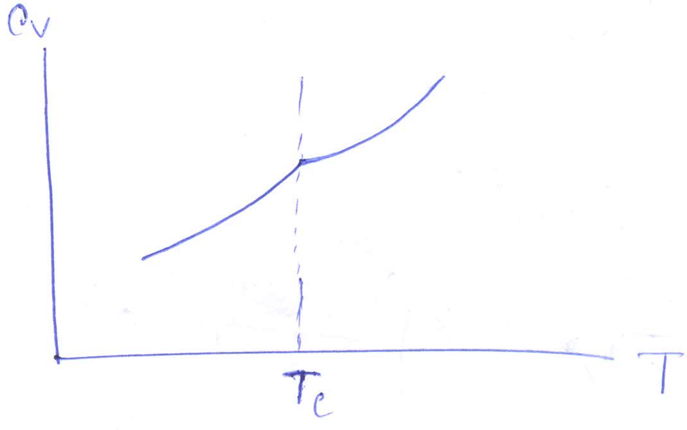
$$f_1(\tau) = f_1(1) + \cancel{K}C_0(\tau - T_c)^2 + \text{----}$$

some constant

$$\frac{\partial}{\partial T} f_1(\tau) = C_0(T - T_c) \longrightarrow 0 \quad \text{as } T \to T_c$$

(as a result) $C_V$ is continuous at $T_c$.

$\frac{\partial C_V}{\partial T}$ is not continuous at $T_c$



H.W.

$$I = \int_0^\infty dy\; y^2 \frac{1}{(e^{y/2}+e^{-y/2})^2} = \int_0^\infty dy\, \frac{y^2 e^y}{(1+e^y)^2}$$

$$= \cancel{-\frac{y^2}{e^y+1}\Big|_0^\infty} + 2\int_0^\infty \frac{y}{e^y+1}\, dy$$

$$= 2\int_0^\infty \frac{y e^{-y}}{1+e^{-y}}\, dy$$

$$= 2\sum_{n=1}^\infty \int_0^\infty y e^{-ny}\, dy$$

$$= 2\sum_{n=1}^\infty \left[\cancel{-\frac{y e^{-ny}}{n}\Big|_0^\infty} + \frac{1}{n}\int_0^\infty e^{-ny}\, dy\right]$$

$$= 2\sum_{n=1}^\infty \left[-\frac{1}{n^2} e^{-ny}\Big|_0^\infty\right]$$

$$= 2\sum_{n=1}^\infty \frac{1}{n^2}$$

$$\therefore \int_{-\infty}^\infty dy\; y^2 \frac{1}{(e^{y/2}+e^{-y/2})^2} = 4\sum_{n=1}^\infty \frac{1}{n^2} = \frac{4\times\pi^2}{12} = \frac{\pi^2}{3}$$

---

H.W. of 30/1/06

2nd order correction :-

$$\frac{\sqrt{\pi}}{4} z\left(1-\frac{z}{2\sqrt{2}}+\cdots\right) = \frac{N}{V}\frac{h^3}{4\pi}(2mkT)^{-3/2}$$

$$\Rightarrow \frac{\sqrt{\pi}}{4} z - \frac{\sqrt{\pi}}{8\sqrt{2}} z^2 = \frac{N}{V}\frac{h^3}{4\pi}(2mkT)^{-3/2}$$

$$\Rightarrow \frac{\sqrt{\pi}}{4} z = \frac{\sqrt{\pi}}{8\sqrt{2}} z^2 + \frac{N}{V}\frac{h^3}{4\pi}(2mkT)^{-3/2}$$

$$\Rightarrow \frac{\sqrt{\pi}}{4} z = \frac{\sqrt{\pi}}{8\sqrt{2}} \times\left(\frac{N}{V}\right)^2\left(\frac{2m\pi kT}{h^2}\right)^{-3} + \frac{N}{V}\frac{h^3}{4\pi}(2mkT)^{-3/2}$$

by successive approxmn method

$$\Rightarrow z = \frac{1}{2\sqrt{2}}\left(\frac{N}{V}\right)^2\left(\frac{2m\pi kT}{h^2}\right)^{-3} + \frac{N}{V}\frac{4}{4\pi\sqrt{\pi}}\left(\frac{h^2}{2m\pi kT}\right)^{3/2}$$

$$z = \frac{1}{2\sqrt{2}}\left(\frac{\bar{N}}{V}\right)^2\left(\frac{h^2}{2m\pi kT}\right)^3 + \frac{\bar{N}}{V}\left(\frac{h^2}{2m\pi kT}\right)^{3/2}$$

$\therefore$ Eqn of state is :-

$$\frac{PV}{\bar{N}kT} = 1 + \frac{1}{4\sqrt{2}}\frac{\bar{N}}{V}\left(\frac{h^2}{2m\pi kT}\right)^{3/2} + \frac{1}{16}\left(\frac{\bar{N}}{V}\right)^2\left(\frac{h^2}{2m\pi kT}\right)^3 + \cdots$$

$$\frac{4\pi^2 h^2}{2m\pi kT} = \frac{2\pi h^2}{m\pi kT}$$

$$\frac{4\sqrt{2}}{4\sqrt{2}}$$

10/2/06

## System of photons in a box

Quanta of electromagnetic radiation

(1) Photons are massless particles

$$e(\vec{p}) = \sqrt{\underset{\substack{\| \\ 0}}{m^2}c^4 + \vec{p}^2 c^2} = c|\vec{p}|$$

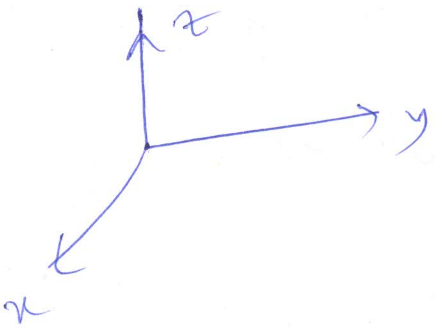
(2) Each photon comes in two polarization states.

Need another quantum number besides momentum to specify the state of a photon.

$s = +1$ or $-1$

→ represent the result of quantizing left circularly polarized and right circularly polarized em wave.

$$(E_x, B_y) \neq 0$$
$$(E_y, B_x) \neq 0$$




$$\sum_{\vec{p}} \to \sum_{s=\pm 1} \sum_{\vec{p}}$$

(3) Number of photons is not conserved.

→ can get emitted ~~or~~ and absorbed by the wall.

→ cannot work with a fixed no. of photons.

Recall standard procedure for grand canonical ensemble.

$$Q = \sum_N Tr\left(e^{-\beta H + \beta \mu N}\right)$$

Determine $\mu$ by requiring that

$$\bar{N} = \frac{\sum_N N \,\mathrm{Tr}\left(e^{-\beta H + \beta \mu N}\right)}{\sum_N \mathrm{Tr}\left(e^{-\beta H + \beta \mu N}\right)}$$

agrees with total no. of particles

$\mu$ was a parameter we introduced by hand, but ultimately $\mu$ was eliminated or $\mu$ had to be determined by demanding $\bar{N} = \frac{--}{--}$ — by solving that eqn.

This procedure will not work bcos u can't fix $\bar{N}$ for photons — we can't determine $\mu$

Recall the canonical partition fn:

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right)$$

$\downarrow$ over all states with arbitrary number of photons

Previously, we restricted the Trace to be taken over a fixed no. of particles — but now it's not true

$$= \mathcal{Q}\ (\mu = 0)$$

This is equiv. to Grand can. ensemble with $\mu = 0$

$\xrightarrow{\text{this implies}}$ For photons we can use grand canonical partition function if we set $\mu = 0$.

$\rightsquigarrow$ A generic feature for all systems for which the number of particles is not conserved.

In case of photons, we don't try to fix $\bar{N}$ — Here $\mu$ is our input parameter & that is equal to zero

This is true for all system where no. of particles of a particular type ~~aren't~~ isn't conserved

In a system where $A + B \to C + D$ Break up the consvd. charges — those that are conserved are det. by $\bar{N} = \frac{--}{--}$ & for those that aren't, we set $\mu = 0$

$\bar{N}$ is still the avg. no. of particles but it no longer is the input parameter — we don't try to match $\bar{N}$ with the total no. of particles in the system

$$\ln Q = -\sum_{s=+1,-1} \sum_{\vec{p}} \ln\left(1 - e^{-\beta e(\vec{p})}\right)$$

$$= -2 \frac{4\pi V}{h^3} \int_0^\infty p^2 \, dp \ln\left(1 - e^{-\beta e(p)}\right)$$

(Here we are in a dangerous situation as by putting $\mu = 0$, we have set $z = 1$ — But still this integral can be written bcos in log $Q$ there is no problem)

$e^{i(k_1 x + k_2 y + k_3 z)}$

$\downarrow$

mom. quant comes from here

$\downarrow$

this part is the same as non-rel. formula

Pair prodn negligible for low temp.

~~$\ln Q$~~

But $e(p) = cp$

$\beta c p = u \qquad p = \frac{u k T}{c}$

$$\ln Q = \frac{8\pi V}{h^3} \cdot \left(\frac{kT}{c}\right)^3 \underbrace{\left[-\int u^2 \, du \ln\left(1 - e^{-u}\right)\right]}_{A \text{ (some number)}}$$

$$\frac{p}{kT} = \frac{\partial}{\partial V} \ln Q = 8\pi \left(\frac{kT}{hc}\right)^3 A$$

$$E = -\frac{\partial \ln Q}{\partial \beta} = kT^2 \frac{\partial}{\partial T} \ln Q$$

$$= \frac{24\pi V}{h^3 c^3} A (kT)^4$$

$$\rho = \frac{E}{V} = \frac{24\pi}{h^3 c^3} (kT)^4 \cdot A$$

$\rho \propto T^4$ Stefan-Boltzmann law

$p = \rho/3 \longrightarrow$ known relation betn pressure & energy density of radiation.

pressure doesn't depend upon volume — this isn't surprise bcos in a normal gas, as $V \downarrow$ ($T$ fixed) $\frac{N}{V} \uparrow$ (density $\uparrow$) — but here as $V \downarrow$, no. of photons $\downarrow$ appropriately as ~~density~~ no. of photons isn't an intrinsic prop. of the system

Any relation which doesn't have an 'h' is a classical relation

$$n(\vec{p}, s) = \frac{e^{-\beta e(\vec{p})}}{1-e^{-\beta e(\vec{p})}} = \frac{e^{-\beta c p}}{1-e^{-\beta c p}}$$

* avg. occupancy no. of a mom. state per polarization

in the thermodyn. limit, it is not reasonable to focus on a single state — we must ask no. of particles in the mom. range $|\vec{p}|$ & $|\vec{p}| + |\delta \vec{p}|$

Total no. of states in momentum range $p \le |\vec{p}| \le p + \delta p = \frac{8\pi V p^2 \delta p}{h^3}$

$$\sum_s \sum_{\vec{p}} \longrightarrow \int \frac{8\pi V}{h^3} p^2 \, dp$$

$\Rightarrow$ Average no. of photons with $p \le |\vec{p}| \le p + \delta p$

$$= \frac{8\pi V}{h^3} p^2 \delta p \, \frac{e^{-\beta c p}}{1-e^{-\beta c p}}$$

Average energy carried by photons in the momentum range $p \le |\vec{p}| \le p + \delta p$

$$= \frac{8\pi V}{h^3} p^2 \delta p \, \frac{e^{-\beta c p}}{1-e^{-\beta c p}} \, cp$$

Multiply avg. # of photons --- with $cp$

Quantum mechanics : (relates)

one side q. mech. & on the other side class. em

$e = h\nu$ → frequency of the wave

energy of photon

~~$p = h/\lambda$~~

$p = h/\lambda$ → wave-length

momentum

$e = pc \Rightarrow \nu = c/\lambda$

sum over s will be non-trivial if you somehow remove the degeneracy & make e energy & s-dep. e.g. charge in an electric field *

Average energy in photons in the wavelength range $(\lambda, \lambda+\delta\lambda)$

$$= \frac{8\pi V}{h^3}\left(h/\lambda\right)^2 h\left(\frac{\delta\lambda}{\lambda^2}\right)\frac{e^{-ch/kT\lambda}}{1-e^{-ch/kT\lambda}} \times ch/\lambda$$

$$\boxed{\begin{array}{l} p \to h/\lambda \\ \delta p \to \frac{h}{\lambda^2}\delta\lambda \end{array}}$$

$$= \underbrace{\frac{8\pi V.ch}{\lambda^5}\frac{e^{-ch/kT\lambda}}{1-e^{-ch/kT\lambda}}}_{V f(\lambda)}\delta\lambda$$

→ Planck distribution law

'−' sign not imp. as you are measuring the magnitude

# High temperature limit: $T \to \infty$

$$f(\lambda) \to \frac{8\pi \cancel{ch}}{\lambda^3}\frac{1}{\cancel{ch}/kT\lambda} = \frac{8\pi kT}{\lambda^4}$$

This formula doesn't have an 'h' & can be derived from class. stat. mech. for em fields — there is a classical origin of the formula

No. of modes of an em field can change — valid in classical picture — but from the pt. of view particles, how to sum over no. of photons not conserved. — so here high temp. limit is class. stat. mech. of fields & not particles

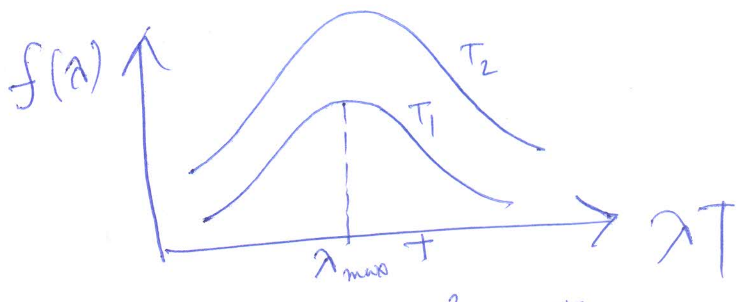
Can't draw analogy that $h \to 0$ limit should give us classical picture

full formula for f:

$$f(\lambda) = \frac{8\pi ch}{\lambda^5}\frac{e^{-ch/\lambda kT}}{1-e^{-ch/\lambda kT}}$$

$$= T^5 \cdot 8\pi ch \underbrace{\left\{\frac{e^{-ch/\lambda kT}}{(\lambda T)^5\left(1-e^{-ch/\lambda kT}\right)}\right\}}_{g(\lambda T)}$$

$g(\lambda T) \to$ a fn. of $\lambda T$




$$\lambda^1_{max} T_1 = \lambda^2_{max} T_2$$

$f(\lambda) = \underbrace{T^5}_{\text{overall multiplicative factor}} \; 8\pi ch \; g(\lambda T) \rightarrow$ so peak of $f$ vs $\lambda T$ at same value of $\lambda T$

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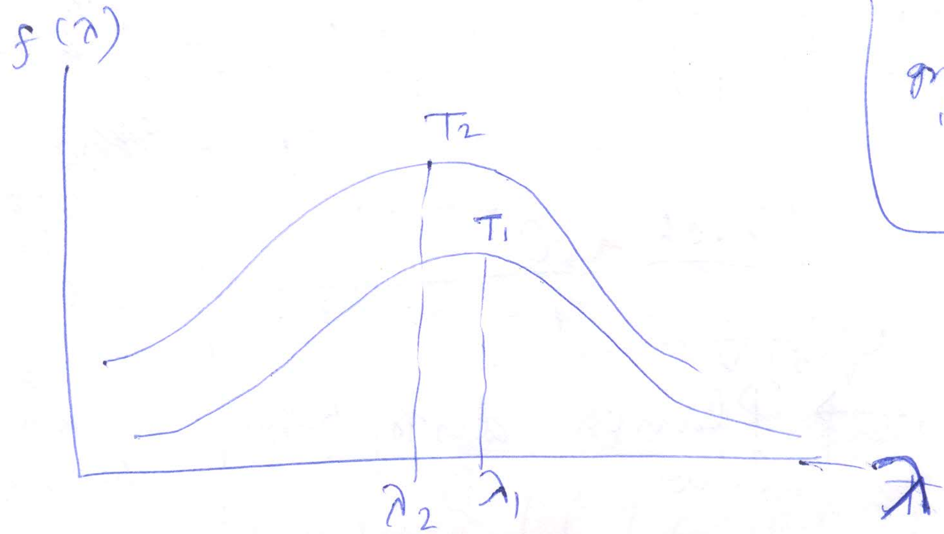


graphs are identical upto a scaling

$$\lambda_1 T_1 = \lambda_2 T_2$$

$$f(\lambda) = T^5 \; 8\pi ch \; g(\lambda T)$$

$$\partial f / \partial \lambda = 0 \Rightarrow g'(\lambda T) = 0$$

$g'(u) = 0$ has a soln. $\underset{\text{fixed no.}}{u = u_0}$

$$g(u) = \frac{1}{u^5} \frac{e^{-ch/ku}}{1 - e^{-ch/ku}}$$

$$\lambda_{max} \cdot T = u_0$$

$\hookrightarrow$ position of maximum

---

Bose-Ein. condensation isn't seen in photons - infinite no. of ~~photons~~ photons in $\epsilon = 0$ state has no physical effect - you may ask whether there is any significant effect from levels close to zero but not zero energy - this will give negligible effect as in the case we dealt with

# Electrons in a magnetic field

Ignore spin

Uniform magnetic field along z-dirn

$$B_x = B_y = 0, \quad B_z = B$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Choose $A_x = -\frac{1}{2} B y, \quad A_y = \frac{1}{2} B x, \quad A_z = 0$

$$\Rightarrow B_z = \partial_x A_y - \partial_y A_x = B$$

$$\boxed{\partial_x = \frac{\partial}{\partial x}}$$

Check $B_x = 0, \ B_y = 0$

---

Consider an electron of charge $-e$ moving in this field

$$H = \frac{1}{2m}\left(\vec{p} + e\vec{A}(\vec{x})/c\right)^2$$

$$= \frac{1}{2m}\left\{ p_z^2 + \left(p_x - \frac{1}{2}\frac{eB}{c} y\right)^2 + \left(p_y + \frac{1}{2} eB/c\, x\right)^2 \right\}$$

$$= \underbrace{\frac{p_z^2}{2m}}_{H_z} + \underbrace{\frac{1}{2m}\left[p_x^2 + p_y^2 + \frac{e^2B^2}{4c^2}(x^2+y^2)\right]}_{H_0} + \frac{eB}{2mc}\underbrace{\left(x p_y - y p_x\right)}_{L_z}$$

$H_z$, $H_0$ & $L_z$ commute with each other

$\Rightarrow$ can be simultaneously diagonalized

$$\boxed{\omega = eB/2mc} \rightarrow \text{defn.} \rightarrow \boxed{K = \frac{e^2B^2}{4mc^2}}$$

Creation & annihilation op.

$$a_x^+ = \frac{1}{\sqrt{2}}\left\{\left(\frac{\hbar}{km}\right)^{1/2} p_x + i\left(\frac{km}{\hbar}\right)^{1/2} x\right\}$$

$$a_x = \frac{1}{\sqrt{2}}\left\{\left(\frac{\hbar}{km}\right)^{1/2} p_x - i\left(\frac{km}{\hbar}\right)^{1/2} x\right\}$$

$a_y$, $a_y^+$ similarly

$$a_x^+ a_y = \frac{1}{2}\left[\frac{\hbar}{km} p_x p_y + \left(\frac{km}{\hbar}\right) xy + i x p_y - i y p_x\right] = \tfrac{1}{2}\left[\left(\frac{\hbar}{km}\right) p_x p_y + \left(\frac{km}{\hbar}\right) xy - L_z/\hbar\right]$$

$$a_y^+ a_x = \tfrac{1}{2}\left[\left(\frac{\hbar}{km}\right) p_x p_y + \left(km/\hbar\right) xy + L_z/\hbar\right]$$

$$\therefore a_x^+ a_y - a_y^+ a_x = -L_z/\hbar \Rightarrow L_z|0\rangle = 0$$

$$\mathcal{H}_0 = \hbar\omega\left(a_x^+ a_x + a_y^+ a_y + 1\right)$$

Ground state

$$a_x|0\rangle = 0 = a_y|0\rangle \Rightarrow L_z|0\rangle = 0$$

(Check this)

Excited states

$$(a_x^+)^{n_1}(a_y^+)^{n_2}|0\rangle$$

$\rightarrow$ not $L_z$ eigenstates

$$[L_z, a_x^+] = i\hbar a_y^+, \quad [L_z, a_y^+] = -i\hbar a_x^+$$

Define : $a_+^+ = \frac{1}{\sqrt{2}}\left(a_x^+ + i a_y^+\right)$

$a_-^+ = \frac{1}{\sqrt{2}}\left(a_x^+ - i a_y^+\right)$

Defining new creation & annihilation operators

(These have the following features) :-

1. $[a_+, a_+^+] = 1$, $[a_-, a_-^+] = 1$

   other commutators vanish

2. $[L_z, a_+^+] = \hbar a_+^+$

   $[L_z, a_-^+] = -\hbar a_-^+$

3. $\mathcal{H}_0 = \hbar\omega\left(a_+^+ a_+ + a_-^+ a_- + 1\right)$

H.W.

$L_z a_+^+|0\rangle - a_+^+ \cancel{L_z|0\rangle}^{0}$

$= \hbar a_+^+|0\rangle$

$\Rightarrow L_z a_+^+|0\rangle$

$= \hbar a_+^+|0\rangle$

Similarly,

$L_z a_-^+|0\rangle$

$= -\hbar a_-^+|0\rangle$

Define $|0\rangle$ such that $a_+|0\rangle = 0 = a_-|0\rangle$

$$(a_+^+)^{N-n_1}(a_-^+)^{n_1}|0\rangle = |s\rangle$$

$N = 1, 2, 3, \ldots$

$n_1 = 0, 1, 2, \ldots, N$

$$\mathcal{H}_0|s\rangle = \hbar\omega\left(N - n_1 + n_1 + 1\right)|s\rangle = (N+1)\hbar\omega|s\rangle$$

$\therefore |s\rangle$ is an ev of $H_0$ with e.value $\hbar\omega(N - n_1 + \tfrac{1}{2} + n_1 + \tfrac{1}{2})$

$$L_z|s\rangle = \hbar\{(N-n_1) - n_1\}|s\rangle = \hbar(N - 2n_1)|s\rangle$$

$\cancel{(\mathcal{H}_0 + L_z)|s\rangle = }$

$\cancel{\left(\mathcal{H}_0 + \frac{eB}{2mc} L_z\right)}$

$$\left(\mathcal{H}_0 + \frac{eB}{2mc} L_z\right) = \left(\mathcal{H}_0 + \omega L_z\right)$$

$$L_z (a_+^+)^2|0\rangle = 2\hbar (a_+^+)^2|0\rangle$$

$$L_z (a_+^+)^3|0\rangle = 3\hbar (a_+^+)^3|0\rangle$$

$$L_z (a_+^+)^n|0\rangle = n\hbar (a_+^+)^n|0\rangle$$

$$(\mathcal{H}_0 + \omega L_z)|s\rangle$$
$$= \hbar\omega\left(N + 1 + (N - 2n_1)\right)|s\rangle$$
$$= \hbar\omega \underbrace{(2N - 2n_1 + 1)}_{(2j+1)}|s\rangle$$

Eigenvalues of $\left(\mathcal{H}_0 + \frac{eB}{2mc} L_z\right)$ are $(2j+1)\omega\hbar$, $j$ = integer. $N - n_1 = j$

$\boxed{j \text{ runs from } 0 \text{ to } N}$

Every $j$ can be obtained in infinite no. of ways.

$$(2j+1)$$
$$\sum_{N=1}^{\infty}\sum_{n_1=0}^{N}\left[2(N-n_1)+1\right]$$
$$= \sum_{N=1}^{\infty}\left(2N^2 - \frac{N(N+1)}{2} + N\right)$$
$$= \infty$$

H. W. $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+)$ & $p = -i\sqrt{\frac{m\hbar\omega}{2}}(a^+ - a)$

$\boxed{a_x \text{ \& } a_y \text{ commute}}$

$\boxed{\text{similarly, } a_x^+ \text{ \& } a_y^+ \text{ commute}}$

$$xp_y - yp_x$$
$$= i\hbar/2\left[(a_y + a_y^+)(a_x^+ - a_x) - (a_x + a_x^+)(a_y^+ - a_y)\right]$$
$$= i\hbar/2\left[a_y a_x^+ - a_y a_x + a_y^+ a_x^+ - a_y^+ a_x - a_x a_y^+ + a_x a_y - a_x^+ a_y^+ + a_x^+ a_y\right]$$
$$= i\hbar\left(a_x^+ a_y - a_y^+ a_x\right)$$

$\therefore L_z|0\rangle = 0 \quad (\because a_y|0\rangle = a_x|0\rangle = 0)$

$[L_z, a_x^+] = i\hbar a_y^+ \Rightarrow L_z a_x^+ - a_x^+ L_z = i\hbar a_y^+$

$\Rightarrow L_z a_x^+|0\rangle - a_x^+ \cancel{L_z|0\rangle}^{0} = i\hbar a_y^+|0\rangle \Rightarrow \boxed{L_z a_x^+|0\rangle = i\hbar a_y^+|0\rangle}$

Similarly, $\boxed{L_z a_y^+|0\rangle = i\hbar a_x^+|0\rangle}$