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Relation between "phases" and "distance" in quantum evolution

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We discuss the dynamical phase and the geometric phase in relation to the geometric distance function for cyclic evolution of quantum states. For all cyclic evolution of quantum states, we have shown that the non-adiabatic geometric phase is the integral of the contracted length of the curve which the system traverses.

1. Introduction

In studying the geometric aspects of quantum evolution, two things are relevant in recent years. One is the geometrical phase discovered by Berry [1] and second is the geometric distance formulated by Anandan and Aharonov [2]. Berry made a rather striking and perceptive observation in the quantal adiabatic theorem [3]. He found that for circuital adiabatic excursions of the system in parametric space the wave function acquires a non-integrable phase which depends on the geometry of the circuit and on the eigenstates under consideration in addition to the usual dynamical phase. In the same year, Simon [4] explained that this geometric phase could be viewed as a consequence of parallel transport of vectors in a curved space appropriate to the quantum system. It appears naturally as due to holonomy in a line bundle over the parameter space. Later on, Aharonov and Anandan (AA) [5] generalised the Berry phase by defining it for any cyclic evolution of the quantum system. The importance of the AA formulation is that it is applicable irrespective of the cyclic and adiabatic condition that is imposed on the Hamiltonian of the system. Furthermore, Samuel and Bhandari [6] formulated Berry's phase for the case of non-unitary and non-cyclic evolution of states by employing Pancharatnam's [7] idea of comparing the phases of two arbitrary polarised light rays based on their interference. Numerous experiments have also supported Berry's profound discovery in the last half decade.

Turning to the second geometric quantity, it is Anandan and Aharonov who have concretised the formulation of the distance function in the projective Hilbert space and given a geometric meaning to it (although various authors [8–11] in the past have defined similar quantities in the literature). It is the distance between quantum states along a given curve C in projective Hilbert space $\mathscr{P}$ as measured by the Fubini–Study metric defined from the inner product of the representative states in the Hilbert space $\mathscr{H}$. It is equal to the time integral of the uncertainty of the energy, and is geometric in the sense that this distance is independent of the particular Hamiltonian used to move the quantum system along a given curve in $\mathscr{P}$; it depends only on the points in $\mathscr{P}$ to which they project.

In this Letter our purpose is to discuss the dynamical phase and the geometric phase in relation to the distance function. We present a new expression for the geometric phase in the case of non-adiabatic and arbitrary cyclic evolution of quantum states. An example is studied to realise our new expression.

2. Dynamical phase and distance function

Let $\{\psi(\lambda)\}$ be a set of normalised vectors belonging to a Hilbert space $\mathscr{H}$ of dimension $N+1$. The λ's form an n-dimensional parameter space $\boldsymbol{\lambda}=(\lambda_1, \lambda_2,$

..., $\lambda_n) \in \mathbb{R}^n$ on which $\{\psi(\lambda)\}$ depends smoothly. Then we can define a projective Hilbert space with dimension one less, i.e. N. It consists of a set of rays of the Hilbert space $\mathscr{H}$, where the rays are defined as the equivalence classes of states differing only in phase. The equivalence relation is $|\psi(\lambda)\rangle \sim |\psi'(\lambda)\rangle$ if $|\psi'(\lambda)\rangle = c|\psi(\lambda)\rangle$ where $0 \neq c \in \mathbb{C}^*$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$ is a multiplicative group of non-zero complex numbers. The projective Hilbert space is $\mathscr{P} = \mathscr{P}_N(\mathbb{C}) = \{\mathscr{H} - \{0\}\}/\mathbb{C}^*$, which is the quantum state space; vectors in $\mathscr{H}$ are projected onto $\mathscr{P}_N(\mathbb{C})$. Physical states are elements of $\mathscr{P}$ and represented as points in $\mathscr{P}$.

Given any two vectors $|\psi_1\rangle = |\psi(\lambda_1)\rangle$, $|\psi_2\rangle = |\psi(\lambda_2)\rangle \in \mathscr{H}$ we can define a distance function from the inner product of vectors in $\mathscr{H}$ as a "physically natural" topology [8] in the set of quantum states and this topology is specified by a numerical-valued, non-negative function $D(\psi_1, \psi_2)$ which measures the distance between states $|\psi_1\rangle$ and $|\psi_2\rangle$. In a mathematical sense it is a metric. It satisfies the identity, symmetry, and triangle axioms. A useful way of defining it is

$$D(\psi_1, \psi_2) = \|\psi_1 - e^{-i\phi}\psi_2\| , \tag{1}$$

and if it is minimised by equating its first derivative with respect to ϕ to zero then we call it the minimum-normed distance function. The value of the arbitrary phase for which the distance function is minimised is called the "minimum phase" and is given by

$$\exp(i\phi_{\min}) = \frac{\langle\psi_1|\psi_2\rangle}{|\langle\psi_1|\psi_2\rangle|} . \tag{2}$$

Thus the minimum-normed distance function which is an appropriate measure of the distance between two quantum states reads as

$$D(\psi_1, \psi_2) = \min_{\phi} \|\psi_i - e^{-i\phi}\psi_2\| .$$

With the help of eq. (2) we can write $D(\psi_1, \psi_2)$ as [9–11]

$$D(\psi_1, \psi_2) = (2 - 2|\langle\psi_1|\psi_2\rangle|)^{1/2} . \tag{3}$$

Eq. (3) gives the minimum realisation for all propositions of the absolute values of the difference between probabilities predicted in the state $|\psi_1\rangle$ and $|\psi_2\rangle$. The meaning of the distance function is that given a state a priori, it defines how close we can then determine the state a posteriori. In particular the distance between two vectors which are infinitesimally close induces a Riemannian metric. Taking $|\psi_1\rangle = |\psi(\lambda)\rangle$ and $|\psi_2\rangle = |\psi(\lambda + d\lambda)\rangle$ and Taylor expanding $|\psi(\lambda + d\lambda)\rangle$ up to second order, we have the metric tensor associated with the infinitesimal distance function. It is given by [11]

$$dD^2(\psi(\lambda), \psi(\lambda + d\lambda)) = g_{ij}\, d\lambda_i\, d\lambda_j , \tag{4}$$

where

$$g_{ij} = \mathrm{Re}\,(\langle\partial_i\psi|\partial_j\psi\rangle) - (i\langle\psi|\partial_i, \psi\rangle)(i\langle\psi|\partial_j\psi\rangle) . \tag{5}$$

This metric is gauge invariant and preserves isometry; hence it is an important quantity in studying the natural geometric structure of the underlying manifold of quantum states.

If we consider a quantum state smoothly dependent on one parameter and let $|\psi(\lambda)\rangle$ be obtainable from $|\psi(0)\rangle$ by a one-parameter family of unitary transformation, then

$$|\psi(\lambda)\rangle = \mathrm{P}\exp\left(i\int_0^{\lambda} A(\lambda')\, d\lambda'\right)|\psi(0)\rangle . \tag{6a}$$

In eq. (6a) P denotes path ordering and $A(\lambda)$ is a Hermitian operator which is the generator of the corresponding parameter. An infinitesimal parametric evolution equation can be written (from (6a)) as

$$i\frac{d}{d\lambda}|\psi(\lambda)\rangle = A(\lambda)|\psi(\lambda)\rangle . \tag{6b}$$

Eq. (4) yields

$$dD^2 = \Delta A^2(\lambda)\, d\lambda^2 , \tag{7}$$

where $\Delta A\ (\lambda)$ is the uncertainty in A. Therefore, the distance along a line in $\mathscr{P}$, as measured by the metric (4) is

$$D = \int \Delta A(\lambda)\, d\lambda . \tag{8}$$

This distance is the same for an infinite number of $A(\lambda)$'s which generate the motion in $\mathscr{P}$.

If a quantum evolution is considered where the state vector at every instant of time belongs to a Hilbert space $\mathscr{H}$ and the evolution of the state $|\psi(t)\rangle$

is governed by the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle , \tag{9}$$

where $H(t)$ is a general time dependent Hamiltonian, after an infinitesimal time dt, the state is $|\psi(t+dt)\rangle$ such that the infinitesimal minimum-normed distance dD between $|\psi(t)\rangle$ and $|\psi(t+dt)\rangle$ is given by

$$dD = [2-2|\langle\psi(t)|\psi(t+dt)\rangle|]^{1/2} .$$

This can be easily evaluated either by directly Taylor expanding $|\psi(t+dt)\rangle$ up to second order and noting that

$$|\langle\psi(t)|\psi(t+dt)\rangle| = 1 - \tfrac{1}{2}\Delta E^2(t)\, dt^2/\hbar^2 + O(dt^3) ,$$

and

$$\Delta E^2(t) = \langle\psi(t)|H^2(t)|\psi)t)\rangle - [\langle\psi(t)|H(t)|\psi(t)\rangle]^2 ,$$

or by using (8) and choosing $\lambda = t$ and $A(\lambda) = H(t)/\hbar$. Thus

$$dD = \Delta E(t)\, dt/\hbar, \quad D = \int \Delta E(t)\, dt/\hbar , \tag{10}$$

which differs from the distance between $\Pi(|\psi(t)\rangle)$ and $\Pi(|\psi(t+dt)\rangle)$ along the curve $\hat{C}$ in $\mathscr{P}$ as measured by the Fubini-study metric only by a numerical factor of two [2]. This distance function is independent of the particular Hamiltonian used to transport the state along a given curve $\hat{C}$ in $\mathscr{P}$. That is to say, on the manifold of quantum states, once the fluctuation in the energy is specified, we can fix the metric even though the specification to the fluctuation in the energy comes from an infinite number of possible Hamiltonians.

To see how the dynamical phase arises in the context of the distance function we go back to (2). From (2) we can argue that the infinitesimal phase associated with the state in transporting a quantum system along a one-parameter family of curves in $\mathscr{H}$ generated by the Hamiltonian, and minimising the distance function, is given by

$$\exp(i\, d\phi_{\min}) = \frac{\langle\psi(t)|\psi(t+dt)\rangle}{|\langle\psi(t)|\psi(t+dt)\rangle|} . \tag{11}$$

The phase $\phi = \phi(t)$ is a sufficiently smooth function of time and it keeps track of the passage of time during the evolution of the state vector. Expanding both sides of (11) up to first order, and noting that the denominator on the right hand side is $1+O(dt^2)$, we have

$$d\phi_{\min} = -\frac{1}{\hbar}\langle\psi(t)|H(t)|\psi(t)\rangle\, dt .$$

During the evolution from an initial state at $t=0$ to a final state at $t=T$ one can integrate both sides to get

$$\phi_{\min}(T) = \phi_{\mathrm{dyn}} = -\frac{1}{\hbar}\int_0^T \langle\psi(t)|H(t)|\psi(t)\rangle\, dt , \tag{12}$$

which is just the dynamical phase. Thus, the dynamical phase can be interpreted as the value of the arbitrary phase in the final state for which the normed distance function is minimised.

3. Geometric phase and distance function

This section is motivated by a remark by Anandan and Aharonov [2]: "we expect it (the distance function) to have a geometric meaning analogous to the geometric phase for closed curve in $\mathscr{P}$." We want to see the extent to which this analogy can be carried further.

Before drawing the analogy, we remark on the basis for it. The dynamics of the system governed by the Hamiltonian $H(t)$ cannot change the geometry of the projective Hilbert space $\mathscr{P}$, as the geometry of the state space is signatured by the metric defined on it. Since the metric is fixed by the dispersion in the generator of the motion $\Delta E(t)$, it is possible in principle to have the same $\Delta E(t)$ by a variety of Hamiltonians that generate the motion. On the other hand, the time dependent physical states in $\mathscr{H}$ are the elements of $\mathscr{P}$, and trace out a closed curve $\hat{C}$ in $\mathscr{P}$ during evolution. In a given metric on a manifold of quantum states, for all cyclic evolutions, there develops a geometric object called the geometric phase (AA phase) in projective Hilbert space. It depends only on the path traced by the rays in $\mathscr{P}$ and does not depend on the rate of traversal of the path. This

phase is also independent of $H(t)$ for a given closed curve $\hat{C}$ in $\mathscr{P}$. As we know in many physical problems the observables of importance are those which remain invariant under a gauge transformation. In this context, the geometric phase and distance are, in fact, gauge invariant. In addition to this, they are invariant under unitary transformations. From all these facts, it appears natural to ask whether some possible link can be established between them. The result of our query is that we are able to show that the non-adiabatic Berry phase for all cyclic evolution (i.e. the AA phase) is the integral of the contracted length of the curve that is traced out by $|\psi(t)\rangle$ in $\mathscr{P}$.

Consider a quantum state that evolves according to (9) and for all time, $|\psi(t)\rangle \in \mathscr{H}$ with $\langle\psi|\psi\rangle = 1$. Then, we can define a cyclic vector for the evolution equation if there is a cyclic time T such that an initial state and a final state differ by a phase factor Φ, i.e.

$$|\psi(T)\rangle = e^{i\Phi}|\psi(0)\rangle .$$

The existence of a cyclic state is assured by the very fact that it is an eigenvector of the unitary evolution operator $U(T)$ and the corresponding eigenvalue is $e^{i\Phi}$. Following Anandan and Aharonov [5], let there be a natural projection map in $\mathscr{P}$, $\Pi: \mathscr{H} \to \mathscr{P}$ defined by $\Pi(|\psi\rangle) = \{|\psi'\rangle : |\psi'\rangle = c|\psi\rangle$, for any complex number $c\}$. Then the cyclic evolution of the state describes a curve $\hat{C}$, $t \to \psi(t)$ in $\mathscr{H}$ that begins and ends on the same ray. That is to say $C: [0, T] \to \mathscr{H}$, with $\hat{C} = \Pi(C)$ being a closed curve in $\mathscr{P}$. Also, $\hat{C}$ is the image of the curve C under the projection map Π. Now a modified state is defined by choosing from the curve $|\psi(t)\rangle$, so that the cyclic vector is easily converted into a periodic form (i.e. a vector which returns to itself after a time T):

$$|\tilde{\psi}(t)\rangle = e^{-if(t)}|\psi(t)\rangle ,$$

with $\langle\tilde{\psi}|\tilde{\psi}\rangle = 1$ and $f(t)$ is any smooth function satisfying $f(T) - f(0) = 0$. Then it trivially follows

$$|\tilde{\psi}(T)\rangle = |\tilde{\psi}(0)\rangle .$$

It has been shown by Anandan and Aharonov that if one identifies

$$-\frac{1}{\hbar}\int_0^T \langle\psi(t)|H(t)|\psi(t)\rangle \, dt$$

as the dynamical phase and removes it from the total phase Φ, then one gets the so-called "non-adiabatic Berry phase" (AA phase),

$$\beta = i\int_0^T \langle\tilde{\psi}(t)|\dot{\tilde{\psi}}(t)\rangle \, dt . \tag{13}$$

The geometric quantity β is independent of the phase that relates the initial state and the final state and of the Hamiltonian for a given projection of the evolution in $\mathscr{P}$. In deriving the expression for β, AA have neither used the adiabatic theorem nor the cyclicity condition of the Hamiltonian. However the cyclic evolution of the state is necessary for its derivation.

Here, we seek to derive the non-adiabatic Berry phase from metric considerations. We emphasize that the distance function (in fact, any quadratic, positive definite function) is of pivotal importance in discussing the geometry of quantum evolution.

Define a quadratic, positive definite function in $\mathscr{P}$ as

$$\Omega^2(t) = \| i|\dot{\tilde{\psi}}(t)\rangle - \dot{\beta}(t)|\tilde{\psi}(t)\rangle \|^2 , \tag{14}$$

where $\dot{\beta}(t)$ is an arbitrary, real frequency. It is easy to see that if we minimise $\Omega(t)$ with respect to an arbitrary frequency $\dot{\beta}(t)$, then the minimum value of $\dot{\beta}$ for which Ω is minimised is given by

$$\dot{\beta}_{\min}(t) = i\langle\tilde{\psi}(t)|\dot{\tilde{\psi}}(t)\rangle , \tag{15}$$

and it may be called the Berry frequency. Hence

$$\beta = \int_0^T \dot{\beta}_{\min}(t) \, dt = i\int_0^T \langle\tilde{\psi}(t)|\dot{\tilde{\psi}}(t) \, dt \tag{16}$$

is the geometric phase as given by (13).

To assign a meaning to the quantity $\Omega(t)$, we can take (see eq. (20)) $\dot{\beta}_{\min}$ to be of the form $\dot{f}(t) + \langle H(t)\rangle/\hbar$, i.e., $\dot{\beta}_{\min}$ is a real frequency with the dynamical frequency removed from $\dot{f}(t)$. Then, Ω is equal to $\Delta E(t)/\hbar$. It has a physical meaning which is paraphrased as the magnitude of the velocity of transportation (speed of transportation) in $\mathscr{P}$ (apart from a factor of two in our convention) by Anandan and Aharonov. Therefore, the geometric phase appears as a result of the principle of minimisation of speed of transportation in $\mathscr{P}$. The quantity β is the time integral of the Berry frequency of which the speed of transportation is minimised during the evolution from time $t = 0$ to $t = T$, whereas the geometric

distance is just the time integral of $\Omega(t)$ during the evolution. In this sense the geometric phase and distance seem to be interrelated.

In the sequel we recognise yet another geometric quantity, which is referred as the "length of the curve" along which the quantum system is transported. Given a closed curve $\hat{C}$ in $\mathscr{P}$, we define the length of the curve in $\mathscr{P}$ which is traced out by the normalised vector $|\psi(t)\rangle$. Such a projective Hilbert space on which one has singled out a specific metric, it is easy (though not trivial) to relate the length and distance during a quantum evolution via the geometric phase. Below we define the length of the curve.

Definition. Let $\psi(t)$ be a curve C: $[t, T]\to\mathscr{H}$. Then choose a section of the curve as $\tilde{\psi}$ which is differentiable along C such that the length of the curve $\tilde{\psi}(t)$ along which the system evolves from point $\tilde{\psi}(0)$ to a point $\tilde{\psi}(T)$ (or from a parameter value $t=0$ to $t=T$) is a number defined as

$$l(\tilde{\psi})|_0^T=\int_0^T\langle\dot{\tilde{\psi}}(t)|\dot{\tilde{\psi}}(t)\rangle^{1/2}\,\mathrm{d}t\,. \tag{17}$$

$|\dot{\tilde{\psi}}\rangle$ is the velocity vector in projective Hilbert space $\mathscr{P}$ of the curve $\tilde{\psi}$ at point t along the path of evolution of the state vector. It is the tangent vector to the curve $\tilde{\psi}(t)$.

The integral (17) exists, since the integrand is continuous. The length of a broken C curve is defined as the (finite) sum of the length of its C pieces. The number $l(\tilde{\psi})|_0^T$ is independent of the parametrisation of its image set, i.e. for a smooth transformation from parameter t to τ where $\mathrm{d}t/\mathrm{d}\tau>0$, the length of the curve remains unaltered. Therefore, the length of the curve is a property of the whole curve and is t-invariant. Hence for an arbitrary parametric evolution of the state, we can define the infinitesimal length of the curve during the infinitesimal time, $\mathrm{d}t$, as

$$\mathrm{d}l=\langle\dot{\tilde{\psi}}(t)|\dot{\tilde{\psi}}(t)\rangle^{1/2}\,\mathrm{d}t\,. \tag{18}$$

Proposition. For an arbitrary cyclic evolution of a quantum system any representative physical state traces a closed curve in $\mathscr{P}$ (generally an open curve in $\mathscr{H}$) such that (i) at each instant of time, $\mathrm{d}l^2$ is greater than $\mathrm{d}D^2$ and (ii) β, the geometric phase, is manifested as the integral of the contracted length of the curve C in $\mathscr{P}$.

Proof. This can be easily seen by directly evaluating $\mathrm{d}l^2$, the square of the infinitesimal length of the curve. Differentiating $|\tilde{\psi}(t)\rangle$ with respect to time (from $|\tilde{\psi}(t)\rangle=\mathrm{e}^{-\mathrm{i}f(t)}|\psi(t)\rangle$) we have

$$\mathrm{d}l^2=\langle\dot{\tilde{\psi}}|\dot{\tilde{\psi}}\rangle\,\mathrm{d}t^2=\dot{f}^2\,\mathrm{d}t^2+\frac{1}{\hbar^2}\langle\psi|H^2|\psi\rangle\mathrm{d}t^2$$

$$+2\frac{\dot{f}}{\hbar}\langle\psi|H|\psi\rangle\,\mathrm{d}t^2\,. \tag{19}$$

On using

$$\dot{f}(t)=-\frac{1}{\hbar}\langle\psi(t)|H(t)|\psi(t)\rangle+\mathrm{i}\langle\tilde{\psi}(t)|\dot{\tilde{\psi}}(t)\rangle\,, \tag{20}$$

we can write $\mathrm{d}l^2$ as

$$\mathrm{d}l^2=\frac{1}{\hbar^2}\left[\langle\psi|H^2|\psi\rangle-(\langle\psi|H|\psi\rangle)^2\right]\mathrm{d}t^2$$

$$+(\mathrm{i}\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle\,\mathrm{d}t)^2\,.$$

Making use of the definition of $\mathrm{d}D^2$ we finally arrive at

$$\mathrm{d}l^2-\mathrm{d}D^2=[\mathrm{i}\langle\tilde{\psi}(t)|\dot{\tilde{\psi}}(t)\rangle\,\mathrm{d}t]^2\,. \tag{21}$$

Because of the normalisation of $|\tilde{\psi}(t)\rangle$ for all time, the quantity $\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle$ is purely imaginary and hence $\mathrm{i}\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle$ is real. In view of the above fact, $\mathrm{d}l^2-\mathrm{d}D^2>0$; proving thereby the first part of the proposition. To prove (ii), we simply note that during a cyclic evolution of the state vector, the geometric phase factor acquired by the system is

$$\beta=\mathrm{i}\int_0^T\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle\,\mathrm{d}t=\int_0^T\sqrt{\mathrm{d}l^2-\mathrm{d}D^2}$$

$$=\int_0^T\sqrt{1-v_{\mathscr{H}}^1/u_{\mathscr{H}}^2}\,\mathrm{d}l\,,$$

$$\beta=\int_0^T\mathrm{d}L\,, \tag{22}$$

where $\mathrm{d}L=\sqrt{1-v_{\mathscr{H}}^2/u_{\mathscr{H}}^2}\,\mathrm{d}l$ and we call it the infinitesimal contracted length of the curve. Thus the non-adiabatic Berry phase is the integral of the contracted length of the curve. The quantity $v_{\mathscr{H}}=\mathrm{d}D/\mathrm{d}t$ has been defined earlier and $u_{\mathscr{H}}=\mathrm{d}l/\mathrm{d}L$ is referred to as the magnitude of rate of change of length along

the curve C. The expression (22) is the main result of this paper. It gives a new insight into the Berry phase. Since the Berry phase depends on two objects in $\mathscr{P}$, namely the length of the curve (a t-invariant quantity) and the distance function (a $H(t)$-invariant quantity), we expect it to naturally depend on the structure of the curve $\hat{C}$ in $\mathscr{P}$. We believe that the expression (22) is more geometric than any other one previously known. It provides an explicit relation between geometric phase and the topology of the curve. During the cyclic evolution, we may regard the excess length of the curve at each instant of time over the distance as accumulating, so that it finally appears as the geometric phase. It is important to note that our interpretation of the non-adiabatic Berry phase in terms of the integral of the contracted length of the curve is independent of the cyclicity of the Hamiltonian and only depends on the cyclicity of the quantum states.

To add a little, we remark that the proof of our proposition also stems from the observation of (14), (15) and (20). After minimising the quantity Ω^2 with respect to $\dot{\beta}$ and using $\dot{\beta}=\dot{f}(t)+\langle H(t)\rangle/\hbar$ is nothing but $\Delta E^2(t)/\hbar^2$, and this in turn again equals $\langle\dot{\tilde{\psi}}|\dot{\tilde{\psi}}\rangle-(\mathrm{i}\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle)^2$ on using $\dot{\beta}=(\mathrm{i}\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle)$. Hence we get (21) and the proof follows.

To validate our interpretation, we supplement it by one more special case where we choose the Hamiltonian to be periodic and the state to be cyclic and obtain an expression similar to that of (22). Also, we use explicitly (22) to calculate the geometric phase for a spin-$\frac{1}{2}$ particle precessing in a magnetic field.

We work in a finite dimensional Hilbert space. Then once the basis is chosen, the quantum state will evolve according to eq. (9) which will be a matrix equation, where H is an $n\times n$ square matrix and $|\psi(t)\rangle$ is a column vector of $n\times 1$. Since the system evolves unitarily, the evolution operator will satisfy the equation

$$\mathrm{i}\hbar\frac{\mathrm{d}U(t)}{\mathrm{d}t}=H(t)U(t)\,. \tag{23}$$

Let us consider a periodic Hamiltonian $H(t)$ with some period $t=T$, i.e. $H(t+T)=H(t)$; then via Floquet's [12,13] theorem, any fundamental matrix of a linear system of the ordinary differential equation (23) can be put in a form

$$U(t)=V(t)\exp(\mathrm{i}Mt)\,,$$

where $U(t)$ is the matrix representation of the time evolution operator and $|\psi(t)\rangle=U(t)|\psi(0)\rangle$. Also U, V and M are $n\times n$ square matrices and V is unitary since U is unitary with M being a real constant. V has the property that $V(0)=V(T)=1$, i.e., periodic in T. One can extract the final state by

$$\begin{aligned}|\psi(T)\rangle&=U(T)|\psi(0)\rangle=\mathrm{e}^{\mathrm{i}\Phi}|\psi(0)\rangle\\&=\mathrm{e}^{\mathrm{i}mT}|\psi(0)\rangle\,,\end{aligned} \tag{24}$$

i.e., $|\psi(0)\rangle$ is an eigenvector of $U(t)$ with eigenvalue $\Phi=mT$ and Φ is real. This shows that the initial state returns to itself after a time $t=T$ modulo a phase factor of Φ, a property which depends only on the unitarity of $U(t)$ and not on the finite dimensionality of $\mathscr{H}$ and the cyclic properties of the Hamiltonian.

To see how the expression (22) comes about in this framework we evaluate the distance function as given by (10). We have

$$\begin{aligned}\mathrm{d}D^2=\frac{1}{\hbar^2}\,[&\langle\psi(0)|U^\dagger H^2U|\psi(0)\rangle\\&-\langle\psi(0)|U^\dagger HU|\psi(0)\rangle^2]\,\mathrm{d}t^2.\end{aligned}$$

Now

$$U^\dagger HU=\mathrm{i}\hbar U^\dagger\frac{\mathrm{d}U}{\mathrm{d}t}=\mathrm{e}^{-\mathrm{i}Mt}(\mathrm{i}\hbar V^\dagger\dot{V}-\hbar M)\mathrm{e}^{\mathrm{i}Mt}\,, \tag{25}$$

and

$$\begin{aligned}U^\dagger H^2U&=(U^\dagger HU)^\dagger(U^\dagger HU)\\&=\mathrm{e}^{-\mathrm{i}Mt}(\hbar^2\dot{V}^\dagger\dot{V}+\hbar^2M^2+\mathrm{i}\hbar^2M\dot{V}^\dagger V\\&\quad-\mathrm{i}\hbar^2MV^\dagger\dot{V})\mathrm{e}^{\mathrm{i}Mt}\,.\end{aligned}$$

Therefore

$$\begin{aligned}&\langle\psi(t)|H(t)|\psi(t)\rangle\\&\quad=\langle\psi(0)|\mathrm{e}^{-\mathrm{i}Mt}(\mathrm{i}\hbar V^\dagger\dot{V}-\hbar M)\mathrm{e}^{\mathrm{i}Mt}|\psi(0)\rangle\\&\quad=\mathrm{i}\hbar\langle\tilde{\psi}(t)|\dot{\tilde{\psi}}(t)\rangle-\hbar m,\end{aligned} \tag{26}$$

where we have used the fact that $|\psi(0)\rangle$ is an eigenstate of M with eigenvalue m, and $|\tilde{\psi}(t)\rangle=V(t)|\psi(0)\rangle$.

Next we evaluate the term $\langle\psi|H^2|\psi\rangle$:

$$\langle\psi|H^2|\psi\rangle=\langle\psi(0)|U^\dagger H^2 U|\psi(0)\rangle$$

$$=\hbar^2\langle\dot{\tilde{\psi}}|\dot{\tilde{\psi}}\rangle+\hbar^2m^2+\mathrm{i}\hbar^2m\langle\dot{\tilde{\psi}}|\tilde{\psi}\rangle-\mathrm{i}\hbar^2m\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle,$$

and using

$$\langle\tilde{\psi}(t)|\tilde{\psi}(t)\rangle=1,$$

$$\langle\dot{\tilde{\psi}}(t)|\tilde{\psi}(t)\rangle=-\langle\tilde{\psi}(t)|\dot{\tilde{\psi}}(t)\rangle$$

one can write

$$\langle\psi(t)|H^2|\psi(t)\rangle$$

$$=\hbar^2\langle\dot{\tilde{\psi}}|\dot{\tilde{\psi}}\rangle+\hbar^2m^2-2\mathrm{i}\hbar^2m\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle\,. \tag{27}$$

Substituting eq. (26) and (27) into $\mathrm{d}D^2$, we have

$$\mathrm{d}D^2=\langle\dot{\tilde{\psi}}|\psi\rangle\,\mathrm{d}t^2-(\mathrm{i}\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle\,\mathrm{d}t)^2\,. \tag{28}$$

Eq. (28) is the same as (22). Hence for cyclic evolution and a periodic Hamiltonian the non-adiabatic Berry phase takes the form

$$\beta=\mathrm{i}\int_0^T\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle\,\mathrm{d}t=\int_0^T\mathrm{d}L\,.$$

Finally as an illustration to the above relation we can consider the elegant example taken by Anandan [4] in his recent work, i.e. a spin-$\frac{1}{2}$ particle undergoing precession in a homogeneous magnetic field. Here $\mathscr{H}$ is spanned by two-dimensional vectors with components $(\cos\frac{1}{2}\theta,\ \sin\frac{1}{2}\theta)$ and $\theta\in[0,\pi]$, (with a suitable basis choosen for the Hamiltonian that describes the system). Due to precession, the wave function rotates by 2π radians about some axis which results in a cyclic motion of every vector of $\mathscr{H}$. Without loss of generality, at every instant of time one can choose the coordinate axis in such a way that the Hamiltonian is given by $H=-\mu B\sigma_3$, with $\varepsilon=\mu B$ being a positive number in units of energy, $B=|\boldsymbol{B}|$ and σ_3 is the Pauli spin matrix. From the evolution equation, $\mathrm{d}\phi=2\varepsilon\,\mathrm{d}t/\hbar$, which gives the infinitesimal angle by which the state is rotated about the instantaneous direction of the magnetic field in an infinitesimal time interval, $\mathrm{d}t$. Then, one can calculate $\mathrm{d}D^2$, which is equal to $\varepsilon^2\sin^2\theta\,\mathrm{d}t^2/\hbar^2$. To calculate $\mathrm{d}l^2$ we have to choose the $V(t)$ matrix properly. An appropriate form of $V(t)$ is

$$V(t)=\exp[-(\mathrm{i}/\hbar)\varepsilon(\mathbb{1}-\sigma_3)t]\,, \tag{29}$$

where $\mathbb{1}$ is a unit matrix. Then,

$$\mathrm{d}l^2=\langle\psi(0)|\dot{V}^\dagger\dot{V}|\psi(0)\rangle\,\mathrm{d}t^2$$

$$=\frac{\varepsilon^2}{\hbar^2}\langle\psi|2(\mathbb{1}-\sigma_3)|\psi\rangle\,\mathrm{d}t^2$$

$$=2\frac{\varepsilon^2}{\hbar^2}(1-\cos\theta)\,\mathrm{d}t^2. \tag{30}$$

Calculation of the geometric phase is trivial:

$$\beta=\int_0^T\sqrt{\mathrm{d}l^2-\mathrm{d}D^2}$$

$$=\int_0^T\sqrt{2(\varepsilon^2/\hbar^2)(1-\cos\theta)-(\varepsilon^2/\hbar^2)\sin^2\theta}\,\mathrm{d}t$$

$$=\frac{\varepsilon}{\hbar}\int_0^T(1-\cos\theta)\,\mathrm{d}t\,,$$

changing the variable from t to ϕ we have

$$\beta=\pi(1-\cos\theta)\,. \tag{31}$$

This can be evaluated alternatively by using

$$\beta=\mathrm{i}\int_0^T\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle\,\mathrm{d}t=\mathrm{i}\int_0^T\langle\psi(0)|V^\dagger\dot{V}|\psi(0)\rangle\,\mathrm{d}t$$

$$=\pi(1-\cos\theta)\,,$$

which is half the solid angle subtended by the orbit of motion in a sphere of unit radius. Also one can see that for $\theta=\pi$ the magnitude of velocity of transportation is zero and hence the geometric phase coincides with the length of the curve during cyclic excursion. Thus the above example provides a firm ground for our interpretation of the geometric phase by relating it to the geometric distance function through the length of the curve. In the future we want to see whether our interpretation of the geometric phase as the integral of the contracted length of the curve holds good for non-Hermitian and non-adiabatic cases.

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