Weak Measurements, Quantum State Collapse and the Born Rule

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N. Gisin, Phys. Rev. Lett. 52 (1984) 1657

Apoorva Patel and Parveen Kumar, arXiv:1509.08253

Axioms of Quantum Dynamics

(1) Unitary evolution (Schrödinger):

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle \ , \quad i\frac{d}{dt}\rho = [H, \rho] \ .$$

Continuous, Reversible, Deterministic.

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(2) Projective measurement (von Neumann):

$$|\psi\rangle \longrightarrow P_i|\psi\rangle/|P_i|\psi\rangle|, \ P_i = P_i^\dagger, \ P_iP_j = P_i\delta_{ij}, \ \sum_i P_i = I.$$

Discontinuous, Irreversible, Probabilistic choice of “i".
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Instead, with Born rule and ensemble interpretation,

$$prob(i) = \langle\psi|P_i|\psi\rangle = Tr(P_i\rho) \ , \quad \rho \longrightarrow \sum_i P_i\rho P_i \ .$$

Pure state evolves to mixed state. Predicted expectation values are averages over many experimental runs.

Quantum Measurement Terminology

$$\begin{tikzcd}[ampersand replacement=\&]
\rho_i = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} \arrow[r, "\text{Decoherence}"] \arrow[dr, "\text{Collapse}"'] \& \rho_r = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \arrow[d, "\text{Quantum jump}"'] \\
\& \rho_f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{tikzcd}$$

The evolution steps involved in the quantum measurement process for a qubit
(ρ_i and ρ_f are pure states, while $\rho_{\rm r}$ is obtained from an entangled state):
(a) Decoherence deterministically entangles the system with its environment,
and drives the off-diagonal reduced density matrix components to zero.
(b) Quantum jump removes the system-apparatus entanglement, and probabilistically
converts the diagonal reduced density matrix into a measurement eigenstate.
(c) Collapse is the overall process that yields measurement eigenstates probabilistically,
and it may or may not go through decoherence.

Terminology (contd.)

Decoherence entangles the observed system degrees of freedom with the unobserved environmental degrees of freedom. Sum over the unobserved degrees of freedom yields a reduced mixed state, with the same structure as the ensemble of statistical mechanics. It quantitatively explains how the off-diagonal elements of ρ decay.

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von Neumann interaction is a particular instance of the decoherence paradigm. It creates perfect entanglement between the measured eigenstates of the system and the pointer basis states of the apparatus.

$$H_{\rm vN} = g\, x_S \otimes p_A \;\; : \quad |x\rangle_S |0\rangle_A \longrightarrow |x\rangle_S |x\rangle_A$$

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Quantum jump is probabilistic. It defines which interactions of a system with its surroundings are measurements.
A measurement interaction is the one where the apparatus cannot remain in a superposition of pointer states.

Weak Measurements

Information about the measured observable is extracted from the system at a slow rate (e.g. by weak coupling).
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New questions:

- How is the projection replaced by a continuous evolution?
- What is the local evolution rule during measurement?
- What is the state if the measurement is left incomplete?
- How is the ensemble to be interpreted?
- How should multipartite measurements be described?

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The answers are important for increasing accuracy of quantum control and feedback. Knowledge of what happens in a particular experimental run (and not the ensemble average) can improve efficiency and stability.

The projective measurement axiom needs to be replaced by a different continuous stochastic dynamics.

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Such a dynamical process exists! Gisin (1984)

Salient Features

A precise ratio of evolution towards the measurement eigenstates and unbiased white noise is needed to reproduce the Born rule as a constant of evolution.

This is reminiscent of the “fluctuation-dissipation theorem" that connects diffusion and viscous damping, implying a common origin for both in molecular scattering.

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This is also an indication that the deterministic and the stochastic contributions to the evolution arise from the same underlying dynamics.

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Technological advances allow us to monitor the quantum evolution during weak measurements. That would help us test the validity of the stochastic measurement process, and then figure out what may lie beyond.

Beyond Quantum Mechanics

Physical:

(1) Hidden variables with novel dynamics may produce quantum mechanics as an effective theory, with extra rules supplementing Schrödinger's equation.

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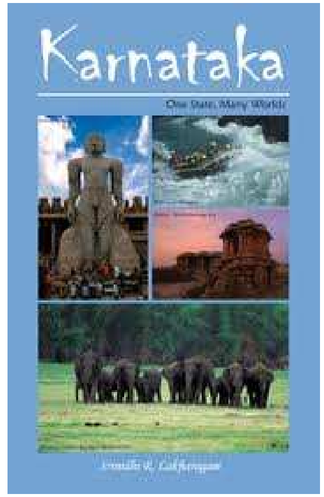
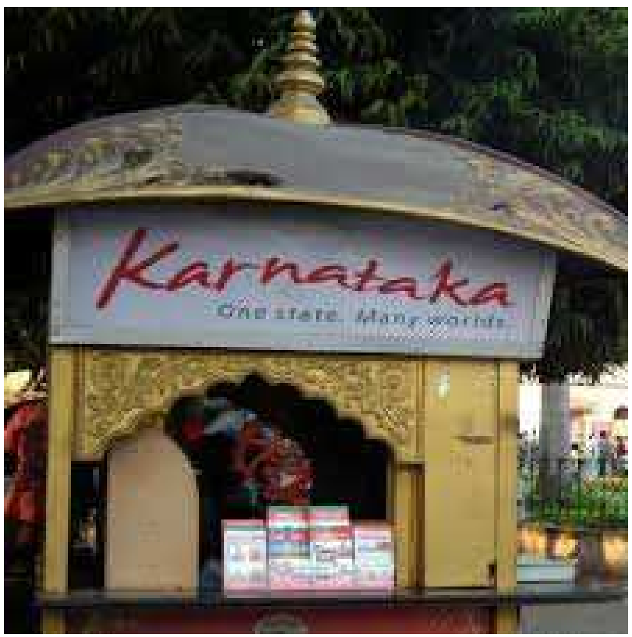
Bypass:

Many worlds interpretation—each evolutionary branch is a different world, and we only observe the measurement outcome corresponding to the world we live in.

Uncountable proliferation of evolutionary branches is highly ungainly.





Karnataka
One state. Many worlds.
Department of Tourism

Quantum Collapse Equation

Let the projective measurement arise from a continuous geodesic evolution, with parameter $s \in [0,1]$:

$$|\psi\rangle \longrightarrow Q_i(s)|\psi\rangle / |Q_i(s)|\psi\rangle| \ , \quad Q_i(s) = (1-s)I + sP_i \ .$$

Then an individual quantum trajectory evolves as

$$\rho \longrightarrow \frac{(1-s)^2\rho + s(1-s)(\rho P_i + P_i\rho) + s^2 P_i\rho P_i}{(1-s)^2 + (2s-s^2)Tr(P_i\rho)} \ , \quad Tr(\rho) = 1 \ .$$

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Expansion around $s = 0$ gives the collapse equation:

$$\boxed{\frac{d}{dt}\rho = g[\rho P_i + P_i\rho - 2\rho\, Tr(P_i\rho)] \ .}$$

$s \rightarrow gt$ in terms of the system-apparatus coupling g, and the “measurement time" t.

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- This nonlinear evolution preserves pure states,

$$\rho^2 = \rho \Longrightarrow \frac{d}{dt}(\rho^2 - \rho) = \rho\frac{d}{dt}\rho + (\frac{d}{dt}\rho)\rho - \frac{d}{dt}\rho = 0 \ ,$$

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in addition to maintaining $Tr(\rho) = 1$.

- Projective measurement is the fixed point of this equation:

$$\frac{d}{dt}\rho = 0 \quad \text{at} \quad \rho^* = P_i\rho P_i / Tr(P_i\rho) \ .$$

Convergence to fixed point makes the measurement consistent on repetition.

$$\boxed{\frac{d}{dt}\rho = g[\rho P_i + P_i\rho - 2\rho Tr(P_i\rho)]}$$

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- In a bipartite setting, $\{P_i\} = \{P_{i_1} \otimes P_{i_2}\}$. The evolution is linear in the projection operators, and $\sum_i P_i = I$. So partial trace over the unobserved environment gives the same equation for the reduced density matrix for the system.

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- For pure states, the collapse equation is:

$$\frac{d}{dt}\rho = -2g\mathcal{L}[\rho]P_i$$

This structure (involving Lindblad operator) hints at an action-reaction relation between processes of decoherence and collapse, possibly following from a conservation law.

Interpretation: When $\mathcal{L}[\rho]P_i$ decoheres the apparatus pointer state P_i (it cannot remain in superposition by definition), there is an equal and opposite effect $-\mathcal{L}[\rho]P_i$ on the system state ρ leading to its collapse.

Ensemble of Quantum Trajectories

The prefered basis $\{P_i\}$ is fixed by the system-apparatus interaction, but a separate criterion is needed to determine which P_i will occur in a particular experimental run.

Quantum jump: The evolution trajectory is chosen at the start of the measurement and remains unaltered thereafter.

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Assign time-dependent real weights $w_i(t)$ to the evolution trajectories for P_i, with $\sum_i w_i = 1$:

$$\frac{d}{dt}\rho = \sum_i w_i \; g[\rho P_i + P_i\rho - 2\rho Tr(P_i\rho)] \; .$$

Ensemble Evolution

The trajectory averaged evolution is:

$$\frac{d}{dt}(P_j \rho P_k) = P_j \rho P_k \; g[w_j + w_k - 2\sum_i w_i Tr(P_i \rho)] \; .$$

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Diagonal projections of ρ fully determine the evolution:

$$\frac{2}{P_j\rho P_k}\frac{d}{dt}(P_j\rho P_k) = \frac{1}{P_j\rho P_j}\frac{d}{dt}(P_j\rho P_j) + \frac{1}{P_k\rho P_k}\frac{d}{dt}(P_k\rho P_k)$$

For one-dimensional projections, $P_j\rho(t)P_j = d_j(t)P_j$,

$$d_j \geq 0\ ,\quad P_j\rho(t)P_k = P_j\rho(0)P_k\left[\frac{d_j(t)d_k(t)}{d_j(0)d_k(0)}\right]^{1/2}\ .$$

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The diagonal projections evolve according to:

$$\frac{d}{dt}d_j = 2g\; d_j(w_j - w_{\rm av}) \; , \quad w_{\rm av} \equiv \sum_i w_i d_i \; .$$

Evolution is restricted to the subspace spanned by all the $P_j\rho(t=0)P_j \neq 0$. Diagonal elements with $w_j > w_{\rm av}$ grow; those with $w_j < w_{\rm av}$ decay. Every $\rho = P_i$ is a fixed point.

All these features are stable under small perturbations of ρ.

Ensemble Evolution (contd.)

Instantaneous Born rule: $w_j = w_j^{IB} \equiv Tr(\rho(t)P_j)$

This is a local and appealing choice for the trajectory weights throughout the measurement process. Then

$$\frac{d}{dt}(P_j\rho P_k) = P_j\rho P_k \; g[w_j^{IB} + w_k^{IB} - 2\sum_i (w_i^{IB})^2] \; .$$

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Instead of heading towards the nearest fixed point, the trajectories can be made to wander around and explore other possibilities by adding noise to their dynamics.

Stochastic noise can be added to the quantum trajectory weights w_i, in a structure similar to the Langevin equation.

Size of noise has to be found, while retaining $\sum_i w_i = 1$.

Single Qubit Measurement

The evolution equations simplify considerably for a qubit with $|0\rangle$ and $|1\rangle$ as the measurement eigenstates:

$$\frac{d}{dt}\rho_{00} = 2g\,(w_0 - w_1)\rho_{00}\rho_{11}\ ,$$

$$\rho_{01}(t) = \rho_{01}(0)\left[\frac{\rho_{00}(t)\rho_{11}(t)}{\rho_{00}(0)\rho_{11}(0)}\right]^{1/2}\ .$$

With $\rho_{11}(t) = 1 - \rho_{00}(t)$ and $w_1(t) = 1 - w_0(t)$, only one independent variable describes evolution of the system.

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Adding unbiased white noise with spectral density S_ξ to the instantaneous Born rule, the trajectory weights become:

$$w_0 - w_1 = \rho_{00} - \rho_{11} + \sqrt{S_\xi}\,\xi\ .$$

$$\langle\xi(t)\rangle = 0\ ,\quad \langle\xi(t)\xi(t')\rangle = \delta(t-t')\ .$$

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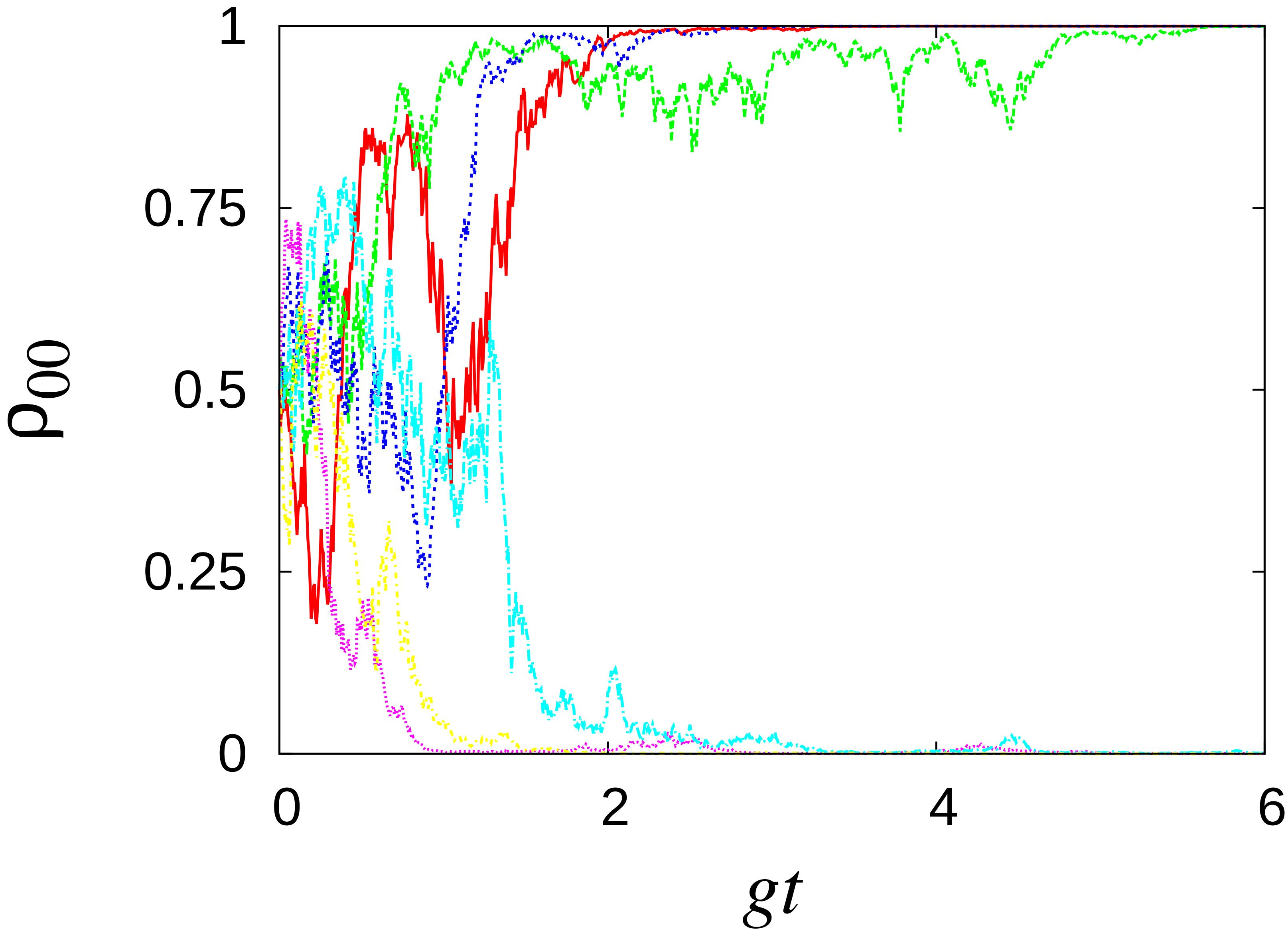
This is a stochastic differential process on $[0,1]$. The fixed points at $\rho_{00} = 0, 1$ are perfectly absorbing boundaries. A quantum trajectory would zig-zag through the interval before ending at one of the two boundary points.



Individual quantum evolution trajectories for the initial state $\rho_{00} = 0.5$, with measurement eigenstates $\rho_{00} = 0, 1$, and in presence of measurement noise satisfying $gS_{\xi} = 1$.

Single Qubit Measurement (contd.)

Let $P(x)$ be the probability that the initial state with $\rho_{00} = x$ evolves to the fixed point at $\rho_{00} = 1$. Then by symmetry,

$$P(0) = 0,\ P(0.5) = 0.5,\ P(1) = 1\ .$$

No noise : $\quad S_\xi = 0 \implies P(x) = \theta(x - 0.5)\ .$

Only noise : $\quad S_\xi \to \infty \implies P(x) = 0.5\ .$

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$$\text{Only noise :} \quad S_\xi \to \infty \implies P(x) = 0.5\ .$$

It is instructive to convert the stochastic evolution equations from the differential Stratonovich form to the Itô form that specifies forward evolutionary increments:

$$d\rho_{00} = 2g\ \rho_{00}\rho_{11}(\rho_{00} - \rho_{11})(1 - gS_\xi)dt + 2g\sqrt{S_\xi}\ \rho_{00}\rho_{11}\ dW\ ,$$

$$\langle dW(t)\rangle = 0\ ,\quad \langle (dW(t))^2\rangle = dt\ .$$

The Wiener increment can be modeled as a random walk.

Single Qubit Measurement (contd.)

Let $P(x)$ be the probability that the initial state with $\rho_{00} = x$ evolves to the fixed point at $\rho_{00} = 1$. Then by symmetry,

$$P(0) = 0,\ P(0.5) = 0.5,\ P(1) = 1\ .$$

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The Wiener increment can be modeled as a random walk.

The first term produces drift in the evolution, while the second gives rise to diffusion. The evolution with no drift, i.e. the pure Wiener process with $gS_\xi = 1$, is rather special:

$\langle d\rho_{00}\rangle = 0 \Leftrightarrow$ Born rule is a constant of evolution.

Single Qubit Measurement (contd.)

Starting at x, one moves to $x+\epsilon$ with some probability, moves to $x-\epsilon$ with the same probability, and stays put otherwise. Balancing the probabilities,

$$P(x) = \alpha(P(x+\epsilon) + P(x-\epsilon)) + (1-2\alpha)P(x) \ .$$

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The general solution, independent of the choice of ϵ, is that $P(x)$ is a linear function of x, which is the Born rule:

$$\boxed{gS_\xi = 1,\ P(0) = 0, P(1) = 1 \implies P(x) = x}$$

Specific choices of g, α, ϵ only alter the rate of evolution, and not the asymptotic outcome.

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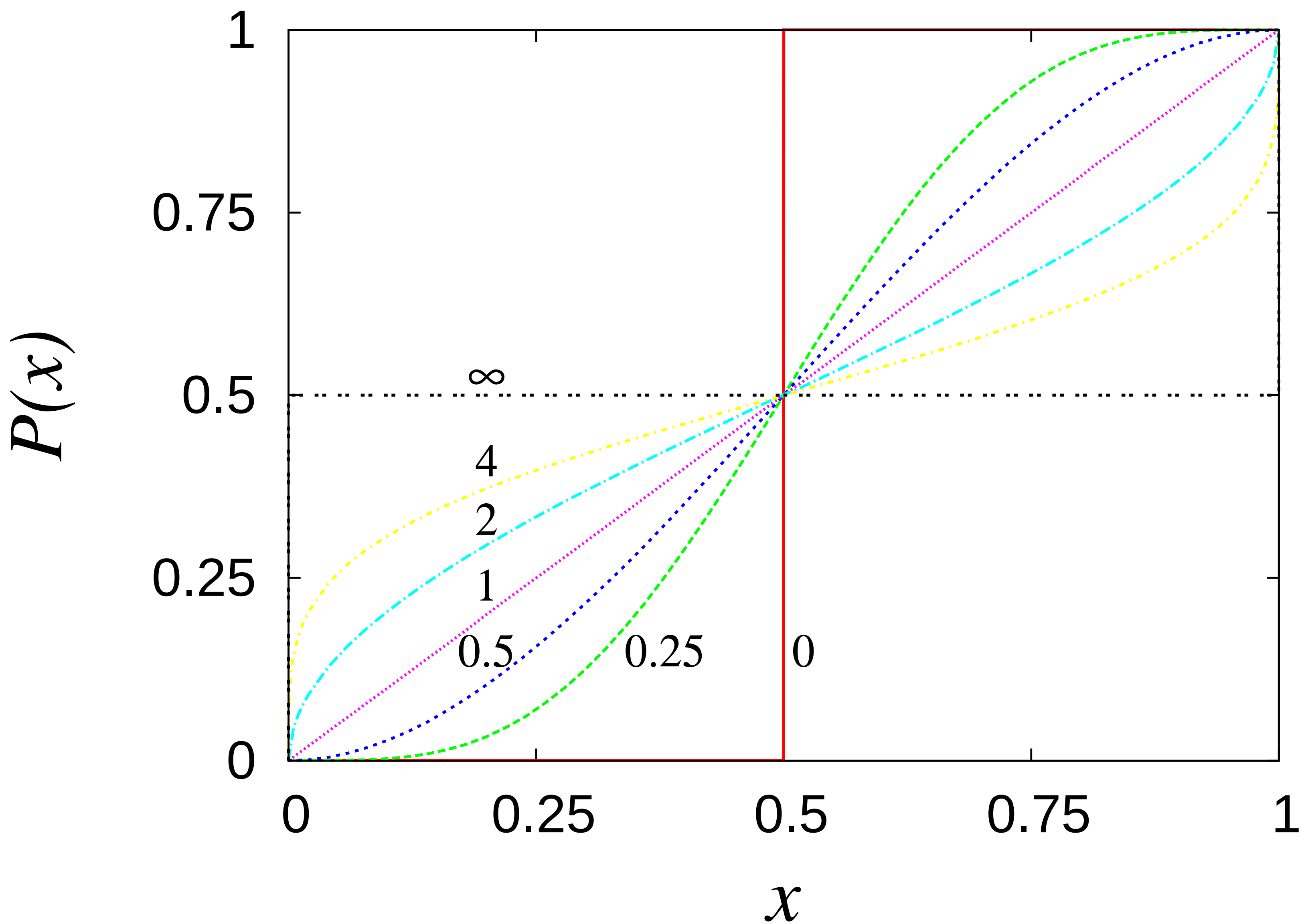
Numerical tests were performed for different values of gS_ξ.

$$\frac{\rho_{00}(t+\tau)}{\rho_{11}(t+\tau)} = \frac{\rho_{00}(t)}{\rho_{11}(t)} e^{2g\tau\overline{w}} \ , \quad \overline{w} = \frac{1}{\tau}\int_t^{t+\tau} (w_0 - w_1) dt \ .$$

With $g\tau \ll 1$, $\overline{w}$ was generated as a Gaussian random number with mean $\rho_{00}(t) - \rho_{11}(t)$ and variance S_ξ/τ.

The data clearly show the special status of $gS_\xi = 1$.



Probability that the initial qubit state $\rho_{00} = x$ evolves to the measurement eigenstate $\rho_{00} = 1$, for different values of the measurement noise. The gS_ξ values label the curves.

Evolution Dynamics

During measurement, the probability distribution $p(\rho_{00}, t)$ of the quantum trajectories evolves according to the Fokker-Planck equation (with $gS_\xi = 1$):

$$\frac{\partial p(\rho_{00},t)}{\partial t} = 2g \frac{\partial^2}{\partial^2 \rho_{00}} \left(\rho_{00}^2 (1-\rho_{00})^2 p(\rho_{00},t)\right) \ .$$

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Let $\tanh(z) = 2\rho_{00} - 1$ map $\rho_{00} \in [0, 1]$ to $z \in (-\infty, \infty)$. Then the two peaks are diffusing Gaussians, with their centres at $\tanh^{-1}(2x - 1) \pm gt$ and variance gt.

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They reach the boundaries only asymptotically.

The formal “measurement time" can be made finite by a (non-unique) change of variables, e.g. $s = \tanh(gt) \in [0,1]$.

$$(1-s^2)\frac{d}{ds}\rho_{00} = 2(\rho_{00} - \rho_{11} + \sqrt{1-s^2}\xi)\rho_{00}\rho_{11} \ .$$

The time-dependent coupling $g \to 1/(1-s^2)$ gives the same equation.

Larger Quantum Systems

Preceding results are valid for binary orthogonal measurements on any quantum system, with the replacement $\rho_{ii} \to Tr(\rho P_i)$.

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Another option for n-dimensional quantum measurements is to use the orthonormal set of weights in the convention of $SU(n)$ Cartan generators ($k = 1, \ldots, n-1$):

$$\sum_{i=0}^{k-1} w_i - k w_k = \sum_{i=0}^{k-1} \rho_{ii} - k\rho_{kk} + \sqrt{\frac{k(k+1)S_\xi}{2}}\xi_k \ ,$$

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where ξ_k are independent white noise terms.

The condition for the evolution to be a pure Wiener process, and hence satisfy the Born rule, remains $gS_\xi = 1$.

Notable Features

- Individual quantum trajectories evolve unitarily, even in presence of noise. Mixed states arise when multiple trajectories with different noise histories are combined.

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- When the Born rule is satisfied, attraction to measurement eigenstates scales as g while the noise scales as $\sqrt{g}$. This relation is local between the system and the apparatus.
- Measurement outcomes are independent of $\rho_{i\neq j}$, and so are unaffected by decoherence. Noise can be added to the phases of $\rho_{i\neq j}$ without spoiling the described evolution of ρ_{ii}. The Born rule imposes no constraint on that noise.

Summary

The quadratically nonlinear quantum Langevin equation for state collapse supplements the Schrödinger evolution:

$$\frac{d}{dt}\rho = \sum_i w_i \; g[\rho P_i + P_i\rho - 2\rho Tr(\rho P_i)] \; .$$

The weights w_i contain attraction towards measurement eigenstates and stochastic white noise. Fixing their ratio, i.e. $gS_\xi = 1$, makes the Born rule a constant of evolution.

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The precise relation between the magnitude of the noise and the system-apparatus coupling governing the collapse time scale, points to a common origin for the two.
The dynamics is closed and local; different interacting system-apparatus pairs can have different couplings.

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The dynamics is closed and local; different interacting system-apparatus pairs can have different couplings.

It is not a universal background dynamics.

It is a challenge to unify the two dynamical contributions in a fundamental underlying theory of quantum measurements.

Open Questions

- What underlying dynamics can simultaneously produce attraction towards measurement eigenstates (geodesic evolution) and irreducible noise (stochastic fluctuations)?

  These features appear in variational principles and path integral framework.

  Non-abelian gauge theories and general relativity also have quadratic nonlinearities.

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- Are there generalisations that would allow the noise to be bypassed or modified under some unusual conditions?

- Incomplete measurements can test the dynamics of quantum state collapse. Experiments would require determination of the density matrix evolution, using weak measurements and with highly suppressed decoherence.

  Such tests are becoming technologically feasible! Superconducting qubit experiments need to be extended to larger quantum systems.

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