Feshbach resonances II

Controlling interactions

D. Angom

Physical Research Laboratory,
Ahmedabad–380 009

School on Physics of Cold Atoms
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Plan of the talk

Recap: the story so far

Theoretical model

Experimental signatures and some more

Applications
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Applications
Racap: Atom-atom scattering

Interatomic potential $\leftrightarrow$ phase shift of partial waves

$$\frac{-1}{2\mu} \left[ \frac{d^2\chi(r)}{r^2} - \frac{l(l+1)}{r^2} - V(r) \right] \chi(r) = \frac{k^2}{2\mu} \chi(r).$$

$\sigma$ of the partial waves

$$\sigma_l = \frac{4\pi}{k^2} (2l + 1) \sin^2 \delta_l(k)$$

For $V(R) \propto R^n$, for $l < (n - 3)/2$ in the limit $k \to 0$

$$a_l = \lim_{k \to 0} \frac{\tan \delta_l}{k^{2l+1}},$$

$\delta_l$ is the $l^{th}$ phase shift. From spin statistics, $l = 0$ and 1 are dominant in identical bosons and fermions, respectively.

Recap: effective interaction

Simplified radial partial waves equation

\[ \frac{-1}{2\mu} \left[ \frac{d^2\chi(r)}{r^2} - \frac{l(l + 1)}{r^2} - V(r) \right] \chi(r) = \frac{k^2}{2\mu} \chi(r). \]

with identical \( \delta_l \) is to replace \( V(r) \) with pseudopotential

\[ V_{ps}\chi_l(r) = \frac{-(2l + 1)!!}{l + 1} \frac{1}{2\mu} \frac{\delta(r) \tan \delta_l}{r^{l+2}} \left( \frac{\partial}{\partial r} \right)^{2l+1} [r^{l+1} \chi_l(r)]. \]

For s-partial wave, in \( k \to 0 \) limit

\[ V_0\chi_0(r) = \frac{1}{2\mu} \frac{\delta(r)}{r^2} a_0 \left( \frac{\partial}{\partial r} \right) [r\chi_0(r)]. \]

Question: What is the form of \( V_{ps} \) we have encountered?

Recap: pseudopotential

For $s$-partial wave, in $k \to 0$ limit

$$V_0 \chi_0(r) = \frac{1}{2\mu} \frac{\delta(r)}{r^2} a \left( \frac{\partial}{\partial r} \right) [r\chi_0(r)].$$

From the definition of delta function $\delta(r) = \delta(r)/(4\pi r^2)$, and for slowly varying wave-functions

$$\left| \frac{d \ln \chi_0(r)}{d \ln r} \right| \ll 1 \Rightarrow \left( \frac{\partial}{\partial r} \right) [r\chi_0(r)] \simeq \chi_0(r).$$

Based on these considerations, the $s$-wave interaction is

$$V_0 = \frac{4\pi}{\mu} a_0 \delta(r).$$

Recap: Feshbach Resonances

Feshbach resonance two potentials overlaps. Magnetic Feshbach resonance relies on
- hyperfine coupling between two $V(R)$,
- Zeeman shift to tune $E_{Th}$.

It arises from atom-atom collisions, which
- transfer free atoms to bound state,
- back to free states.

Recap: Feshbach Resonances

An intriguing aim of experiments with magnetically trapped cesium atoms is to observe quantum collective effects in weakly interacting Bose gas, the so called Bose-Einstein condensates (BEC).

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Recap: the story so far

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Theoretical description

Hilbert space of two interacting atoms is separable into close $Q$ and open $P$ spaces

\[
(E - H_{PP})\psi_P = H_{PQ}\psi_Q, \\
(E - H_{QQ})\psi_Q = H_{QP}\psi_P. 
\]

From the second equation

\[
\psi_Q = \frac{1}{E - H_{QQ}}H_{QP}\psi_P. 
\]

Using this in first equation

\[
\left[ H_{PP} + H_{PQ}\frac{1}{E - H_{QQ}}H_{QP} \right] \psi_P = H_{\text{eff}}\psi_P = E\psi_P. 
\]

Feshbach resonance arises from the second term

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Molecular Hamiltonian

Effective Hamiltonian of two atoms

\[ H = \frac{\mathbf{p}^2}{2\mu} + \sum H^{\text{int}} + V^c. \]

Appropriate, internal Hamiltonian of atom

\[ H^{\text{int}} = a_{\text{hf}} \mathbf{s} \cdot \mathbf{i} + (\gamma e s_z - \gamma i_z) B_z. \]

For two Hydrogen atoms, electronic Hamiltonian

\[ H_e = \left( -\frac{1}{r_1} \nabla^2 + \frac{1}{r_{A1}} + \frac{1}{r_{A2}} \right) + \left( -\frac{1}{r_2} \nabla^2 + \frac{1}{r_{B1}} + \frac{1}{r_{B2}} \right) + \frac{1}{r_{12}} + \frac{1}{R}, \]

central (Coulomb) potential between atoms (weak field)

\[ V^c = H_{PQ} = \frac{1}{r_{A2}} + \frac{1}{r_{B1}} + \frac{1}{r_{12}} + \frac{1}{R} = V_0(r)P_0 + V_1(r)P_1. \]
Recall internal Hamiltonian of atoms

\[ H^{\text{int}} = a_{hf} s \cdot i + (\gamma_e s_z - \gamma i_z) B_z. \]

Recast the first term

\[
\begin{align*}
    a_{hf} s \cdot i &= a_{hf} (s_1 + s_2) \cdot (i_1 + i_2) + a_{hf} (s_1 - s_2) \cdot (i_1 - i_2) \\
    &= V_{+}^{hf} + V_{-}^{hf}.
\end{align*}
\]

Strong field limit, near hyperfine avoided crossing \( H_{PQ} = V_{-}^{hf} \). It is

- negligible at large interatomic distances,
- antisymmetric in \( s_1 \) and \( s_2 \), couples singlet and triplet states,
Single channel: part 1

Spectral decomposition of green operator in the $H_{\text{eff}}$ is

$$\frac{1}{E^+ - H_{QQ}} = \sum_m \frac{\langle \phi_m | \phi_m \rangle}{E - \epsilon_m} + \int d\epsilon \frac{|\phi(\epsilon) \rangle \langle \phi(\epsilon) |}{E^+ - \epsilon}$$

For $E$ close to a bound state $|\phi_B\rangle$

$$\frac{1}{E^+ - H_{QQ}} \approx \frac{\langle \phi_m | \phi_m \rangle}{E - \epsilon_m} \Rightarrow (E - H_{PP})\psi_P = \frac{H_{PQ} |\phi_B\rangle \langle \phi_B | H_{QP} |\psi_P\rangle}{E - \epsilon_B}$$

Formal solution with incoming state $|\Phi_i^+\rangle$ is

$$|\psi_P\rangle = |\psi_i^+\rangle + \frac{1}{E^+ - H_{PP}} \frac{H_{PQ} |\phi_B\rangle \langle \phi_B | H_{QP} |\psi_P\rangle}{E - \epsilon_B}.$$
Single channel: part 2

Projecting the formal solution $|\psi_P\rangle$ on $\langle \phi_B | H_{QP}$, we obtain

$$
\langle \phi_B | H_{QP} | \psi_P \rangle = \frac{(E - \epsilon_B) \langle \phi_B | H_{QP} | \psi_i^+ \rangle}{E - \epsilon_B - \langle \phi_B | H_{QP} \frac{1}{E+ - H_{PP}} H_{PQ} | \phi_B \rangle}.
$$

With this relation, $|\psi_P\rangle$ in terms of states in $P$ and $Q$ space is

$$
|\psi_P\rangle = |\psi_i^+\rangle + \frac{1}{E+ - H_{PP}} H_{PQ} |\phi_B\rangle \\
\times \frac{\langle \phi_B | H_{QP} | \psi_i^+ \rangle}{E - \epsilon_B - \langle \phi_B | H_{QP} \frac{1}{E+ - H_{PP}} H_{PQ} | \phi_B \rangle}.
$$

Single channel: part 3

From $S$ matrix transition probability to exit channel $j$ is

$$S_{ji} = S_{ji}^0 - 2\pi i \frac{\langle \psi_j^- | H_{PQ} | \phi_B \rangle \langle \phi_B | H_{QP} | \psi_i^+ \rangle}{E - \epsilon_B - \langle \phi_B | H_{QP} \frac{1}{E^+ - H_{PP}} H_{PQ} | \phi_B \rangle}.$$ 

Define, $\sqrt{2\pi} \langle \phi_B | H_{QP} | \psi_i^+ \rangle = e^{i\varphi_B i} \Gamma^{1/2}$ and complex energy shift

$$\langle \phi_B | H_{QP} \frac{1}{E^+ - H_{PP}} H_{PQ} | \phi_B \rangle = \Delta - i \Gamma / 2,$$

Identical entry and exit ($i = j$) channels

$$S_{ii} = S_{ii}^0 \left( 1 + \frac{i\Gamma}{E - \epsilon_B - \Delta + i\Gamma / 2} \right).$$

Resonance energy is

$$\epsilon_{res} = \epsilon_B + \mathcal{R}(\Delta - i\Gamma / 2),$$

The $S$ matrix is

$$S_{ii} = S_{ii}^0 \left[ 1 - \frac{i\Gamma}{E - \epsilon_{\text{res}} - \mathcal{I}(\Delta + i\Gamma/2)} \right].$$

At low energy scattering $E \to 0$, the scattering length is

$$a(B) = a_0 - \frac{C}{\epsilon_{\text{res}}}.$$ 

Near the resonance field $B_0$, $\epsilon_{\text{res}} = \Delta \mu (B - B_0)$, so

$$a(B) = a_0 - \frac{C}{\Delta \mu (B - B_0)},$$

where, $\Delta \mu = [2\mu_{\text{at}}(B_0) - \mu_{\text{mol}}(B_0)]$.

Recap: the story so far

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Applications
First observation

Feshbach resonances ⇆ enhanced trap loss of atoms.

Experimental procedure
- trap atoms optically.
- vary magnetic field.
- measure number of atoms.
- Feshbach resonance ⇆ higher loss rate.

Ketterle and collaborators reported first observation in BEC of $^{23}\text{Na}$. Since then, it has been observed in $^{85}\text{Rb}$, $^{133}\text{Cs}$, $^7\text{Li}$, etc.

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Observation in alkali atoms

$p$-wave Feshbach resonance

Optical Feshbach resonance

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Recap: the story so far

Theoretical model

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Applications
Many-body effects

Ground state of condensate with effective interaction, GP equation

\[ \left[ \frac{1}{2m} \nabla^2 + V(r) + U_0 |\psi(r)|^2 \right] \psi(r) = \mu \psi(r). \]

Many-body effects incorporated through the mean field term

\[ U_0 |\psi(r)|^2 = \frac{4\pi}{\mu} a |\psi(r)|^2. \]

Stable BEC occur when this is mildly repulsive. Effective interaction is tunable with Feshbach resonance. In particular, magnetic Feshbach resonance is widely used.
Applications of Feshbach Resonances

There are several remarkable achievements in exploring many-body physics with the use of Feshbach resonances. Few examples are:

- BEC-BCS cross over and fermion superfluidity,
- quasi-1D systems \(\leftrightarrow\) Luttinger liquid, Tonks-Girardeau gas,
- Bosenova,
- optical lattices \(\leftrightarrow\) superfluid-Mott-insulator transition,
- solitons,
- noninteracting condensates,
- dipolar condensates \(\leftrightarrow\) suppress contact interaction to enhance dipolar interaction.
Coupled Gross-Pitaevskii Equation

\[ i\hbar \frac{\partial \psi_1}{\partial t} = \left[ -\frac{\hbar^2}{2m_1} \nabla^2 + V_1(\mathbf{r}) + U_{11}N_1 |\psi_1|^2 + U_{12}N_2 |\psi_2|^2 \right] \psi_1, \]

\[ i\hbar \frac{\partial \psi_2}{\partial t} = \left[ -\frac{\hbar^2}{2m_2} \nabla^2 + V_2(\mathbf{r}) + U_{22}N_2 |\psi_2|^2 + U_{12}N_1 |\psi_1|^2 \right] \psi_2. \]

- \( \psi_1 \equiv \psi_1(\mathbf{r}, t) \): condensate wave function for species 1
- \( \psi_2 \equiv \psi_2(\mathbf{r}, t) \): condensate wave function for species 2
- \( U_{ii} = 4\pi \hbar^2 a_{ii}/m_i \): intraspecies interaction
- \( U_{ij} = 2\pi \hbar^2 a_{ij}/m_{ij} \): interspecies interaction
- \( m_{ij} = \frac{m_i m_j}{(m_i + m_j)} \)

\(^1\)C. Pethick & H. Smith, *Bose-Einstein Condensation in Dilute Gases* (2008)
Phase Separation

- **Miscible Regime**
  \[ U_{11} U_{22} - (U_{12})^2 > 0 \]
  Coexistence of both the species in some regions of space – partially overlapping wave function

- **Immiscible Regime**
  \[ U_{11} U_{22} - (U_{12})^2 < 0 \]
  No coexistence of species in any region of space – separated wavefunction

Rayleigh-Taylor Instability (RTI)

- Instability of an interface when a lighter fluid supports a heavier one in a gravitational field
- Can also occur when a lighter fluid pushes a heavier one
- Leads to turbulent mixing of the two fluids as the perturbations at the interface grow exponentially \(^3\)

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RTI in binary Bose-Einstein Condensates

- To initiate RTI\(^4\) in TBEC, we consider harmonic trapping potential.
- We choose the initial state of TBEC to be in immiscible (phase-separated) domain.
- Species with stronger intraspecies repulsive interaction surrounds the other.
- In analogy to normal fluids, species with stronger intraspecies repulsive interaction may be considered to be the lighter fluid.

Phase-Separated Pan-Cake Shaped TBEC

The trapping potential is

\[ V_i(x, y, z) = \frac{m_i \omega^2}{2} (x^2 + \alpha_i^2 y^2 + \lambda_i^2 z^2) \]

- \( \omega \): radial trap frequency
- \( \alpha_i, \lambda_i \): anisotropy parameters
- For pancake shaped trap: \( \lambda_i >> 1 \)
- For simplicity of analysis, we consider, \( \alpha_1 = \alpha_2 = \alpha \), \( \lambda_1 = \lambda_2 = \lambda \).
Details

In the initial state, at $t = 0$

- We consider a system of $^{85}\text{Rb}-^{87}\text{Rb}$ atoms
- $a_{11} = 460 a_o$, $a_{22} = 99 a_o$, $a_{12} = a_{21} = 214 a_o$ \(^5\)
- $N_1 = 5 \times 10^5$ and $N_2 = 10^6$
- $\alpha = 1$, $\omega_x = \omega_y = 2\pi \times 8\text{Hz}$
- $\lambda = 11.25$

We initiate RTI in this system by

- Decreasing $a_{11} = 460a_o$ to $a_{11} = 55a_o$ between $t = 0$ ms and $t = 200$ ms.
- After $t = 100$ ms, $a_{11}$ is kept fixed. The system is let to evolve freely for another $t = 200$ ms.
- Phase separation condition is maintained throughout the process.
- The outer species tends to come inside the inner species.
- In the process, the circular interface develops instability and grows into mushroom shape pattern.
Development of mushroom pattern

After $t = 358$ ms, $t = 378$ ms, $t = 400$ ms \(^6\)

Development of various patterns

For $\alpha = 1.8, 2.0, 3.0$
Helmholtz equation

- 2-dimensional Helmholtz equation in Cartesian coordinates \((x, y)\) transformed to elliptic cylindrical coordinates \((u, \nu)\)

\[
x = ax \cosh u \cos \nu, \\
y = a \sinh u \sin \nu.
\]

- Helmholtz equation in elliptic cylindrical coordinates

\[
\frac{1}{a^2 (\sinh^2 u + \sin^2 \nu)} \left( \frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial \nu^2} \right) + k^2 F = 0,
\]

where, \(F\) is the solution of the form \(F = U(u)V(\nu)\)
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Thank you