Irreducibility of Polynomials Whose Coefficients are Integers

R. Thangadurai
Harish-Chandra Research Institute
Chhatnag Road, Jhunsi
Allahabad 211019 India
E-mail: thanga@hri.res.in

Abstract. In this article, we give an account for testing the irreducibility of a given polynomial with integer coefficients over the field of rational numbers. Apart from the traditional tests like Eisenstein criterion and irreducibility over prime finite field, we study the recent criteria like those of Ram Murty, Chen *et al.*, Filaseta and so on.

1. Introduction

A polynomial is said to be *reducible* over a given field if it is expressible as a product of lower degree polynomials with coefficients in this field. Otherwise, it is said to be *irreducible*.

In this article, we shall concentrate on the polynomials whose coefficients are integers and their irreducibility over the field of rational numbers (which is denoted by $\mathbb{Q}$). Let $\mathbb{Z}[X]$ be the ring of polynomials with integer coefficients. Let

$$f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \mathbb{Z}[X] \quad (1)$$

of degree n with $a_i \in \mathbb{Z}$, $a_n \neq 0$ and the greatest common divisor $(a_0, a_1, \ldots, a_n) = 1$. If $f(X)$ is reducible over $\mathbb{Q}$, then, by Gauss lemma, we can, as well, assume that the factors of $f(X)$ also have integer coefficients.

We are interested in the question of deciding whether a given polynomial is irreducible or not. Consequently, a simple test or criterion which would give this information is desirable. Unfortunately, no such criterion which will apply to all the classes of polynomials has yet been devised; but a number of tests, or *irreducibility criteria* have been found so far which give valuable information for some particular classes of polynomials.

Throughout the article, unless otherwise specified, the irreducibility of $f(X)$ will be over $\mathbb{Q}$.

The most popular irreducibility criterion is due to Eisenstein [12] which states that:

Criterion # 1. *If there exists a prime number p such that $p \nmid a_n$, $p | a_i$ for all $i = 0, 1, \ldots, n-1$ and $p^2 \nmid a_0$, then, $f(X)$ is irreducible over $\mathbb{Q}$.*

Such a polynomial $f(X)$ is called *Eisenstein polynomial*. It often happens that this criterion is not directly applicable to a given polynomial $f(X)$, but it may be applicable to $f(X+a)$ for some constant a. So we try various values of a, hoping to transform $f(X)$ into a polynomial that satisfies the conditions of the criterion.

Notice that Eisenstein's criterion essentially reduces the problem of factoring a polynomial of degree n to a problem of factoring n integers, the coefficients of the transformed polynomial, to see if they share a suitable common prime divisor. Obviously as we try various transformations we will produce polynomials with larger and larger coefficients, so the computational task of computing and then checking the factorizations of those coefficients can be significant.

Moreover, a simple transformation that allows us to apply Eisenstein's criterion may not even exist. Indeed, Algebraic Number Theory predicts for what prime p the above criterion works. In fact, the primes p that make the above criterion work is really a special type of prime called *totally ramified prime* in the finite extension of $\mathbb{Q}$, the field obtained by attaching a root of $f(X)$. This type of primes are rare and hence, this criterion cannot be applied to test the irreducibility of all polynomials.

By Probabilistic Galois Theory, it is known that almost all the polynomials with integer coefficients are irreducible polynomials. Therefore, it is reasonable to look for more such criteria to prove irreducibility of a given polynomial.

However, if we are willing to factor some large integers, there are other criteria even easier than Eisenstein's, and these always work. These criteria, in the literature, are not as popular as Eisenstein criterion. The purpose of this article is to give an exposition of these criteria to test the irreducibility of a given polynomial with integer coefficients. For older results we refer to the exposition of Dorwart [9].

2. Preliminaries

We define

$$u = u(f) := \#\{m \in \mathbb{Z} \mid f(m) = \pm 1\}.$$

Clearly, u counts the number of times f is a unit at the integral arguments.

If $f(X)$ assumes the values ± 1 at $X = b_i$ for integers b_i ($i = 1, 2, \ldots, m$), then,

$$f(X) = r(X)\prod_{i=1}^{m}(X - b_i) \pm 1 \quad \text{where} \quad r(X) \in \mathbb{Z}[X].$$

Remark 2.1. *(Dorwart and Ore, [10]) If $f(X)$ takes the value $+1$ (respectively, -1) at $m > 3$ distinct integers, then $f(X)$ cannot take the value -1 (respectively, $+1$). For, let $b_1, b_2, \ldots, b_m$ be the integers such that $f(b_i) = 1$ for all i. Then*

$$f(x) = (x - b_1)(x - b_2)\cdots(x - b_m)g(x) + 1 \qquad (2)$$

for some $g(x) \in \mathbb{Z}[x]$. Suppose that b_{m+1} is an integer such that $f(b_{m+1}) = -1$. Then, from the equation (2), we get

$$-1 = (b_{m+1} - b_1)(b_{m+1} - b_2)\cdots(b_{m+1} - b_m)g(b_{m+1}) + 1$$

which would imply

$$(b_{m+1} - b_1)(b_{m+1} - b_2)\cdots(b_{m+1} - b_m)g(b_{m+1}) = -2.$$

Therefore, the differences $b_{m+1} - b_i$ can take the values ± 1 and ± 2 only. Thus, $m \leq 3$.

Remark 2.2. *If $f(X)$ is of degree n, then $u(f) \leq n$, whenever $n \geq 4$. For, by Remark 2.1, we see that $f(X)$ cannot take $+1$ as well as -1 as its value. If $f(X)$ takes the value $+1$, then it can take $+1$ for at most n distinct integers m_i's, as these m_i are the roots of the polynomial $f(X) - 1$ which is of degree n again. Hence, $u(f) \leq n$.*

When $n \leq 3$, f can take values $+1$ and -1. Hence, we can take the trivial bound for $u(f)$ as twice the degree of f; i.e., $u(f) \leq 2n$.

Remark 2.3. *If $f(X)$ is of degree $n \geq 8$, and it takes values ± 1 at $m > n/2$ distinct integers, then $f(X)$ must be irreducible. By Remark 2.1, it is clear that $f(X)$ can take either $+1$ or -1 as its value, but not both. Assume that $f(X)$ takes the value $+1$ for $m > n/2 > 3$ distinct integers. If possible, suppose that $f(X) = g(X)h(X)$ where $g(X)$ and $h(X)$ are non-trivial factors of $f(X)$. Since the degree of f is the sum of the degrees of g and h, it is clear that one of the factors, say, g will have degree $\geq n/2$. Since $n/2 > 3$, by Remark 2.1, g cannot take both the values $+1$ and -1. So, we assume that g takes the value $+1$ only. But, whenever $f(a) = 1$, we have $g(a)h(a) = 1$ which implies $g(a) = 1 = h(a)$ and in no way $h(X)$ can take the value -1, as g cannot take the value -1. However, since f takes the value 1 for $m > n/2$ distinct integers, $h(X)$ also takes the value 1 for $m > n/2$ distinct integers, which is a contradiction to that fact that the degree $n(h)$ of h satisfies $n(h) \leq n/2$. Hence, $f(X)$ is irreducible.*

Definition 2.1. *Brown and Graham [4]) A polynomial $g(X) \in \mathbb{Z}[X]$ is said to be fat if*

$$\ell(g) := u(g) - n(g) > 0,$$

where $n(g)$ is the degree of $g(X)$.

Remark 2.4. *If $f(X)$ is a fat polynomial, then it is clear that f has to assume both the values $+1$ and -1. If not, then $+1$ (respectively, -1) is assumed by more than the degree of $f(X)$ distinct number of integers which is not possible. Therefore, by Remark 2.1, we have $n \leq 3$. That is, if $f(X)$ is a fat polynomial, then its degree is less than or equal to 3.*

Remark 2.5. *Dorwart and Ore [10] proved that if $f(X)$ is a fat polynomial of degree n, then $f(X) = \pm h_i(\pm X + a)$, where the polynomials $h_i(X)$, $i = 1, 2, \ldots, 5$ are listed below.*

$$h_1(X) = X(X-1)(X-3) + 1, \quad n = 3, \quad u(f) = 4.$$
$$h_2(X) = (X-1)(X-2) - 1, \quad n = 2, \quad u(f) = 4.$$
$$h_3(X) = 2X(X-2) + 1, \quad n = 2, \quad u(f) = 3.$$
$$h_4(X) = 2X - 1, \quad n = 1, \quad u(f) = 2.$$
$$h_5(X) = X - 1, \quad n = 1, \quad u(f) = 2.$$

We define

$$P(f) := \#\{n \in \mathbb{Z} \mid f(n) = \pm p \text{ where } p \text{ is a prime number}\}$$

to be the number of times f assumes prime values upto units in $\mathbb{Z}$ at the integer arguments. Note that $P(f) = \infty$ for many

$f(X) \in \mathbb{Z}[X]$. For example, if we consider $f(x) = ax + b$ with $(a, b) = 1$, then by Dirichlet's prime number theorem, it is known that $P(f) = \infty$.

Remark 2.6. *(Stäckel, 1918, [24]) If $P(f) > 2n$, then $f(X)$ is irreducible. For, suppose $f(X) = g(X)h(X)$ where $g, h \in \mathbb{Z}[X]$ and degrees of $g, h > 1$. Suppose that $p = f(n) = g(n)h(n)$ is a prime. Then either $g(n)$ or $h(n)$ is ± 1. Thus, to know how many prime values f can assume, it is enough to know the number of integer solutions to the equation $r(x) = \pm 1$. Clearly, the equation $r(X) = 1$ can have at most $n(r)$ distinct integer solutions. Therefore $r(X) = \pm 1$ can have at most twice of the degree of $r(X)$ distinct integer solutions. Thus, by noting that the sum of the degrees of g and h is n, we have, $P(f) \leq 2n$, which is a contradiction.*

For example, suppose that $f(X) = 3X^2 + 11X + 121 \in \mathbb{Z}[X]$. It turns out that $f(X)$ is not an Eisenstein polynomial. Note that $f(-7) = 191, f(-6) = 163, f(-1) = 113, f(3) = 181$ and $f(5) = 251$. Since 113, 163, 181, 191 and 251 are all prime numbers, by Remark 2.6, we conclude that $f(X)$ is irreducible over $\mathbb{Q}$.

Remark 2.7. *(References [18] and [19]) Note that Remark 2.6 can be improved under some additional assumptions. Indeed, in Remark 2.6, it suffices to conclude that if we can find $n + 1$ integers $m_1, m_2, \ldots, m_{n+1}$ such that $|m_i - m_j| > 2$ and $f(m_i)$ is a prime or a unit, then $f(X)$ has to be irreducible. This is because of the following observation. Suppose $f(X) = g(X)h(X)$. Therefore, $g(m_i) = \pm 1$ or $h(m_i) = \pm 1$. Using the given condition, first we claim that $g(X) = \pm 1$ (respectively, $h(X) = \pm 1$) can have at most $d = n(g)$ (respectively, $n(h)$), degree of g, solutions in $\mathbb{Z}$. For, if $g(X) = b_d X^d + b_{d-1} X^{d-1} + \cdots + b_0$ and $g(m_1) = 1$ and $g(m_2) = -1$, then we have*

$$b_d(m_1^d - m_2^d) + b_{d-1}(m_1^{d-1} - m_2^{d-1}) + \cdots + b_1(m_1 - m_2) = 2.$$

This implies that $(m_1 - m_2)$ divides 2 - a contradiction to the fact that $|m_i - m_j| > 2$. Hence $g(X) = \pm 1$ (respectively, $h(X) = \pm 1$) can have at most $n(g)$ (respectively, $n(h)$) solutions. Therefore, the solutions of $g(X) = \pm 1$ and $h(X) = \pm 1$ together will not exceed the value $n(g) + n(h) = n$. Hence, if $f(X)$ assumes more than $n + 1$ prime values or unit, it cannot be reducible, because of the above reason.

For example, consider the polynomial $f(X) = X^6 - 3X^5 - 87X^4 + 118X^3 - 33X^2 + 21X - 1$. In this case, we can easily compute the values

$$\begin{aligned} f(-22) &= 107187629, \\ f(-8) &= -58601, \\ f(-4) &= -23269, \\ f(0) &= -1, \\ f(12) &= 634859, \\ f(18) &= 19888469, \\ f(30) &= 588786929. \end{aligned}$$

Each of the arguments $-22, -8, \ldots, 30$ differs by more than 2 from the others, and each of the 7 values of $f(k)$ is a prime or a unit, so it follows that $f(x)$ is irreducible over the integers (and therefore over the rationals).

Remark 2.8. *(J. Brillhart, 1980, [12]) If $f(X)$ assumes a prime value for a sufficiently large integer, then $f(X)$ is irreducible. For, suppose $f(X)$ is reducible and hence $f(X) = g(X)h(X)$, where $g, h \in \mathbb{Z}[X]$. Let $\{b_i\}$ and $\{b'_j\}$ be all the integer roots of the polynomials $g(X) \pm 1$ and $h(X) \pm 1$ respectively. Define $M_1 = \max_i |b_i|$ and $M_2 = \max_j |b'_j|$. Let $M := \max\{M_1, M_2\}$. We can take this M to be the desired large integer. Since $f(m)$ is a prime for an integer m with $|m| > M$, then, clearly, either $g(m)$ or $h(m)$ is ± 1 which is impossible, from the definition of M. Therefore, $f(X)$ is irreducible.*

In particular, if $P(f) = \infty$, then, $f(X)$ must be irreducible.

The converse, unfortunately, is not true. That is, if $f(X)$ is irreducible, we cannot expect, in general, that $P(f) = \infty$. In fact, we give an example of an irreducible polynomial f with $P(f) = 0$ of any degree $n > 1$. If we take

$$f(X) = X^n + 105X + 12,$$

then, it is irreducible by Eisenstein criterion. However, we have $P(f) = 0$. For, since $f(X) = X(X^{n-1} + 105) + 12$, for any integer value m, we have, $f(m)$ is even. Hence, if at all it can produce primes p, then the only possibility is $p = 2$. However, if we consider

$$g_{\pm}(X) = f(X) \pm 2$$

then, $g_{\pm}(X)$ is irreducible by Eisenstein criterion and hence, $P(f) = 0$.

In the above counter example, we had 2 as a common divisor of $f(m)$ for every $m \in \mathbb{Z}$. Hence, it is reasonable to have the following definition.

Definition 2.2. *Let $f(X) \in \mathbb{Z}[X]$. The fixed divisor of f, denoted by d_f, is the largest integer d such that $d | f(n)$ for all $n \in \mathbb{Z}$.*

For example, if $f(X) = X^2 + 9X - 4$, then $d_f = 2$.

Conjecture 1. (Bunyakovsky, [5]) If $f(X) \in \mathbb{Z}[X]$ is irreducible, then $P(g) = \infty$ where $g(X) = d_f^{-1} f(X)$.

The only case, for which Conjecture 1 is known to be true, is $f(X) = aX + b$ by Dirichlet Prime Number Theorem (See for instance, [1]). Otherwise, Conjecture 1 remains completely unsolved.

We define the following 'heights' of $f(X)$;

$$H_1 = \max_{0 \le i \le n-1} \left| \frac{a_i}{a_n} \right|$$

and

$$H_2 = \max_{0 \le i \le n-1} \left| \frac{a_i}{a_n} \right|^{1/(n-i)}.$$

If one has bounds for the roots of $f(X)$ in the complex plane, then we can prove some irreducibility criteria using these bounds. Hence, first, we give some bounds for any complex root of $f(X)$ in the following three lemmas.

Lemma 2.1.1. *Let $f(X)$ be a polynomial as defined in (1). Suppose $a_{n-k} = 0$ for all $k = 1, 2, \ldots, r$ where $0 \le r \le n-1$. If $\alpha \in \mathbb{C}$ is a root of $f(X)$, then*

$$|\alpha| < H_1^{1/(r+1)} + 1.$$

Remark 2.9. *If $r = 0$ in the statement of Lemma 2.1.1, then, we have $a_{n-1} \neq 0$. This case was proved by Cauchy [6], Brillhart [2] and Ram Murty [23].*

Proof. Let $\alpha \in \mathbb{C}$ be a root of $f(X)$. Since α is a root of $f(X)$ and $a_{n-k} = 0$ for all $k = 1, 2, \ldots, r$, we have

$$-a_n \alpha^n = a_{n-r-1} \alpha^{n-r-1} + \cdots + a_1 \alpha + a_0.$$
$$\implies -\alpha^n = \frac{a_{n-r-1}}{a_n} \alpha^{n-r-1} + \cdots + \frac{a_1}{a_n} \alpha + \frac{a_0}{a_n}.$$

Therefore,

$$|\alpha|^n \le H_1(|\alpha|^{n-r-1} + \cdots + |\alpha| + 1) = H_1 \left(\frac{|\alpha|^{n-r} - 1}{|\alpha| - 1} \right). \tag{3}$$

If $|\alpha| \le 1$, then clearly, $|\alpha| < H_1^{\frac{1}{r+1}} + 1$ as $H_1 > 0$. If $|\alpha| > 1$, then by (3), we have,

$$|\alpha|^n (|\alpha| - 1) \le H_1(|\alpha|^{n-r} - 1) < H_1 |\alpha|^{n-r}$$
$$\implies |\alpha|^r (|\alpha| - 1) < H_1.$$

Since

$$(|\alpha| - 1)^{r+1} \le |\alpha|^r (|\alpha| - 1),$$

we have,

$$(|\alpha| - 1)^{r+1} < H_1$$

and hence we get $|\alpha| < H_1^{1/(r+1)} + 1$. •

Lemma 2.1.2. *([21], Page 53, Lemma 4) Let $f(X)$ be a polynomial as defined in (1). If $\alpha \in \mathbb{C}$ is a root of $f(X)$, then*

$$|\alpha| < 2H_2.$$

Proof. Set $b_i = a_i/a_n$ for all $i = 0, 1, 2, \ldots, n-1$. Let

$$c := \max_{0 \le i \le n-1} \left\{ |b_i|^{1/(n-i)} \right\} \quad \text{and} \quad \eta = \frac{\alpha}{c}.$$

To prove the lemma, it is enough to prove $|\eta| < 2$, as $c = H_2$.

By definition, we have, $|a_i/a_n| \le c^{n-i}$ for all $i = 0, 1, 2, \ldots, n-1$. Then we have,

$$a_n \left(\eta^n + \frac{b_{n-1}}{c} \eta^{n-1} + \cdots + \frac{b_0}{c^n} \right)$$
$$= a_n \left(\frac{1}{c^n} \alpha^n + \frac{b_{n-1}}{c^n} \alpha^{n-1} + \cdots + \frac{b_0}{c^n} \right)$$
$$= \frac{1}{c^n} (a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_0)$$
$$= \frac{1}{c^n} f(\alpha) = 0.$$

Since $a_n \neq 0$, we have,

$$\eta^n + \frac{b_{n-1}}{c} \eta^{n-1} + \cdots + \frac{b_0}{c^n} = 0.$$

Since $|b_i| \le c^{n-i}$, we get,

$$|\eta|^n \le 1 + |\eta| + |\eta|^2 + \cdots + |\eta|^{n-1}. \tag{4}$$

If $|\eta| \ge 2$, then by above inequality (4) we have

$$|\eta|^n \le \frac{|\eta|^n - 1}{|\eta| - 1} < \frac{|\eta|^n}{|\eta| - 1}$$

which implies $|\eta| < 2$, a contradiction. Hence $|\eta| < 2$. That is, $|\alpha| < 2c$. Since $c = H_2$, we get the result. •

Lemma 2.1.3. *(Ram Murty, [23]) Let $f(X)$ be a polynomial as defined in (1). Assume that $a_n \geq 1$, $a_{n-1} \geq 0$ and $|a_i| \leq M$ for $i = 0, 1, \ldots, n-2$ and for some $M > 0$. If $\alpha \in \mathbb{C}$ is a root of $f(X)$, then α satisfies either*

$$\Re(\alpha) \leq 0 \quad or \quad |\alpha| < \frac{1+\sqrt{1+4M}}{2},$$

where $\Re(z)$ means real part of $z \in \mathbb{C}$.

Proof. If $a_{n-1} = 0$, then, by Lemma 2.1.1, we have $|\alpha| < \sqrt{H_1} + 1$ where $H_1 \leq M/a_n$. Hence, it is an easy verification that, in this case, we get,

$$|\alpha| < \sqrt{H_1} + 1 \leq \frac{1+\sqrt{1+4M}}{2}.$$

Thus, we can assume that $a_{n-1} \geq 1$.

Let $z \in \mathbb{C}$ such that $|z| > 1$ and $\Re(z) > 0$. Then first we claim that

$$\left|\frac{f(z)}{z^n}\right| > 0 \text{ whenever } |z| \geq \frac{1+\sqrt{1+4M}}{2}.$$

For,

$$\begin{aligned}\left|\frac{f(z)}{z^n}\right| &= \left|a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right| \\ &\geq \left|a_n + \frac{a_{n-1}}{z}\right| - \left(\frac{|a_{n-2}|}{|z|^2} + \cdots + \frac{|a_0|}{|z|^n}\right) \\ &\geq \Re\left(a_n + \frac{a_{n-1}}{z}\right) - M\left(\frac{1}{|z|^2} + \cdots + \frac{1}{|z|^n}\right) \\ &> 1 - \frac{M}{|z|^2 - |z|} = \frac{|z|^2 - |z| - M}{|z|^2 - |z|},\end{aligned}$$

as $a_n, a_{n-1} \geq 1$ and $\Re(z) > 0$. Hence,

$$\left|\frac{f(z)}{z^n}\right| > 0 \quad \text{whenever} \quad \frac{|z|^2 - |z| - M}{|z|^2 - |z|} \geq 0.$$

But, whenever $|z| \geq \frac{1+\sqrt{1+4M}}{2}$, we have $\frac{|z|^2 - |z| - M}{|z|^2 - |z|} \geq 0$. Thus we conclude that

$$\left|\frac{f(z)}{z^n}\right| > 0 \text{ whenever } |z| \geq \frac{1+\sqrt{1+4M}}{2}$$

which proves our claim.

To end the proof of the lemma, we assume that $\Re(\alpha) > 0$. Therefore, we have to prove

$$|\alpha| < \frac{1+\sqrt{1+4M}}{2}.$$

Also, $\alpha \neq 0$ as $\Re(\alpha) > 0$. If $|\alpha| < 1$, then there is nothing to prove, as $M \geq 0$ and $\frac{1+\sqrt{1+4M}}{2} \geq 1$.

Suppose $|\alpha| \geq \frac{1+\sqrt{1+4M}}{2}$. Therefore, by the above claim, we get $\frac{|f(\alpha)|}{|\alpha|^n} > 0$, which is a contradiction to the fact that $f(\alpha) = 0$, as α is a root of $f(X)$. Hence, we get the result. •

3. Irreducibility Criteria

Criterion # 2. It is an easy observation that if $f(X)$ is reducible in $\mathbb{Z}[X]$, then it is reducible over $\mathbb{F}_p[X]$, where $\mathbb{F}_p$ is the finite field of p elements. Hence, we have the following criterion.

If $f(X)$ is irreducible over $\mathbb{F}_p[X]$ for some prime number p, then $f(X)$ is irreducible over $\mathbb{Q}$.

Converse is not true. That is, if $f(X)$ is irreducible over $\mathbb{Q}$, then it is not necessarily irreducible over $\mathbb{F}_p$ for some prime number p.

For example, if $f(X) = X^4 + 1$, then it is irreducible over $\mathbb{Q}$, by Eisenstein criterion. But, it is reducible over $\mathbb{F}_p$ for every prime number p. When $p = 2$, clearly, $f(X) = X^4 + 1 \equiv (X^2+1)^2 \pmod 2$ and hence it is reducible over $\mathbb{F}_2$. Let $p \geq 3$ be any prime. Then, p satisfies, $p^2 \equiv 1 \pmod 8$. That is, $8|(p^2-1)$ and hence

$$(X^8 - 1)|(X^{p^2-1} - 1) \implies X(X^4+1)(X^4-1)|(X^{p^2} - X).$$

That is, $f(X)$ is a factor of the polynomial $X^{p^2} - X$. The splitting field K of the polynomial $X^{p^2} - X$ over $\mathbb{F}_p$ is $\mathbb{F}_{p^2}$. Clearly, $[K : \mathbb{F}_p] = 2$.

If $f(X)$ is irreducible over $\mathbb{F}_p$ for some prime $p \geq 3$, then $K_1 = \mathbb{F}_p(\alpha)$, where α is a root of $f(X)$, is an non-trivial field extension of $\mathbb{F}_p$ of dimension 4. Also, since $f(X)$ is a factor of $X^{p^2} - X$, K_1 is an intermediate field of K over $\mathbb{F}_p$. Hence, we have,

$$[K_1 : \mathbb{F}_p]|[K : \mathbb{F}_p] \implies 4|2,$$

which is absurd. Hence $f(X)$ is reducible over $\mathbb{F}_p$ for all primes p.

More generally, Driver, Leonard and Williams [11] gave a necessary and sufficient condition for an 4th degree polynomial with integer coefficients which is irreducible over $\mathbb{Q}$; but reducible over $\mathbb{F}_p$ for every prime number p. Also, another recent result of Guralnick, Schacher and Sonn, [17] states that for any composite integer $n \geq 4$, there exists an irreducible polynomial $f(X) \in \mathbb{Z}[X]$ of degree n which is reducible over $\mathbb{F}_p$ for every prime p.

Now, we give a criterion involving $P(f)$. More precisely, we have the following criterion which was proved by Ore [22].

Criterion # 3. If $P(f) \geq n+3$ where n is the degree of $f(X)$, then $f(X)$ is irreducible over $\mathbb{Q}$.

Since the original proof is not easily available, we present the proof here for all polynomials of degree ≥ 7 for simplicity.

Proposition 3.1. *If $P(f) + 2u \geq n + 4$, then $f(X)$ is irreducible.*

Proof. If possible, we assume that $f(X) = g(X)h(X)$ where $g, h \in \mathbb{Z}[X]$ and degrees of g (say $n(g)$) and h (say $n(h)$) are ≥ 1. Without loss of generality we may assume that $\ell(g) \geq \ell(h)$.

Claim. $\ell(g) + \ell(h) \geq P(f) + 2u - n.$ (5)

For each $m \in \mathbb{Z}$ such that $f(m)$ is a prime number, we have either $g(m)$ or $h(m)$ must be a unit. While for each $m \in \mathbb{Z}$ such that $f(m)$ is a unit, we have $g(m)$ and $h(m)$ is a unit. Therefore, $u(g) + u(h) \geq P(f) + 2u$. Therefore, we have

$$\ell(g) + \ell(h) = u(g) - n(g) + u(h) - n(h) = u(g) + u(h) - n \geq P(f) + 2u - n$$

as claimed.

Since by assumption $P(f) + 2u \geq n + 4$, we have $P(f) + 2u - n \geq 4$. Therefore, by the claim, we get, $\ell(g) + \ell(h) \geq 4$. If $\ell(g) > 0$ and $\ell(h) > 0$, then by the definition, we have g and h are fat polynomials. Hence by Remark 2.4, we have $n(g), n(h) \leq 3$ which is not possible as its sum is ≥ 7. Therefore, only $g(X)$ is fat. Since $h(X)$ is not fat and $n \geq 7$, we have $n(h) \geq 4$ and $\ell(h) \leq 0$. Also, since $\ell(g) + \ell(h) \geq 4$, we have $\ell(g) = u(g) - n(g) \geq 4$ which would imply $u(g) \geq 4 + n(g)$, which is not possible because $n(g) \leq 3$ and $u(g) \leq 2n(g)$. Thus this contradiction shows that $f(X)$ has to be irreducible. •

Corollary 3.1. *If $P(f) \geq n + 2$ and $u \geq 1$, then $f(X)$ is irreducible.*

Proof of criterion # 3. If $P(f) \geq n+4$, then by Proposition 3.1, clearly, $f(X)$ is irreducible. So, it is enough to assume that $P(f) = n + 3$.

Suppose we assume that $f(X) = g(X)h(X)$ where $g, h \in \mathbb{Z}[X]$ of positive degree. By (5) and $P(f) = n + 3$, it is clear that either $\ell(g)$ or $\ell(h)$ is positive. Since the degree of f is ≥ 7, it is clear that exactly one of the factors must be fat. Hence, either g or h coming from the list stated in Remark 2.5; but not both.

Without loss of generality, we may assume that g is fat and h is not fat. Therefore, $\ell(g) \geq 1$ and $\ell(h) \leq 0$ and hence $\ell(g) + \ell(h) \leq \ell(g)$. However, by (5), we know that $\ell(g) + \ell(h) \geq P(f) + 2u - n \geq n + 3 - n = 3$. Thus, we arrive at $\ell(g) \geq 3$, which would implies $u(g) \geq n + 3$. Since g is coming from the list stated in Remark 2.5, we conclude that $u(g) = n + 1$, which is a contradiction to $u(g) \geq n + 3$. Hence $f(X)$ is irreducible. •

The following Conjecture (which is still open) says that the criterion # 3 is tight.

Conjecture 2. (Chen, *et al.* [8]) For each $n \geq 2$, there does exist a polynomial $f(X) \in \mathbb{Z}[X]$ which is reducible and $P(f) = n + 2$.

When $n = 2$, take $f(X) = X(X - 4)$ which has $P(f) = 4$. When $n = 3$, consider $f(X) = (X - 5)(1 + X(X - 3))$ which has $P(f) = 5$.

Define

$$P^+(f) = \#\{n \in \mathbb{Z} \ / \ f(n) > 0 \text{ is prime}\}.$$

Clearly $P^+(f)$ counts the number of positive prime values that f assumes at distinct integral arguments. Recently Chen *et al.* [8] proved the following theorem.

The following Theorem gives another criterion for irreducibility which is similar to Criterion # 3.

Theorem 3.2. (Chen, *et al.* [8]) *If $f(X)$ is reducible, then $P^+(f) \leq n$. On the other hand, there is a reducible polynomial $f \in \mathbb{Z}[X]$ for which $P^+(f) = n$.*

Criterion # 4. If we can find an integer m which is bigger than the 'height' of the given polynomial $f(X)$ and $f(m)$ is prime, then $f(X)$ is irreducible.

Since $f(X)$ is a given polynomial, we know its coefficients and therefore we can compute H_1 and H_2 as defined in section 2. Also, we can able to compute r as defined in Lemma 2.1.1. Put

$$H = \min\{H_1^{1/(r+1)} + 1, 2H_2\}.$$

Theorem 4.1. *If $f(X)$ is a polynomial as defined in (1) and if there exists an integer $m \geq H+1$ such that $f(m)$ is a prime number, then, $f(X)$ is irreducible.*

Proof. Let $f(X)$ be defined as in (1). Let $\alpha \in \mathbb{C}$ be a root of $f(X)$. By Lemmas 2.1.1 and Lemma 2.1.2, we have

$$|\alpha| < H_1^{1/(r+1)} + 1 \text{ and } |\alpha| < 2H_2.$$

Hence we get

$$|\alpha| < H.$$

Suppose $f(X)$ is reducible, say $f(X) = g(X)h(X)$ where $g(X)$ and $h(X)$ in $\mathbb{Z}[X]$ are of positive degree. Since $f(m)$ is a prime for some integer $m \geq H+1$, we have either $g(m)$ or $h(m)$ is ± 1. Without loss of generality, we may assume that $g(m) = \pm 1$. Write, $g(X) = c\prod_i (X - \alpha_i)$ where $\alpha_i \in \mathbb{C}$ roots of $g(X)$ and c is the leading coefficient of g. Since α_i are the roots of f, we have $|\alpha_i| < H$. Therefore,

$$1 = |g(m)| = |c|\prod_i |m - \alpha_i|$$
$$\geq \prod_i (m - |\alpha_i|) > \prod_i (m - H) \geq 1,$$

a contradiction. Hence $f(X)$ must be irreducible. •

Remark 4.2. *Theorem 4.1 with $r = 0$ is the best possible in the following sense. Consider the reducible polynomial $f(X) = (X-9)(X^2+1) = X^3 - 9X^2 + X - 9$ having $H_1 = 9$ and hence $H = H_1 + 1 = 10$. Though $f(10) = 101$ a prime, it is a reducible polynomial.*

Remark 4.3. *(Reference, [18]) By assuming the truth of the Conjecture 1, it is always possible to use Theorem 4.1 to proof the irreducibility of $f(X)$. On the other hand, it won't always be easy. For example, the first prime value of the polynomial $f(X) = X^{12} + 488669$ occurs with $X = 616980$ and has 70 decimal digits.*

Corollary 4.1. *Let p be any prime number and $b \geq 6$ be any integer. Suppose p is written in base b as follows:*

$$p = a_n b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0;$$
$$a_i \in \{0, 1, 2, \ldots, b-1\},\ a_n \neq 0 \text{ and } a_{n-1} \in \{0, 1\}.$$

Then the polynomial $f(X) = a_n X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$ is irreducible.

Proof. Assume that $a_{n-1} = 0$. Then, clearly, $H_1 \leq b-1$ and $r = 1$. Therefore, $H \leq 1 + \sqrt{b-1}$. If we can prove that $b > H+1$, then, since $f(b)$ is a prime number, by Theorem 4.1, we can conclude that $f(X)$ is irreducible. So, it is enough to show that $b > 2 + \sqrt{b-1}$ for all $b \geq 3$. Indeed, a trivial calculation reveals this fact and hence the result.

Now, assume that $a_{n-1} = 1$. Therefore, $H_2 \leq \sqrt{b-1}$ and hence, $H \leq 2\sqrt{b-1}$. So, it is enough to prove, $b - 1 > 2\sqrt{b-1}$ for all $b \geq 6$, which is true. Hence, by Theorem 4.1, $f(X)$ is irreducible. •

The prime number $104729 = 10^5 + 4\times 10^3 + 7\times 10^2 + 2\times 10 + 9$ in usual decimal system. Consider the digit polynomial $f(X) = X^5 + 4X^3 + 7X^2 + 2X + 9$. Clearly, $H_1 = 9$ and $r = 1$. Therefore, we have $10 > \sqrt{H_1} + 2 = 3 + 2 = 5$ such that $f(10)$ is a prime. By Theorem 4.1, we see that $f(X)$ is irreducible. In fact, the following more general result is true.

Theorem 4.2. (Brillhart, Filaseta and Odlyzko, [3] and Ram Murty, [23]) *Let $b \geq 2$ and p be a prime written in base b*

$$p = a_n b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0.$$

Then the digit polynomial $f(X)$ defined in (1) is irreducible.

The case when $b = 10$, Theorem 4.2 was proved by Cohen (see [3]). We prove the above theorem for all $b \geq 3$ using Lemma 2.1.3. The case $b = 2$ is slightly technical and we leave the proof here.

Proof. Suppose the digit polynomial $f(X) = g(X)h(X)$ with $g(X)$ and $h(X)$ are non-constant polynomials in $\mathbb{Z}[X]$. Since $f(b)$ is prime, we have either $g(b)$ or $h(b)$ is ± 1. Without loss of generality, we may assume that $g(b) = \pm 1$. Write, $g(X) = c\prod_i (X - \alpha_i)$ where $\alpha_i \in \mathbb{C}$ roots of $g(X)$ and c is the leading coefficient of g. Since α_i are the roots of f, and $0 \leq a_i \leq b-1$, we have $M = b-1$ in Lemma 2.1.3. Therefore, either $\Re(\alpha_i) \leq 0$ or

$$|\alpha_i| < \frac{1 + \sqrt{1 + 4(b-1)}}{2}.$$

So,

$$1 = |g(b)| \geq \prod_i |b - \alpha_i|.$$

Note that $\alpha_i \neq 0$. If $\Re(\alpha_i) \leq 0$, then $|b - \alpha_i| > b$. If $|\alpha_i| < \frac{1+\sqrt{1+4(b-1)}}{2}$, then $|\alpha_i| < b - 1$ and hence $(b - |\alpha_i|) > 1$. Hence in both the cases, we have $|b - \alpha_i| > 1$. Hence, we get,

$$1 = |g(b)| > 1$$

which is absurd and hence $f(X)$ is irreducible. •

Other criteria. Here we state the other known criteria which work for special types of polynomials.

Theorem 5.1. (Filaseta, [14]) *Let $f(X)$ be a polynomial defined as in (1). Assume that $a_i \geq 0$ and $b > 1$ is an integer such that $f(b)$ is a prime number. Let $N_1 = \pi/\sin^{-1}(1/b)$ and $N_2 = \pi/\tan^{-1}(1/b)$.*

(i) If $n < N_1$, then $f(X)$ is irreducible.
(ii) There exists a polynomial $g(X) \in \mathbb{Z}[X]$ of degree $m \geq N_2$ such that $g(X)$ is reducible and $g(b)$ is a prime number.

Theorem 5.2. (Filaseta, [14]) *Let $f(X)$ be a polynomial defined as in (1). Suppose $0 \leq a_i \leq a_n 10^{30}$. If $f(10)$ is a prime, then f is irreducible.*

Note that Theorem 5.1 will imply that for a polynomial $f(x)$ with non-negative coefficients of degree ≤ 31, if it happens that $f(10)$ is a prime, then $f(x)$ is irreducible. To show the sharpness of the theorem, Filaseta [14] gives the following example. He considers the reducible polynomial

$$\begin{aligned} g(X) = X^{32} &+ 130X^2 \\ &+ 5603286754010141567161572637720X \\ &+ 6109104104761309555986010605952 9, \end{aligned}$$

of degree 32 and having non-negative integer coefficients. It is an easy computation that $g(10) = 21712390858771451123147583244972 9$ is a prime number. This also shows that the upper bound 10^{30} on the coefficients in Theorem 5.2 is rather sharp.

Further, one can compute $g(11)$ and $g(12)$ and conclude that they are composite numbers. Since $H_1 = 6109104104761309555986010605952 9$ and $H_1^{1/31} = 10.6$, by Theorem 4.1, it is clear that $g(m)$ is composite for all $m \geq 13$.

Theorem 5.3. (Filaseta, [13]) *Let $b > 2$ be an integer and $1 \leq w < b$ be any integer. Let p be any prime such that*

$$\begin{aligned} wp = a_n b^n + a_{n-1} b^{n-1} + \cdots a_1 b + a_0; \\ 0 \leq a_i \leq b-1, a_n \neq 0. \end{aligned}$$

Then the polynomial $f(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$ is irreducible.

Remark 5.1. *Theorem 5.3 is further improved when $b = 10$ by Filaseta [14] as follows. Let $f(X)$ be a polynomial as defined in (1) together with $0 \leq a_i \leq 5.79 \times 10^7$. If $f(10) = wp$ for some $w \in \{1, 2, \ldots, 9\}$ and for some prime p, then f is irreducible.*

Similar to Theorem 5.3, we can improve Theorem 4.1 as follows.

Theorem 5.4. (K. Girstmair, [16]) *Let $f(X)$ be the polynomial as in (1) and let H be defined as in Theorem 4.1. If d and m are positive integers such that $m \geq H + d$ and*

$$f(m) = \pm d \cdot p$$

for some prime number p not dividing d, then $f(X)$ is irreducible.

Proof. One can prove this fact as we have proved Theorem 4.1. We shall omit the proof here. •

In [20], Lipka proved that if $f(X)$ is a polynomial as defined in (1) and $a_0 = b_0 p^k$ where $b_0 \neq 0$ and p is a large enough prime, then $f(X)$ is irreducible over $\mathbb{Q}$.

More recently, Finch and Jones [15], have characterized irreducible polynomials of 4th degree having coefficients from the set $\{-1, 0, 1\}$. Also, Chahal and Ram Murty [7] studied the converse of Conjecture 1 in the number field set-up. More precisely, suppose that K is a finite extension of $\mathbb{Q}$. Let $\mathcal{O}_K$ be its ring of integers. Then they proved the following theorem:

Theorem 5.5. (Chahal and Ram Murty, [7]) *Suppose that $f(X) \in \mathcal{O}_K[X]$ is a polynomial with coefficients from $\mathcal{O}_K$ with non-zero discriminant. If $f(X)$ represents an irreducible element of $\mathcal{O}_K$ infinitely often, then either $f(X)$ is an irreducible polynomial or $f(X) = g(X)h(X)$, where $h(X)$ is an irreducible polynomial and $g(X)$ is a linear factor.*

Acknowledgment: I am thankful to Prof. N. Raghavendra for the reference [21] for Lemma 2.1.2.

References

[1] Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, (1976).

[2] J. Brillhart, Note on Irreducibility Testing, *Math. of Computation*, **35** (1980) No. 152, 1379–1381.

[3] J. Brillhart, M. Filaseta and A. Odlyzko, On an irreducibility theorem of A. Cohen, *Canad. J. Math.*, **33** (1981), No. 5, 1055–1059.

[4] W. S. Brown and R. L. Graham, An irreducibility criterion for polynomials over the integers, *Amer. Math. Monthly,* **76** (1969), 795–797.

[5] V. Bunyakovsky, Sur les diviseurs numeriques invariables des fonctions rationelles entieres, *Mem. Acad. Sci. St. Petersberg*, **6** (1857) 305–329.

[6] A. L. Cauchy, Exercises de mathématiques, *J. Ecole Poly.*, **25** (1837), 176.

[7] J. S. Chahal and M. Ram Murty, Irreducible polynomials over number fields, *The Riemann zeta function and related themes*, 39–43, Ramanujan Math. Soc. Lect. Notes Ser., 2, Ramanujan Math. Soc., Mysore, 2007.

[8] Y. G. Chen, G. Kun, G. Pete, I. Z. Ruzsa and A. Timar, Prime values of reducible polynomials, II, *Acta Arith.*, **104.2** (2002), 117–127.

[9] H. L. Dorwart, Irreducibility of Polynomials, *Amer. Math. Monthly*, **42** (1935), No. 6, 369–381.

[10] H. L. Dorwart and O. Ore, Criteria for the irreducibility of polynomials, *Annals of Math.* **34** (1933), No. 1, 81–94.

[11] E. Driver, P. A. Leonard and K. S. Williams, Irreducible quartic polynomials with factorizations modulo p, *Amer. Math. Monthly*, **112** (2005), No. 10, 876–890.

[12] F. G. M. Eisenstein, Uber die Irreductibilitat und einige andere Eigenschaften der Gleichung, von welcher die Theilung der ganzen Lemniscate abhangt, *J. reine angew. Math.*, **39**, (1850), 166–167.

[13] M. Filaseta, A further generalization of an irreducibility theorem of A. Cohn, *Canad. J. Math.*, **34** (1982), 1390–1395.

[14] M. Filaseta, Irreducibility criteria for polynomials with non-negative coefficients, *Canad. J. Math.*, **40** (1988), No. 2, 339–351.

[15] C. Finch and L. Jones, On the irreducibility of $\{-1, 0, 1\}$-quadrinomials, *Integers*, **6** (2006), # A 16.

[16] K. Girstmair, On an irreducibility criterion of M. Ram Murty, *Amer. Math. Monthly*, **112** (2005), No. 3, 269–270.

[17] R. Guralnick, M. Schacher, J. Sonn, Irreducible polynomials which are locally reducible everywhere, *Proc. Amer. Math. Soc.*, **133** (2005), No. 11, 3171–3177.

[18] http://www.mathpages.com/home/kmath406.htm

[19] B. Kirkels, Irreducibility certificates for polynomials with integer coefficients, M. Sc Thesis, Radboud Universiteit Nijmegen, The Netherlands, (2004). URL address:
`http://www.math.ru.nl/~bosma/students/kirkels/mscthesis.pdf`

[20] S. Lipka, Uber die Irreduzibilität von Polynomen, *Math. Ann.*, **118** (1941) 235–245.

[21] M. S. Narasimhan, R. R. Simha, R. Narasimhan and C. S. Seshadri, Riemann surfaces, Mathematical Pamphlets, 1, Tata Institute of Fundamental Research, Bombay, (1963).

[22] O. Ore, Einige Bemerkungen über Irreduzibilität, Jahresber. Deutsch. Math.-Verein., **44** (1934), 147–151.

[23] M. Ram Murty, Prime numbers and irreducible polynomials, *Amer. Math. Monthly*, **109** (2002), No. 5, 452–458.

[24] P. Stäckel, Arithmetische Eigenschaften ganzer Funktionen, *J. reine angew. Math.*, **148** (1918), 101–112.

A Classical Proof of the Fundamental Theorem of Algebra Dissected

R. B. Burckel
Department of Mathematics
Kansas State University
Manhattan, Kansas 66506, U.S.A

As is well-known, there is a plethora of proofs of the Fundamental Theorem of Algebra (FTA) within classical function theory. Elegant as they are, they often intimidate the undergraduate teacher who feels that his students lack the necessary background. But one of these proofs, when stripped to its essentials (which this note aims to do), is fully accessible to second-semester calculus students. It uses only

(1) the trivial fact that a polynomial is asymptotic to its lead term,
(2) the relation between $D_1 f(x+iy)$ and $D_2 f(x+iy)$for a complex-differentiable function f
[that is, the Cauchy-Riemann equation $D_2 = iD_1$], and
(3) Fubini's theorem on inverting the order of integrations of a *continuous* function on a rectangle.

Here is how it goes:

Suppose that $p(z)$ is a monic polynomial of degree $n \geq 1$ having coefficients in $\mathbb{C}$, the complex numbers, but having no zero in $\mathbb{C}$. Then $P(z) := p(z)\overline{p(\overline{z})}$ is one of degree $2n$, which moreover is *positive* at every real number. For appropriately large positive integer N we have

$$\frac{1}{|P(z)|} < \frac{2}{|z|^{2n}} \qquad \forall z \in \mathbb{C} \quad \text{with} \quad |z| \geq N. \tag{1}$$

The five magic numbers for the sequel (whose *post facto* genesis the reader will immediately perceive at the end) are

$$A := \int_{-1}^{1} \frac{1}{P(x)}\,dx, \quad \text{a positive real number,} \tag{2}$$

$$Y := \left(\frac{4\pi}{A}\right)^{\frac{1}{2n-1}} + N, \tag{3}$$

$$X := \left(\frac{8Y}{A}\right)^{\frac{1}{2n}} + N, \tag{4}$$

$$B := \int_{-X}^{X} \frac{1}{P(x)}\,dx > A, \quad \text{since} \quad X > 1 \quad \text{and}$$
$$P(x) > 0 \quad \text{for all real} \quad x, \tag{5}$$

$$C := \int_{-X}^{X} \frac{1}{P(x+iY)}\,dx. \tag{6}$$

Since $Y > N$, from (1) we get

$$|C| \leq \frac{2}{Y^{2n}} \int_{-X}^{X} \frac{1}{[(x/Y)^2+1]^n}\,dx \leq \frac{2}{Y^{2n}} \int_{-X}^{X} \frac{1}{(x/Y)^2+1}$$
$$dx = \frac{4}{Y^{2n-1}} \tan^{-1}(X/Y) < \frac{4}{Y^{2n-1}} \frac{\pi}{2} \overset{(3)}{<} \frac{A}{2}. \tag{7}$$

Next note that

$$C - B = \int_{-X}^{X} \left[\frac{1}{P(x+iY)} - \frac{1}{P(x)}\right]$$
$$dx = \int_{-X}^{X} \int_{0}^{Y} D_2\left[\frac{1}{P(x+iy)}\right] dy\,dx$$
$$= \int_{-X}^{X} \int_{0}^{Y} iD_1\left[\frac{1}{P(x+iy)}\right] dy\,dx \quad \text{by Cauchy-Riemann}$$
$$= \int_{0}^{Y} i\left[\frac{1}{P(X+iy)} - \frac{1}{P(-X+iy)}\right] dy, \quad \text{by Fubini.}$$

Since $X > N$, it follows from this equality and (1) that

$$|C - B| \leq \int_{0}^{Y} \frac{4}{(X^2+y^2)^n}\,dy < \frac{4}{X^{2n}}\,Y \overset{(4)}{<} \frac{A}{2}. \tag{8}$$

From (7) and (8) follows $|B| < A$, contrary to (5). Hence no such polynomial p exists.

If students know about Leibniz' rule for differentiating under the integral sign (an easy consequence of Fubini, to be sure) and are comfortable with the complex exponential function, this proof can be made even shorter. See SANTOS[2007]

Remarks on History

The FTA has a rich history, really a microcosm of the history of post-Renaissance mathematics and paralleling the crystallization of complex numbers (as, obviously, without a proper grounding of the latter a genuine proof of the former is not possible). Two surveys of this history are PETROVA[1974] and GILAIN [1991].

By general consensus Gauss is considered to have given the first logically unimpeachable proof, although the first of four he gave in his lifetime, in his 1799 dissertation, contained a gap first filled in 1920. A little (stress "little") irony for the Prince of Mathematicians, as the prolog of that dissertation was devoted to pinpointing the errors in all prior proof claims. Since Jean d' Alembert's earlier proof was later rehabilitated (after $\mathbb{C}$ was secure), the French still refer to the FTA as "le théorème de d' Alembert". See BALTUS[2004].

The number and variety of proofs of the theorem is astounding. References to over 100 (prior to 1907!) will be found in the encyclopedia article of NETTO & LEVAVASSEUR [1907]. There are analytic proofs, topological proofs, and algebraic proofs. The short book of FINE & ROSENBERGER [1977] treats each kind, building up all necessary background first. The pedagogical treatment of NEUBRAND[1985] evaluates various proofs of the FTA (giving some in detail) for their classroom suitability.

An excellent (non-pareil) short history of both $\mathbb{C}$ and the FTA is chapter 3 and 4 (by Reinhold Remmert) of the

8-authored book *Numbers*. One of the several historically significant proofs of the FTA there, the analytic one by Argand (with some non-trivial modern embellishments) is probably the most elementary of all; it uses (as it must!) the compactness of bounded closed subsets of $\mathbb{C}$, but neither differentiation, integration, nor the complex exponential function (i.e., the cyclometric functions), hence is perhaps the most suitable for an undergraduate analysis class. Minimalist proofs are always esthetically appealing, but it seems especially appropriate to avoid the complex exponential, a *transcendental* function whose theory lies deeper than that of the FTA, in proving a result about *polynomials*. Remmert also gives the beautiful algebraic proof of Laplace, based on symmetric multinomials – the key idea behind the modern proofs via Galois theory and the Sylow theorems (see, e.g., HOROWITZ [1966] for the latter).

Two novel analytic proofs, modern in chronology but classic in spirit, also very suitable for the classroom, are REDHEFFER[1957] and LAZER[2006]. Readers who are aware of the (apparently indispensable) role of the FTA in producing eigenvalues (as zeros of characteristic polynomials) for linear transformations on finite-dimensional $\mathbb{C}$-vector spaces, should give themselves the pleasure of reading DERKSEN[2003], where the process is reversed and the FTA is proved via linear algebra.

Finally, we should note that the FTA closes the door on further finite field extensions of $\mathbb{C}$. More precisely (as Gauss himself observed and Weierstrass later proved), the only *field* that is algebraic over $\mathbb{C}$ is $\mathbb{C}$ itself. The elegant simple proof is in *Numbers*, p. 118.

Acknowledgment: The author thanks his friend G. P. Youvaraj for the invitation to contribute to a journal named for so famous a mathematician.

References

BALTUS, C., "D' Alembert's proof of the fundamental theorem of algebra," *Historia Math.* 31 (2004), 414–428. **MR** 2005g: 01019.

DERKSEN, H., "The fundamental theorem of algebra and linear algebra," *Amer. Math. Monthly* 110 (2003), 620–623. **MR** 2004g: 12002.

EBBINGHAUS, H.D., *et al., Numbers*, Graduate Texts in Mathematics vol. 123 (translated from German by H.L.S. Orde), Springer-Verlag (1990), New York. **MR** 91h:00005.

FINE, B. & ROSENBERGER, G., *The Fundamental Theorem of Algebra*, Undergraduate Texts in Mathematics, Springer-Verlag (1997), New York. **MR** 98h:30001.

GILAIN, CH., "Sur l'histoire du théorème fondamental de l'algèbre: théorie des équations et calcul intégral," *Arch. Hist. Exact Sci.* 42 (1991), 91–136. **MR** 92j:01034.

HOROWITZ, L., "A proof of the 'fundamental theorem of algebra' by means of Galois theory and 2-Sylow groups," *Nieuw Archief voor Wiskunde* (3) 14 (1966), 95–96. **MR** 33#5617.

LAZER, A.C., "From the Cauchy-Riemann equations to the fundamental theorem of algebra," *Math. Magazine* 79 (2006), 210–213.

NETTO, E.& LEVAVASSEUR, R., "Les fonctions rationnelles" *Encyclopédie des sciences mathématiques pures et appliquées,* t.I, v.2, fasc. 1, 1–232. Gauthier-Villars (1907), Paris. **MR** 95f:00029a.

NEUBRAND, M., *Didaktik-Zahlen-Algebra:Mathematik-didaktische Überlegungen am Fundamentalsatz der Algebra,* Band 17 of Texte zur mathematisch-naturwissenschaftlich-technischen Forschung und Lehre, Verlag Barbara Franzbecker(1985), Bad Salzdetfurth und Hildesheim. **MR** 86k:00014.

PETROV, S.S., "Sur l'histoire des démonstrations analytiques du théorème fondamental de l'algèbre," *Historia Math.* 1 (1974), 255–261. **MR** 57# 5609.

REDHEFFER, R., "The fundamental theorem of algebra," *Amer. Math. Monthly* 64 (1957), 582–585. **MR** 19, p.537.

SANTOS, J.C., "The fundamental theorem of algebra deduced from elementary calculus," *Math. Gazette* 91 (2007), 302–303.

Planar Harmonic Mappings

S. Ponnusamy
Department of Mathematics
Indian Institute of Technology Madras
Chennai 600 036, India
E-mail: samy@iitm.ac.in

Antti Rasila
Institute of Mathematics
Helsinki University of Technology
P. O. Box 1100, FIN-02015 TKK, Finland
E-mail: antti.rasila@tkk.fi

Abstract. The theory of univalent analytic functions (or conformal mappings) has a rich history and classical applications of conformal mappings to problems in mathematical physics deal with solutions of Laplace equations. The history of the theory goes back to over a century and continues its presence till date. We are interested to know whether the classical results on conformal mappings can be extended in some way to harmonic mappings because of its interesting links with geometric function theory, minimal surfaces, and locally quasiconformal mappings. There are many surveys on planar harmonic mappings. The purpose of the series of proposed articles is to introduce the topic of planar harmonic mappings from basic to recent results and open problems.

The first part of the series begins with basic facts about harmonic mappings that are required for a better understanding of the topic that follows and to provide later a wealth of up to date information on this theory. Some of the key results will be presented with proofs and some of the theorems will be stated merely with an indication where a proof can be obtained.

Keywords. Harmonic, Analytic, Univalent and Convex Functions

2000 Mathematics Subject Classification: 30C45

1. Introduction to Harmonic Functions

Let Ω be a domain (open and connected) in the complex plane $\mathbb{C}$. A real-valued function $u\colon \Omega \to \mathbb{R}$ is *harmonic* if $u \in C^2(\Omega)$ (continuous first and second partial derivatives in Ω) and satisfies the Laplace equation in Ω:

$$\Delta u = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

A solution of $\Delta u = 0$ is called a (real) harmonic function or a potential function.

Definition 1.1. *A complex-valued function $f\colon \Omega \to \mathbb{C}$, $(x, y) \mapsto (u, v)$, is planar harmonic if the two coordinate functions u and v are (real) harmonic in Ω.*

1.2. Differential Operators of $\partial/\partial z$ and $\partial/\partial \overline{z}$

A convenient notation is to treat the pair of conjugate complex variables $z := x + iy$ and $\overline{z} := x - iy$ as two independent variables by writing

$$x = \frac{z+\overline{z}}{2},\ y = \frac{z-\overline{z}}{2i}.$$

This leads to the following differential operators:

$$\frac{\partial}{\partial z} := \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \overline{z}} := \frac{\partial}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

In view of this observation, we may treat $f(x+iy)$ as a function of z and $\overline{z}$, and so

$$f(x+iy) = u\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) + iv\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right).$$

Consequently, for a complex-valued function $f = u+iv$ with continuous partial derivatives, we may use the formal notations

$$f_x = u_x + iv_x \quad \text{and} \quad f_y = u_y + iv_y.$$

Then

$$f_z = \frac{1}{2}\left(f_x - if_y\right) = \frac{1}{2}[(u_x+v_y) - i(u_y - v_x)],$$

$$f_{\overline{z}} = \frac{1}{2}\left(f_x + if_y\right) = \frac{1}{2}[(u_x-v_y) + i(u_y + v_x)],$$

and

$$|f_z|^2 - |f_{\overline{z}}|^2 = u_x v_y - u_y v_x,$$

where subscripts denote partial derivatives. The motivation for these notations is two-fold. We start by observing that

$$\frac{\partial(z)}{\partial z} = 1 = \frac{\partial(\overline{z})}{\partial \overline{z}}; \quad \frac{\partial(\overline{z})}{\partial z} = 0 = \frac{\partial z}{\partial \overline{z}}$$

and thus,

$$\begin{aligned} f_{\overline{z}} = 0 &\iff f_x = -if_y \\ &\iff u_x = v_y \quad \text{and} \quad u_y = -v_x. \end{aligned} \tag{1.3}$$

The following properties are easy to verify.

- The operators $\partial/\partial z$ and $\partial/\partial \overline{z}$ are linear and have the usual properties of differential operators. For example, the product and quotient rules hold:

$$(fg)_z = fg_z + gf_z \quad \text{and} \quad \left(\frac{f}{g}\right)_z = \frac{gf_z - fg_z}{g^2},$$

  and similarly for $\partial/\partial\overline{z}$.
- The two derivatives are connected by the property

$$\overline{(f_z)} = \left(\overline{f}\right)_{\overline{z}}.$$

One of the fundamental theorems in the complex function theory concerns with necessary and sufficient conditions for analyticity. We have an equivalent formulation of this result.

Definition 1.4. *A (planar) harmonic function $f = u+iv$ is analytic on a domain Ω if and only if u and v are harmonic conjugates; i.e. $u, v \in C^2(\Omega)$ satisfy the Cauchy–Riemann equations:*

$$u_x = v_y, \; u_y = -v_x \;\; on \;\; \Omega.$$

The most important examples of harmonic functions arise naturally from the Cauchy Riemann equations. The intimate connection comes from the following result.

Theorem 1.5. *Real and imaginary parts of an analytic function in an open set are harmonic thereat. In particular, every analytic function is harmonic.*

Clearly, real and imaginary parts of a harmonic function are not necessarily conjugates. Moreover, the most natural way of passing from harmonic to an analytic function is remembered in the following:

Theorem 1.6. *Let $u(z)$ be a real-valued harmonic function in a simply connected domain Ω. Then there exists an analytic function $f(z)$ such that* $\text{Re}\, f(z) = u(z)$ *on* Ω.

For basic results concerning the theory of analytic functions and examples, exercises and related applications, we refer to the standard texts such as Ahlfors [1], advanced texts such as [4], and the recent books of Ponnusamy [18], and Ponnusamy and Silverman [19].

We use the following notations: for $a \in \mathbb{C}$ and $\delta > 0$,

$$\mathbb{D}(a;\delta) = \{z \in \mathbb{C} : |z-a| < \delta\},$$

$$\overline{\mathbb{D}}(a;\delta) = \{z \in \mathbb{C} : |z-a| \leq \delta\},$$

$$\partial\mathbb{D}(a;\delta) = \{z \in \mathbb{C} : |z-a| = \delta\}$$

denote the open disk (about a), the closed disk, and the circle, respectively. Further, we let $\mathbb{D}(0;\delta) = \mathbb{D}_\delta$ and $\mathbb{D}_1 = \mathbb{D}$, the open unit disk. Notations such as $\partial\mathbb{D}$ and $\overline{\mathbb{D}}$are defined in the obvious way.

1.7. Orthogonality of Level Curves

Let $u(x, y)$ be a real-valued function in a planar domain Ω. The set of all points (x, y), which are the solution of $u(x, y) = c$ (where c is a real constant), is called a *level set or level curve* of u. For example, if $u(x, y) = x^2 + y^2$ then the level curves of this functions are simply circles centered at (0, 0). The level set corresponding to $c = 0$ is simply the single point, namely the origin.

An important property of harmonic functions and their conjugates, as far as applications are concerned, relates to orthogonal curves. Here are some basic examples.

(i) Let $f(z) = z$. Then, $u(x, y) = x$ and $v(x, y) = y$. Consider one parameter families of their respective level curves:

$$\begin{cases} \gamma_\alpha = \{(x, y) : x = \alpha\}, & \text{and} \\ \Gamma_\beta = \{(x, y) : y = \beta\} \end{cases}$$

Clearly, for each α, γ_α is perpendicular to every Γ_β.

(ii) Let $f(z) = z^2$. Then, $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. We have that

$$\begin{cases} \gamma_\alpha = \{(x, y) : x^2 - y^2 = \alpha\}, & \text{and} \\ \Gamma_\beta = \{(x, y) : 2xy = \beta\}. \end{cases}$$

Again, we note that each curve in the family $\{\gamma_\alpha : \alpha \in \mathbb{R}\}$ is perpendicular to every curve in the other family $\{\Gamma_\beta : \beta \in \mathbb{R}\}$ and conversely.

(iii) Let $f(z) = 1/z$, $z \in \mathbb{C}\backslash\{0\}$. Then, with $\alpha, \beta \in \mathbb{R}\backslash\{0\}$,

$$u(x, y) = \frac{x}{x^2 + y^2} = \frac{1}{\alpha} \iff \left| z - \frac{1}{2\alpha} \right| = \frac{1}{2|\alpha|},$$

and

$$v(x, y) = -\frac{y}{x^2 + y^2} = \frac{1}{\beta} \iff \left| z + \frac{i}{2\beta} \right| = \frac{1}{2|\beta|}$$

so that the corresponding level curves are nothing but the circles

$$\begin{cases} \gamma_\alpha = \partial\mathbb{D}(1/2\alpha; 1/2|\alpha|), & \text{and} \\ \Gamma_\beta = \partial\mathbb{D}(-i/2\beta; 1/2|\beta|). \end{cases}$$

These two families are orthogonal to each other.

In view of the above examples, it is natural to ask, whether is it always the case that every analytic function possesses this property? More precisely, given an analytic function $f = u + iv$, is it always the case that the family of level curves of u is orthogonal to the family of level curves of v? The answer is yes. To see this, let $f = u + iv$ be an analytic function in a domain Ω and $f'(z) \neq 0$ in Ω. Consider two level curves passing through a point $(a, b) \in \Omega$:

$$u(x, y) = \alpha_1 \quad \text{and} \quad v(x, y) = \beta_1,$$

where $\alpha_1 = u(a, b)$ and $\beta_1 = v(a, b)$. Then

$$\begin{cases} u_x(x, y)\, dx + u_y(x, y)\, dy = 0, & \text{and} \\ v_x(x, y)\, dx + v_y(x, y)\, dy = 0 \end{cases}$$

so that the *gradient* m_1 of the level curve of u at (a, b) is

$$m_1 = \left.\frac{dy}{dx}\right|_{(a,b)} = -\left.\frac{u_x}{u_y}\right|_{(a,b)} = -\frac{u_x(a, b)}{u_y(a, b)}.$$

Similarly, the gradient m_2 of the level curve of v at (a, b) is

$$m_2 = -\frac{v_x(a, b)}{v_y(a, b)}.$$

Recall that the *normal vector* to $\gamma_{\alpha_1} = \{(x, y) : u(x, y) = \alpha_1\}$ at the point $(a, b) \in \Omega$ is the gradient vector grad $u = u_x\mathbf{i} + u_y\mathbf{j}$ of u at this point. Thus, by the virtue of the Cauchy–Riemann equations, we have $m_1 m_2 = -1$ showing that the two level curves through (a, b) must be *orthogonal* since their tangents are perpendicular at (a, b). Since (a, b) is arbitrary, we have established that the two families of level curves are mutually orthogonal to each other. In particular, we have the following

Proposition 1.8. *The level curves of the real and imaginary parts of an analytic function are orthogonal families.*

What happens when $f'(z) = 0$ in the above discussion?

If f is an analytic function defined on a domain Ω, then by the open mapping theorem, $f(\Omega)$ is a domain, and if Ω is a simply connected domain then so is $f(\Omega)$. The function f is said to be *conformal* at a point $z_0 \in \Omega$ if f preserves the angle at z_0 between any pair of smooth curves γ_1 and γ_2 passing through z_0. That is the angle between the image curves Γ_1 and Γ_2 at the image point $w_0 = f(z_0)$ is the same as that between the curves γ_1 and γ_2 at z_0. If the analytic function f is conformal at every point of Ω, then we say that f is conformal in Ω. Thus, a conformal mapping is simply an angle-preserving (i.e. both sense and magnitude) homeomorphism of some domain onto another.

Then, we have the following celebrated theorem due to Riemann.

Theorem 1.9 (Riemann Mapping Theorem). *There exists a unique conformal map f of $\mathbb{D}$ onto a simply connected domain (except the whole complex plane $\mathbb{C}$) such that $f(z_0)$ and* $\arg f'(z_0)$ *take given values.*

It is important that solutions of the Laplace equation remain invariant if the original domain is subject to a conformal mapping. Consequently, complicated domains can be transformed into more convenient ones without having any change in the Laplace equation. Thus, one aims at developing a relationship between a harmonic function $\phi(x, y)$ on Ω (called a physical plane) and the corresponding harmonic function $\Phi(u, v)$ in Ω' (called a model plane) such that ϕ at $(x, y) \in \Omega$ has the relation

$$\phi(x, y) = \Phi(u(x, y), v(x, y))$$

where $f(z) = u(x, y) + iv(x, y)$ is conformal on Ω with $f(\Omega) = \Omega'$. But then we ask, how do conformal maps help us to solve boundary value problems? We state the following result which actually shows that the Laplace equation remains invariant under conformal maps (see Figure 1).

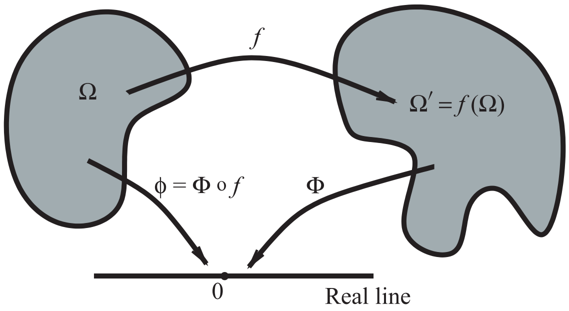

Figure 1. Description for invariance of Laplacian under conformal maps

Theorem 1.10. *Assume that $\phi \in C^2(\Omega)$ and $f = u + iv$ is a conformal mapping of Ω onto $\Omega' = f(\Omega)$. Then, for $\phi(x, y) = \Phi(u, v)$ with $u = u(x, y)$ and $v = v(x, y)$, we have*

$$\Delta\phi = |f'(z)|^2 \Delta\Phi.$$

In particular, $\Delta\phi = 0$ on Ω if and only if $\Delta\Phi = 0$ on Ω'.

Proof. The proof follows by applying the rule of change of variables. Indeed, we have

$$\phi_x = \Phi_u u_x + \Phi_v v_x,$$
$$\phi_y = \Phi_u u_y + \Phi_v v_y$$

so that

$$\phi_{xx} = (\Phi_{uu} u_x + \Phi_{uv} v_x) u_x + (\Phi_{vu} u_x + \Phi_{vv} v_x) v_x + \Phi_u u_{xx} + \Phi_v v_{xx}$$
$$\phi_{yy} = (\Phi_{uu} u_y + \Phi_{uv} v_y) u_y + (\Phi_{vu} u_y + \Phi_{vv} v_y) v_y + \Phi_u u_{yy} + \Phi_v v_{yy}.$$

Then, by addition, we easily obtain the desired conclusion.

□

An analytic function f defined on a domain Ω is said to be univalent (or one-to-one or schlicht) on Ω if $f(z)$ assumes different values for different values of z so that the equation $w = f(z)$ has at most one root in Ω for every complex w. For an analytic function $f(z)$ to be univalent in a small neighborhood of a point $z_0 \in \Omega$, it is necessary and sufficient that $f'(z_0) \neq 0$. We remark that local univalence at all points of a domain is however insufficient for the univalence in that domain. For instance, $f(z) = e^z$ is locally univalent in $\mathbb{C}$ but is not univalent in $|z| < R$ if $R > \pi$. At this place it is important to observe that one does not require the analyticity for defining univalence. For instance, $f(z) = \overline{z}$ is univalent in $\mathbb{C}$ although it is nowhere analytic.

Another important and basic result concerning conformal mapping is the following.

Theorem 1.11. *Suppose that f is analytic in a domain Ω. If $f'(z_0) \neq 0$ at $z_0 \in \Omega$, then f is conformal at $z = z_0$. The converse is also true.*

For instance,

- $1 + e^z$ is conformal everywhere on $\mathbb{C}$
- z^2 is conformal everywhere except at the origin
- $\cos(\pi z)$ is conformal everywhere except at integer points
- $z + 1/z$ is conformal at all values of $\mathbb{C}$ except at $0, \pm 1$.

Conformal mappings have been successfully used to solve two-dimensional Poisson equations which appear in problems related to electric fields with space charges, and thermal fields with heat generation, to mention but a few applications.

The *Dirichlet problem* deals with the following question: Given a domain D, and a function $F : \partial D \to \mathbb{R}$, does there exist a function u that is harmonic in D such that $u = F$ on the boundary ∂D? The solution to this problem has immediate applications in the fluid mechanics. Further, since there is a large stockpile of analytic functions, the above theorem is helpful in finding closed-form solution to many Dirichlet problems, especially in solving Dirichlet problem in a region Ω once the Dirichlet problem in the image region Ω' is known. Likewise, we are interested in presenting importance of harmonic mappings rather than conformal mappings as in many situations conformality is not required. We end this subsection with a couple of simple examples which involve with solutions of the Laplace equation where the solution takes prescribed values on certain contours. For instance, to find a harmonic function ϕ on the vertical strip

$$\Omega = \{z : a \le \operatorname{Re} z \le b\} \quad (a < b),$$

with $\phi(a, y) = A$ and $\phi(b, y) = B$, a natural choice is to set

$$u(x, y) = ax + b.$$

It is easy to see that the required solution is given by

$$\phi(x, y) = A + \frac{A - B}{a - b}(x - a).$$

Here is another problem which follows from the derivation of the equation governing the steady-state temperature distribution $\phi(x, y)$.

Example 1.12. *Suppose that we wish to determine electrostatic potential ϕ on the domain Ω between the circles $|z| = 1$ and $|z - 1/2| = 1/2$ such that (see Figure 2)*

$$\phi(x, y) = -10 \ \text{on} \ |z| = 1 \ \text{and}$$
$$\phi(x, y) = 20 \ \text{on} \ |z - 1/2| = 1/2.$$

Clearly, our aim is to find a harmonic function ϕ on Ω satisfying the given boundary conditions. In applying Theorem 1.10 to find such a ϕ, one must know a conformal map which transforms Ω onto the image the domain Ω' for which many explicit solutions to a Dirichlet problem are known. For instance, in our problem we may transform Ω into a horizontal infinite strip by choosing $i \mapsto 0$, $-1 \mapsto 1$, $1 \mapsto \infty$. This can be done by a Möbius transformation. Indeed, the well-known invariance property of the cross ratio immediately yields that (see for example [18, 19])

$$f(z) = (1 - i)\frac{z - i}{z - 1}.$$

Note that 1 is common to the both the circles so that $f(\partial\mathbb{D}) = \mathbb{R} \cup \{\infty\}$ and $f(\partial\mathbb{D}(1/2, 1/2))$ is the extended line with ∞, namely, the line $v = 1$ with the point at ∞. The boundary conditions are transformed to

$$\Phi(u, v) = -10 \ \text{on} \ v = 0, \quad \text{and} \quad \Phi(u, v) = 20 \ \text{on} \ v = 1.$$

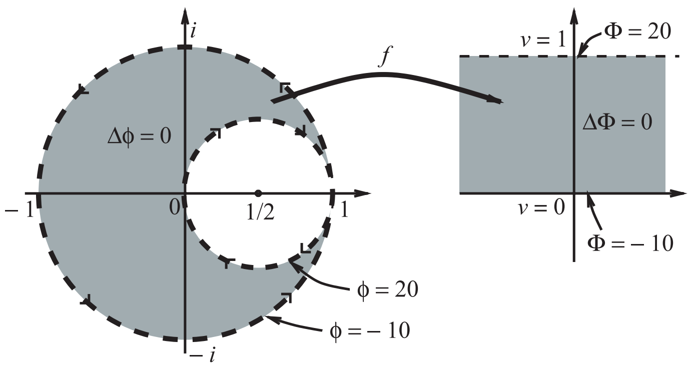

Figure 2. A conformal mapping between a domain and a strip

Now, we introduce $\Phi(u, v) = a + bv$. Using the transformed new boundary conditions, we compute a and b and obtain

$$\Phi(u, v) = 30v - 10 = 30\,\mathrm{Im}\,(f(z)) - 10.$$

This gives

$$\phi(x, y) = 30\left(\frac{1 - (x^2 + y^2)}{(1 - x)^2 + y^2}\right) - 10$$

which is a desired solution to our problem. •

When $\log z$ is suitably restricted, it becomes analytic and hence, we have a pair of two harmonic functions (real and imaginary parts), i.e. $\log z = \ln|z| + i \arg z$. We observe that $\ln|z|$ is constant on circles centered at the origin. In view of this observation and the idea of the above example, it is easy to find a steady state temperature distribution ϕ in Ω consisting of points outside of the two circles $|z - 5/2| = 1/2$ and $|z| = 1$ such that ϕ equals 30 on the unit circle $|z| = 1$ and ϕ vanishes on the circle $|z - 5/2| = 1/2$. We leave this problem as a simple exercise.

1.13. Canonical Representation

Recall from (1.3) that the Cauchy–Riemann equations in cartesian form can be equivalently written in a single concise equation: $f_{\overline{z}} = 0$. Often this is referred to as the complex form of Cauchy–Riemann equations. Moreover, it is a simple exercise to see that the *Laplacian* of f becomes

$$\Delta f = 4\frac{\partial}{\partial \overline{z}}\left(\frac{\partial f}{\partial z}\right) = 0, \quad \Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}},$$

which is referred to as the complex form of Laplace equation. Thus, we have equivalent formulation of Definition 1.1.

Definition 1.14. *We say that a complex valued function f is harmonic if and only if $f \in C^2(\Omega)$ with $\Delta f = 4f_{z\overline{z}} = 0$.*

There is a close interrelation between analytic functions and harmonic functions. For example, if we use the formula $\Delta f = 4f_{z\overline{z}}$, then we conclude the following:

Proposition 1.15. *f is necessarily independent of $\overline{z}$ for analytic functions f whereas f_z is independent of $\overline{z}$ for planar harmonic functions f.*

From this proposition, we observe that the function f with continuous partial derivatives is harmonic in a planar domain

Ω if and only if f_z is analytic in Ω. Also, if f is analytic in Ω then $f_z(z) = f'(z)$ in Ω. We now present our first basic result for harmonic functions.

Lemma 1.16 (Canonical Representation). *Let Ω be a simply connected domain in $\mathbb{C}$. Then $f: \Omega \to \mathbb{C}$ is a (planar) harmonic function if and only if the function f has the representation $f = h + \overline{g}$, where h and g are analytic in Ω. The representation is unique up to an additive constant. We call the functions h and g the analytic and the co-analytic parts of f, respectively.*

Proof. Set $f = u + iv$, where u and v both are harmonic in a simply connected plane domain Ω. Then there exist analytic functions F and G on Ω such that

$$u = \operatorname{Re} F = \frac{F + \overline{F}}{2} \quad \text{and} \quad v = \operatorname{Im} G = \frac{G - \overline{G}}{2i}.$$

This observation gives the representation

$$f = \frac{F + \overline{F}}{2} + \frac{G - \overline{G}}{2} = \left(\frac{F + G}{2}\right) + \overline{\left(\frac{F - G}{2}\right)} := h + \overline{g},$$

where both h and g are clearly analytic in Ω, and $\overline{g}$ denotes the function $z \mapsto \overline{g(z)}$.

Alternately, suppose that f is harmonic. Then, f_z is analytic in the simply connected domain Ω, and so we may define h by $h' = f_z$, where h is analytic in Ω and h is determined up to an additive constant. To obtain the desired representation for known f and h, we define g by

$$g = \overline{f - h} = \overline{f} - \overline{h}$$

so that $f = h + \overline{g}$. It suffices to show that g is analytic in Ω, i.e. $g_{\overline{z}} = 0$. Now

$$g_{\overline{z}} = \frac{\partial}{\partial \overline{z}}(\overline{f} - \overline{h}) = \overline{f_z} - \overline{h_z} = \overline{f_z} - \overline{h'} = 0 \ \text{ in } \ \Omega,$$

and thus, g is analytic in Ω. The converse part is obvious.

The uniqueness follows from the fact that a function which is both analytic and anti-analytic[1] is constant. □

Remark 1.17. *If the function f in Lemma 1.16 is real-valued, then f may be expressed as $f = h + \overline{h} = 2\operatorname{Re} h$ so that $2h$ represents the analytic completion of f and is unique up to an additive imaginary constant.* •

[1]Conjugate of an analytic function is called an anti-analytic function.

1.18. Composition Rule for Harmonic Functions

In the linear space $\mathcal{H}(\Omega)$ of analytic functions in Ω, analytic functions are preserved under product and composition rules, but harmonic functions are not. For example, the functions x and x^2 show that the product of two harmonic functions is not necessarily harmonic.

Proposition 1.19.

(1) *If $f: \Omega \to \mathbb{C}$, $g: f(\Omega) \to \mathbb{C}$ are harmonic functions, $g \circ f$ is not necessarily harmonic.*
(2) *If $f: \Omega \to \mathbb{C}$ is analytic and $g: f(\Omega) \to \mathbb{C}$ is harmonic, then $g \circ f$ is harmonic.*
(3) *If $f: \Omega \to \mathbb{C}$ is harmonic and $g: f(\Omega) \to \mathbb{C}$ is analytic, then $g \circ f$ is not necessarily harmonic.*

Proof. We leave the proof as an exercise to the reader. □

At this place, it is important to emphasize that the class of harmonic functions is not conformally invariant. In particular, inverse or square of a harmonic function need not be harmonic.

2. Harmonic Mappings

A complex-valued harmonic function $f: \Omega \to \mathbb{C}$ is said to be a *harmonic mapping* if it is univalent (one-to-one) in Ω, i.e. $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in \Omega$ with $z_1 \neq z_2$.

Thus, (planar) harmonic mappings are univalent complex-valued functions whose real and imaginary parts are not necessarily conjugate, i.e. do not need to satisfy the Cauchy–Riemann equations. Thus, every conformal or anti-conformal mapping f on a domain Ω is a harmonic mapping. In particular, the class of (planar) harmonic mappings on the unit disk $\mathbb{D}$ includes the subclass of univalent functions that are also analytic in $\mathbb{D}$, a popular topic in geometric function theory (see [7, 9, 17]).

Example 2.1. *Consider $f: \mathbb{C} \to \mathbb{C}$ by $f(z) = 4x + i4xy$. Then, f is a harmonic function. Also, we easily have the decomposition*

$$f = h + \overline{g}, \quad h(z) = 2z + z^2, \ g(z) = 2z - z^2.$$

Is this a harmonic mapping? If not, how about the same function when it is restricted to the right half-plane $\Omega = \{z : \operatorname{Re} z > 0\}$? Does the inverse exist on Ω? Must the inverse be a harmonic mapping on Ω?

2.2. Jacobian and Local Univalence

The Jacobian of a function $f = u + iv$ at a point z is defined to be

$$J_f(z) = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - u_y v_x,$$

which may be expressed equivalently in terms of z- and $\overline{z}$-derivatives

$$J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2.$$

If f is analytic on Ω, then $f_{\overline{z}}(z) = 0$ on Ω and so $f_z(z) = f'(z)$. Thus, the Jacobian takes the form

$$J_f(z) = (u_x)^2 + (v_x)^2 = |f'(z)|^2.$$

Let f be an univalent function defined on a domain Ω and belong to $C^1(\Omega)$ such that $J_f(z) \neq 0$ in Ω. Then f is said to be a *diffeomorphism* or more accurately, a C^1-diffeomorphism of Ω onto its range. We remark that, if $f : \Omega \to \mathbb{C}$ is a diffeomorphism, then either $J_f(z) > 0$ everywhere in Ω or $J_f(z) < 0$ throughout the domain Ω. This follows from the fact that Ω is connected and $J_f : \Omega \to \mathbb{R}$ is a continuous and zero-free function, i.e. the set $\{J_f(z) : z \in \Omega\}$ is a connected set of real numbers that does not contain zero and hence, $J_f(\Omega)$ is either a subset of $(-\infty, 0)$ or a subset of $(0, \infty)$. When J_f is positive in Ω, then the diffeomorphism f is called *orientation-preserving mapping* or *sense-preserving mapping*. A diffeomorphism with a negative Jacobian is said to be *orientation-reversing mapping* or *sense-reversing mapping*. We see that the conjugate $\overline{f}$ of a diffeomorphism $f : \Omega \to \mathbb{C}$ is also a diffeomorphism, i.e. the one for which

$$J_{\overline{f}}(z) = -J_f(z).$$

Therefore, $\overline{f}$ is orientation-reversing when f is orientation-preserving, and vice versa. For example, in the unit disk $\mathbb{D}$,

(1) $f(z) = z$ is sense-preserving, as $J_f(z) = 1$ in $\mathbb{D}$.
(2) $f(z) = (1 + z)^2$ is sense-preserving, as $J_f(z) = |2(1 + z)|^2 > 0$ in $\mathbb{D}$.
(3) $f(z) = \overline{z}$ is sense-reversing, as $J_z(z) = -1 < 0$ in $\mathbb{D}$.

A well-known classical result for analytic functions states that an analytic function f is locally univalent at z_0 if and only if $J_f(z_0) \neq 0$ (see for example, [19, Theorems 11.2 and 11.3]). In 1936, Hans Lewy [16] showed that this remains true for harmonic functions.

Theorem 2.3 (Lewy's Theorem). *A harmonic function f is locally univalent in a neighborhood of z_0 if and only if $J_f(z_0) \neq 0$. That is, f is locally univalent in a domain Ω if and only if $J_f(z) \neq 0$ throughout Ω.*

An immediate consequence of Lemma 1.16 and Theorem 2.3 is the following.

Corollary 2.4. *Let f be a complex-valued harmonic function on a simply connected domain Ω with the decomposition $f = h + \overline{g}$. Then f is locally univalent and sense-preserving in Ω if and only if $|h'(z)| > |g'(z)|$ in Ω. Equivalently, f is a sense-preserving local homeomorphism if and only if $J_f(z) > 0$.*

The set of all critical points of a C^1-function consists of those points where the Jacobian vanishes. Thus, for a harmonic function f, the set of critical points consists of those points for which f is not locally univalent.

Remark 2.5. *Lewy's theorem does not hold for harmonic mappings in higher dimensions ($n \geq 3$). The following example is due to Wood [23]. Consider $f : \mathbb{R}^3 \to \mathbb{R}^3$ by*

$$f(x, y, z) = (x^3 - 3xz^2 + yz, y - 3xz, z).$$

The three coordinate functions $u = x^3 - 3xz^2 + yz$, $v = y - 3xz$, $w = z$ are harmonic as they satisfy the 3-dimensional Laplace equation: $\Delta u = 0 = \Delta v = \Delta w$, where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Thus, the function f is harmonic in $\mathbb{R}^3$. The Jacobian of the given function is

$$J_f(x, y, z) = \begin{vmatrix} 3x^2 - 3z^2 & -3z & 0 \\ z & 1 & 0 \\ -6xz + y & -3x & 1 \end{vmatrix} = 3x^2.$$

To find the inverse function, we need to solve x and y in terms of u, v, w. Substituting $z = w$, the expression for u and v becomes

$$u = x^3 + w(y - 3xw) \quad \text{and} \quad v = y - 3xw.$$

Using the second equation, the first one may be rewritten as

$$u = x^3 + vw \quad \text{or} \quad x = \sqrt[3]{u - vw}$$

and so $v = y - 3xw$ gives

$$y = v + 3w\sqrt[3]{u - vw}.$$

Thus, the inverse function $f^{-1}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$f^{-1}(u, v, w) = (\sqrt[3]{u - vw}, v + 3w\sqrt[3]{u - vw}, w).$$

Thus, the given function is a homeomorphism of $\mathbb{R}^3$ but the Jacobian vanishes on the plane $x = 0$. □

2.6. Harmonic Mappings on the Plane $\mathbb{C}$

For entire functions (analytic in $\mathbb{C}$), it is well-known that the only univalent analytic self mappings of $\mathbb{C}$ are the linear mappings of the form $f(z) = a_0 + a_1 z$, where a_0, a_1 are constants with $a_1 \neq 0$ (see for example [18] and [19]). It is natural to ask the harmonic analog of this result.

Theorem 2.7. *[6] The only harmonic mappings of $\mathbb{C}$ onto $\mathbb{C}$ are the affine mappings $f(z) = \alpha z + \gamma + \beta\overline{z}$, where α, β and γ are complex constants and $|\alpha| \neq |\beta|$.*

Proof. Let f map $\mathbb{C}$ harmonically onto $\mathbb{C}$. Then f has the form

$$f = h + \overline{g},$$

where h and g are entire functions, and we may assume without loss of generality that f is sense-preserving. As f is sense-preserving, we have $|g'(z)| < |h'(z)|$ in $\mathbb{C}$ and so g'/h', being a bounded entire function, reduces to a constant (by Liouville's theorem). Consequently, $g'(z) \equiv bh'(z)$ so that integration gives

$$g(z) = bh(z) + c$$

for some complex constants b and c with $|b| < 1$. Thus, f reduces to the form

$$f(z) = h(z) + \overline{b}\,\overline{h(z)} + \overline{c}.$$

Setting $w = h(z)$, we may write

$$f(z) = F(h(z)) = (F \circ h)(z) \text{ with } F(w) = w + \overline{b}\overline{w} + \overline{c}.$$

Note that F is an (invertible) affine mapping. It follows that $h = F^{-1} \circ f$ is analytic and maps $\mathbb{C}$ univalently onto $\mathbb{C}$, and so $h(z) = a_0 + a_1 z$, where a_0, a_1 are complex constants with $a_1 \neq 0$. This shows that f has the form

$$f(z) = a_0 + a_1 z + \overline{b}(\overline{a_0} + \overline{a_1 z}) + \overline{c},$$

which is the affine mapping in the desired form. □

Theorem 2.7, in particular, shows that *there exists no harmonic mapping of $\mathbb{C}$ onto a proper sub domain Ω of $\mathbb{C}$.* Also, we remark that the simplest example of sense-preserving harmonic mapping on the plane that is not necessarily conformal is an affine mapping

$$f(z) = \alpha z + \beta\overline{z}, \quad |\alpha| > |\beta| > 0.$$

Example 2.8. *For $n \geq 1$, consider (compare with Lemma 1.16)*

$$f_n(z) = z + \frac{n}{n+1}\overline{z}.$$

Then, each f_n is a harmonic mapping in $\mathbb{C}$ (and in particular, in the unit disk $\mathbb{D}$). Note that $\{f_n\}$ converges uniformly to $f(z) = z + \overline{z}$ which is clearly not a harmonic mapping, see Figure 3. What does this mean? •

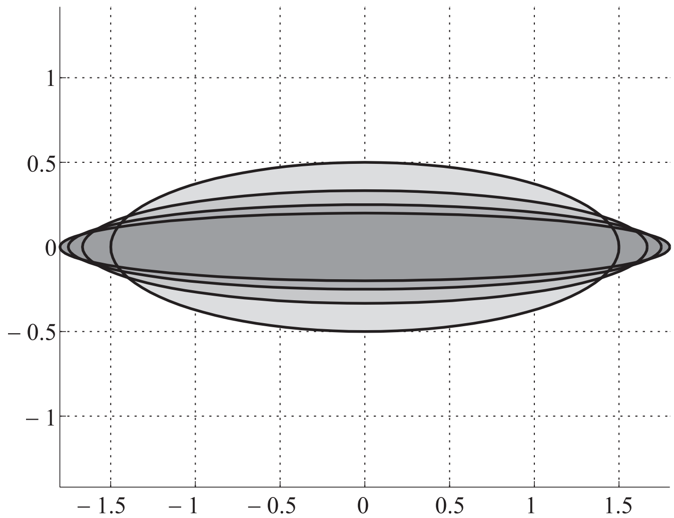

Figure 3. Image of unit disk under $f_n(z) = z + (n/(n+1))\overline{z}$ for $n = 1, 2, 3, 4$

Example 2.9. *Consider*

$$f(z) = z - \frac{1}{\overline{z}} + 2\ln|z|.$$

Then it is easy to see that f is a harmonic mapping on the exterior $\Omega = \mathbb{C} \setminus \overline{\mathbb{D}}$ of the unit disk $\mathbb{D}$ onto the punctured complex plane $\mathbb{C} \setminus \{0\}$. Note that $f(\partial\mathbb{D})$ is simply the origin. We write

$$f = h + \overline{g}, \quad h(z) = z + \log z \text{ and } g(z) = -\frac{1}{z} + \log z$$

and note that h and g are not (globally) analytic on $\mathbb{C} \setminus \overline{\mathbb{D}}$ which is not simply connected in $\mathbb{C}$. So, we require an analog of Lemma 1.16 for multiply connected domains. □

2.10. Connections to Quasiconformal Mappings

Let $f : \Omega \to f(\Omega)$ be a sense-preserving C^1-diffeomorphism, $z_0 \in \Omega$, and $w = f(z) = u(z) + iv(z)$. By using the properties of the differential operators from advanced calculus, we obtain

$$du = u_x dx + u_y dy, \quad dv = v_x dx + v_y dy,$$

which may be written in the form

$$dw = df = f_z dz + f_{\bar{z}} d\bar{z}.$$

Since f is a diffeomorphism, it is locally linear (in this case at z_0). Indeed, the affine map L defined by

$$L(z) := f_z(z_0)dz + f_{\bar{z}}(z_0)d\bar{z}$$

sends a circle with center 0 in the dz-plane onto an ellipse in the dw-plane, with major axis of length $L = |f_z(z_0)| + |f_{\bar{z}}(z_0)|$ and minor axis of length $l = |f_z(z_0)| - |f_{\bar{z}}(z_0)|$. It follows that

$$(|f_z| - |f_{\bar{z}}|)|dz| \le |dw| \le (|f_z| + |f_{\bar{z}}|)|dz|$$

where both the limits are attained. The differential dw maps the circle $|dz| = r$ onto the ellipse, see Figure 4. The ratio $D_f(z)$ between the major and the minor axes is

$$D_f := \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \ge 1.$$

This quantity $D_f(z)$ is called *dilatation* of f at the point $z \in \Omega$.

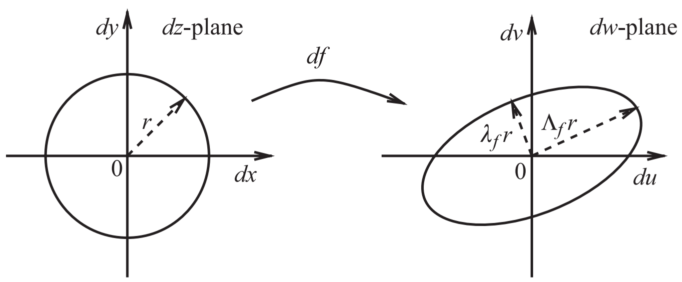

Figure 4. Image of the circle $|dz| = r$ under the differential map dw.

Definition 2.11. *A sense-preserving diffeomorphism f is said to be K-quasiconformal, if $D_f(z) \le K$ throughout the given region Ω, where $K \in [1, \infty)$ is a constant.*

We define the quantity $d_f(z)$ as follows:

$$d_f := \frac{|f_{\bar{z}}|}{|f_z|} < 1,$$

because, for sense-preserving maps, $|f_z| > |f_{\bar{z}}|$. Thus, we have

$$D_f(z) = \frac{1 + d_f(z)}{1 - d_f(z)} \quad \text{and} \quad d_f(z) = \frac{D_f(z) - 1}{D_f(z) + 1}. \tag{2.12}$$

We may use D_f as a measure of the local distortion of the mapping f at z. Now we define *complex dilatation* μ_f by

$$\mu_f = \frac{f_{\bar{z}}}{f_z} \quad \text{with} \quad |\mu_f| = d_f. \tag{2.13}$$

Thus, by (2.12) and (2.13), the two dilatations μ_f and D_f are related by

$$|\mu_f(z)| = \frac{D_f(z) - 1}{D_f(z) + 1}.$$

If $\partial_\alpha f(z)$ denotes the directional derivative of a C^1-mapping f in a direction making an angle α with the positive x-direction, then

$$\begin{aligned}
\partial_\alpha f(z) &= \lim_{r\to 0} \frac{f(z + re^{i\alpha}) - f(z)}{re^{i\alpha}} \\
&= e^{-i\alpha} \lim_{r\to 0} \left\{ \frac{u(x + r\cos\alpha, y + r\sin\alpha) - u(x, y)}{r} \right. \\
&\qquad \left. + i\frac{v(x + r\cos\alpha, y + r\sin\alpha) - v(x, y)}{r} \right\}.
\end{aligned}$$

Adding and subtracting terms $u(x, y + r\sin\alpha)$ and $v(x, y + r\sin\alpha)$ in the numerators of the first and the second terms on the right, respectively, we obtain

$$\begin{aligned}
\partial_\alpha f(z) &= e^{-i\alpha}[(u_x \cos\alpha + u_y \sin\alpha) \\
&\qquad + i(v_x \cos\alpha + v_y \sin\alpha)] \\
&= e^{-i\alpha}[f_x \cos\alpha + f_y \sin\alpha] \\
&= e^{-i\alpha}[(f_z + f_{\bar{z}})\cos\alpha + i(f_z - f_{\bar{z}})\sin\alpha] \\
&= e^{-i\alpha}[f_z e^{i\alpha} + f_{\bar{z}} e^{-i\alpha}] \\
&= f_z + f_{\bar{z}} e^{-2i\alpha} \\
&= f_z \left[1 + \frac{f_{\bar{z}}}{f_z} e^{-2i\alpha}\right] \quad \text{for } f_z \ne 0.
\end{aligned}$$

Hence, for a sense-preserving C^1-map f between planar domains, we have

$$\Lambda_f(z) := \max_\alpha |\partial_\alpha f(z)| = |f_z(z)| + |f_{\bar{z}}(z)| \quad \text{and}$$

$$\lambda_f(z) := \min_\alpha |\partial_\alpha f(z)| = |f_z(z)| - |f_{\bar{z}}(z)|.$$

It is an easy exercise to see that if $z_1 \neq 0$, then

$$|z_1 + z_2| = |z_1| + |z_2| \Longleftrightarrow \arg(z_2/z_1) = 0$$

and similarly,

$$|z_1 + z_2| = |z_1| - |z_2| \Longleftrightarrow \arg(z_2/z_1) = \pm\pi.$$

Therefore the maximum is attained provided that

$$\arg\left(\frac{f_{\overline{z}}}{f_z}e^{-2i\alpha}\right) = 0, \quad \text{i.e.} \quad \alpha = \frac{1}{2}\arg\left(\frac{f_{\overline{z}}}{f_z}\right).$$

Thus, the maximum corresponds to the direction

$$\arg dz = \alpha = \frac{1}{2}\arg \mu_f$$

and the minimum in the orthogonal direction

$$\arg dz = \beta = \alpha \pm \frac{\pi}{2}.$$

We define the quantity $\omega := \nu_f$ of f by

$$\omega(z) := \nu_f(z) = \frac{\overline{f_{\overline{z}}(z)}}{f_z(z)}.$$

The function $\nu_f(z)$ is called the second complex dilatation which turns out to be more natural than the first complex dilatation μ_f. Because $|\nu_f| = |\mu_f|$, f is quasiconformal if and only if $|\nu_f(z)| \leq k < 1$. Finally, we remark that

$$D_f \leq K \Leftrightarrow d_f \leq k := \frac{K-1}{K+1}.$$

The mapping f is conformal if and only if $\mu_f(z) = 0$ on Ω, i.e. $f_{\overline{z}}$ vanishes identically on Ω. From our notation, $\partial_\alpha f(z)$ is then independent of α so that

$$\partial_\alpha f(z) = f_z(z) = f'(z).$$

This is equivalent to the dilatation quotient being identically equal to 1. Thus, f is conformal if and only if $D_f = 1$ and $\mu_f = 0$. If K is defined by

$$K = \sup_{z\in\Omega} D_f = \sup_{z\in\Omega} \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}$$

is bounded, then we call f a K-quasiconformal mapping of Ω to $f(\Omega)$. We call K the maximal dilatation of f and it is a finite number ≥ 1. Sometimes, f is called a K-quasiconformal mapping with the Beltrami coefficient μ_f.

The simplest example of quasiconformal mapping is given by the affine mapping

$$f(z) = az + b\overline{z} \quad (a, b \in \mathbb{C},\ |b| < |a|);$$

for $f_z = a$, $f_{\overline{z}} = b$, $|\mu_f| = |b|/|a|$ and f maps the unit circle to an ellipse, and the ratio of the major and minor axes of this ellipse is

$$K = \frac{|a| + |b|}{|a| - |b|} = \frac{1 + |\mu_f|}{1 - |\mu_f|}$$

As another example, we consider

$$f(z) = \begin{cases} z & \text{if } z \in \mathbf{U} = \{w : \operatorname{Re}(w) > 0\}, \\ x + iky & \text{if } z \in \mathbb{C} \setminus \mathbf{U},\ k > 1. \end{cases}$$

Then $f_z = 1$ and $f_{\overline{z}} = 0$ if $z \in \mathbf{U}$. If $z \in \mathbb{C} \setminus \mathbf{U}$, then we may rewrite f as

$$f(z) = \frac{z + \overline{z}}{2} + ik\left(\frac{z - \overline{z}}{2i}\right)$$

so that

$$f_z = \frac{1+k}{2} \quad \text{and} \quad f_{\overline{z}} = \frac{1-k}{2}.$$

Thus,

$$D_f = \frac{(1+k)/2 + (k-1)/2}{(1+k)/2 - (k-1)/2} = k$$

and hence, f is a K-quasiconformal mapping of $\mathbb{C}$ with $K = k$.

What is important about quasiconformal mappings? We first recall the following facts:

(1) Quasiconformal maps are considered to be a natural generalization of conformal maps.
(2) In many of the results on conformal mappings, one requires just the quasi-conformality and hence, it is of interest to know when conformality is necessary and when it is not.
(3) Quasiconformal maps behave less rigidly than conformal maps and thus can be used as a tool in complex analysis.
(4) Later development shows that the class of quasiconformal mappings play an important role in the study of elliptic partial differential equations.
(5) Extremal problems in quasiconformal mappings lead to analytic functions connected with regions or Riemann surfaces.
(6) The family of conformal mappings degenerates to Möbius transformations when generalized to several variables, but the family of quasiconformal mappings is interesting in the higher dimensions also (see [21])

There are many areas in which quasiconformal mappings are used. As a general reference to the theory of quasiconformal mappings in plane, we refer to [2, 14, 15] and we would consider the topic at a later stage. For the higher dimensional theory, we refer to the books of Väisälä [21] and Vuorinen [22] although at this stage it is too early for us to discuss higher dimensional results. However, it might be useful to recall the problem of Grötzsch with two simple examples of quasiconformal mappings in the plane.

Problem 2.14. *Let **Q** be a square and **R** be a rectangle not a square. In 1928, H. Grötzsch, posed the following problem: Does there exist a conformal mapping of **Q** onto **R** which maps vertices onto vertices?*

Because of the extra condition that the vertices mapping onto vertices, the Riemann mapping theorem does not guarantee existence of such a mapping and so the problem of Grötzsch is interesting in itself. It is this extra condition that led us to the development of the theory of quasiconformal mappings. Actually, there does not exist a conformal mapping of **Q** onto **R** taking vertices onto vertices and hence, the question is to find most nearly conformal mapping of this kind, and this needs a measure of approximate conformality.

2.15. Solutions of Elliptic Partial Differential Equation

Suppose that f is a harmonic mapping defined on a simply connected domain Ω. Then, because f has the form $f = h+\overline{g}$ and f is univalent in Ω, we have

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2 = |h'|^2 - |g'|^2 \neq 0.$$

Thus, f is either sense-preserving or sense-reversing. In the first case, $J_f(z) > 0$, i.e. $|h'(z)| > |g'(z)|$ throughout Ω. Note that in the second case, $\overline{f}$ is sense-preserving. In the case of sense-preserving harmonic mappings, we have the following

Theorem 2.16. *Let $f \in C^2(\Omega)$ with $J_f(z) > 0$ on Ω. Then the function f is harmonic on Ω if and only if f is the solution of the elliptic partial differential equation*

$$\overline{f_{\overline{z}}(z)} = \omega(z) f_z(z), \quad z \in \Omega, \tag{2.17}$$

for some analytic function ω on Ω with $|\omega(z)| < 1$ on Ω.

Proof. $\Rightarrow$: Suppose that $f\ (= h+\overline{g})$ is a harmonic function with $J_f(z) > 0$ on Ω. Then, as $f_z(z) = h'(z) \not\equiv 0$ and $\overline{f_{\overline{z}}(z)} = g'(z)$, we can define a function $\omega(z)$ by

$$\omega(z) := \nu_f(z) = \frac{g'(z)}{h'(z)} = \frac{\overline{f_{\overline{z}}(z)}}{f_z(z)}$$

which is analytic on Ω and $|\omega(z)| < 1$ on Ω, because $J_f = |h'|^2 - |g'|^2 > 0$. The desired form (2.17) follows from the last relation.

$\Leftarrow$: Conversely, suppose that f is a C^2-solution of (2.17) with $J_f(z) > 0$ on Ω. Differentiating the equation (2.17) with respect to $\overline{z}$, one finds

$$\overline{f_{\overline{z}z}(z)} = \omega(z) f_{z\overline{z}}(z) + \omega_{\overline{z}}(z) f_z(z)$$

so that $\overline{f_{\overline{z}z}} = \omega f_{z\overline{z}}$ as $\omega_{\overline{z}} = 0$. Further, as $|\omega(z)| < 1$, we have $f_{z\overline{z}}(z) = 0$ and therefore, f is harmonic on Ω. □

Note that if the second complex dilatation ω of a harmonic mapping f on the domain Ω satisfies $|\omega(z)| \leq k < 1$ in Ω, then f is a quasiconformal with the maximum dilatation $K = (1+k)/(1-k)$; i.e. f is a K-quasiconformal.

Corollary 2.18. *Sense-preserving harmonic mappings are locally quasiconformal.*

Example 2.19. *For $n \geq 2$, consider the function*

$$f(z) = z - \frac{1}{n}\overline{z}^n.$$

For each $n \geq 2$, the function f is harmonic and has the second complex dilatation $\omega(z) = -z^{n-1}$ (apply for example, Theorem 2.16). To verify its univalence in $\mathbb{D}$, suppose $f(z_1) = f(z_2)$ for $z_1, z_2 \in \mathbb{D}$. Then, we have

$$n(z_1 - z_2) = \overline{z_1}^n - \overline{z_2}^n = (\overline{z_1} - \overline{z_2}) \times (\overline{z_1}^{n-1} + \overline{z_1}^{n-2} z_2 + \cdots + \overline{z_2}^{n-1}).$$

By taking absolute values on both sides, we see that this is impossible unless $z_1 = z_2$ because $|\overline{z_1}^{n-1} + \overline{z_1}^{n-2} z_2 + \cdots + \overline{z_2}^{n-1}| < n$. Thus, f is a harmonic mapping of the open unit disk $\mathbb{D}$ onto the domain bounded by a hypocycloid of $n+1$ cusps, inscribed in the circle $|w| = (n+1)/n$. Note also that $F(z) = z + \frac{1}{n} z^n$ is analytic and univalent in $\mathbb{D}$ because $|F'(z) - 1| < 1$ in for all $z \in \mathbb{D}$. Thus, F is a normalized univalent function in $\mathbb{D}$. The images of f and F for $n = 2$ are illustrated in Figures 5 and 6 whereas the images of f and F for $n = 4$ are illustrated in Figures 7 and 8. □

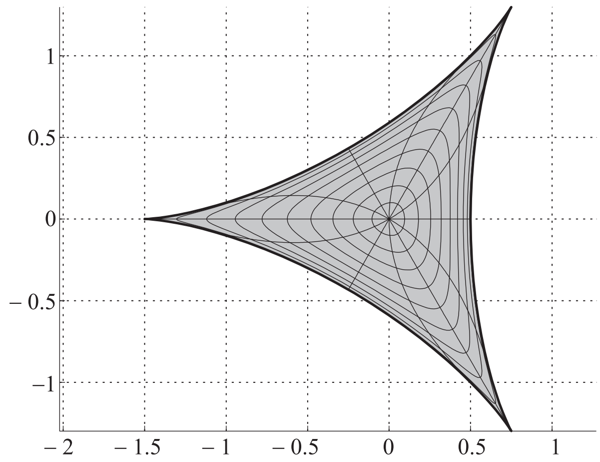

Figure 5. Image of $\mathbb{D}$ under $f(z) = z - \frac{1}{2}\bar{z}^2$

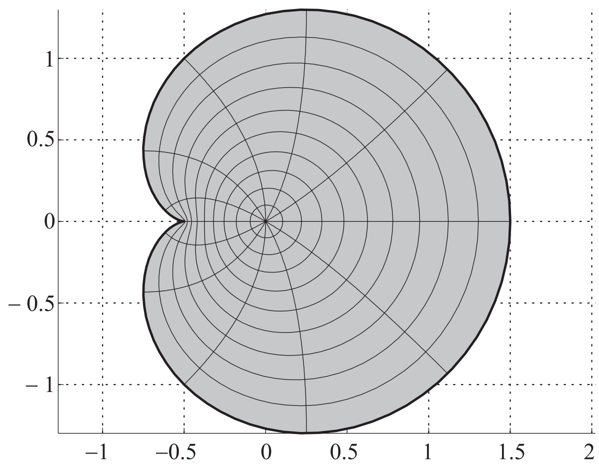

Figure 6. Image of $\mathbb{D}$ under $F(z) = z + \frac{1}{2}z^2$

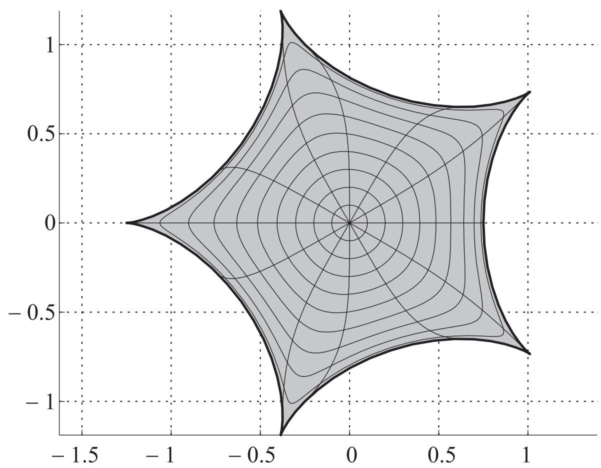

Figure 7. Image of $\mathbb{D}$ under $f(z) = z - \frac{1}{4}\bar{z}^4$

2.20. Normalizations

Suppose that f is a harmonic mapping of a proper simply connected domain Ω of $\mathbb{C}$. By the Riemann mapping theorem

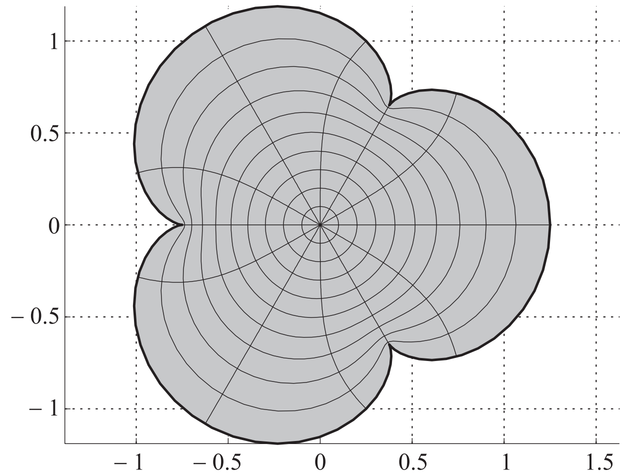

Figure 8. Image of $\mathbb{D}$ under $F(z) = z + \frac{1}{4}z^4$

there is a conformal mapping φ of $\mathbb{D}$ onto Ω. It follows that the composition $f \circ \varphi$ maps $\mathbb{D}$ harmonically onto Ω. As a consequence of this observation, we may assume that Ω is the unit disk and that f is sense-preserving in $\mathbb{D}$. We next observe that, because $J_f = |h'|^2 - |g'|^2 > 0$ for the sense-preserving harmonic mapping $f = h + \overline{g}$, $f_z(0) = h'(0) \neq 0$, and so we arrive at the normalized form of f:

$$\frac{f(z) - f(0)}{f_z(0)}.$$

Thus, the analytic and co-analytic parts of the harmonic mapping $f = h + \overline{g}$ in $\mathbb{D}$ may be written, respectively,

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

Definition 2.21. *Let $\mathcal{S}_H$ denote the class of all complex-valued (sense-preserving) harmonic mappings that are normalized on the unit disk $\mathbb{D}$. That is,*

$$\mathcal{S}_H = \{f : \mathbb{D} \to \mathbb{C} : f \text{ is harmonic and univalent with } f(0) = 0 = f_z(0) - 1\}.$$

We note that $\mathcal{S}_H$ reduces to $\mathcal{S}$, the class of normalized univalent analytic functions in $\mathbb{D}$ whenever the co-analytic part of f is zero, i.e. $g(z) \equiv 0$ in $\mathbb{D}$. Furthermore, for $f = h + \overline{g} \in \mathcal{S}_H$, $|b_1| < 1$ (because $J_f(0) = 1 - |g'(0)|^2 = 1 - |b_1|^2 > 0$) and therefore, the function

$$F = \frac{f - \overline{b_1 f}}{1 - |b_1|^2} \tag{2.22}$$

is also in $\mathcal{S}_H$. Note that this function is obtained by applying an affine mapping to f. Thus, we may sometimes restrict our attention to the subclass of functions f in $\mathcal{S}_H$ for which $b_1 = f_{\bar{z}}(0) = 0$. We define

$$\mathcal{S}_H^0 = \{f \in \mathcal{S}_H : f_{\bar{z}}(0) = 0\}.$$

Note that the condition $f_{\bar{z}}(0) = 0$ is equivalent to $\omega(0) = 0$ or $g(z) = O(z^2)$ for z near the origin, where $\omega(z)$ is the second complex dilatation of f. Clearly,

$$\mathcal{S} \subsetneq \mathcal{S}_H^0 \subsetneq \mathcal{S}_H.$$

Although both $\mathcal{S}_H$ and $\mathcal{S}_H^0$ are known to be normal families [i.e. every sequence of functions in $\mathcal{S}_H$ (resp. $\mathcal{S}_H^0$) has a subsequence that converges locally uniformly in $\mathbb{D}$] only $\mathcal{S}_H^0$ is compact with respect to the topology of locally uniform convergence (see [6]). Indeed, $\mathcal{S}_H$ is not a compact family because it is not preserved under passage to locally uniform limits. The limit function is necessarily harmonic in $\mathbb{D}$, but it need not be univalent. For example, the sequence of affine mappings f_n defined in Example 2.8 demonstrates that $\mathcal{S}_H$ is not a compact family.

Identifying $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$ and points in $\mathbb{R}^2$ as 2×1 column matrix, we may rewrite the equation (2.22) in matrix form as

$$(1 - |b_1|^2)\begin{pmatrix} \operatorname{Re} F \\ \operatorname{Im} F \end{pmatrix} = \begin{pmatrix} 1 - \overline{b_1} & 0 \\ 0 & 1 + \overline{b_1} \end{pmatrix}\begin{pmatrix} \operatorname{Re} f \\ \operatorname{Im} f \end{pmatrix}$$

and so, pre-multiplying by the inverse of the 2×2 matrix gives

$$\begin{pmatrix} \operatorname{Re} f \\ \operatorname{Im} f \end{pmatrix} = \begin{pmatrix} 1 + \overline{b_1} & 0 \\ 0 & 1 - \overline{b_1} \end{pmatrix}\begin{pmatrix} \operatorname{Re} F \\ \operatorname{Im} F \end{pmatrix}.$$

Simplifying the last relation reveals that the transformation (2.22) is one-to-one and its inverse is given by

$$f = F + \overline{b_1 F}.$$

This observation enables us to derive a number of properties about $\mathcal{S}_H$ and $\mathcal{S}_H^0$, [6].

2.23. Method of shearing

One of the forms of constructing harmonic mappings, introduced by Clunie and Sheil-Small [6], is known as *"shear construction."* This method produces certain planar harmonic mappings by adjoining functions in $\mathcal{S}$ with co-analytic parts that are related to or derived from analytic parts. Moreover, in this method one produces a harmonic mapping of $\mathbb{D}$ with a specified dilatation onto a domain in one direction by "shearing" a conformal mapping along parallel lines.

Definition 2.24. *A domain Ω is convex in the direction $e^{i\alpha}$, if for every fixed complex number a, the set $\Omega \cap \{a + te^{i\alpha} : t \in \mathbb{R}\}$ is either connected or empty; i.e. every line parallel to the line through 0 and $e^{i\alpha}$ has a connected intersection with Ω.*

In particular, it is convenient to consider domains convex in the direction of the real axis (denoted by CHD). Thus, a domain Ω is a CHD if every line parallel to the real axis has a connected intersection with Ω. Clearly, a domain Ω in $\mathbb{C}$ is convex if and only if it is convex in every direction.

Definition 2.25. *A function f defined in $\mathbb{D}$ is said to be CHD if its range is CHD; i.e. if the intersection of $f(\mathbb{D})$ with each horizontal line is connected.*

For example, the Koebe function $k(z) = z/(1 - z)^2$, and the convex function $c(z) = z/(1 - z)$ are univalent in $\mathbb{D}$ and CHD. The following lemma due to Clunie and Sheil-Small [6] is a basis for the construction of harmonic mappings that are convex in the direction of the real axis, in particular.

Lemma 2.26. *For analytic functions h and g, assume that $f = h + \overline{g}$ is harmonic and locally univalent in the unit disk $\mathbb{D}$. Then f is a univalent mapping of $\mathbb{D}$ onto a* CHD *domain if and only if $h - g$ is a conformal mapping of $\mathbb{D}$ onto a* CHD *domain.*

Remark 2.27.

(1) *From Lemma 2.26, we note that the imaginary part of the analytic function $h - g$ equals the imaginary part of the harmonic function $f = h + \overline{g}$ so that the domain is changed or cut or stretched in the real direction. In view of this observation, the method is known as "shearing".*

(2) *Lemma 2.26 indeed holds for simply connected domains rather than for functions defined on the unit disk $\mathbb{D}$.*

(3) *From the definition, it is clear that the harmonic mapping $f = h + \overline{g}$ has a convex range if and only if the range is convex in every direction. That is equivalent to saying that the range of every rotation $e^{i\alpha} f(z)$ $(= e^{i\alpha}h + \overline{e^{-i\alpha}g})$ is CHD, for $0 \le \alpha < 2\pi$.* •

One can use the technique of shear construction for constructing interesting examples of harmonic mappings by shearing conformal mappings with a prescribed dilatation $\omega(z)$. As a simple demonstration, let ϕ be a conformal mapping of the unit disk $\mathbb{D}$ onto a CHD domain with $\phi(0) = 0$, and let $f = h + \overline{g}$ be a sense-preserving harmonic function, where h and g are analytic in $\mathbb{D}$. Then the dilatation $\omega = g'/h'$ is analytic with $|\omega(z)| < 1$ in $\mathbb{D}$. According to the shear technique, the construction of harmonic mappings proceeds by letting $h - g = \phi$. This gives the pair of linear differential equations

$$h'(z) - g'(z) = \phi'(z) \quad \text{and} \quad \omega(z)h'(z) - g'(z) = 0.$$

Solving for $h'(z)$ and $g'(z)$, and then integrating with the normalization $g(0) = h(0) = 0$, we arrive at the formulas for h and g explicitly:

$$h(z) = \int_0^z \frac{\phi'(t)}{1 - \omega(t)}\,dt \quad \text{and}$$

$$g(z) = \int_0^z \frac{\omega(t)\phi'(t)}{1 - \omega(t)}\,dt = h(z) - \phi(z).$$

So, the shear construction produces the harmonic mapping f defined by

$$f(z) = h(z) + \overline{g(z)} = 2\mathrm{Re}\, h(z) - \overline{\phi(z)}$$

and f maps $\mathbb{D}$ onto a CHD domain. Various choices of the dilatation $\omega(z)$ and the conformal mapping $\phi(z)$ produce a number of harmonic mappings, as demonstrated by a number of examples below (see the paper by Greiner [10] for many more examples).

Example 2.28. *Note that $z - z^3/3$ maps $\mathbb{D}$ onto a domain which is convex in the direction of the real axis and so is the harmonic function $f = h + \overline{g}$, where*

$$h - g = z - \frac{1}{3}z^3 \quad \textit{and} \quad \frac{g'(z)}{h'(z)} = z^2 \ (= \omega(z)).$$

As $h'(z) - g'(z) = 1 - z^2$ and $g'(z) = z^2 h'(z)$, solving these two equations yields that

$$h'(z) = 1 \quad \textit{and} \quad g'(z) = z^2.$$

As $h(0) - g(0) = 0$, i.e. $g(0) = h(0)$, it follows that

$$h(z) = z \quad \textit{and} \quad g(z) = \frac{1}{3}z^3$$

which gives

$$f(z) = h(z) + \overline{g(z)} = z + \frac{1}{3}\overline{z}^3.$$

The harmonic mapping f maps $\mathbb{D}$ onto the interior of the region bounded by the hypocycloid with 4 cusps. Also, we note that $h(z) - g(z)$ is a conformal mapping of $\mathbb{D}$ onto the interior of an epicycloid with 2 cusps. The geometric behaviors of $f(z) = z + \frac{1}{3}\overline{z}^3$ and $F(z) = z - \frac{1}{3}z^3$ are illustrated in Figures 9 and 10. •

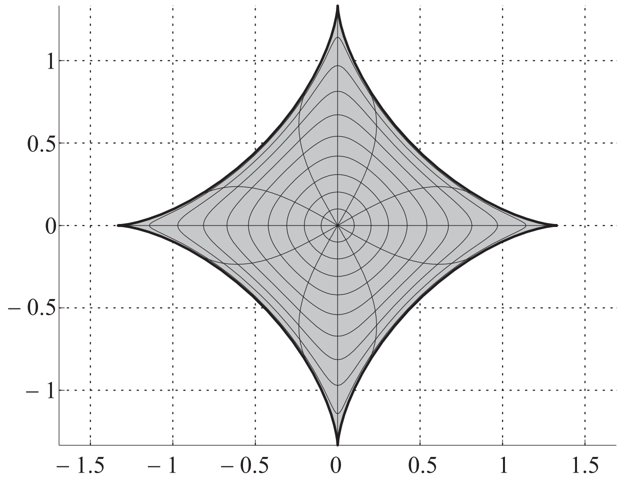

Figure 9. Image of $\mathbb{D}$ under $f(z) = z + \frac{1}{3}\overline{z}^3$

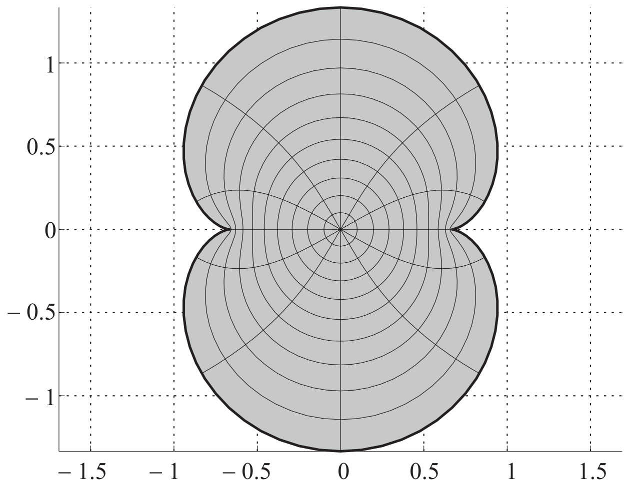

Figure 10. Image of $\mathbb{D}$ under $F(z) = z - \frac{1}{3}z^3$

Example 2.29. *Let $h - g = k$ with the dilatation $\omega(z) = z$, where $k(z) = z/(1 - z)^2$ is the Koebe function. This gives the pair of differential equations*

$$h'(z) - g'(z) = \frac{1 + z}{(1 - z)^3} \quad \textit{and} \quad g'(z) = zh'(z).$$

Using these, we easily see that

$$h'(z) = \frac{1+z}{(1-z)^4} \quad \text{and} \quad g'(z) = \frac{z(1+z)}{(1-z)^4}.$$

Integrating with $g(0) = h(0) = 0$, we easily arrive at the formulas for h and g explicitly. So,

$$f(z) = h(z) + \overline{g(z)} = \mathrm{Re}\left(\frac{z + \frac{1}{3}\overline{z}^3}{(1-z)^3}\right) + i\,\mathrm{Im}\left(\frac{z}{(1-z)^2}\right)$$

which maps $\mathbb{D}$ onto the slit plane $\mathbb{C}\setminus\{u+iv : u \leq -1/6,\ v = 0\}$. For a proof of the range, we refer to the original article [6]. •

In view of the Remark **??**, we can state the following equivalent form of Lemma 2.26.

Theorem 2.30. *For analytic functions h and g, assume that $f = h + \overline{g}$ is harmonic and locally univalent in the unit disk $\mathbb{D}$. Then f is a univalent mapping of $\mathbb{D}$ onto a convex domain if and only if for each α $(0 \leq \alpha < 2\pi)$ the analytic function $e^{i\alpha}h - e^{-i\alpha}g$ is univalent and maps $\mathbb{D}$ onto a* CHD *domain.*

Corollary 2.31. *If $f = h + \overline{g}$ is a convex harmonic mapping on $\mathbb{D}$, then the analytic function $h - e^{-2i\alpha}g$ is univalent and maps $\mathbb{D}$ onto a domain that is convex in the direction $e^{i\alpha}$ for all $0 \leq \alpha < \pi$.*

Proof. Assume that $f = h+\overline{g}$ is a convex harmonic mapping on $\mathbb{D}$. By Theorem 2.30, the analytic function $e^{i\alpha}h - e^{-i\alpha}g$ is (univalent) convex in the direction of real axis. By rotating, we see that $h - e^{-2i\alpha}g$ is convex in the direction of $e^{i\alpha}$ for every α, $0 \leq \alpha < \pi$. □

2.32. Harmonic Univalent Polynomials

Harmonic univalent polynomials form a subtopic of the theory of harmonic mappings. Specifically a harmonic polynomial is a function $f = h+\overline{g}$, where h and g are analytic polynomials in z. The degree of f is defined as the larger of the degrees of h and g. Finding a method of constructing sense-preserving univalent harmonic polynomials is another important problems, see [20]. Such polynomials in general have the above form with the normalization $h(0) = g(0) = 0 = g'(0)$ and the dilatation ω is a finite Blaschke product. However, very little is known about harmonic mappings whose dilatation is not a Blaschke product. Very few explicit examples of such mappings are known. In any case, it is appropriate to include one more simple example of a harmonic univalent polynomial (see also Examples 2.8 and 2.19) although we would continue our discussion on this topic at a later stage.

Example 2.33. *Let $f = h + \overline{g}$, where*

$$h(z) = z + \alpha\frac{z^2}{2} \quad \text{and} \quad g(z) = \beta\frac{z^2}{2}$$

where α and β are complex constants lying in the closed unit disk $|z| \leq 1$ such that $|\alpha| + |\beta| = 1$. We see that

$$f(z) = z + \alpha\frac{z^2}{2} + \overline{\beta}\frac{\overline{z}^2}{2}$$

is a harmonic mapping on $\mathbb{D}$. To verify the univalence of f in $\mathbb{D}$, we suppose that $f(z_1) = f(z_2)$ for $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$. Then, we have

$$0 = (z_1 - z_2)\left[1 + \frac{\alpha}{2}(z_1 + z_2)\right] + \frac{\overline{\beta}}{2}(\overline{z_1} - \overline{z_2})(\overline{z_1} + \overline{z_2})$$

or

$$0 = 1 + \frac{\alpha}{2}(z_1 + z_2) + \frac{\overline{\beta}}{2}\left(\frac{\overline{z_1} - \overline{z_2}}{z_1 - z_2}\right)(\overline{z_1} + \overline{z_2})$$

which is a contradiction. This is because, for $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$,

$$0 = \left|1 + \frac{\alpha}{2}(z_1 + z_2) + \frac{\overline{\beta}}{2}\left(\frac{\overline{z_1} - \overline{z_2}}{z_1 - z_2}\right)(\overline{z_1} + \overline{z_2})\right|$$

$$> 1 - |\alpha| - |\beta| = 0,$$

which is false. Thus, for $z_1 \neq z_2$, we must have $f(z_1) \neq f(z_2)$ and hence, f is a harmonic mapping on $\mathbb{D}$. The images of f for various of choices of α and β are illustrated in Figures 11–15. •

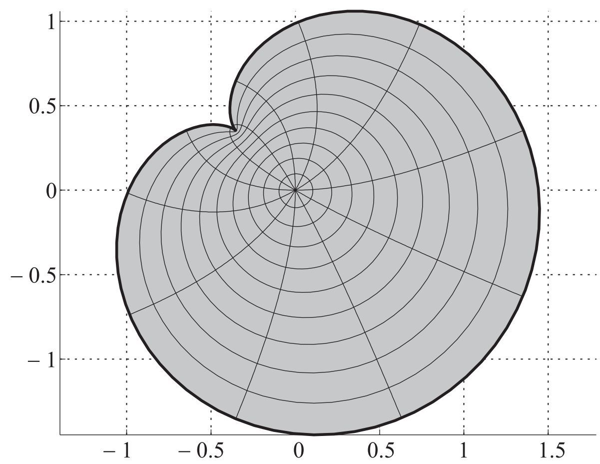

Figure 11. Image of $\mathbb{D}$ under $f(z) = z + \frac{e^{i\pi/4}}{2}z^2$

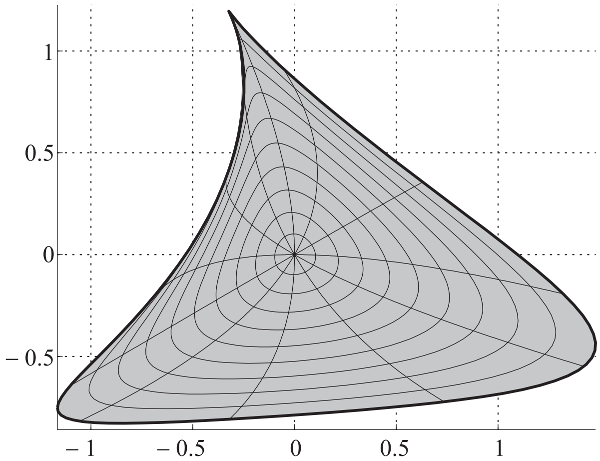

Figure 12. Image of $\mathbb{D}$ under $f(z) = z + \frac{3e^{i\pi/4}}{8}z^2 + \frac{3e^{-i\pi/3}}{8}\overline{z}^2$

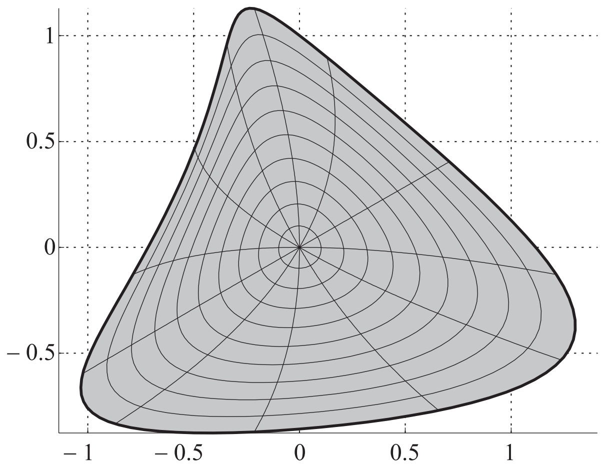

Figure 13. Image of $\mathbb{D}$ under $f(z) = z + \frac{e^{i\pi/4}}{4}z^2 + \frac{e^{-i\pi/3}}{4}\overline{z}^2$

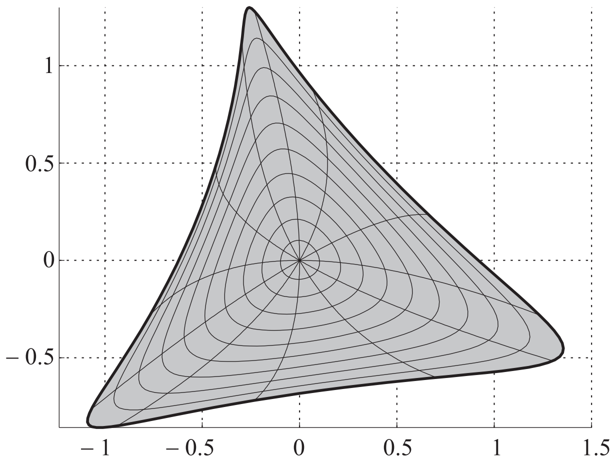

Figure 14. Image of $\mathbb{D}$ under $f(z) = z + \frac{e^{i\pi/4}}{8}z^2 + \frac{3e^{-i\pi/3}}{8}\overline{z}^2$

Harmonic mappings can be considered as close relatives of conformal mappings. However, in contrast to conformal mappings, harmonic mappings are not at all determined (up to normalizations) by their image domains. Thus, it is natural to study the class $\mathcal{S}_K(\Omega, \Omega')$ of harmonic sense-preserving mappings of a domain Ω onto another domain Ω'. For the case $\Omega = \Omega' = \mathbb{D}$, we refer to the work of Choquet [5], Heine [12] and Hall [11]. For Ω, a proper sub domain of $\mathbb{C}$ and Ω', a strip, we refer to the work of Hengartner and Schober [13]. For a recent survey of harmonic mappings, we refer to the paper of Bshouty and Hengartner [3]. However, we shall return to discuss on these and other related articles with some important questions.

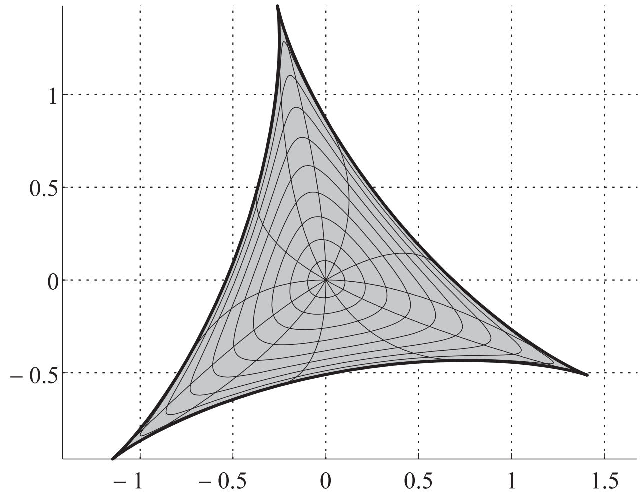

Figure 15. Image of $\mathbb{D}$ under $f(z) = z + \frac{e^{-i\pi/3}}{2}\overline{z}^2$

3. Exercises

(1) Distinguish the difference between the real-analytic and analytic function in an open set Ω.

(2) Must every analytic function be harmonic?

(3) When can every harmonic function be analytic? Let $f(z) = x^2 - y^2 + ix$. Is f harmonic on $\mathbb{C}$? Is f^2 harmonic on $\mathbb{C}$?

Hint: Set $f^2(z) = (x^2 - y^2)^2 - x^2 + 2x(x^2 - y^2)i := U + iV$. Then $\Delta V = 8x$.

(4) Give an example of C^1-function that is not a harmonic mapping. How about the function $f(z) = x\cos yz + iy$?

(5) Show that neither the reciprocal $1/f$ nor the inverse f^{-1} (when they exist) of a harmonic function is in general harmonic.

(6) Consider the automorphism of the unit disk: $\phi_a(z) = e^{i\theta}(z-a)/(1-\overline{a}z)$ $(|a| < 1)$. If $f : \mathbb{D} \to \mathbb{D}$ is a harmonic function, must $\phi_a \circ f$ be harmonic in $\mathbb{D}$?

(7) Show that the composition of a harmonic mapping with an affine mapping is harmonic, i.e. if f is harmonic, then so is $\alpha f + \beta \overline{f} + \gamma$, $|\alpha| \neq |\beta|$.

(8) Verify whether the composition of a harmonic function with a conformal premapping is a harmonic function.

(9) Suppose that $f(z) = \ln|z| + iy$. Determine Ω, analytic functions h and g so that the decomposition $f(z) = h(z) + \overline{g(z)}$ Is possible. Also determine the corresponding second complex dilatation $\omega(z)$ which is analytic in Ω with $|\omega(z)| < 1$ for $z \in \Omega$.

(10) For $|\alpha| \leq 2$, let $f_\alpha(z) = z - \frac{1}{\overline{z}} + \alpha \ln|z|$. Show that each f_α is sense-preserving harmonic mapping of the exterior of the unit disk onto the punctured plane.

Hint: See [13].

(11) State and prove the composition rule for $(f \circ g)_z$ and $(f \circ g)_{\overline{z}}$.

(12) Suppose that f is a harmonic function in an open set $\Omega \subseteq \mathbb{C}$ such that $J_f(z_0) \neq 0$ at some point $z_0 \in \Omega$. Show that f is either sense-preserving or sense-reversing in some neighbourhood of z_0.

(13) Suppose that f is a sense-preserving harmonic mapping of a domain Ω. Must $|f_z| > 0$ on Ω? Must the second complex dilatation $\omega(z)$ be an analytic mapping of Ω into the unit disk $\mathbb{D}$?

(14) Suppose that f is an sense-preserving (reversing) harmonic diffeomorphism and g is analytic (anti-analytic). Verify whether the Jacobian of $f \circ g$ does not change sign, i.e. $J_{f \circ g} > 0$ (< 0).

(15) Modify Remark 2.5 to obtain a harmonic homeomorphism of $\mathbb{R}^n$ with Jacobian vanishing on a hyperplane. Conclude that Lewy's theorem is false for all $n \geq 3$.

(16) If $f(z) = z + z^2 - \overline{z}$, determine the critical points of f in $\mathbb{C}$.

(17) Verify whether the following functions are harmonic in the upper half-plane $\Omega = \{z = x + iy : y > 0\}$:

(a) $f(z) = \arg z + iy$

(b) $f(z) = xy + iy$.

Are these mappings take Ω onto Ω. If not find $f(\Omega)$.

(18) Must $f(z) = z + \mathrm{Re}\,(e^z)$ be a harmonic mapping in $\mathbb{C}$?

(19) Let $f = h + \overline{g}$ with $h + g = z/(1-z)$ and with the dilatation $\omega(z) = -z$ (Note that $z/(1-z)$ is convex in every direction). Compute the formulas for h and g explicitly and determine $f(\mathbb{D})$.

(20) Does Example 2.9 follow from Lemma 1.16 or Corollary 2.4? If yes, explain. If not, could you find an analog of Corollary 2.4 applicable to Example 2.9?

(21) Define $f(z) = z|z|^{\frac{1}{k}-1}$ for $k \geq 1$. Is f a quasiconformal mapping of $\mathbb{C}$ with $K = k$?

(22) Let $f = h + \overline{g}$ with $h - g = z$. Assume first the dilatation $\omega(z) = z$ and then $\omega(z) = z^2$. Using the shear construction, determine f in each case. Find also $f(\mathbb{D})$.

(23) Let $f = h + \overline{g}$ be a CHD domain. Must f_α defined by $f_\alpha = h + e^{i\alpha}\overline{g}$ convex in the direction $e^{i\alpha}$?

References

[1] L. V. AHLFORS, Complex Analysis, 2nd ed., McGraw-Hill, New York, 1966.

[2] L. V. AHLFORS, Lectures on Quasiconformal Mappings, Second edition. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. University Lecture Series, 38. American Mathematical Society, Providence, RI, 2006. viii+162 pp.

[3] D. BSHOUTY AND W. HENGARTNER, Univalent harmonic mappings in the plane, In: Handbook of Complex Analysis: Geometric Function Theory, Vol. 2 (2005), 479–506, Edited by R. Kühnau (ISBN: 0-444-51547-X), Elsevier.

[4] R. B. BURCKEL, An introduction to Classical Complex Analysis, Vol. I, Academic Press, 1979.

[5] G. CHOQUET, Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques, *Bull. Sci. Math.* **69** (1945), 156–165.

[6] J. G. CLUNIE AND T. SHEIL-SMALL, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A.I.* **9** (1984), 3–25.

[7] P. L. DUREN, Univalent Functions (Grundlehren der mathematischen Wissenschaften 259, New York, Berlin, Heidelberg, Tokyo), Springer-Verlag, 1983.

[8] P. DUREN, Harmonic Mappings in the Plane, Cambridge University Press, 2004.

[9] A. W. GOODMAN, Univalent Functions, Vols. I and II. Mariner Publishing Co. Tampa, Florida, 1983.

[10] P. GREINER, Geometric properties of harmonic shears, *Comput. Methods Funct. Theory* **4** (2004), no. 1, 77–96.

[11] R. R. HALL, On an inequality of E. Heinz, *J. Analyse Math.* **42** (1983), 185–198.

[12] E. HEINZ, Über die Lösungen der Minimalflächengleichung, Nachr. Akad. Wiss. Göttingen, *Math.-Phys. K1.* (1952), 51–56.

[13] W. HENGARTNER, AND G. SCHOBER, Univalent harmonic mappings, *Trans. Amer. Math. Soc.* **299** (1987), 1–31.

[14] O. LEHTO, Univalent functions and Teichmüller spaces, Graduate Texts in Mathematics, 109. Springer-Verlag, New York, 1987.

[15] O. LEHTO AND K. I. VIRTANEN, Quasiconformal mappings in the plane, Second edition. Translated from the German by K. W. Lucas. Die Grundlehren der mathematischen Wissenschaften, Band 126. Springer-Verlag, New York-Heidelberg, 1973.

[16] H. LEWY, On the nonvanishing of the Jacobian in certain one-to-one mapppings, *Bull. Amer. Math. Soc.* **42** (1936), 689–692.

[17] CH. POMMERENKE, Univalent Functions, Gottingen: Vandenhoeck & Ruprecht, 1975.

[18] S. PONNUSAMY, Foundations of Complex Analysis, Alpha Science International Publishers, UK, 2005.

[19] S. PONNUSAMY AND H. SILVERMAN, Complex Variables with Applications, Birkhäuser, Boston, 2006.

[20] T. J. SUFFRIDGE, Harmonic univalent polynomials, *Complex Variables Theory Appl.* (2)**35** (1998), 93–107.

[21] J. VÄISÄLÄ, Lectures on n-dimensional quasiconformal mappings, Lecture Notes in Math. Vol. 229, Springer-Verlag, 1971.

[22] M. VUORINEN, Conformal geometry and quasiregular mappings. (Monograph, 208 pp.). Lecture Notes in Math. Vol. 1319, Springer-Verlag, 1988.

[23] J. C. WOOD, Lewy's theorem fails in higher dimensions, *Math. Scand.* **69** (1991), 166.

Refresher Course in Functional Analysis

Ramanujan Institute for Advanced Study in Mathematics
University of Madras

Sponsored by

UGC-Academic Staff College 2007–2008

There will be a refresher course (batch No. XXIII), **for college teachers**, in Functional Analysis, from 14-11-2007 to 04-12-2007, at Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chepauk, Chennai 600 005. In this course, Functional Analysis will be developed from the basics. Applications of Functional Analysis will be discussed after developing the basic theory. Experts in Functional Analysis from leading institutions will deliver lectures for this refresher course.

For Applications and Other Information, Please Contact:

Dr. E. H. Ilamathian (Director)
UGC – Academic Staff College
University of Madras, Chennai 600 005
Phone: 25368778 Ext: 269, 343
Email: `ugcascuomyahoo.com`

If You Have Further Questions, You May Also Contact:

G. P. Youvaraj (Coordinator)
RIASM, University of Madras, Chennai 600 005
Phone: 25360357 Ext: 20
Email: `youvarajgpyahoo.com`

Combinatorics and Statistical Mechanics

January 14–July 4, January 2008

Venue: Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom.

The past half-decade has seen an increasing interaction between combinatorialists, probabilists, computer scientists and theoretical physicists concerned broadly with the study of "probability theory on graphs" or "statistical mechanics on graphs". The programme will build on this cross-fertilization.

Topics: Zeros of combinatorial polynomials, including the chromatic, flow, reliability and Tutte polynomials; Markov-chain Monte Carlo methods; combinatorial identities and their applications in statistical mechanics; use of methods from statistical mechanics and quantum field theory in combinatorial enumeration; correlation inequalities; and phase transitions in combinatorial structures.

For Further Information Contact:

E-mail: `info@newton.cam.ac.uk.`
Home page: `http://www.newton.cam.ac.uk/programmes/CSM/`

CMFT WORKSHOP 2008 (First Announcement)

CMFT-Workshop 2008, Guwahati, Assam, India
3rd January, 2008–10th January, 2008

Institute of Advanced Study in Science and Technology (IASST), Guwahati 35

Objectives of the Workshop: Complex Analysis is one of the central mathematical disciplines, with very important ramifications into many branches of pure and applied sciences. For instance, the relevance of complex numbers in all kinds of engineering is reflected in the existence and continuous development of many algorithms, based on theoretical complex analysis, to help in finding solutions of numerical problems occurring in modern areas like cryptology, data compression, image processing, brain imaging etc.

The more theoretical aspects of complex analysis have always been given high importance within mathematical research in India. In this workshop not only the purely mathematical but also the computational and algorithmic aspects will be emphasized. This is important also for the future teaching of the subject at Indian institutions of higher studies.

The workshop will consist of various short courses (4–5 lectures each) in subjects of particular relevance in modern complex analysis, among them:

The universality of the Riemann Zeta function;
Riemann-Hilbert problems (analytic and discrete);
Numerical conformal mappings (theory and algorithms);
Normal and quasi-normal families (re-scaling methods with applications);
Conformal metrics: the interplay with differential geometry and PDEs;
Integral transforms and moment problems.

About CMFT: CMFT (Computational Methods and Function Theory) started as a series of international conferences of high standard, aiming at closer cooperation between pure and applied complex analysis. The conferences are held every four years in various places around the world, thereby also fulfilling the second main aim, namely to give scientists with limited international contacts a chance to link themselves into the international community of mathematicians in the field. Besides, an inter-national journal CMFT has been founded in 2001, also dedicated to the goals mentioned above. See `http://www.cmft.de` for more information.

To further support the aims of CMFT regional workshops are also organized. The workshop to be held in Guwahati, in north-east India, belongs to this series. It is meant for Ph.D. students and research workers in complex analysis from India and neighbouring countries. The resource persons for the workshop are internationally renowned mathematicians who are experts in specific subjects within the general area.

CMFT-Workshop 2008, Guwahati: The Guwahati CMFT-Workshop will be hosted by the Institute of Advanced Study in Science and Technology (IASST). The venue will be the Don Bosco Institute at Kharghuli, Guwahati. Financial support from different funding agencies has been applied for. For more details and further information, contact the Convenor of the Local Organizing Committee at the address below:

Proforma for Application: An application form containing name, present affiliation, age, tel/fax/E-mail, postal address, educational qualifications name & contact details of one referee, whether financial support required, brief reasons for wanting to attend, should be sent to the Convenor or the Joint Convenor of the ACC at the addresses overleaf, preferably by E-mail, to arrive on or before the 15 October, 2007. Selected participants will be informed by mid November.

Number of Participants: 50–60

Registration Fee: Rs. 1500/- per participant

Limited funds are likely to be available to deserving participants for TA/DA to attend the workshop. Information about travel to Guwahati and accommodation will be sent later to the selected participants.

International Scientific Committee:

R. Balasubramanian, Institute of Mathematical Sciences, Chennai
S. Ruscheweyh, University of Wuerzburg, Germany
E. B. Saff, Vanderbilt University, USA

Academic Coordination Committee (ACC):

M. Bhattacharjee, University of Wuerzburg, Germany
S. Ponnusamy, IIT Madras, India (Joint Convenor)
S. Ruscheweyh, University of Wuerzburg, Germany (Convenor)
B. C. Tripathy, IASST, Guwahati, India

Resource Persons:

R. Balasubramanian, India
R. Fournier, Canada
Daniela Kraus, Germany
Lisa Lorentzen, Norway
N. Papamichael, Cyprus
S. Ponnusamy, India
O. Roth, Germany
F. Rønning, Norway
S. Ruscheweyh, Germany
J. Steuding, Germany

Contact:

S. Ponnusamy (Joint Convenor)
IIT Madras Department of Mathematics
Chennai 36, India
E-mail: `samy@iitm.ac.in`

Stephan Ruscheweyh (Convenor)
University of Wuerzberg
Institute of Mathematics 97074
Wuerzburg, Germany
E-Mail: ruscheweyh@mathematik.uni-wuerzburg.de

B. C. Tripathy (Convenor, Local)
Institute of Advanced Study in Science & Technology,
Paschim Boragaon
Mathematical Sciences Division
Guwahati 781 035,
Assam, India
E-Mail: tripathybc@yahoo.com
Phone: 9864087231

Ramanujan Mathematical Society

"RMS Lecture Notes Series in Mathematics"
Field Office:- 8/1,13th Cross, Adipampa Road
V. V. Mohalla, Mysore 570 002
India

Sponsored by

The Department of Science and Technology
Government of India, New Delhi

RMS LECTURE NOTES SERIES provides an outlet for publication of a vast amount of recent and relevant materials generated through important developments taking place in Mathematics, Mathematical Sciences (Pure Mathematics, Operations Research, Statistics & Theoretical Computer Science) and allied areas anywhere in the world, especially

- Proceedings of conferences and instructional workshops which focus on a topic of current research and
- Monographs which consist of notes from seminars, lecture series, tutorials, courses and exposition of important topics by eminent scholars for which there are no readily available textbooks.

The manuscripts submitted for possible publication in this series will undergo a screening process as the emphasis of the series rests on standards and high quality of mathematical content. Though it is not mandatory, it is expected that the proposal will meet a part of the cost of printing. This will facilitate to give around 40 copies of the proceedings to the editor(s) free of cost.

Individual Coordinators/institutions where events such as those mentioned above have taken place, or are about to take place in the near future, are encouraged to submit two hard copies of the manuscripts in LATEX pdf formats along with a Compact Disc (CD) copy at the address cited above.

Distribution and/or marketing rights for the volumes under the series would be exclusively with the Ramanujan Mathematical Society (RMS).

National Conference on Applied Mathematics (NCOAM 2008)

3–4 January, 2008

Organizer:

Department of Mathematics
B. S. Abdur Rahman Crescent Engineering College
Vandalur, Chennai 600 048

Important Dates:

Submission of Full paper : 30/09/2007
Confirmation of selection : 15/10/2007
Camera Ready paper and
Conference registration : 31/10/2007

Address for Correspondence:

The Co-ordinator
NCOAM 2008
Department of Mathematics
B. S. Abdur Rahman Crescent Engineering College
Vandalur, Chennai 600 048
Phone No: 22751347/48/75 Extn. 393
E-mail: ncoam 2008@crescentcollege.org
Web site: www.crescentcollege.org

International Conference on Functional Analysis And its Applications (FAA 2007)

November 28–December 1, 2007

Call for Papers: Researchers/Teachers who wish to present papers in the conference are encouraged to send their abstract in any area of Functional Analysis (a few are listed below) preferably in Tex or MS word pdf file by E-mail to `faa2007@scottchristian.org` latest by October 25, 2007. Abstracts sent by post will also be accepted.

Important Dates:

Abstract submission : October 25, 2007

Registration : October 31, 2007

Submission of the full tex
of the paper in Tex format : December 31, 2007

For More Details Contact:

J. R. V. Edward
Organizing Secretary, FAA 2007
Department of Mathematics
Scott Christian College (Autonomous)
Nagercoil 629 003, Tamil Nadu, India.
E-mail Id: `faa2007@scottchristian.org`
`jrvedward@gmail.com`
Cell no: (0)9443605698

Home page:

`http://www.scottchristian.org/faa2007/home.asp`