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A Note on Maximal Acceleration.

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Abstract. – It is found that the maximal acceleration of a quantum particle is directly related to the speed of transportation in the projective Hilbert space. The minimum space-time uncertainty is inversely proportional to the maximal acceleration of the particle. It is argued that the most accurate clocks are those which can be maximally accelerated.

It is said [1] that a geometric reformulation of quantum mechanics may lead to a new physical theory for proper description of gravity, consistent with quantum phenomena. As Minkowski viewed the special theory of relativity (STR) in a geometric way by relating it to the Riemannian geometry with zero curvature, so Einstein was enabled to view the general relativity as a Riemannian space-time geometry with nonzero curvature. A similar reformulation of quantum theory may relate to some quantum geometry of space-time which will describe the quantum theory of gravity. At this juncture we recall that the central theme of the STR is the existence of a unique velocity equal to both the propagation of light in the vacuum, and the limiting velocity for the nonzero mass particles. The fundamental assumption of STR is that light propagates rectilinearly in the vacuum with a constant speed c irrespective of inertial frames and with the help of the principle of causality it prohibits the possibility of travelling elementary particles with a velocity greater than that of light. The limiting speed c is very important in fixing the geometry of Minkowski space-time. In a similar way what would fix the geometric structure of the quantum space-time? A possible candidate for this may be the acceleration of the particle. But for this to be so, the next question naturally arises: is there an upper limit for the acceleration of the particle? Classically neither in STR nor in the general theory of relativity there exists any such limit.

Caianiello [2] has reported a few years ago that quantum-mechanical rules give rise to an upper bound for the acceleration of the particle. Assuming that the particles are «extended objects» with λ being the linear dimension of the particle, he found that the maximal acceleration goes as c^2/λ. In a more general scheme the author [3] has shown that, if the state of the quantum system is Gaussian in nature and consequently minimum position-momentum uncertainty relation is satisfied, then the acceleration of the particle is equal to $c^2/\Delta x$, where Δx is the uncertainty in the position coordinate of the particle. Various consequences of the above result have been discussed in ref. [3]. Also it has been shown that when a charged (either electrically or gravitationally) particle undergoes acceleration, the energy loss goes as the square of the maximal acceleration.

This note brings an interesting fact, where we show that the maximal acceleration of the particle is related to the magnitude of the velocity of transportation in projective Hilbert space $\mathscr{P}$. These at first glance seem to be two unrelated things. The speed of transportation gives the rate at which the state of the system moves in the projective Hilbert space $\mathscr{P}$ at a given time.

Let S be the quantum system characterised by the normalised vectors $\Psi \in H$. If $\hat{A}$ and $\hat{B}$ are any two physical observables corresponding to the system S, it is well known [4] that

$$\Delta A \,\Delta B \geqslant \frac{1}{2} \left|\langle \psi \,|[A,B]|\, \psi\rangle\right| \tag{1}$$

for $|\psi\rangle$, $\hat{A}|\psi\rangle$ and $\hat{B}|\psi\rangle \in D(A) \cap D(B)$.

For $\hat{B}$ to be the Hamiltonian of the system, we have

$$\Delta A \,\Delta H \geqslant \frac{1}{2} \left|\langle \psi(t) |[A,H]|\, \psi(t)\rangle\right| . \tag{2}$$

If $\hat{A}$ is an observable of the system that does not depend on time explicitly, then (2) can be written as

$$\Delta A \,\Delta H \geqslant \frac{\hbar}{2} \left| \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi(t) |A|\, \psi(t)\rangle \right| . \tag{3}$$

Next taking $\hat{A}$ to be the velocity operator of the particle defined through the equation $\hat{v} = (1/i\hbar)[\hat{x}, \hat{H}]$, we have

$$\Delta v \,\Delta H \geqslant \frac{\hbar}{2} \left| \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi \,|v|\, \psi\rangle \right| . \tag{4}$$

The quantity on the right-hand side of the above equation is of importance here, because it is the acceleration of the particle if we make use of the Ehrenfest theorem. Therefore

$$\Delta v \,\Delta H \geqslant \frac{\hbar}{2} a, \qquad \text{or} \qquad a \leqslant \frac{2}{\hbar} \Delta v \,\Delta H, \tag{5}$$

where $a = \left|(\mathrm{d}/\mathrm{d}t)\langle \psi \,|v|\, \psi\rangle\right|$.

From the special theory of relativity the limiting speed of any particle ensures that the uncertainty in the velocity of the particle cannot exceed its maximum attainable velocity [5], *i.e.*

$$\Delta v \leqslant v_{\max} \leqslant c .$$

Therefore we arrive at the relation

$$a \leqslant \frac{2}{\hbar} c \,\Delta H . \tag{6}$$

We want to give a geometric meaning to the above inequality. From the study of geometric aspects of quantum evolution [6], we know that if $|\psi(t)\rangle$ satisfies Schrödinger equation and evolves according to it, then after an infinitesimal time $\mathrm{d}t$ the state is at $|\psi(t + \mathrm{d}t)\rangle$ such that the infinitesimal normed distance function [7] between these two states in $\mathscr{P}$ is

$$\mathrm{d}D(\psi(t), \psi(t+\mathrm{d}t)) = (2 - 2\,|\langle\psi(t)\,|\,\psi(t+\mathrm{d}t)\rangle|)^{1/2}. \tag{7}$$

This induces a Reimannian metric on the manifold of quantum states. Equation (7) can be simplified by Taylor expanding $|\,\psi(t+\mathrm{d}t)\rangle$ up to second order and observing that

$$|\langle\psi(t)\,|\,\psi(t+\mathrm{d}t)\rangle| = 1 - \frac{1}{2}\mathrm{d}t^2\,\frac{\Delta H^2}{\hbar^2} + O(\mathrm{d}t^3), \tag{8}$$

where $\Delta H^2 = \langle\psi\,|H^2\,|\,\psi\rangle - \langle\psi\,|\,H\,|\,\psi\rangle^2$. Then it is easy to see that $\mathrm{d}D$ is given by

$$\mathrm{d}D = \Delta H\frac{\mathrm{d}t}{\hbar}. \tag{7a}$$

This distance function is a geometric quantity, because it is independent of the particular Hamiltonian used to transport the system along a given curve in $\mathscr{P}$, but depends only on the fluctuation in the energy. The useful quantity, the speed of transportation is defined by

$$v_H = \frac{\mathrm{d}D}{\mathrm{d}t} = \frac{\Delta H}{\hbar}. \tag{9}$$

The evolution of the system in the projective Hilbert space completely determines ΔH and hence v_H. Detailed information of the Hamiltonian is not required, which assigns a geometric meaning to the quantity v_H. Therefore from (6) we find

$$a \leqslant 2cv_H. \tag{10}$$

Thus the maximal acceleration that a particle can undergo is limited by the speed of transportation in $\mathscr{P}$ and is given by

$$a_{\max} = 2cv_H. \tag{11}$$

This proves what we have promised earlier. Equation (11) is a very important result although its derivation is an elementary one. The maximal acceleration is a geometric property of the evolution of the quantum system. Since it is independent of the particular Hamiltonian, this means that the maximal acceleration is independent of the generator of the time translation. The Hamiltonian of the system governs the dynamics, so it takes care of the environment in which the system is embedded. Our result shows that in a changing environment we can accelerate a particle to a particular value by keeping the dispersion in energy at a constant value. An important theorem follows from eq. (11).

Theorem: if the wave function of the whole system is an eigenfunction of the total Hamiltonian, then the system cannot be accelerated.

Since v_H is zero if the quantum state is in the eigenstate of the Hamiltonian, the maximal acceleration is also zero and hence the system cannot be accelerated. This fact is consistent with the known results of quantum theory. This gives a conceptual understanding of Bohr's postulate for hydrogen atoms. The electrons in the hydrogen atom as long as they are in the eigenstates do not radiate energy. The reason being that their acceleration is zero and hence they cannot radiate energy.

Finally we wish to give a lower bound to the uncertainty of the space-time geodesic. It will be shown that the minimum uncertainty in the space-time sheet is inversely proportional to the maximal acceleration of the test particle (a quantum-mechanical clock). Towards the end we answer a very subtle question: which kind of clocks are the most accurate ones?

Consider a net of timelike geodesic which is as tight as possible and uncertainty in space-time measurements can be performed purely via time measurements on timelike geodesic [8]. The clock which measures time can be realised by considering a set of test particles subject to quantum-mechanical laws. In what follows we define the uncertainty in the space-time s as

$$\Delta s = \left| \frac{\mathrm{d}s}{\mathrm{d}x} \right| \Delta x, \tag{12}$$

where $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ is the uncertainty spatial coordinate of the particle. On using eq. (3) and $\hat{A}$, the position coordinate of the particle, we have

$$\Delta x \, \Delta H \geqslant \frac{\hbar}{2} \left| \frac{\mathrm{d}\langle x \rangle}{\mathrm{d}t} \right|. \tag{13}$$

Defining $|\mathrm{d}\langle x \rangle / \mathrm{d}t| = v$, the velocity of the particle we have

$$\Delta x \geqslant \frac{\hbar}{2} \frac{v}{\Delta H}. \tag{14}$$

Therefore

$$\Delta s \geqslant \left| \frac{\mathrm{d}s}{\mathrm{d}x} \right| \frac{\hbar}{2} \frac{v}{\Delta H}. \tag{12a}$$

On writing $|\mathrm{d}s/\mathrm{d}x| = (1/v)|\mathrm{d}s/\mathrm{d}t|$, where $\mathrm{d}s/\mathrm{d}t = [g_{\mu\nu}(\mathrm{d}x^{\mu}/\mathrm{d}t)(\mathrm{d}x^{\nu}/\mathrm{d}t)]^{1/2}$ and $g_{\mu\nu}$ is the Reimannian metric tensor, we have

$$\Delta s \geqslant \frac{1}{2} \left| \frac{\mathrm{d}s}{\mathrm{d}t} \right| \frac{1}{v_H}. \tag{15}$$

Thus eq. (15) specifies the uncertainty in space-time and the minimum uncertainty in s is given by

$$\Delta s_{\min} = \frac{1}{2} \left| \frac{\mathrm{d}s}{\mathrm{d}t} \right| \frac{1}{v_H}, \tag{15a}$$

which is inversely proportional to the speed of transportation in the projective Hilbert space $\mathscr{P}$. Making use of eq. (11) we can rewrite eq. (15a) as

$$\Delta s_{\min} = \left| \frac{\mathrm{d}s}{\mathrm{d}t} \right| \frac{c}{a_{\max}}. \tag{16}$$

Hence, if the test particle can be accelerated to a much higher value, then the minimum uncertainty in space-time can be reduced to a greater extent. To answer the above-raised question, we write eq. (15) in the following form:

$$\frac{\Delta s}{|\mathrm{d}s/\mathrm{d}t|} \geqslant \frac{1}{2v_H} \quad \text{or} \quad r_s \geqslant \frac{c}{a_{\max}}, \tag{17}$$

where $r_s = \Delta s / |\mathrm{d}s/\mathrm{d}t|$.

Here r_s is the inaccuracy with which the clock will measure the time, while moving along the geodesic. Thus it is clear that the most accurate clocks are those which can be accelerated to a greater extent, because in that case the inaccuracy in time measurement is lowered.

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