Topological insulators and helical edge states

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Plan of the talk

1. Introduction
   - Quantum Hall effect
   - Two dimensional topological insulators

2. Helical Luttinger liquids
   - Charge and spin fractionalisation in HLL
   - Charge fractionalisation in normal LL
   - Measurement issues

3. Edge states in a magnetic field
   - Spin transistor effect in a magnetic field

4. Conclusion
Electrons and atoms in the quantum world form many different states of matter - crystals, magnets, superconductors, etc.

Classification of all these quantum states is through principle of spontaneous symmetry breaking - e.g., crystal breaks translational symmetry, ferromagnet breaks spin symmetry, superconductivity breaks gauge symmetry, etc.

Recent years, new way of classifying phases depending on topological quantum numbers.
Quantum Hall effect discovered in eighties

Hall conductance quantised \( \sigma_{xy} = ne^2/h \)
Quantum Hall effect

Problem is of electrons moving on a 2-dim surface in the presence of a magnetic field in the perpendicular direction - no electron-electron interactions

Solved in quantum mechanics course - leads to degenerate single particle Landau levels - degeneracy is finite for a given area

When the degenerate states at a given Landau level are filled, gap to next level
What does this mean for conduction of electrons through the sample?

Semi-classical picture - closed orbits in the interior of the sample and hopping orbits at the edge of the sample

Electrons at the edges carry current
Uni-directional flow dictated by the sign of the magnetic field
Upper edge supports forward movers and lower edge backward movers
Because of spatial separation of forward and backward movers, no possibility of back-scattering due to impurities. Explains robustness and accuracy of the quantisation of the Hall current - impervious to disorder or impurity scattering.

Well-understood phenomenon by now.
Yet another way of understanding quantisation of current - Physically measured current is related to a ‘topological invariant’

Topological invariant = quantity that does not change under continuous deformation (change of some parameter)

Explains why the quantisation is so accurate, even in the presence of disorder
Can argue that where the system evolves from IQHE to ordinary insulator, the system cannot remain insulating. Else, the invariant cannot change. Hence, it implies conducting edge states.
Hence, understand how edge states and topology are related.

Started the idea of classifying phases of matter through topology.
Two dimensional topological insulators

Generalise the edge states to have two species at each edge
One going forwards and one backward, but with different spins

Spatial separation of the edge states implies no back-scattering unless spin can change
More so, in time-reversal invariant systems with half-integer spin, Kramer’s theorem implies all states are doubly degenerate.

Relation to topology
Kramers degeneracy point cannot be removed under any continuous deformation of band structure, so long as time reversal symmetry exists.
Two dimensional topological insulators

Belongs to different topological class than those of ordinary insulators

Protects against back-scattering even when impurity can change spin (spin-orbit coupling) as long as time-reversal symmetry unbroken

Hence, no back-scattering and hence quantised conductances
But doubling number of edge states implies back-scattering allowed and edge states no longer topologically protected to be massless.

Unlike quantum Hall effect, where topological quantum number was integer, for topological insulators, topological quantum number is Chern parity - it is only a $\mathbb{Z}_2$ invariant.
To find real materials that are topological insulators
Need to look at its band structure

If its band structure has odd number of edge states in the gap, it is a topological insulator
Two dimensional topological insulators

First predicted for graphene with spin-orbit coupling (to make it an insulator), but gap is very small and hence, requires very low temperatures

Kane and Mele
Zhang et al argued that materials where ordering of conduction and valence bands get inverted by spin-orbit coupling, will be topological insulators

Zhang, Bernevig and Hughes
By explicitly solving for the band structure of Mercury Telluride $HgTe$ quantum well, they showed that for thickness greater than some $d_c$, states get inverted and $HgTe$ is a topological insulator.

Amazingly, $HgTe$ was first predicted and then found to be a topological insulator.
As already mentioned, these insulators have conducting edge states like in the QHE

Edge state excitations have spin-charge separation (more later)
No magnetic field required. Opposite spin excitations move in opposite directions
No Hall current, but net ordinary two-terminal current, because one spin species at each edge contributes
Experimentally measured $2e^2/h$ Hall plateau in zero magnetic field

Konig et al, Science, 2007
Proposed three terminal geometry

Sourin Das and S.R, PRL 106, 236403, 2011
Choose spin quantisation axis of electrons in edge state to be \( \hat{z} \) axis. Spin of electrons in polarized STM tip chosen in \( \hat{x}-\hat{z} \) plane, forming an angle \( \theta \) with \( \hat{z} \) axis as shown. \( \hat{y} \)-axis points out of screen.

If spin of STM tip in tune with spin of edge electron, it implies uni-directional injection locally.

No electron-electron interactions, implies charge and spin current only to the right (left) - i.e., highly asymmetric flow of current.
In presence of electron-electron interactions, due to scattering, both right and left-movers
But asymmetry survives, can be measured

Uni-directional injection of electrons possible and hence left-right asymmetry of charge and spin currents
Leads to spin and charge fractionalisation and even a spin amplification effect

If polarisation of STM tip at angle $\theta$ with spin projection of edge electrons, specific computable $\theta$ dependence
Lightning review of bosonisation

Turns out that most convenient way to describe motion of interacting electrons in one-dimension is in terms of free bosons (which are some kind of collective excitations of the electrons).

Strength of electron-electron interactions in fermionic theory parametrised in bosonic theory by $K$

$K = 1$ for free fermions and $K < 1$ for repulsive electron-electron interactions (short-range Coulomb).
The most significant difference from electron-like behaviour is that now the charge and the spin of the electron can move independently as two independent bosonic excitations with different velocities - spinons and holons.
The edge state model in bosonised language

\[ H_0 = v \int_{-L/2}^{L/2} dx \left[ K(\partial_x \Phi)^2 + K^{-1}(\partial_x \Theta)^2 \right] \]

\[ \Phi = (\phi_{R\uparrow} + \phi_{L\downarrow})/2, \Theta = (\phi_{R\uparrow} - \phi_{L\downarrow})/2 \]

\[ \psi_{R\uparrow}(x) \sim \frac{1}{2\pi \alpha} e^{ik_F x} e^{i\phi_{R\uparrow}(x)}, \psi_{L\downarrow}(x) \sim \frac{1}{2\pi \alpha} e^{-ik_F x} e^{i\phi_{L\downarrow}(x)} \]

\[ \alpha \text{ and } K, \text{ short distance cut-off and Luttinger parameter} \]

Note left-movers are spin down and right-movers are spin up - hallmark of HLL
The model for the STM tip

\[ H_t = t \left[ \psi_{i\alpha}^\dagger(x = 0) \chi_\beta(x = 0) + \text{h.c.} \right], \]

where \( i = R, L \) are right and left movers and \( \alpha, \beta \) are spin indices. \( \psi_{i\alpha} \) and \( \chi_\beta \) denote electron destruction operator in HLL and STM tip.
In absence of Luttinger interactions, injection of spin parallel or anti-parallel to HLL spin orientation implies current only at left lead or right lead.

For general orientation of spin of injected electron, one gets $\theta$ dependence.
But with Luttinger interactions, also $K$ dependence

\[
\begin{align*}
< I_{tR} > &= \frac{(1 + K \cos \theta)}{2} I_0 \\
< I_{tL} > &= \frac{(1 - K \cos \theta)}{2} I_0
\end{align*}
\]

\[
I_0 = \frac{2e^2}{h} |t^2| \frac{(T/\Lambda)^\nu}{(\hbar v_F)^2 \Gamma(\nu + 1)} \times V
\]

$\nu$ is the Luttinger tunneling exponent given by $\nu = -1 + (K + K^{-1})/2$. $T \gg T_L, T_V$
We have assumed $T \gg T_L, T_V$ - i.e., long wires and small bias

Note $\theta = 0, \pi$, chiral injection

But we have charge current at left and right

Their measurement can confirm HLL nature of edge state
Note also $l_t(\theta) = l_{tL}(\theta) + l_{tR}(\theta)$ independent of $\theta$
Hence, two terminal tunnel current independent of $\theta$
Need three terminal measurement to study helical nature
Charge and spin fractionalisation in HLL

Spin currents

Measure spin current density at left and right leads due to insertion of electron

Spin given by

\[ S = \int_{-L/2}^{L/2} dx \, s(x) = \int_{-L/2}^{L/2} dx \left( \psi_\alpha^\dagger \sigma_\alpha \beta \psi_\beta / 2 \right) \]

So spin current can be defined as \( dS/dt = -i[S, H_t] \).
\[
\begin{align*}
\dot{S}_X(\theta) &= \frac{1}{2} \left[ \cos(\theta/2) l_t(\theta = \pi) + \sin(\theta/2) l_t(\theta = 0) \right], \\
\dot{S}_Y(\theta) &= \frac{1}{2} \left[ \cos(\theta/2) H_t(\theta = \pi) - \sin(\theta/2) H_t(\theta = 0) \right], \\
\dot{S}_Z(\theta) &= \frac{1}{2K} \left[ l_{tR}(\theta) - l_{tL}(\theta) \right].
\end{align*}
\]
Charge and spin fractionalisation in HLL

Expectation value of $\dot{S}_Y$ is zero, as expected

Spin current (vector) at right and left leads point in different direction than injected current

$$S_{R/L}(\theta) = \left[ \frac{K \mp \cos \theta}{2K \sin \theta} \hat{Z} \pm \frac{1}{2K} \hat{X} \right] l_{R/L}(\theta)$$

Non-linear function of $K$

Total spin current in direction of injected spin

$$\langle \frac{dS}{dt} \rangle = (\hat{Z} \cos \theta + \hat{X} \sin \theta) \frac{l_0}{2}$$
To understand charge fractionalisation, create single right moving electron at $x = 0$

$$\left[ \tilde{\rho}_{R/L}(x), \psi_R^\dagger(0) \right] = \frac{1 \pm K}{2} \delta(x) \psi_R^\dagger(0)$$

implies excitation of charge $(1 \pm K)/2$ in right and left going chiral densities creates right and left moving excitations of charge $(1 + K)/2$ and $(1 - K)/2$
Splitting of current into left and right chiral components for $\theta = 0$ (right chiral injection) agrees exactly with this

$$< I_{tR} > = \frac{(1 + K \cos \theta)}{2} l_0$$

$$< I_{tL} > = \frac{(1 - K \cos \theta)}{2} l_0$$
Similarly for spin fractionalisation, define spin density

\[ s_Z(x) = (1/2)(\psi_\uparrow(x)^\dagger \psi_\uparrow(x) - \psi_\downarrow(x)^\dagger \psi_\downarrow(x)) \]

\[ = (1/2K)(\tilde{\rho}_R(x) - \tilde{\rho}_L(x)) \]

implies \[ s_{Z,R/L} = \pm (1/2K)\tilde{\rho}_{R/L}(x) \]

Create right-moving electron at \( x = 0 \)

\[ \left[ s_{Z,R/L}(x), \psi_R^\dagger(0) \right] = \frac{1}{2} \left( \frac{1 \pm K}{2K} \right) \delta(x) \psi_R^\dagger(0). \]

implies spin excitations of spin \((1 \pm K)/2K\) in units of electron spin

\( K \)-dependent fractionalisation of spin of injected electron
Note here also that splitting of the $\hat{Z}$-component of the spin current operator

$$\langle \dot{S}_z(\theta) \rangle_{R/L} = \pm \frac{\langle l_{tR/L}(\theta) \rangle}{2K} = \pm \left( \frac{1 \pm K \cos \theta}{2K} \right) \frac{l_0}{2}$$

for $\theta = 0$ (right chiral injection) agrees exactly with the splitting of the electron spin evaluated from the commutator
Spin current at the L and R ends (units of electron spin)

\[
\langle \dot{S}_Z(\theta) \rangle_{R/L} = \pm \frac{\langle I_{tR/L}(\theta) \rangle}{2K} = \pm \left( \frac{1 \pm K \cos \theta}{2K} \right) \frac{l_0}{2}
\]

Note that \((1 + K)/2K > 1\) for \(K < 1\) implies amplification of the spin current!

Spin current opposite in sign at two ends (helical nature)
Fraction of spin at one end can be larger than inserted spin due to \(K\) in denominator
(compensated, of course, at the other end)
Measurement of charge fractionalisation in normal LL

Double quantum wire setup with three terminal geometry

Steinberg, Barak, Yacoby, Pfeiffer, West, Halperin and Le Hur, Nat. Phys. 4, 116 (2008)

At $B = B^+$, left-moving electrons are injected into lower wire and more electrons are detected at left contact and less at right.

At $B = B^-$, right-moving electrons are injected and more electrons are detected at right and less at left.
Need upper wire to inject electrons at one Fermi point
Injection from a point-like source injects electrons at both
Fermi points - i.e., both left-movers and right-movers
Questions of whether chiral injection really achieved

Need magnetic field to enable the two 1D dispersions of
the upper and lower wires to overlap to enable tunneling
between co-propagating electrons

At right and left extremeties of the lower wire

\[ \langle l \rangle = l_0 (1 \pm K)/2 \]

Detected value of \( K \sim 0.4 - 0.5 \)
Advantages of our set-up
Possibility of chiral injection from a point-like source because of helical nature of edge states

Need polarised scanning tunneling microscopy - currently possible
Calculations can also be easily extended to the case with partial polarisation $P$

For application to graphene, gap due to intrinsic spin-orbit coupling very small, hence need extremely small temperatures
For topological insulators like $HgTe$ quantum wells, time-reversed states are between total angular momentum states. But as long as the edges do not have spin flip scattering, each edge state with well-defined chirality comes with well-defined spin projection. Hence our analysis applicable.

But if spin symmetry broken (although time reversal is maintained) then our analysis is not applicable.
Question of whether fractional charge can be measured experimentally has been answered positively. But can this be interpreted as the charge of the LL excitations?

Gapless system, hard to define sharp local charges. But if fluctuations of the charge relative to ground state fluctuations are zero, then the charge is defined as sharp.


Charges of \((1 \pm K)/2e\) interpreted as property of electron injection.
Injection of free electron and then adiabatically turning on interaction or looking for excitations after turning on an electric field, gives different results for the fractionalisation.

Hence, interpretation of fractional charge is different from that of the $e/3$ charge of FQHE with filling fraction $1/3$. Gapped system and also, charge is quantised in that case.

Similar analysis for sharpness and better understanding of spin fractionalisation required.
Spin transistor effect in a magnetic field

Schematic diagram of the setup

\[ \vec{B} = B_X \hat{X} + B_Z \hat{Z} \]

Work in progress by Abhiram Soori, Sourin Das and S.R
In a Datta-Das transistor, emitter emits electrons with spin in particular direction, and collector collects spin in particular direction. In between spin precesses, due to gate electrode, which hence controls the current.

Similarly for edge electrons, ‘emitter’ and ‘collector’ spin is fixed by details of the 2 D topological insulator. In between spin precesses by application of a magnetic field, which hence controls current.
We find transmission resonances similar to the resonances in the Datta-Das transistor.
Behaviour of spin (expectation value of spin) as it traverses the magnetic patch can also be tracked.
Spin vectors form a cone

Resonances when spin is pointing in the $+Z$ direction at $x = -L/2$ and $x = +L/2$
Summary

Have given an introduction to topological insulators

Have proposed three terminal polarised STM measurement for helical edge states of topological insulators
Uni-directional injection difficult to achieve for non-helical Luttinger liquids
- required momentum resolved tunneling from finite length wires
Advantage for HLL: can achieve uni-directional insertion of electron by aligning its spin (parallel/antiparallel) with that of the electrons in the HLL

Fractional charge and fractional spin can be measured at the right and left leads
Helical edge states in magnetic field seem to show Datta-Das type resonances. Currently looking at experimental consequences and details.
Future direction

Need better understanding of spin fractionalisation

Extension of STM probe to 3D topological insulators with 2D surface states

Other aspects of topological insulators and HL, including curved edges, junctions with superconductors, Majorana fermions, etc