QUANTUM STATES, ENTANGLEMENT and CLOSED TIMELIKE CURVES

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Introduction

• GTR allows non-trivial geometry of space-time which can generate closed timelike curves (CTCs).

• A closed time like curve connects back on itself, (for example, in the presence of a spacetime wormhole) that could link a future spacetime point with a past spacetime point.

• CTC may allow time travel. Locally the arrow of time leads forward, but globally the observer may return to an event in the past.

• Existence of CTCs lead to the many paradoxes including the “grandfather paradox”.

• Classical description cannot distinguish counterintuitive effects from unphysical effects.

• Classical spacetime models do not take into account quantum effects.

• Is it possible to have a model where one can have a consistent way of describing quantum physics in the presence of CTCs?

• Deutsch (1991) proposed a quantum computational model in the presence of CTCs. This provides a method for finding self-consistent solutions of CTC interactions in quantum theory.

Power of CTC

- Access to CTCs could enhance the computational power. CTC assisted classical computer can factor composite numbers efficiently [Brun (2003)].
- CTC-assisted quantum computer (QC) may solve NP-complete problems, which is an impossibility for a QC [Bacon (2004)].
- Access to CTCs can perfectly distinguish nonorthogonal quantum states [Brun *et al* (2009)].
- Consequence: Cryptographic scheme can be broken if quantum message interacts with a CTC system.

• Power of CTCs for distinguishing nonorthogonal input states and speeding up otherwise hard computations has been re-investigated [Bennett *et al* (2009)].

• The CTC-assisted evolution is not linear. The output of such computation on a mixture of inputs is not a convex combination of its output on the pure components of the mixture.

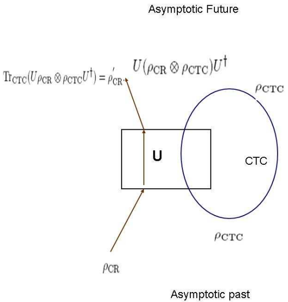
• Nature of density operator (proper vs improper mixture).
• Nature of Entangled sates.
• Cloning of arbitrary state in the presence of CTC.

Deutsch model of CTC

- Causality-respecting (CR) quantum system interacts with another system having a world line in the region of a closed timelike curve (CTC) via a unitary operator.
- It is assumed that states are *similar to the density matrices that we have in standard quantum mechanics*. Before the interaction the combined state is $\rho_{\mathrm{CR}} \otimes \rho_{\mathrm{CTC}}$.
- Unitary interaction:

$$\rho_{\mathrm{CR}} \otimes \rho_{\mathrm{CTC}} \longrightarrow U(\rho_{\mathrm{CR}} \otimes \rho_{\mathrm{CTC}})U^{\dagger}$$

INTERACTION OF CR and CTC SYSTEMS



• For each initial mixed state $\rho_{\rm CR}$ of the CR system, there exists a CTC system $\rho_{\rm CTC}$ such that after the interaction we must have

$$\rho'_{\rm CTC} = {\rm Tr}_{\rm CR}(U\rho_{\rm CR} \otimes \rho_{\rm CTC}U^{\dagger}) = \rho_{\rm CTC}.$$

• This is the self-consistency condition. Since the time travelled system follows a CTC, travelling back in time will make the input density matrix equal to its output density matrix. Solution to this equation is a fixed point.

• The final state of the CR quantum system is

$$\rho'_{\rm CR} = {\rm Tr}_{\rm CTC}(U\rho_{\rm CR} \otimes \rho_{\rm CTC}U^{\dagger}).$$

- These are the two basic conditions which govern the unitary dynamics of the CTC and CR quantum systems.

- Note that $\rho_{\rm CTC}$ depends nonlinearly on $\rho_{\rm CR}$ and hence the output of the CR system is a nonlinear function of the input density matrix.

- Thus, in the presence of CTC the map from initial to final density matrices for the CR system is nonlinear [Cassidy (1995)].

Nature of Quantum states

- What should be the nature of the density operator of a quantum system that traverses a CTC? Is it similar in all respect to the causality-respecting quantum system?

- Can we always purify ρ_{CTC} that interacts with a CR system independent of anything else?

- Within the Deutsch model, we show that it is not possible to purify a mixed state of a CTC quantum system for arbitrary ρ_{CR} and U.

Proper vs Improper Mixture

• Proper mixture ρ is a convex combination of subensembles of pure states, i.e., $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ with $\sum_k p_k = 1$.

• Improper mixture corresponds to the result of tracing out of a pure projector of a composite system such that $\rho = \rho_S = \mathrm{Tr}_A(|\Psi\rangle_{\mathrm{SA}}\langle\Psi|)$, where $|\Psi\rangle_{\mathrm{SA}} = \sum_k \sqrt{p_k}|\psi_k\rangle_S|\phi_k\rangle_A$.

• There is no way to differentiate a proper mixture from improper. Given any density matrix we can always purify it in an enlarged Hilbert space (in infinite number of ways).

• Purified state is independent of interaction and other extraneous system with which ρ_{S} is going to interact.

No-Purification Theorem

Theorem

For arbitrary U on $\mathcal{H}_{\mathrm{CR}} \otimes \mathcal{H}_{\mathrm{CTC}}$ *and* $|\psi\rangle_{\mathrm{CR}}\langle\psi|$ *on* $\mathcal{H}_{\mathrm{CR}}$ *if the CTC density matrix satisfies the consistency condition* $\rho_{\mathrm{CTC}} = \mathrm{Tr}_{\mathrm{CR}}(U\rho_{\mathrm{CR}} \otimes \rho_{\mathrm{CTC}}U^{\dagger})$ *then there does not exist general purification for* ρ_{CTC}*, such that* $\rho_{\mathrm{CTC}} = \mathrm{Tr}_{\mathrm{CTC}'}[|\Phi\rangle\langle\Phi|]$*, where* $|\Phi\rangle_{\mathrm{CTC,CTC}'} = \sum_k \sqrt{p_k}|k\rangle_{\mathrm{CTC}}|b_k\rangle_{\mathrm{CTC}'}$ *for all* ρ_{CR} *and U.*

Proof: Let $\rho_{\rm CR} = |\psi\rangle_{\rm CR}\langle\psi|$ and $\rho_{\rm CTC} = \sum_k p_k |k\rangle_{\rm CTC}\langle k|$. They undergo unitary evolution: $\rho_{\rm CR} \otimes \rho_{\rm CTC} \to U(\rho_{\rm CR} \otimes \rho_{\rm CTC})U^{\dagger}$.

• Assume that purification is possible for the CTC system. Then, it can be thought of as a part of a pure entangled state $|\Phi\rangle$ in $\mathcal{H}_{\rm CTC} \otimes \mathcal{H}_{\rm CTC'}$.

• $|\Phi\rangle$ can be written in the Schmidt decomposition form as (up to local unitary in the Hilbert space of $\rm CTC'$)

$$|\Phi\rangle_{\rm CTC,CTC'} = \sum_k \sqrt{p_k}|k\rangle_{\rm CTC}|b_k\rangle_{\rm CTC'},$$

where p_k are the Schmidt coefficients, $|k\rangle$ and $|b_k\rangle$ are orthonormal.

• In the pure state picture the unitary evolution is equivalent to

$$|\psi\rangle_{\mathrm{CR}}\langle\psi| \otimes |\Phi\rangle_{\mathrm{CTC,CTC'}}\langle\Phi| \rightarrow$$

$U \otimes I(|\psi\rangle_{\mathrm{CR}}\langle\psi| \otimes |\Phi\rangle_{\mathrm{CTC,CTC'}}\langle\Phi|)U^{\dagger} \otimes I$,
where U acts on the Hilbert space of CR and CTC and I acts on $\mathrm{CTC'}$.

• If the purification holds then

$$\mathrm{Tr}_{\mathrm{CTC'}}[U \otimes I(|\psi\rangle_{\mathrm{CR}}\langle\psi| \otimes |\Phi\rangle_{\mathrm{CTC,CTC'}}\langle\Phi|)U^{\dagger} \otimes I] =$$

$= \sum_k p_k U(|\psi\rangle_{\mathrm{CR}}\langle\psi| \otimes |k\rangle_{\mathrm{CTC}}\langle k|)U^{\dagger}$.

• Thus, $U(|\psi\rangle_{\mathrm{CR}}\langle\psi| \otimes \sum_k p_k |k\rangle_{\mathrm{CTC}}\langle k|)U^\dagger = \sum_k p_k U(|\psi\rangle_{\mathrm{CR}}\langle\psi| \otimes |k\rangle_{\mathrm{CTC}}\langle k|))U^\dagger$.

• Taking the partial trace over the CR system and using the consistency condition we have

$$\sum_k p_k |k\rangle_{\mathrm{CTC}}\langle k| = \mathrm{Tr}_{\mathrm{CR}}[\sum_k p_k U(|\psi\rangle_{\mathrm{CR}}\langle\psi| \otimes |k\rangle_{\mathrm{CTC}}\langle k|))U^\dagger].$$

• Define the action of the unitary operator on the CR system and the orthonormal basis of the CTC system in the Schmidt decomposition form:

$U(|\psi\rangle_{\rm CR}|k\rangle_{\rm CTC}) = \sum_m \sqrt{f_m^k(\psi)}|a_m^k(\psi)\rangle_{\rm CR}|u_m^k(\psi)\rangle_{\rm CTC}$, where $f_m^k(\psi)$'s are the Schmidt coefficients with $\sum_m f_m^k(\psi) = 1$ for all k, $|a_m^k(\psi)\rangle_{\rm CR}$ and $|u_m^k(\psi)\rangle_{\rm CTC}$ are orthonormal.

- Therefore, we have

$$\sum_k p_k |k\rangle\langle k| = \sum_{km} p_k f_m^k(\psi)|u_m^k(\psi)\rangle\langle u_m^k(\psi)|.$$

- The eigenvalues of $\rho_{\rm CTC}$ are given by

$$p_n = \sum_{km} p_k f_m^k(\psi)|\langle n|u_m^k(\psi)\rangle|^2$$

- Thus, p_n's depend on $\rho_{\rm CR}$ and U.

- However, the spectrum of $\rho_{\rm CTC'}$ is independent of $\rho_{\rm CR}$ and U (it does not interact with anything and can be far away from the region where CR and CTC systems interact).

- Since we must have the equal spectrum for any pure bipartite entangled state, i.e., Spec($\rho_{\rm CTC}$) $=$ Spec($\rho_{\rm CTC'}$), the purification assumption and consistency condition violate this.

- For any non-trivial interaction between the CR and the CTC to happen, our assumption that the purification is possible for $\rho_{\rm CTC}$ must be wrong.

• Hence, there is no purification of a CTC mixed state for arbitrary CR system and arbitrary interaction.

• Even if we imagine that CTC can be purified with a CR system, then similar arguments will go through and we will get a contradiction

• Thus, *there is no universal 'Church of the larger Hilbert space' (either in CR or CTC world) where a CTC mixed state can be purified* if it has to undergo arbitrary interaction with arbitrary CR system in a non-trivial way.

• One may argue that for a fixed known ρ_{CR} and U one can purify ρ_{CTC}. But then the pure entangled state depends on ρ_{CR} and U, i.e., $|\Phi\rangle = |\Phi(\psi, U)\rangle$.

• In ordinary quantum theory if we have two systems (say) with density matrices ρ and ρ_{S} and they interact via $\rho \otimes \rho_{\mathrm{s}} \to U(\rho \otimes \rho_{\mathrm{s}})U^{\dagger}$, then we can always purify ρ_{S} such that $\rho_S = \mathrm{Tr}_A(|\Psi\rangle_{\mathrm{SA}}\langle\Psi|)$, where $|\Psi\rangle_{\mathrm{SA}}$ is the purified state and it does not depend on ρ and U.

• The purification of a density matrix in ordinary quantum theory is universal. However, in the case of CTC quantum theory it is not so.

• There can be a special unitary and $\rho_{\rm CTC}$ when the purification may be possible.

• When $f_m(k) = 1/d$ for all k, m and $p_n = 1/d$ for all n ($\rho_{\rm CTC} = I/d$), then Spec($\rho_{\rm CTC}$) = Spec($\rho_{\rm CTC'}$) holds and purification of CTC density matrix is possible for arbitrary $\rho_{\rm CR}$.

• If the unitary operator U is a controlled unitary operation then purification is possible.

Nature of Entanglement

• If the state of a CR system and a CTC system is already in a pure entangled state $|\Phi\rangle_{\mathrm{CR,CTC}} = \sum_k \sqrt{p_k}|a_k\rangle_{\mathrm{CR}}|k\rangle_{\mathrm{CTC}}$, then the CTC subsystem will be an improper mixture.

• Question: can we always create such an entangled state between a CR and CTC system using any unitary? This may not be the case.

• Suppose we start with $\rho_{\mathrm{CR}} \otimes \rho_{\mathrm{CTC}} = |\psi\rangle_{\mathrm{CR}}\langle\psi| \otimes |\phi\rangle_{\mathrm{CTC}}\langle\phi|$ for some $|\psi\rangle \in \mathcal{H}_{\mathrm{CR}}$ and for some $|\phi\rangle \in \mathcal{H}_{\mathrm{CTC}}$.

• After evolution $\rho_{\mathrm{CTC}} = \sum_k p_k|k\rangle\langle k| \neq |\phi\rangle_{\mathrm{CTC}}\langle\phi|$. This shows that the consistency condition is not satisfied

- If we start with the initial ρ_{CR} as a pure state and ρ_{CTC} as a mixed state, then the question is how can an overall mixed state (CR and CTC) evolve to a pure state $|\Phi\rangle_{\mathrm{CR,CTC}}$ via unitary transformation. In either case we reach a contradiction.

- How to create a pure entangled state between a CR system and a CTC system?

- First create entanglement between two CR systems (in our world) and then swap half of the CR subsystem with a CTC system whose density matrix is same as the reduced density matrix of the CR system.

How to Create Entanglement?

• Let us consider two causality-respecting quantum systems in a pure entangled state $|\Psi\rangle_{\mathrm{CR,CR'}} = \sum_k \sqrt{p_k}|a_k\rangle_{\mathrm{CR}}|k\rangle_{\mathrm{CR'}}$. Let the initial state of $\rho_{\mathrm{CTC}} = \sum_k p_k|k\rangle_{\mathrm{CTC}}\langle k|$.

• Let the $\mathrm{CR'}$ and the CTC system undergo swap operation.

• After interaction the state of the CR system and the CTC system is in a pure entangled state $|\Psi\rangle_{\mathrm{CR,CTC}} = \sum_k \sqrt{p_k}|a_k\rangle_{\mathrm{CR}}|k\rangle_{\mathrm{CTC}}$ with the fixed point solution $\rho_{\mathrm{CTC}} = \sum_k p_k|k\rangle_{\mathrm{CTC}}\langle k|$.

Cloning of arbitrary state

- SWAP can clone an arbitrary state in the presence of CTC!

- Let $\rho_{\mathrm{CR}} = |\psi\rangle\langle\psi|$ and ρ is the state of the CTC system (blank state). They interact via a unitary $U = U_{\mathrm{SWAP}}$:
$|\psi\rangle\langle\psi| \otimes \rho \rightarrow U_{\mathrm{SWAP}}(|\psi\rangle\langle\psi| \otimes \rho)U_{\mathrm{SWAP}} = \rho \otimes |\psi\rangle\langle\psi|$

- By the kinematic consistency condition
$\rho'_{\mathrm{CTC}} = \rho = |\psi\rangle\langle\psi|$.

- Thus, $|\psi\rangle\langle\psi| \otimes \rho \rightarrow |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|$.

- It seems that CTC has knowledge about all possible states in the universe!

No-Hiding Theorem, CTC and Information Pop-up

• No-Hiding theorem: If the original information disappears from one subsystem, then it moves to the remainder of the Hilbert space with no information stored in the bipartite correlations.

• [Braunstein-Pati (2007)]: If $|\psi\rangle\langle\psi| \to \sigma = \sum_k p_k |k\rangle\langle k|$, then the unitary version must be $U(|\psi\rangle \otimes |A\rangle) = \sum_k \sqrt{p_k}|k\rangle \otimes |q_k\rangle \otimes |\psi\rangle$.

• In the presence of CTC, we will see that before information disappears from one subsystem, it can pop-up in CTC system!

- Up to SWAP operations, the hiding map in the presence of CTC can lead to popping of quantum information.

- Apply SWAP_{23}, SWAP_{12} and $I \otimes U_{23}$

$$|\psi\rangle\langle\psi|\otimes|A\rangle\langle A|\otimes\rho \rightarrow |\psi\rangle\langle\psi|\otimes\rho\otimes|A\rangle\langle A| \rightarrow \rho\otimes|\psi\rangle\langle\psi|\otimes|A\rangle\langle A| \rightarrow$$

$$\rho \otimes U(|\psi\rangle\langle\psi| \otimes |A\rangle\langle A|)U^\dagger$$

- Apply the hiding map:

$$U(|\psi\rangle \otimes |A\rangle) = \sum_k \sqrt{p_k}|k\rangle \otimes |q_k\rangle \otimes |\psi\rangle$$

- Final state of CR and CTC:
$\rho \otimes \sum_k \sqrt{p_k p_l} |k\rangle\langle l| \otimes |q_k\rangle\langle q_l| \otimes |\psi\rangle\langle\psi|$

- Kinematic condition implies:
$\rho_{\mathrm{CTC}}^{\mathrm{final}} = \rho_{\mathrm{CTC}}^{\mathrm{initial}} = \rho = \sum_k p_k |q_k\rangle\langle q_k| \otimes |\psi\rangle\langle\psi|$.

- Thus, the initial states of CTC has information about the input pure state $|\psi\rangle$. Before quantum information disappears it pops up in CTC system.

- CTC has knowledge about past, present and future of every quantum system!

Summary

- If a CTC mixed state interacts with a CR system and satisfies the kinematic consistency condition then it cannot be regarded as a subsystem of a pure entangled state.

- There is no universal ‘Church of the larger Hilbert space’ for CTC quantum system.

- In quantum theory with CTC there can be two kinds of mixtures. This reveals the true nature of the density operators in quantum theory with CTCs.

• In standard quantum theory, there is no way to distinguish a proper mixture from an improper.

• If CTC can help in distinguishing ‘proper’ from ‘improper’ mixtures then it may lead to signalling.

• In essence our result brings out a very fundamental and important difference between the density matrix of the CR system and the CTC system.

• Nature of entanglement between causality-respecting system and CTC system can be different.

• A swap operation in the presence of CTC can clone an arbitrary quantum state.

• CTC can help to erase quantum information completely.

• Up to SWAP operation, hiding map in the presence of CTC can lead to popping of quantum information.

• It seems CTC system has knowledge about every possible quantum state in the universe!

• Our results will add new insights to quantum information science in the presence of closed time like curves.

• *Purification of Mixed State with Closed Timelike Curve is not Possible*
A. K. Pati, I. Chakrabarty, P. Agrawal, arXiv:1003.4221 (2010).

• *Cloning and Popping of Quantum Information in the Presence of CTC*
A. K. Pati (2011).

Thank You