Meeting on Quantum Information Processing and Applications

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Application of Majorization Theory to Gaussian Quantum Information

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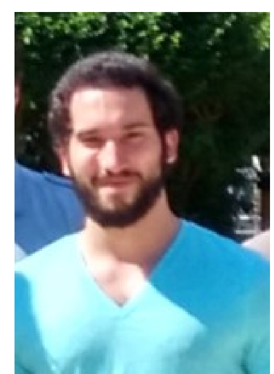


Michael Jabbour

Christos Gagatsos

Outline

- Majorization theory (introduction)

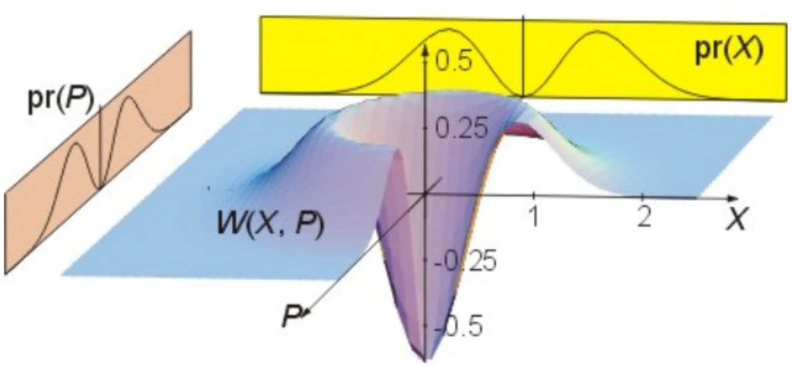
  … provide a means to compare probability distributions

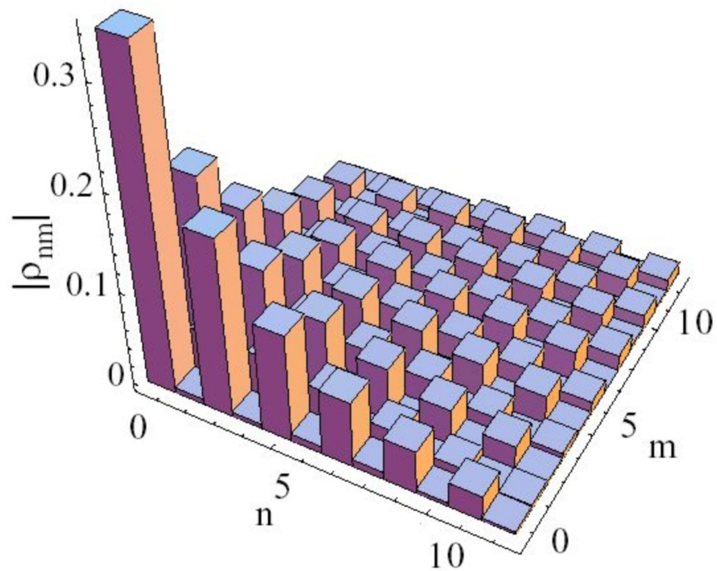
  … extension to quantum information theory (esp. entanglement)

- Majorization in continuous-variable quantum information

  … interplay between phase- and state-space representations

  … Wigner functions and symplectic formalism are well suited to describe optical states and transformations, but majorization requires diagonalization of states !





- Selected applications to Gaussian quantum information

1 … intrinsic majorization relations in fundamental Gaussian optical components (beam-splitters & squeezers)

[with O. Oreshkov, C. Navarrete-Benlloch, S. Lloyd and J. H. Shapiro]

2 … interconversion between Gaussian entangled states (may require non-Gaussian LOCCs)

3 … Gaussian bosonic channels, esp. entropy conjectures (applications to channel capacities)

[with V. Giovannetti and A. S. Holevo]

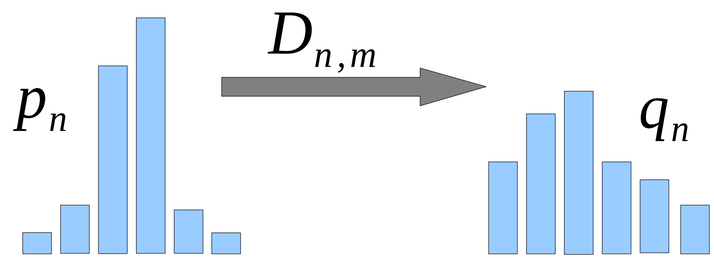
Majorization Theory

(pre)-order relation for probability distributions

$\xrightarrow{D_{n,m}}$

p_n majorizes q_n

$\mathbf{p} \succ \mathbf{q}$



with p_n, q_n probability distributions

if and only if

$\Longleftrightarrow$ $\mathbf{p}$ can be converted to $\mathbf{q}$ by applying a random permutation

$$q_n = \sum_m D_{n,m} p_m \qquad D_{n,m} \text{ is } \textit{doubly-stochastic} \text{ matrix}$$

p is "*more ordered*" than **q**

or

$$\Longleftrightarrow \quad \sum_{n=0}^{m} p_n^{\downarrow} \geq \sum_{n=0}^{m} q_n^{\downarrow} \qquad \forall m \geq 0$$

or

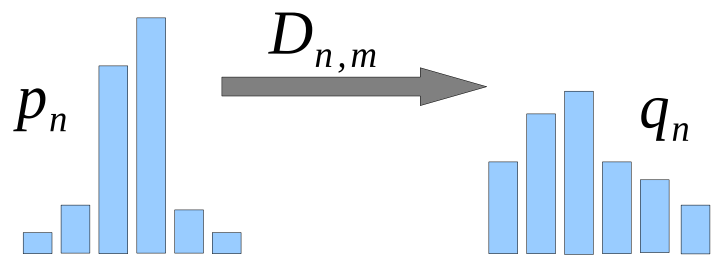
$$\Longleftrightarrow \quad \sum_n f(p_n) \leq \sum_n f(q_n) \qquad \forall f(x) \text{ concave function}$$

e.g., Shannon entropy: $f(x) = -x \log(x)$

$$\mathbf{p} \succ \mathbf{q} \quad \Longrightarrow \quad H(\mathbf{p}) \leq H(\mathbf{q})$$

Shannon entropy can only increase

Majorization vs. Renyi entropies



$$H_\alpha(\mathbf{p}) = \frac{1}{1-\alpha} \log \underbrace{\sum_n p_n{}^\alpha}$$

use $f_\alpha(x) = x^\alpha$ $\quad F_\alpha(\mathbf{p}) = \sum_n f_\alpha(p_n)$

- $0<\alpha<1$ $\quad f_\alpha(x) = x^\alpha$ concave function

  $\Rightarrow F_\alpha(\mathbf{p}) \le F_\alpha(\mathbf{q}) \Rightarrow H_\alpha(\mathbf{p}) \le H_\alpha(\mathbf{q})$

- $\alpha>1$ $\quad f_\alpha(x) = x^\alpha$ convex function

  $\Rightarrow F_\alpha(\mathbf{p}) \ge F_\alpha(\mathbf{q}) \Rightarrow H_\alpha(\mathbf{p}) \le H_\alpha(\mathbf{q})$

$\mathbf{p} \succ \mathbf{q} \Rightarrow H_\alpha(\mathbf{p}) \le H_\alpha(\mathbf{q}) \quad \forall \alpha$ all Renyi entropies increase

$\not\Leftarrow$ catalysis effects may be observed !

Quantum application : comparing density operators

$$\xrightarrow{M}$$

$\hat{\rho}$ majorizes $\hat{\sigma}$

$$\hat{\rho} \succ \hat{\sigma}$$

$$\equiv$$

$$\xrightarrow{D_{n,m}}$$

p_n majorizes q_n

$$\mathbf{p} \succ \mathbf{q}$$

where p_n are the eigenvalues of $\hat{\rho}$
q_n $\hat{\sigma}$

$$\hat{\rho} \succ \hat{\sigma}$$

more mixed

M is a "*disordering*" CP map (convex mixture of unitaries)

$$\hat{\rho} \succ \hat{\sigma} \iff \hat{\sigma} = \sum_i \lambda_i U_i \hat{\rho} U_i^\dagger \qquad 0 \le \lambda_i \le 1 \qquad \sum_i \lambda_i = 1$$

- $\hat{\rho}$ can be transformed into $\hat{\sigma}$ by applying a random unitary
- $S(\rho) \le S(\sigma)$ entropy can only increase

Quantum application : interconversion of pure bipartite states

M. Nielsen, G. Vidal, 2000

$$\xleftarrow{\text{LOCC}}$$

$|\psi\rangle$ majorizes $|\phi\rangle$

$|\psi\rangle \succ |\phi\rangle$

$$\equiv$$

$$\xrightarrow{D_{n,m}}$$

p_n majorizes q_n

$\mathbf{p} \succ \mathbf{q}$

where $|\psi\rangle = \sum_n \sqrt{p_n}\, |e_n\rangle\, |f_n\rangle$

$|\phi\rangle = \sum_n \sqrt{q_n}\, |e_n{}'\rangle\, |f_n{}'\rangle$

- $|\phi\rangle$ can be converted to $|\psi\rangle$ by applying a deterministic LOCC
- $E(|\psi\rangle) \leq E(|\phi\rangle)$ entanglement can only decrease

Trick: $\rho_A = tr_B(|\psi\rangle\langle\psi|) \begin{cases} = \sum_n p_n \underbrace{|e_n\rangle\langle e_n|}_{\text{orthonormal}} & \text{eigenbasis representation} \\ = \sum_n q_n \underbrace{|\zeta_n\rangle\langle\zeta_n|}_{\text{not orthonormal}} & \text{... possible iff } \mathbf{p} \succ \mathbf{q} \end{cases}$

Trick : unitary freedom in ensemble realizing a density operator

$$\hat{\rho} \;=\; \sum_n p_n \, |e_n\rangle\langle e_n| \;=\; \sum_n q_n \, |\zeta_n\rangle\langle\zeta_n|$$

possible iff $\exists U$ s.t. $\sqrt{q_i}\,|\zeta_i\rangle \;=\; \sum_j \underbrace{U_{i,j}}_{\text{unitary}} \sqrt{p_j}\,|e_j\rangle$

then $\displaystyle q_i \underbrace{\langle\zeta_i|\zeta_i\rangle}_{1} \;=\; \sum_{j,k} U_{i,j}\, U^*_{i,k} \sqrt{p_j}\,\sqrt{p_k}\, \underbrace{\langle e_k|e_j\rangle}_{\delta_{j,k}}$

then $\displaystyle q_i \;=\; \sum_j \underbrace{|U_{i,j}|^2}_{D_{i,j}} p_j$ $D_{i,j}$ doubly stochastic

$\boxed{\begin{array}{c} p_n \text{ majorizes } q_n \\ \mathbf{p} \succ \mathbf{q} \end{array}}$

Explicit conversion LOCC

LOCC $\longleftarrow$

$|\psi\rangle$ majorizes $|\phi\rangle$

$|\psi\rangle \succ |\phi\rangle$

$$|\psi\rangle = \sum_n \sqrt{p_n}\, |e_n\rangle\, |f_n\rangle$$

$$|\psi\rangle = \sum_n \sqrt{q_n}\, |\zeta_n\rangle\, |f_n{''}\rangle$$

LOCC $\uparrow$ $\quad \updownarrow$ U

$$|\phi\rangle = \sum_n \sqrt{q_n}\, |e_n{'}\rangle\, |f_n{'}\rangle$$

above trick, provided p_n majorizes q_n that is $|\psi\rangle \succ |\phi\rangle$

POVM: $A_m = \sum_n \omega^{nm} |\zeta_n\rangle\langle e_n{'}|$ with $\omega = e^{i2\pi/d}$ and $\sum_m A_m^+ A_m = I$

$$(A_m \times I)\, |\phi\rangle = \sum_n \sqrt{q_n}\, \omega^{nm}\, |\zeta_n\rangle\, |f_n{'}\rangle \equiv |\phi_m\rangle$$ depends on outcome m

Conditional U: $B_m = \sum_n \omega^{-nm} |f_n{''}\rangle\langle f_n{'}|$ conditional on m

$$(I \times B_m)\, |\phi_m\rangle = \sum_n \sqrt{q_n}\, |\zeta_n\rangle\, |f_n{''}\rangle \equiv |\psi\rangle$$ deterministic LOCC

Other quantum applications

- Majorization-based separability criterion

$$\hat{\rho}_{AB} \text{ separable} \Rightarrow \hat{\rho}_A \otimes \hat{0} \succ \hat{\rho}_{AB} \quad \text{and} \quad \hat{0} \otimes \hat{\rho}_B \succ \hat{\rho}_{AB}$$

more disordered globally than locally

M. Nielsen, J. Kempe, 2001
T. Hiroshima, 2003

… yields a necessary condition for separability
hence, a sufficient condition for entanglement

… stronger than entropic condition

$$S(\hat{\rho}_A) \leq S(\hat{\rho}_{AB}) \quad \text{and} \quad S(\hat{\rho}_B) \leq S(\hat{\rho}_{AB})$$

Horodecki's, 1996
Cerf, Adami, 1997

- Majorization-based uncertainty relations

$$p \otimes q \prec Q$$

Puchala, Rudnicki, Życzkowski, 2013

… implies entropic uncertainty relations $H(p) + H(q) \geq B$

Using majorization theory in quantum optics ?

- Possible in infinite-dimensional (countable) Fock space

  … majorization theory remains applicable for infinite prob. distr.
  … infinite matrices (D stochastic, D^T sub-stochastic)

- Hard problem because phase-space vs. state-space interplay

  … optical states / transformations in <u>phase space</u>
  … channels / entropies / majorization in <u>state space</u>

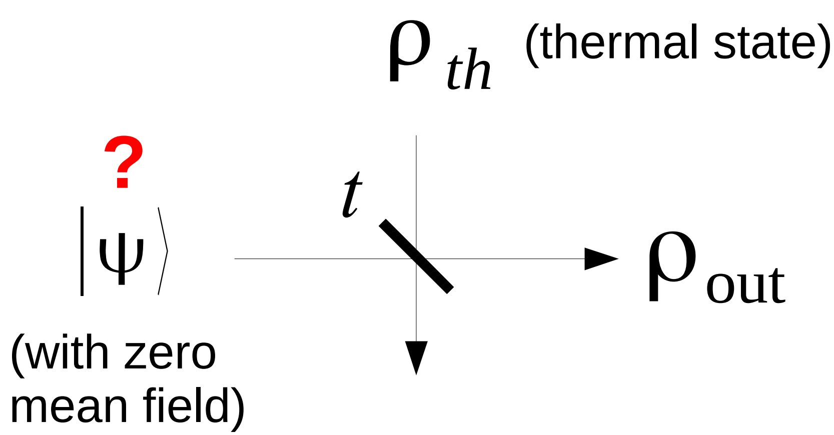
- First application of majorization theory in quantum optics :

  … attacking Gaussian minimum entropy conjectures
  (capacities of Gaussian bosonic channels)

R. Garcia-Patron, C. Navarrete-Benlloch,
S. Lloyd, J. H. Shapiro, and N. J. Cerf, PRL (2012).

Motivation / example

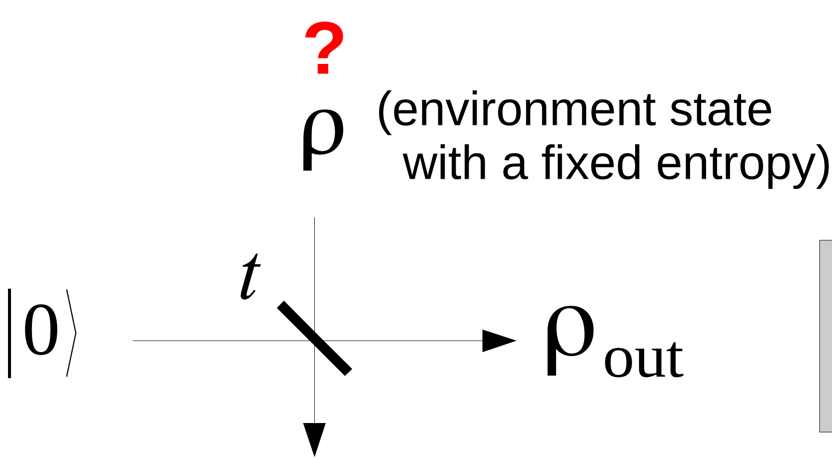
Two entropy conjectures in which majorization does (or might) play a key role



"Least random input ?"

$$S(\rho_{out}) \text{ minimum if } |\psi\rangle = |0\rangle$$

(vacuum state minimizes the disorder)



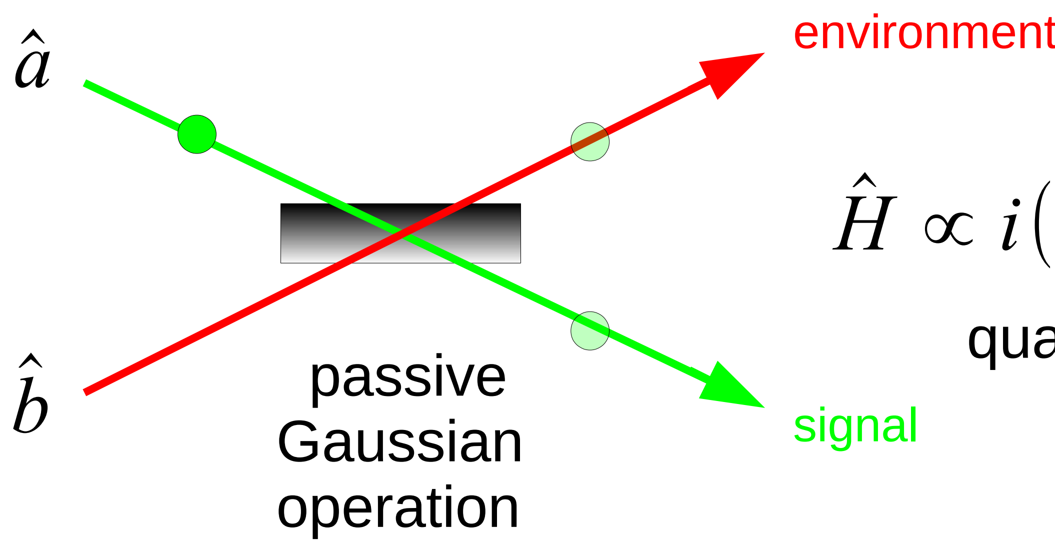
"Least randomizing channel ?"

$$S(\rho_{out}) \text{ minimum if } \rho = \rho_{th}$$

(thermal state with same entropy as ρ)

Two fundamental (Gaussian) optical components

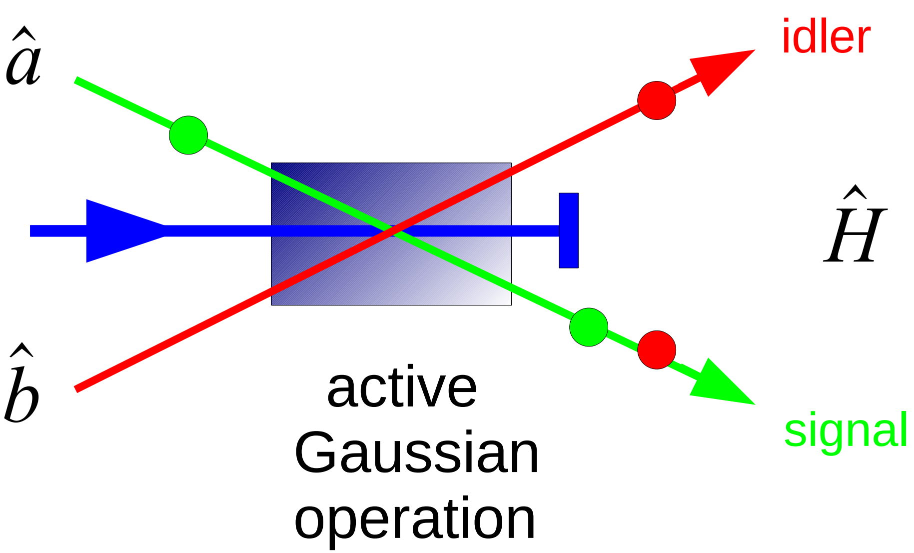
Beam splitter



$$\hat{H} \propto i\left(\hat{a}\,\hat{b}^{\dagger} - \hat{a}^{\dagger}\,\hat{b}\right)$$

quadratic

Two-mode
squeezer

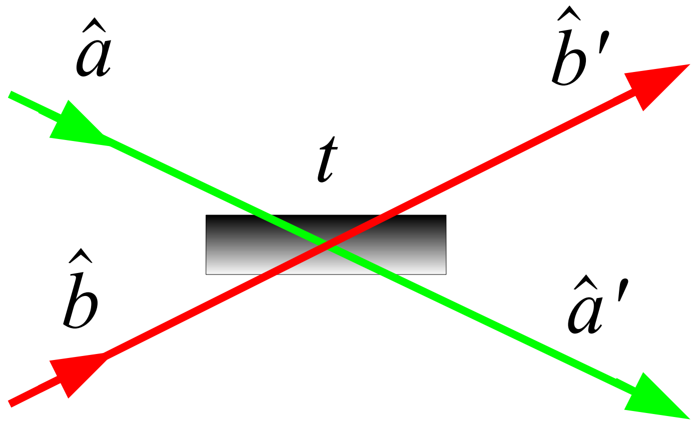


$$\hat{H} \propto i\left(\hat{a}\,\hat{b} - \hat{a}^{\dagger}\,\hat{b}^{\dagger}\right)$$

quadratic

(Linear) canonical transformations in phase space

Beam splitter



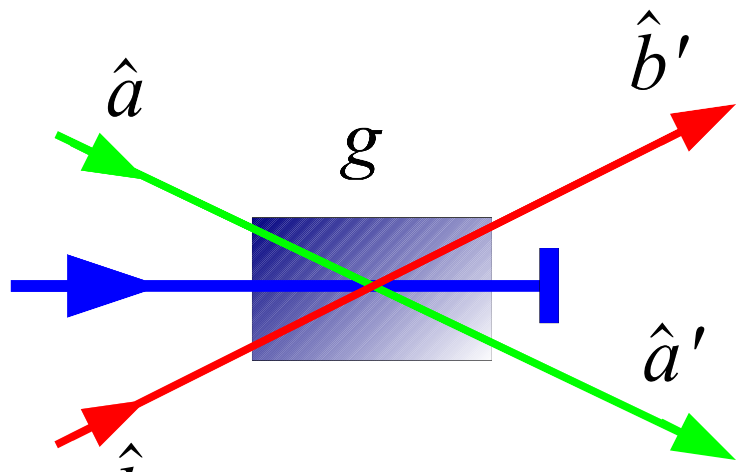
t = transmittance

ROTATION

$$\begin{cases} \hat{a}' = t\,\hat{a} + r\,\hat{b} \\ \hat{b}' = -r\,\hat{a} + t\,\hat{b} \end{cases}$$

with $\begin{cases} t = \cos\theta \\ r = \sin\theta \end{cases}$

Two-mode squeezer



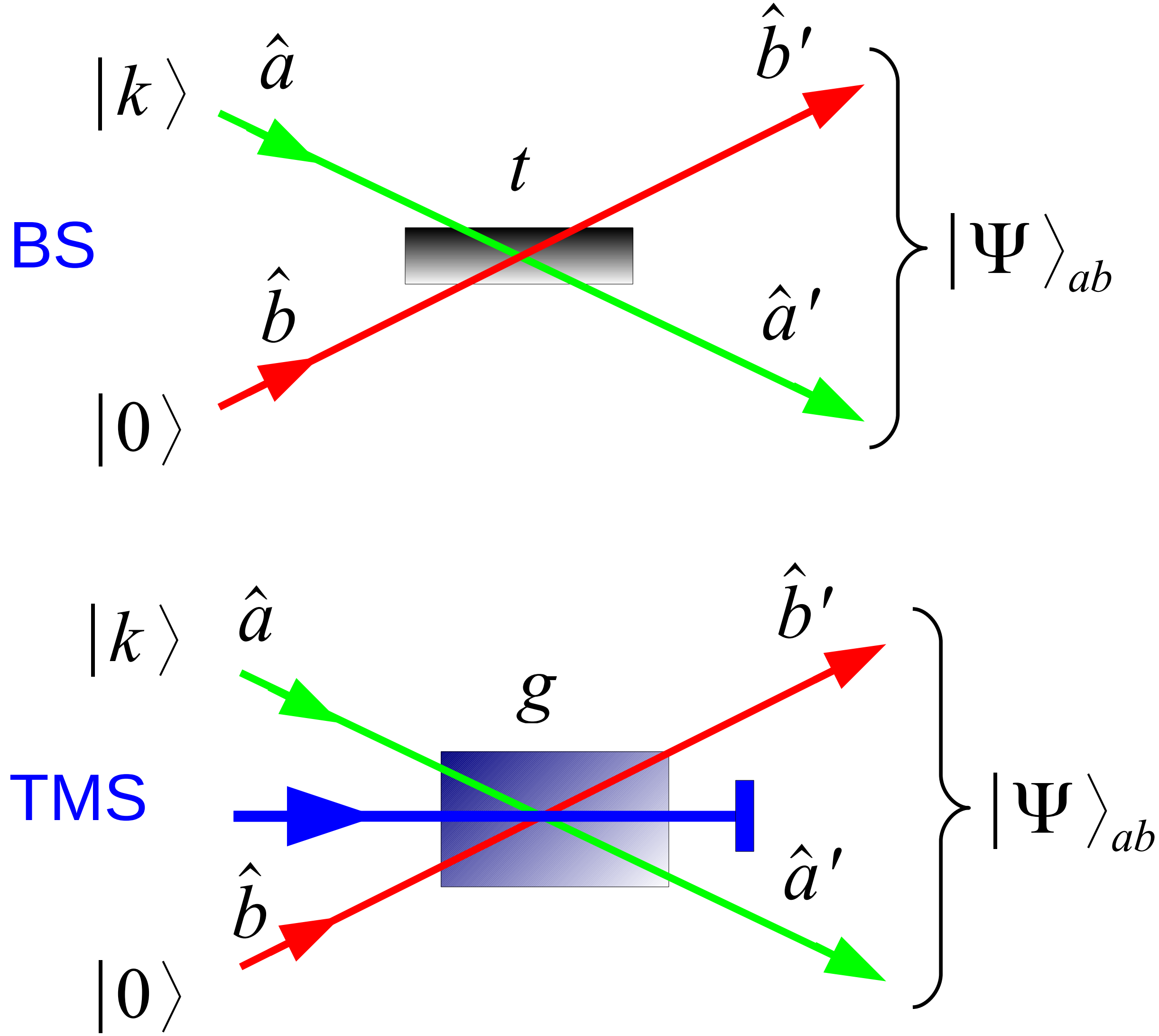
g = gain

AREA-PRESERVING SQUEEZE MAP

$$\begin{cases} \hat{a}' = g\,\hat{a} + s\,\hat{b}^{\dagger} \\ \hat{b}' = s\,\hat{a}^{\dagger} + g\,\hat{b} \end{cases}$$

with $\begin{cases} g = \cosh r \\ s = \sinh r \end{cases}$

BS & TMS examined under majorization theory



Majorization relations on states $|\Psi\rangle_{ab}$

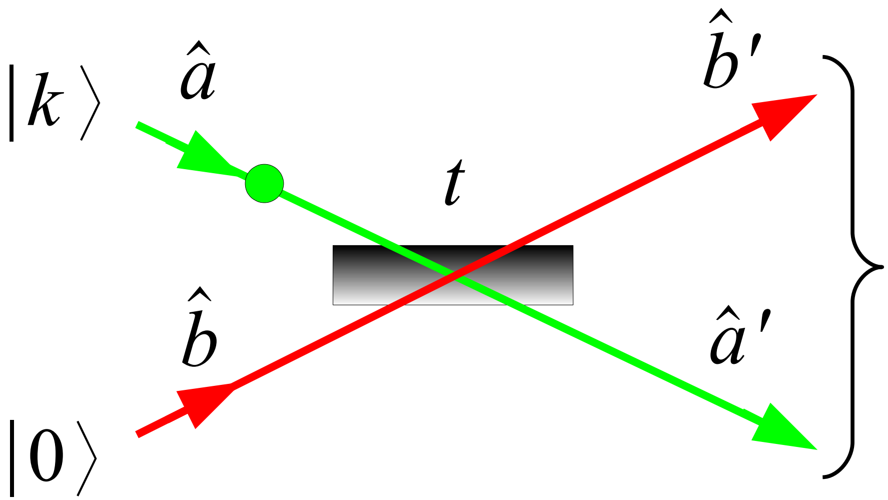
for Fock state inputs $|k\rangle \Rightarrow |k+1\rangle$

or for varying t or g

… implies monotonicity of the generated entanglement (or reduced output entropy)

Majorization ladder in a BS

(simplest case)



finite superposition

$$|\Psi^{(k)}\rangle = \sum_{i=0}^{k} \sqrt{p_i^{(k)}}\, |i\rangle_a |k-i\rangle_b$$

with $p_i^{(k)} = \binom{k}{i} t^{2i} r^{2(k-i)}$

$$p_i^{(k+1)} = \binom{k+1}{i} t^{2i} r^{2(k+1-i)}$$

use Pascal identity

$$= \left[\binom{k}{i-1} + \binom{k}{i}\right] t^{2i} r^{2(k+1-i)}$$

$$= t^2 p_{i-1}^{(k)} + r^2 p_i^{(k)}$$ with $t^2 + r^2 = 1$

$$\boldsymbol{p}^{(k+1)} = \underbrace{\begin{pmatrix} r^2 & 0 & 0 & \cdots \\ t^2 & r^2 & 0 & \cdots \\ 0 & t^2 & r^2 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}}_{\text{stochastic matrix } \mathbf{D}} \boldsymbol{p}^{(k)}$$

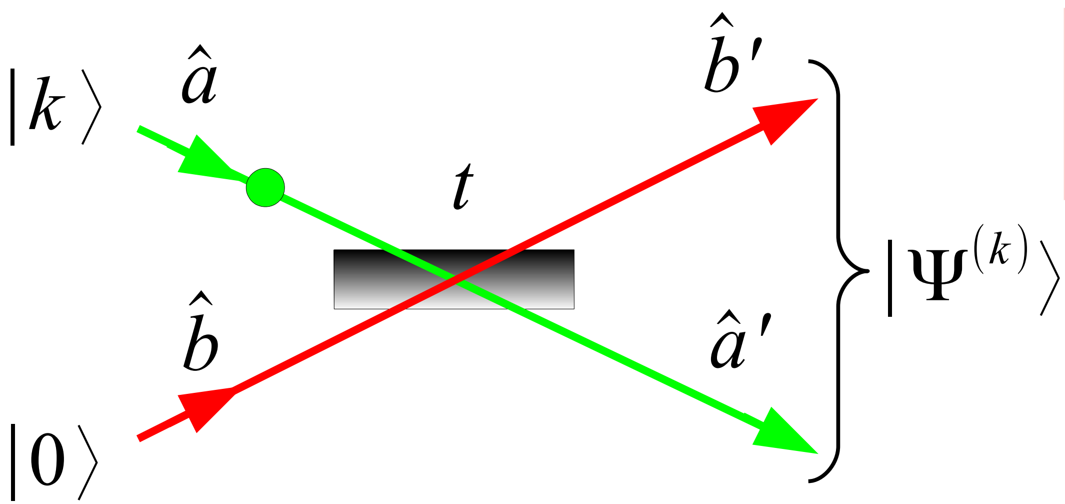
ladder of majorization relations

$$\mathbf{p}^{(k)} \succ \mathbf{p}^{(k+1)}$$

$$|\Psi^{(k)}\rangle \succ |\Psi^{(k+1)}\rangle$$

$$|\Psi^{(k)}\rangle \xleftarrow{\textbf{LOCC}} |\Psi^{(k+1)}\rangle$$

$$E\left(|\Psi^{(k)}\rangle\right) \leq E\left(|\Psi^{(k+1)}\rangle\right)$$

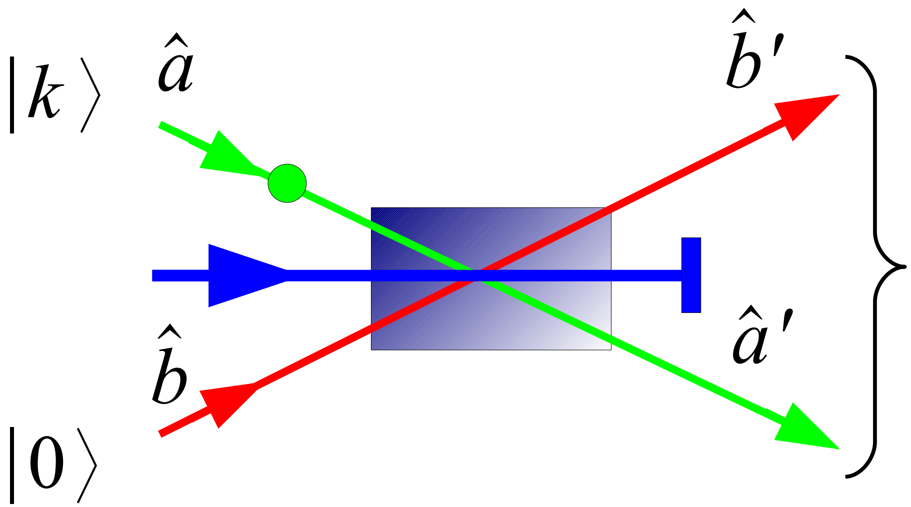


the more input photons,
the stronger the entanglement

C. N. Gagatsos, O. Oreshkov, and N. J. Cerf, PRA (2013).

Majorization ladder in a TMS

(related to the capacity
of Gaussian bosonic channels)



infinite superposition

$$|\Psi^{(k)}\rangle = \sum_{i=0}^{\infty} \sqrt{p_i^{(k)}}\, |i+k\rangle_a |i\rangle_b$$

with $p_i^{(k)} = (1-\lambda^2)^{k+1} \binom{i+k}{i} \lambda^{2i}$

and $\lambda = \tanh r$

r = squeezing param.

$$p_i^{(k+1)} = (1-\lambda^2)^{k+2} \binom{i+k+1}{i} \lambda^{2i}$$

use Pascal identity

$$= (1-\lambda^2)^{k+2} \left[\binom{i+k}{i-1} + \binom{i+k}{i} \right] \lambda^{2i}$$

different

$$= \lambda^2\, p_{i-1}^{(k+1)} + (1-\lambda^2)\, p_i^{(k)} \quad \text{with } 0 \le \lambda^2 \le 1$$

$$\boldsymbol{p}^{(k+1)} = (1-\lambda^2)\underbrace{\begin{pmatrix} 1 & 0 & 0 & \cdots \\ \lambda^2 & 1 & 0 & \cdots \\ \lambda^4 & \lambda^2 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}}_{\text{stochastic matrix } \mathbf{D}} \boldsymbol{p}^{(k)}$$

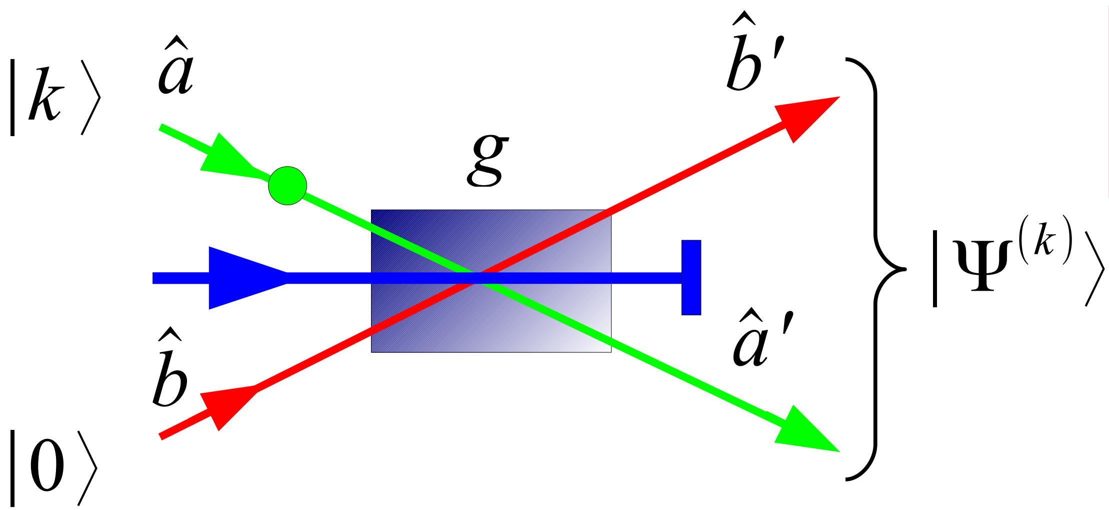
ladder of majorization relations

$$\mathbf{p}^{(k)} \succ \mathbf{p}^{(k+1)}$$

$$|\Psi^{(k)}\rangle \succ |\Psi^{(k+1)}\rangle$$

$$|\Psi^{(k)}\rangle \xleftarrow{\text{LOCC}} |\Psi^{(k+1)}\rangle$$

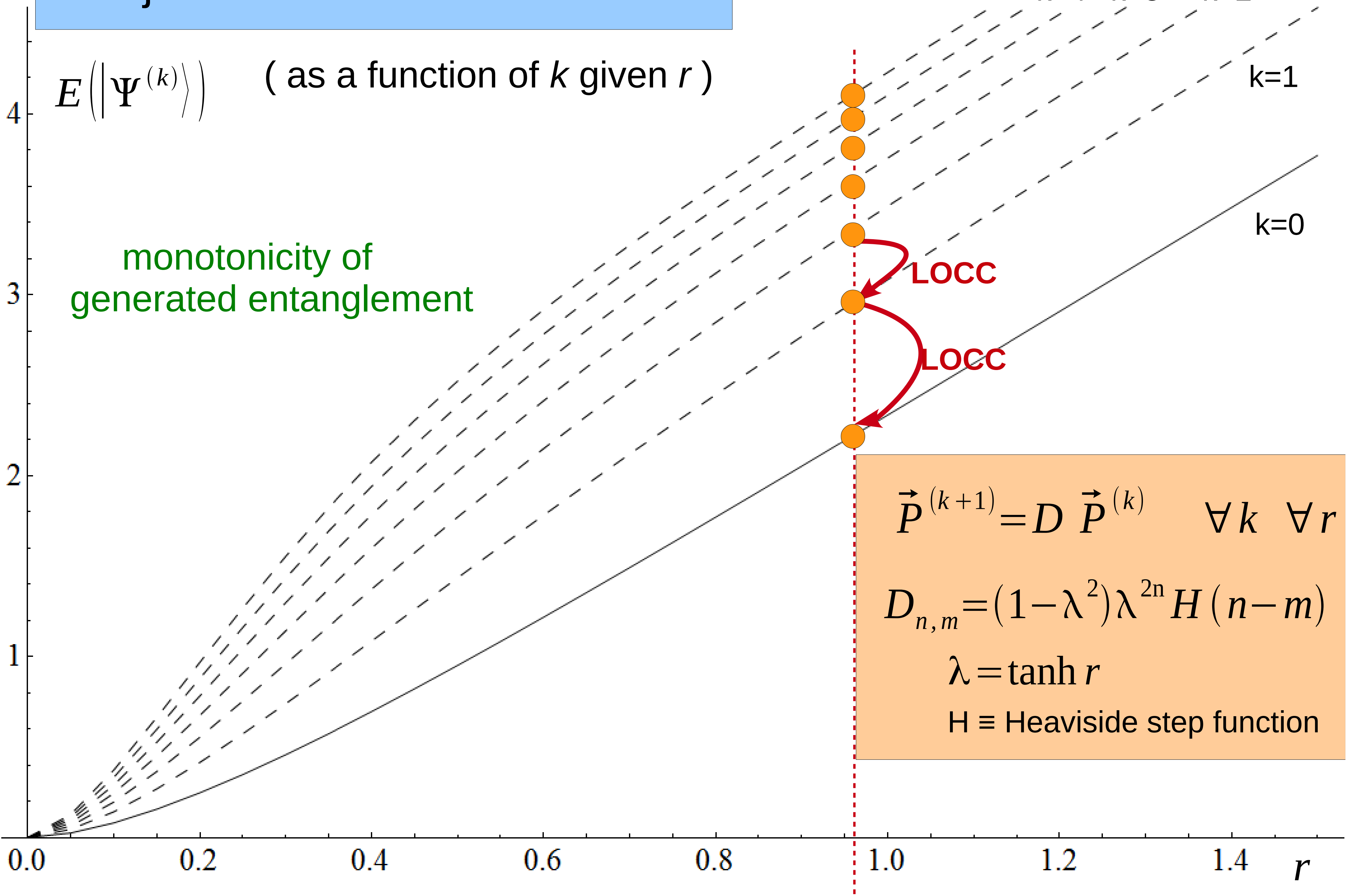
$$E\left(|\Psi^{(k)}\rangle\right) \leq E\left(|\Psi^{(k+1)}\rangle\right)$$



vacuum state “beats”
all other Fock states

R. Garcia-Patron, C. Navarrete-Benlloch,
S. Lloyd, J. H. Shapiro, and N. J. Cerf, PRL (2012).

Majorization relations in a TMS



$$\vec{P}^{(k+1)} = D\ \vec{P}^{(k)} \qquad \forall k \quad \forall r$$

$$D_{n,m} = (1-\lambda^2)\lambda^{2n} H(n-m)$$

$$\lambda = \tanh r$$

H ≡ Heaviside step function

Explicit LOCC

$$|\Psi_k\rangle = \sum_{n=0}^{\infty} \sqrt{p_n(k)}\ |n+k\rangle|n\rangle$$

$$|\Psi_{k+1}\rangle = \sum_{n=0}^{\infty} \sqrt{p_n(k+1)}\ |n+k+1\rangle|n\rangle$$

Alice applies a POVM

$$\begin{cases} A_{YES} = \sum_{m=0}^{\infty} \sqrt{\dfrac{(1-\lambda^2)\,p_m(k)}{p_m(k+1)}}\ |m+k\rangle\langle m+k+1| \\ A_{NO} = \sum_{m=0}^{\infty} \sqrt{\dfrac{\lambda^2\, p_{m-1}(k+1)}{p_m(k+1)}}\ |m+k\rangle\langle m+k+1| \end{cases}$$

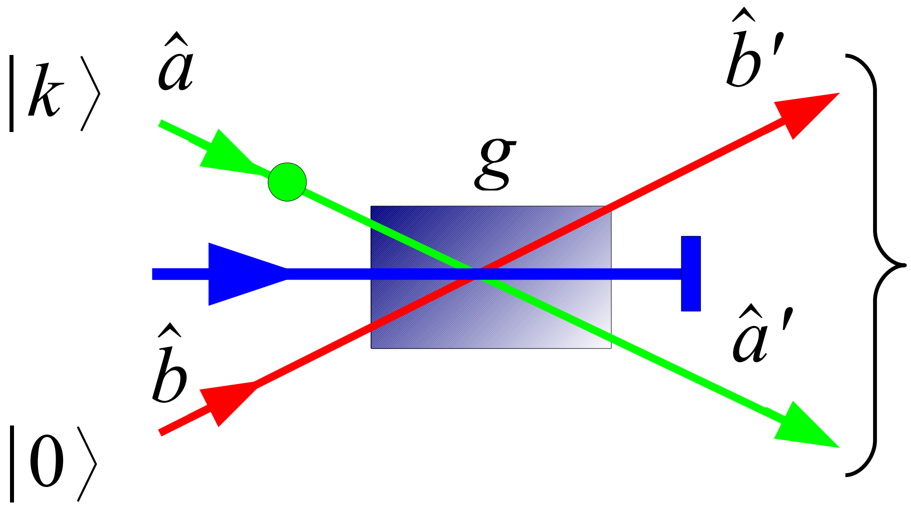
$$(A_{YES}\times 1)\,|\Psi_{k+1}\rangle = \sqrt{(1-\lambda^2)}\ \sum_{n=0}^{\infty} \sqrt{p_n(k)}\ |n+k\rangle|n\rangle = \sqrt{(1-\lambda^2)}\ \boxed{|\Psi_k\rangle}\ \text{YES}$$

$$(A_{NO}\times 1)\,|\Psi_{k+1}\rangle = \sqrt{\lambda^2}\ \sum_{n=0}^{\infty} \sqrt{p_n(k+1)}\ |n+k+1\rangle\boxed{|n+1\rangle} \rightarrow \sqrt{\lambda^2}\ \boxed{|\Psi_{k+1}\rangle}\ \text{NO}$$

If "NO" she communicates it to Bob who applies $U = \sum_{m=0}^{\infty} |m\rangle\langle m+1|$ and then they start a new round again

Parametric majorization relations in a TMS

(as a function of gain)



$|\Psi^{(k)}(g)\rangle$ for $g \rightarrow g+\epsilon \quad (\epsilon > 0)$

$$\mathbf{p}^{(k)}(g) \succ \mathbf{p}^{(k)}(g+\epsilon)$$

$$|\Psi^{(k)}(g)\rangle \succ |\Psi^{(k)}(g+\epsilon)\rangle$$

increasing the gain gives rise to a monotonous increase of entanglement

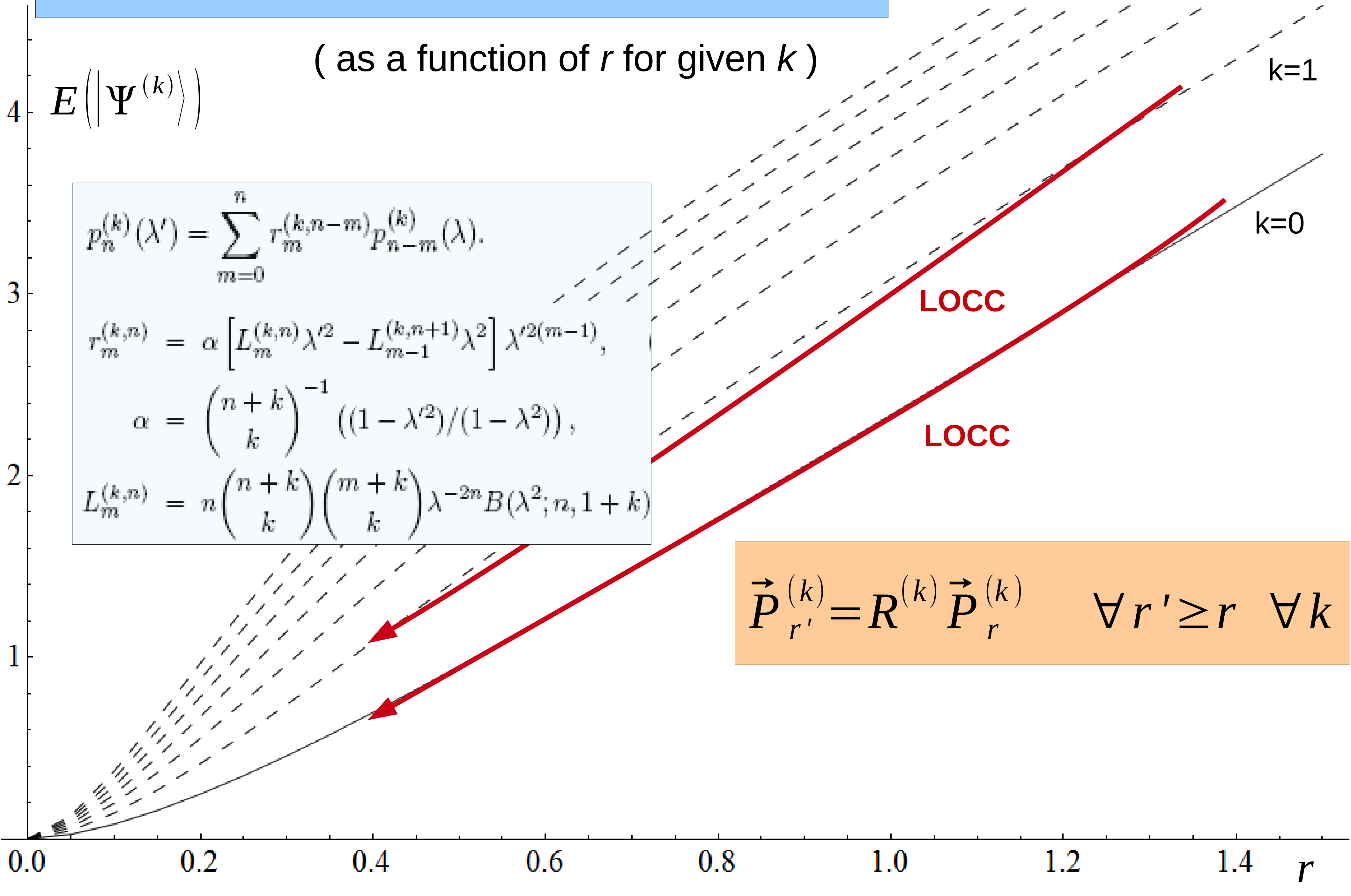
$$|\Psi^{(k)}(g)\rangle \xleftarrow{\text{LOCC}} |\Psi^{(k)}(g+\epsilon)\rangle$$

$$E\left(|\Psi^{(k)}(g)\rangle\right) \leq E\left(|\Psi^{(k)}(g+\epsilon)\rangle\right)$$

R. Garcia-Patron, C. Navarrete-Benlloch,
S. Lloyd, J. H. Shapiro, and N. J. Cerf, PRL (2012).

Parametric majorization relations in a TMS

(as a function of *r* for given *k*)



$$p_n^{(k)}(\lambda') = \sum_{m=0}^{n} r_m^{(k,n-m)} p_{n-m}^{(k)}(\lambda).$$

$$r_m^{(k,n)} = \alpha\left[L_m^{(k,n)}\lambda'^2 - L_{m-1}^{(k,n+1)}\lambda^2\right]\lambda'^{2(m-1)},$$

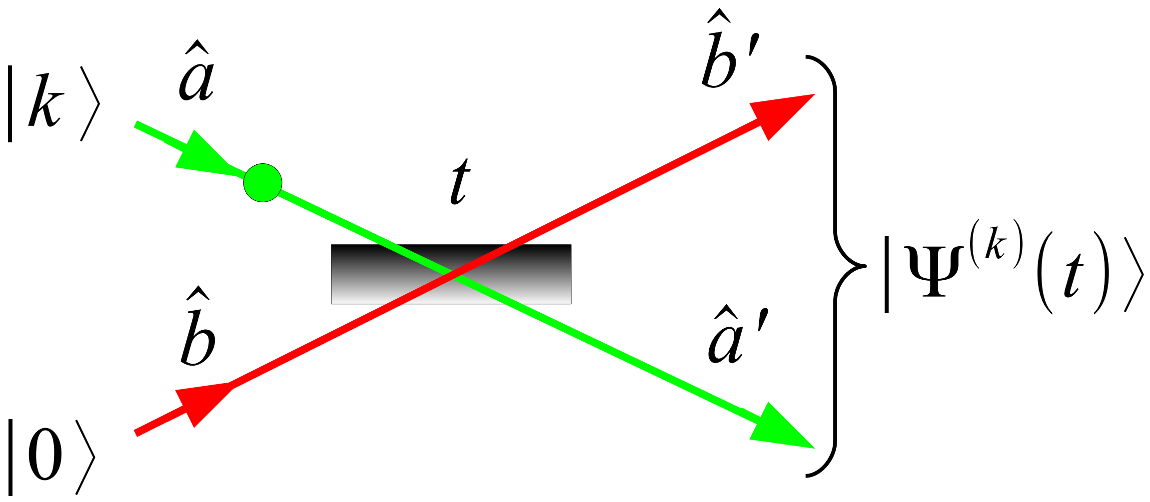
$$\alpha = \binom{n+k}{k}^{-1}\left((1-\lambda'^2)/(1-\lambda^2)\right),$$

$$L_m^{(k,n)} = n\binom{n+k}{k}\binom{m+k}{k}\lambda^{-2n}B(\lambda^2;n,1+k)$$

$$\vec{P}_{r'}^{(k)} = R^{(k)}\vec{P}_r^{(k)} \qquad \forall r' \geq r \quad \forall k$$

Parametric majorization relations in a BS

(as a function of transmittance)



for $0 \leq t \leq t+\epsilon \leq 1/2$

$$\boldsymbol{p}^{(k)}(t) \succ \boldsymbol{p}^{(k)}(t+\epsilon)$$

$$|\Psi^{(k)}(t)\rangle \succ |\Psi^{(k)}(t+\epsilon)\rangle$$

sometimes yes ...

$$|\Psi^{(k)}(t)\rangle \xleftarrow{\text{LOCC}} |\Psi^{(k)}(t+\epsilon)\rangle$$

$$E(|\Psi^{(k)}(t)\rangle) \leq E(|\Psi^{(k)}(t+\epsilon)\rangle)$$

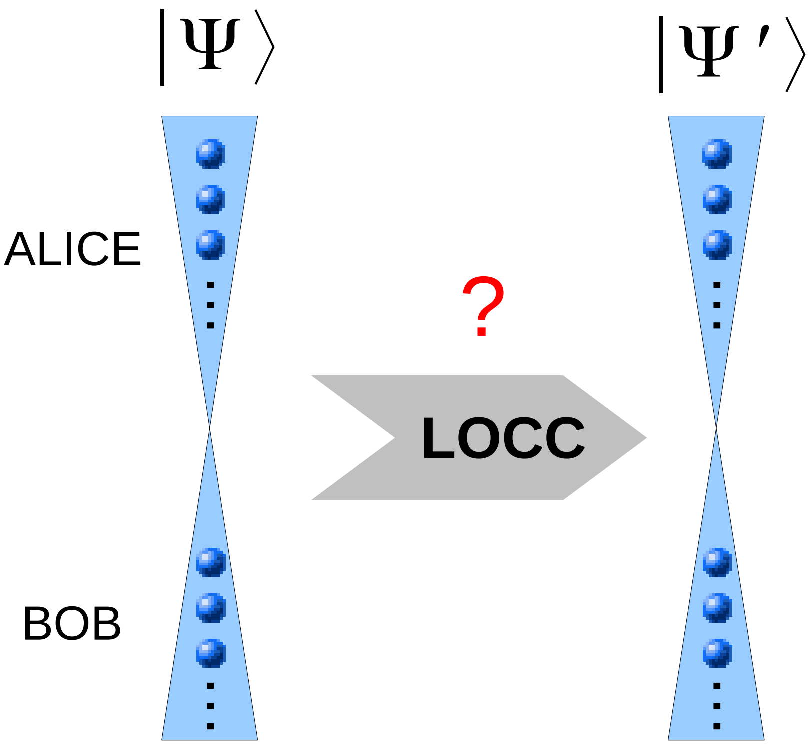
More complicated situation :
increasing the transmittance gives rise to a monotonous entanglement increase only in some regions for *t*

C. N. Gagatsos, O. Oreshkov, and N. J. Cerf, PRA (2013).

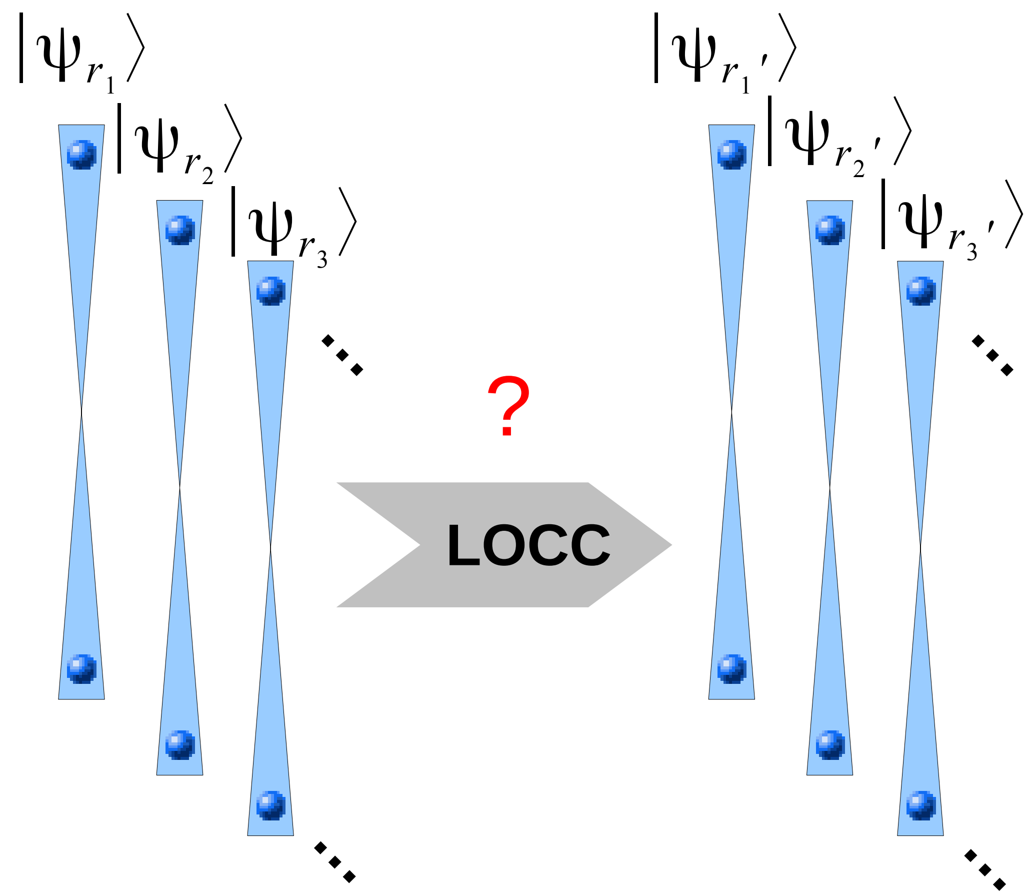
… otherwise, catalysis effects may also be exhibited !

2 Interconversion of entangled Gaussian states

N x N pure Gaussian states



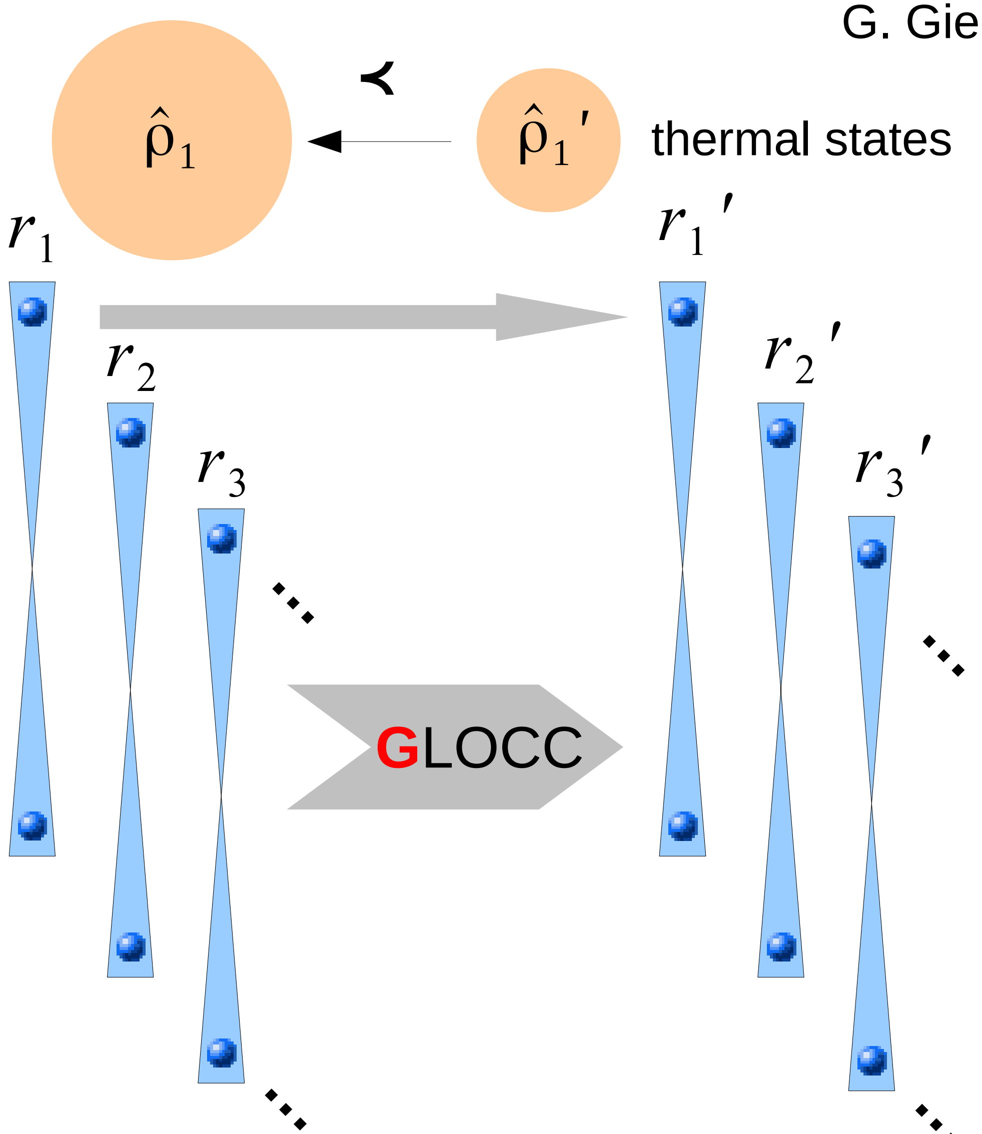
… use normal form decomposition



product of EPR states

Necess. & suffic. condition for conversion by GLOCC

G. Giedke, J. Eisert, J.I. Cirac, M.B. Plenio (2003).



iff

$$\begin{cases} r_1 \geq r_1{}' \\ r_2 \geq r_2{}' \\ r_3 \geq r_3{}' \\ \vdots \end{cases}$$

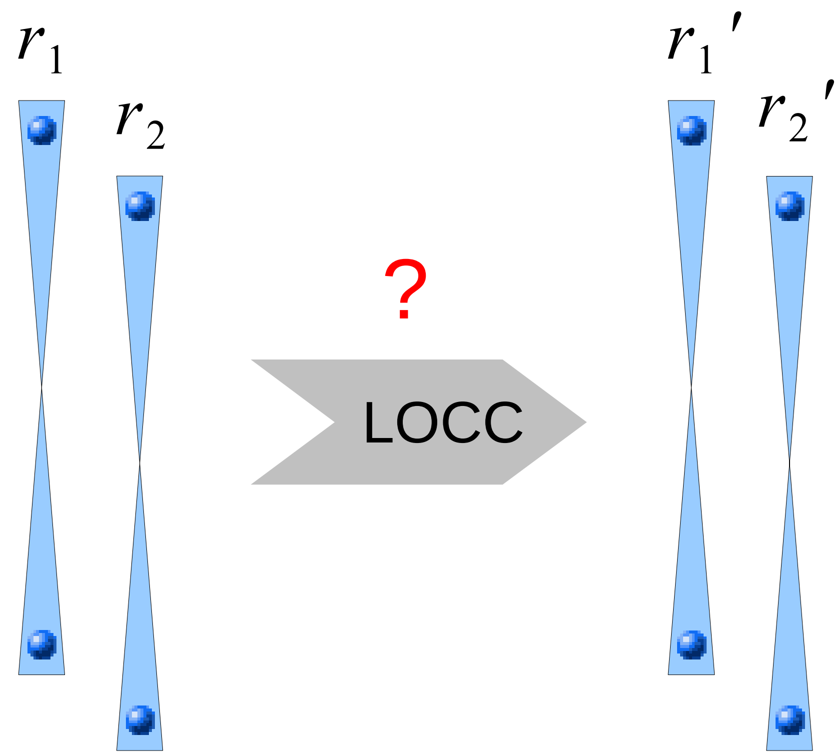
simple explanation in terms of parametric majorization for TMS

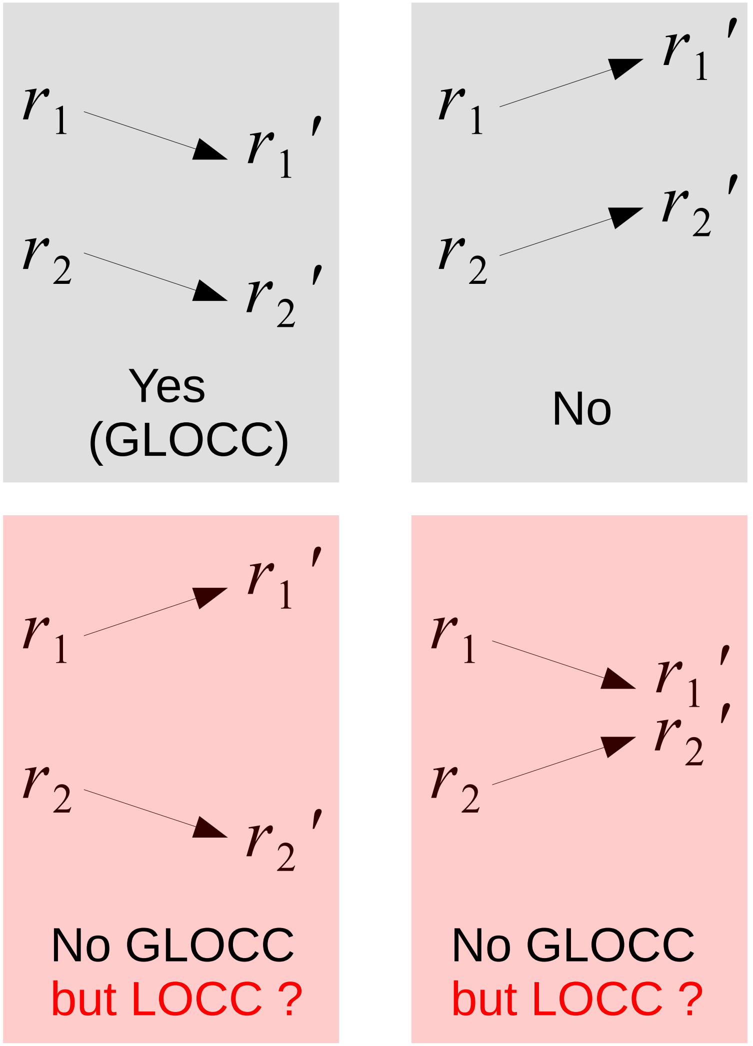
$\Rightarrow$ individual GLOCC's

Conversion with non-Gaussian LOCC ?

(special case N=2)

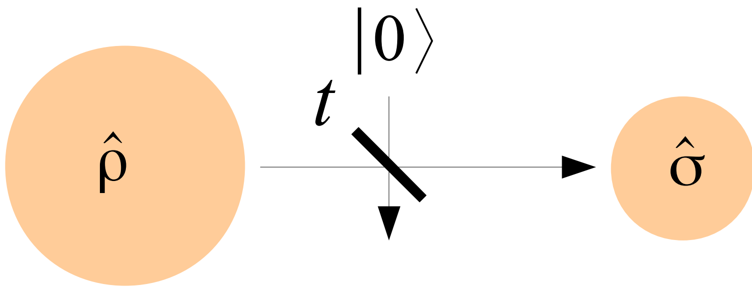
$r_1 \geq r_2$ $\qquad$ $r_1' \geq r_2'$

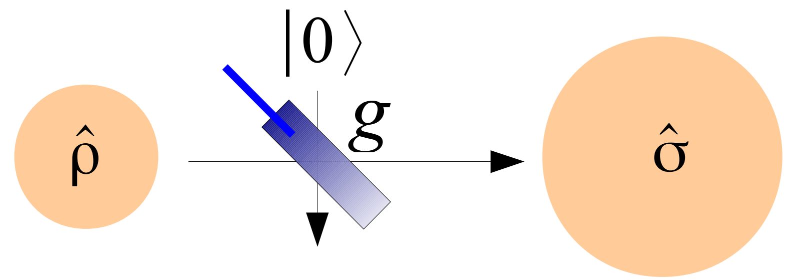




2 interesting possibilities
when GLOCC are impossible
but LOCC may do the trick !

Solution : majorization relations between Fock-diag states





$$\sum_n p_n |n\rangle\langle n| \nsucc \sum_n q_n |n\rangle\langle n| \qquad \sum_n p_n |n\rangle\langle n| \succ \sum_n q_n |n\rangle\langle n|$$

$\mathbf{q} = D_t\ \mathbf{p}$ … no majorization

$$\sum_{\text{rows}} D_t = 1/t^2 \geq 1 \qquad \sum_{\text{columns}} D_t = 1$$

$\mathbf{q} = D_g\ \mathbf{p}$ … majorization

$$\sum_{\text{rows}} D_g = 1/g^2 \leq 1 \qquad \sum_{\text{columns}} D_g = 1$$

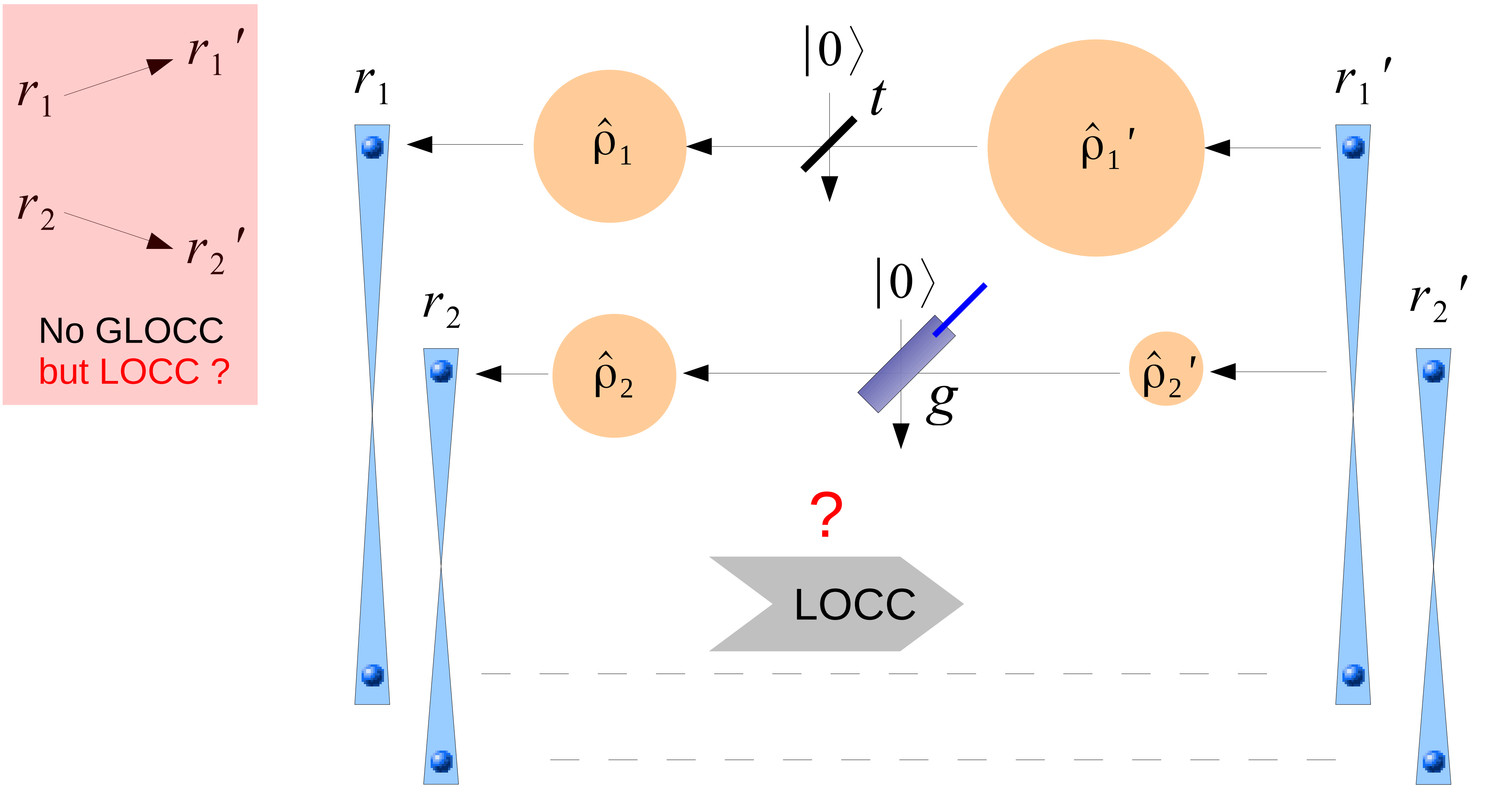
Tensor product of these channels ?

$$\mathbf{q_1} \otimes \mathbf{q_2} = D_{t,g}\ \ \mathbf{p_1} \otimes \mathbf{p_2}$$

$$\sum_{\text{rows}} D_{t,g} = \frac{1}{t^2 g^2} \gtrless 1 \qquad \sum_{\text{columns}} D_{t,g} = 1$$

Majorization $\hat{\rho}_1 \otimes \hat{\rho}_2 \succ \hat{\sigma}_1 \otimes \hat{\sigma}_2$

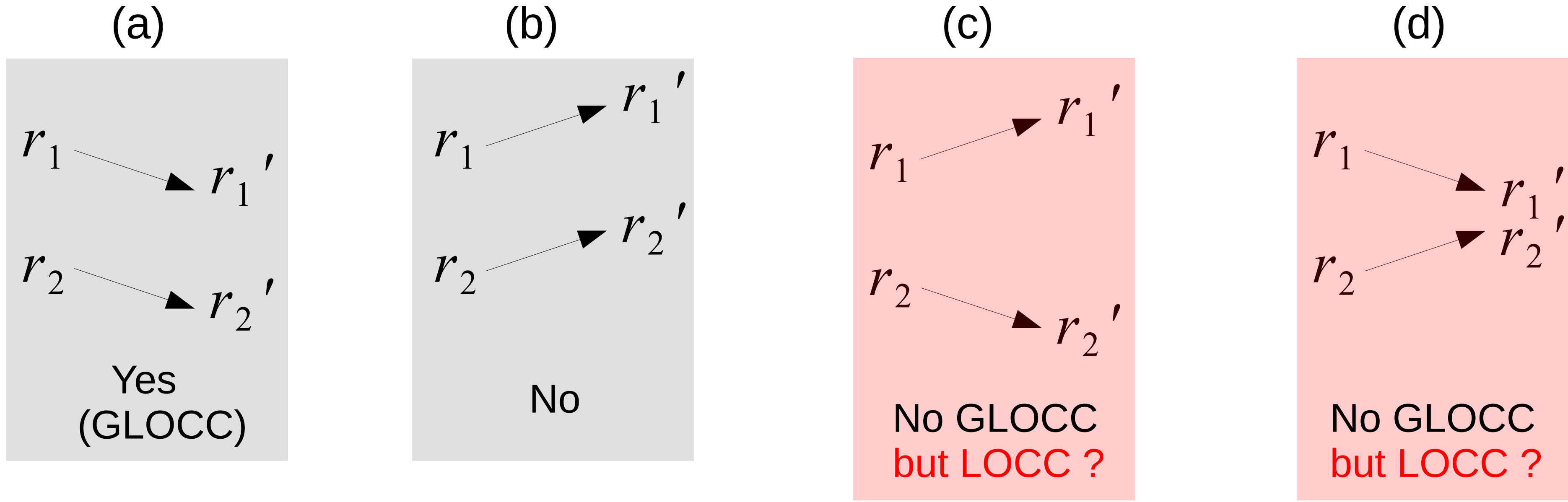
if $g \geq 1/t$



LOCC possible iff $\hat{\rho}_1 \otimes \hat{\rho}_2 \prec \hat{\rho}_1' \otimes \hat{\rho}_2'$ if $g \geq 1/t$ (sufficient condition)

… translates into a condition on squeezing param.: $$\frac{\sinh(r_1+r_2) \pm \sinh(r_1-r_2)}{\sinh(r_1'+r_2') \pm \sinh(r_1'-r_2')} \geq 1$$

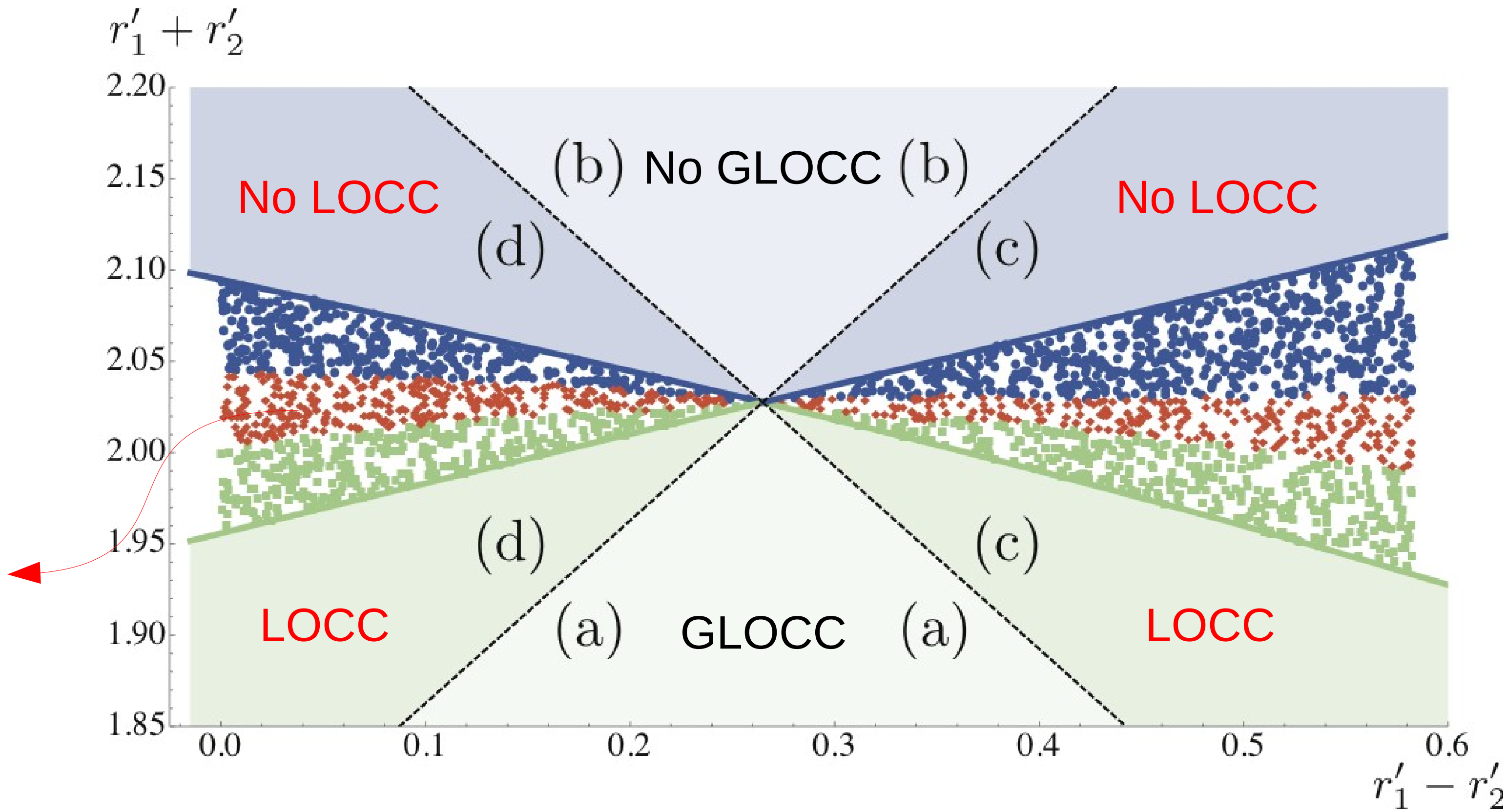
M. Jabbour, R. Garcia-Patron, and N. J. Cerf, PRA (2015).



$r_1 = 1.15 \;\; (\sim 10\,\mathrm{dB})$

$r_2 = 0.88 \;\; (\sim 7\,\mathrm{dB})$

INCOMPARABLE STATES



M. Jabbour, R. Garcia-Patron, and N. J. Cerf, PRA (2015).

3 Application to bosonic Gaussian channels

$$\rho \Longrightarrow \boxed{\text{map } M \begin{cases} K \\ N \end{cases}} \Longrightarrow M[\rho]$$

- Corresponds to linear CP maps $\rho \to M[\rho]$

  s.t. $M[\rho]$ Gaussian if ρ Gaussian

- M fully characterized by two matrices K, N (single-mode case)

  $\vec{r} \to K\vec{r}$ $\qquad$ $\vec{r}$ = coherent vector

  $\gamma \to K\gamma K^T + N$ $\qquad$ γ = covariance matrix

  real $\qquad$ real & symmetric

- M completely positive $\Longleftrightarrow$

  one-mode case

  $$\boxed{N \geq 0 \qquad det\, N \geq (det\, K - 1)^2}$$

  uncertainty principle

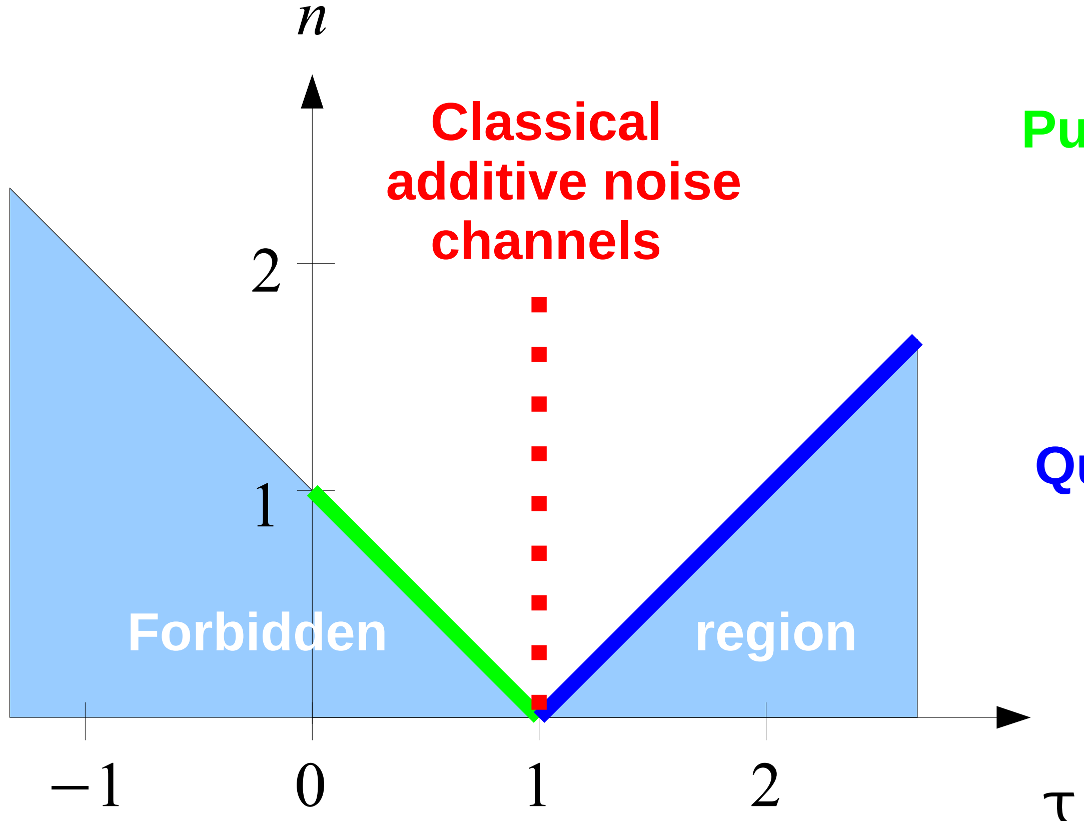
Phase-insensitive Gaussian channels

$$\begin{cases} K = \mathrm{diag}\left(\sqrt{\tau}, \sqrt{\tau}\right) \\ N = \frac{1}{2}\mathrm{diag}\left(n, n\right) \end{cases}$$

$$\rho \;\Longrightarrow\; \boxed{M \begin{cases} \tau = \text{transmission} \\ n = \text{noise variance} \end{cases}} \;\Longrightarrow\; M[\rho]$$

$$\vec{r} \rightarrow \sqrt{\tau}\,\vec{r} \qquad \vec{r} = \text{coherent vector}$$

$$\gamma \rightarrow \tau\,\gamma + n\,I \qquad \gamma = \text{covariance matrix}$$



Pure loss channels

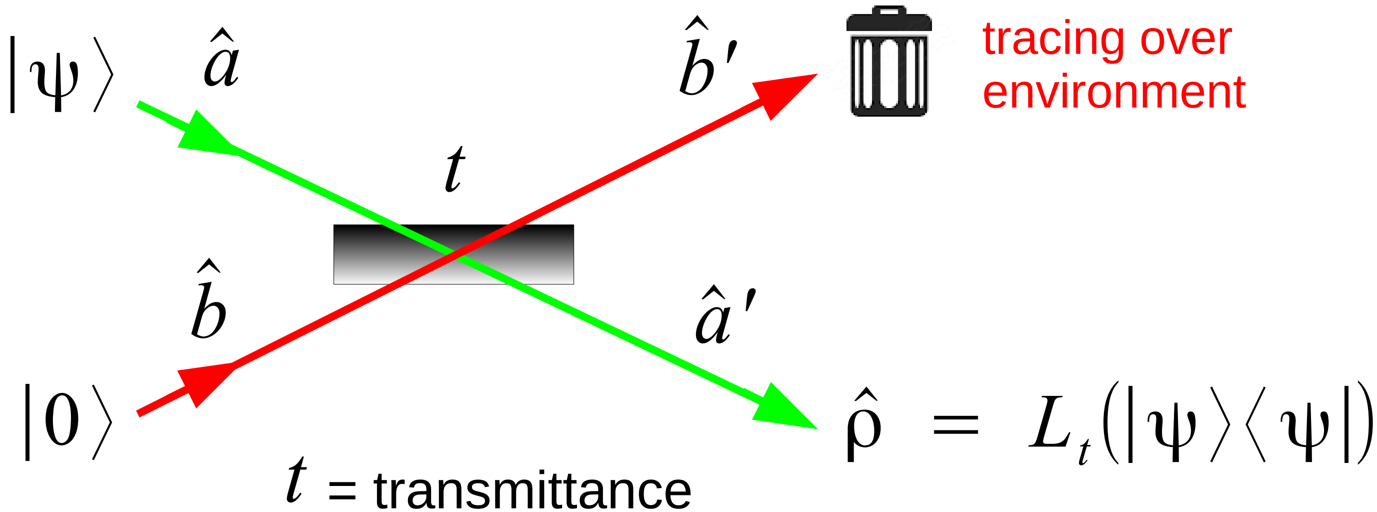
$$\begin{cases} \tau = t^2 \leq 1 \\ n = 1 - t^2 \end{cases}$$

Quantum-limited (ideal) amplifiers

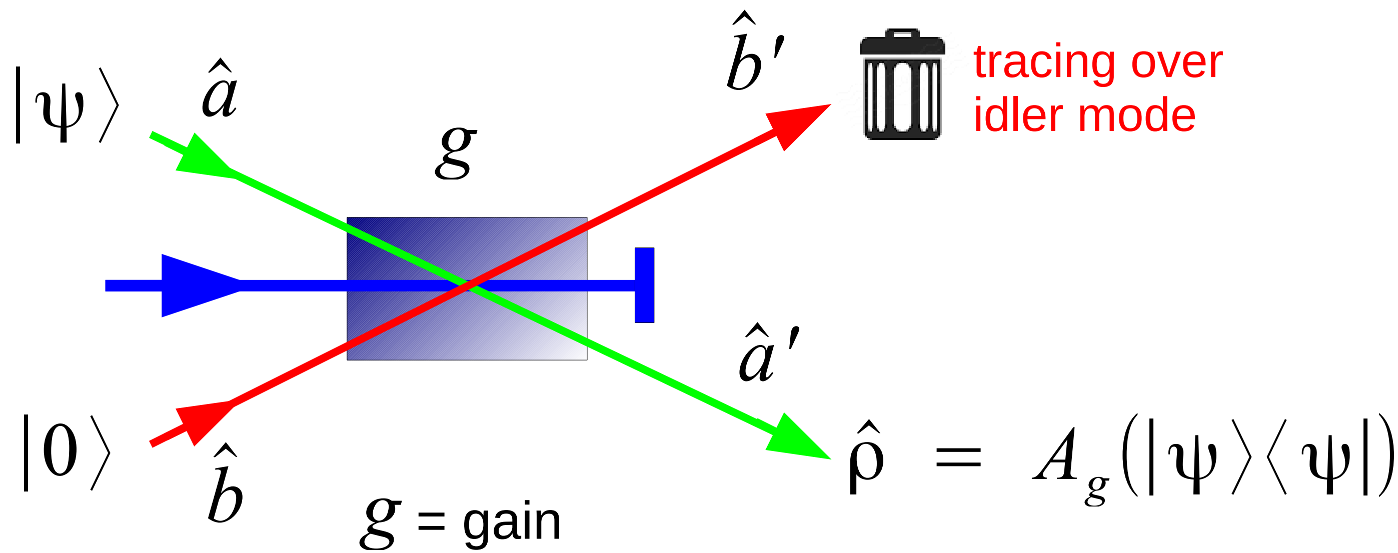
$$\begin{cases} \tau = g^2 \geq 1 \\ n = g^2 - 1 \end{cases}$$

Stinespring dilation of pure-loss / ideal-ampli channels

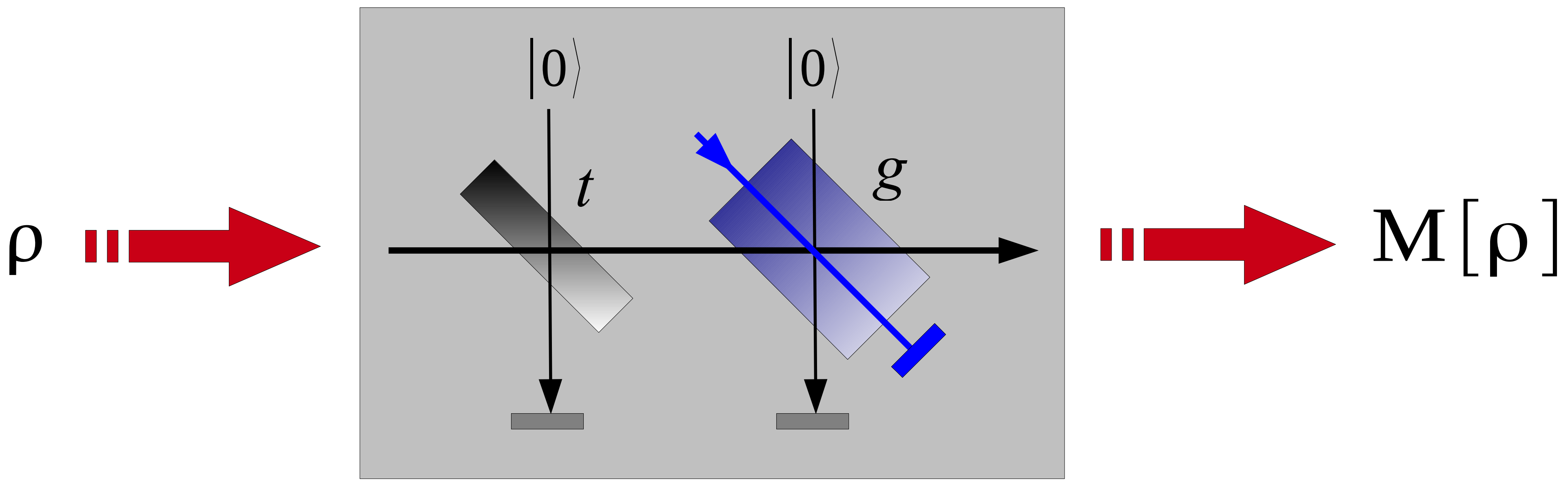
BS ≡ pure loss channel



TMS ≡ quantum limited amplifier



Decomposition of all phase-insensitive Gaussian channels



$$\begin{cases} \text{transmission} \quad \tau = t^2\, g^2 \\ \text{noise variance} \quad n = g^2(1-t^2)+(g^2-1) \end{cases}$$

Capacity of bosonic Gaussian channels

Yuen and Ozawa, 1993

Holevo and Werner, 1998

$$\rho_\alpha \Longrightarrow \mathrm{M} \begin{cases} \tau = \text{transmission} \\ n = \text{noise variance} \end{cases} \Longrightarrow \mathrm{M}[\rho_\alpha]$$

- continuous encoding $\{p(\alpha), \rho_\alpha\}$ such that $\int d^2\alpha \; p(\alpha)\, \rho_\alpha = \rho$
- energy constraint on average input state ρ $\longrightarrow$ $\mathrm{Tr}(\rho\, \hat{n}) = \nu$

$$C^{(1)}(\mathrm{M}) = max_\rho \; \tilde{\chi}(\rho, \mathrm{M})$$

with $\tilde{\chi}(\rho, \mathrm{M}) \equiv S(\mathrm{M}[\rho]) - \underbrace{min_{\{p_\alpha, \rho_\alpha\}} \int d^2\alpha \; p(\alpha)\, S(\mathrm{M}[\rho_a])}_{\geq\, min_\rho S(\mathrm{M}[\rho])}$

$$C^{(1)}(\mathrm{M}) \leq \underbrace{max_\rho \; S(\mathrm{M}[\rho])}_{\substack{\text{for fixed energy,} \\ \text{achieved by a thermal state} \\ S(\mathrm{M}[\rho_{therm}])}} - \underbrace{min_\rho \; S(\mathrm{M}[\rho])}_{S(\mathrm{M}[\Phi_0])}$$

Φ_0 = **pure state**

minimizing output entropy

Minimum output entropy

$$C^{(1)}(\mathrm{M}) \le S\left(\mathrm{M}[\rho_{therm}]\right) - S\left(\mathrm{M}[\Phi_0]\right)$$

V. Giovannetti *et al.*, 2004

Gaussian minimum entropy conjecture: $\Phi_0 = |0\rangle\langle 0|$

- Gaussian coherent-state encoding saturates this upper bound
- Regularization (*n* uses of the channel) comes for free if conjecture holds

$$C(M) = C^{(1)}(M) = g[\tau\nu + n] - g[n]$$

Holevo and Werner, 1998

Provided conjecture is true

with ν = mean photon number

and $g[x] = (x+1)\log(x+1) - x\log(x)$

Proof of Gaussian conjecture

$$\Phi_0 = |0\rangle\langle 0| \Rightarrow \mathrm{M} \begin{cases} \tau = \text{transmission} \\ n = \text{noise variance} \end{cases} \Rightarrow \mathrm{M}[\Phi_0]$$

least disordered output state

- **Reduction to amplifier** & **majorization ladder** with Fock states proves $\Phi_0 = |0\rangle\langle 0|$ only for rotation-invariant input states

R. Garcia-Patron, C. Navarrete-Benlloch, S. Lloyd, J. H. Shapiro, and N. J. Cerf, PRL (2012).

- Vacuum $\Phi_0 = |0\rangle\langle 0|$ achieves **global minimum entropy**

$$S(\mathrm{M}[\Phi_0]) \leq S(\mathrm{M}[\rho]) \qquad \forall \rho$$

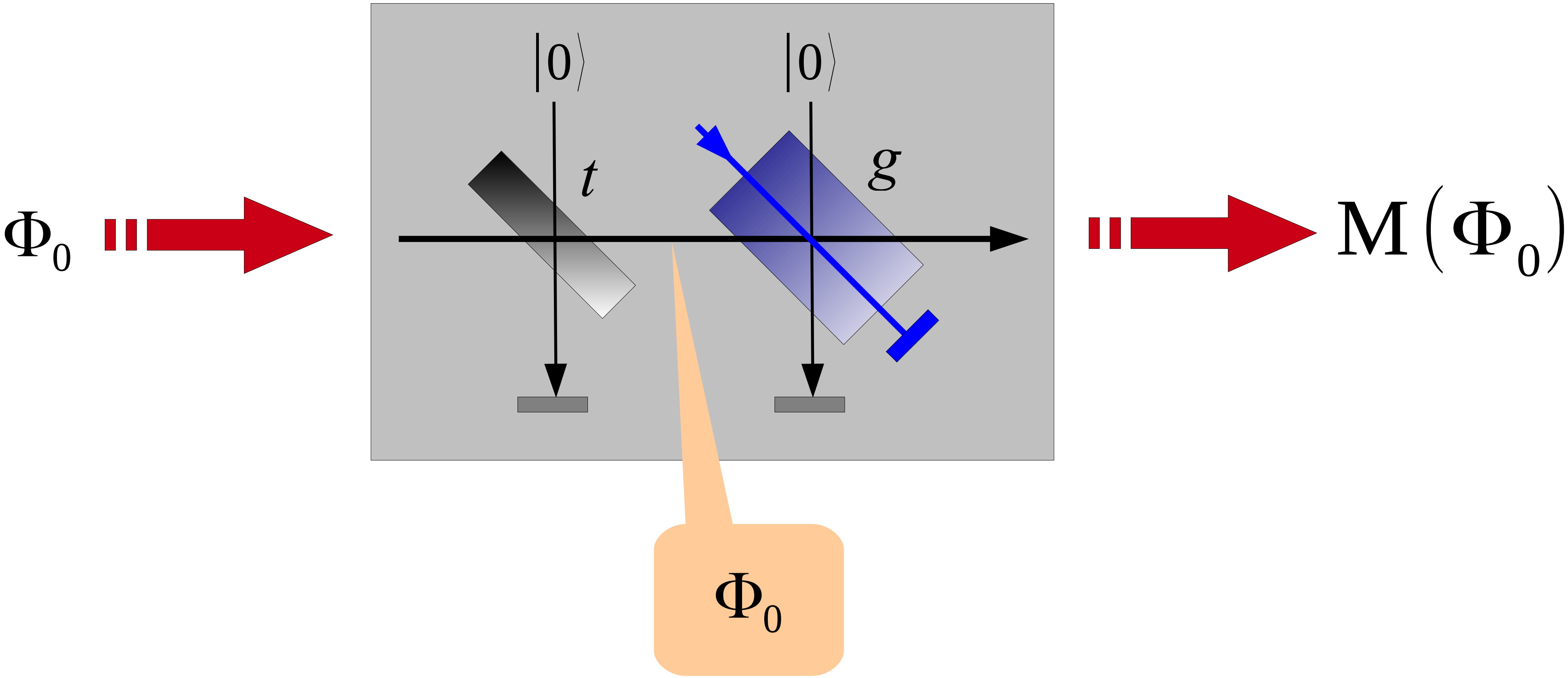
V. Giovannetti, R. Garcia-Patron, N. J. Cerf, and A. S. Holevo, Nature Phot. (2014).

- Vacuum $\Phi_0 = |0\rangle\langle 0|$ actually yields **global majorizing state**

$$\mathrm{M}[\Phi_0] \succ \mathrm{M}[\rho] \qquad \forall \rho$$

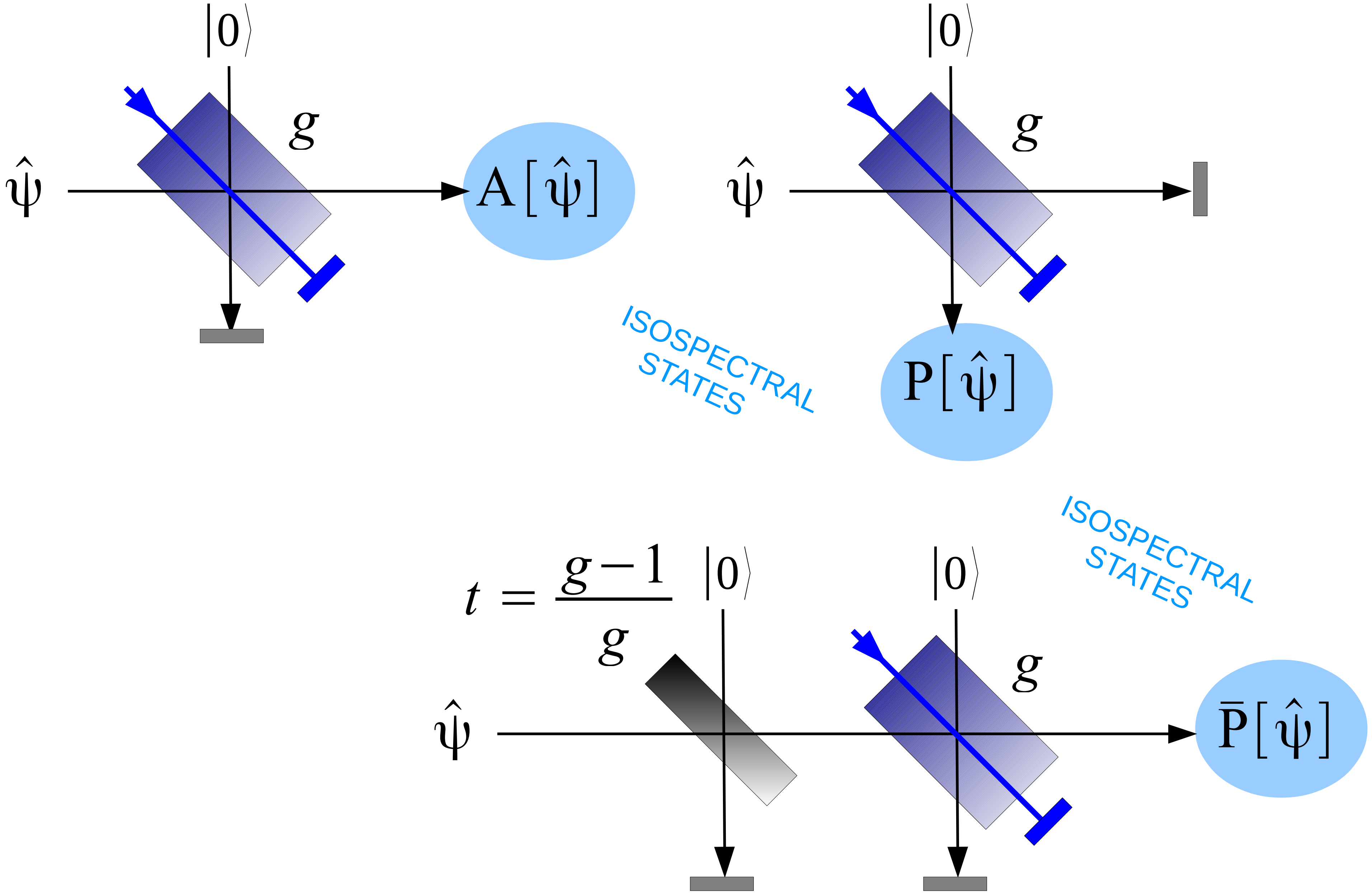
A. Mari, V. Giovannetti, and A. S. Holevo, Nature Comm. (2014).

Reduction to quantum-limited amplifier

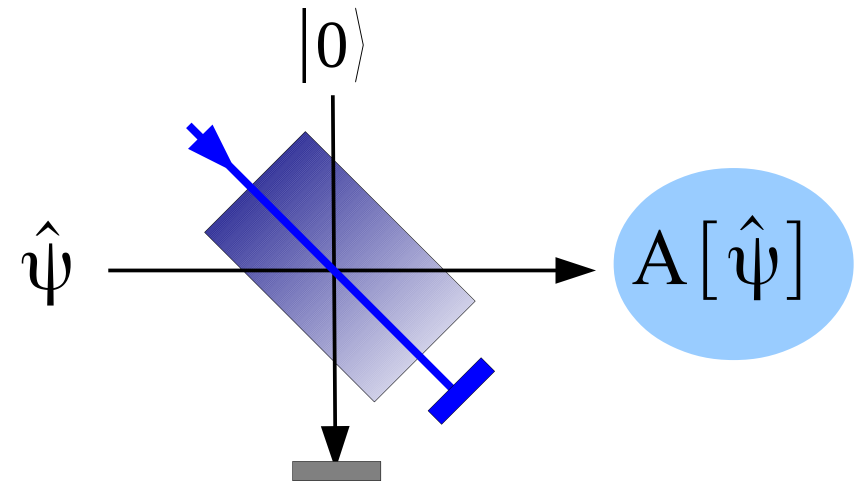


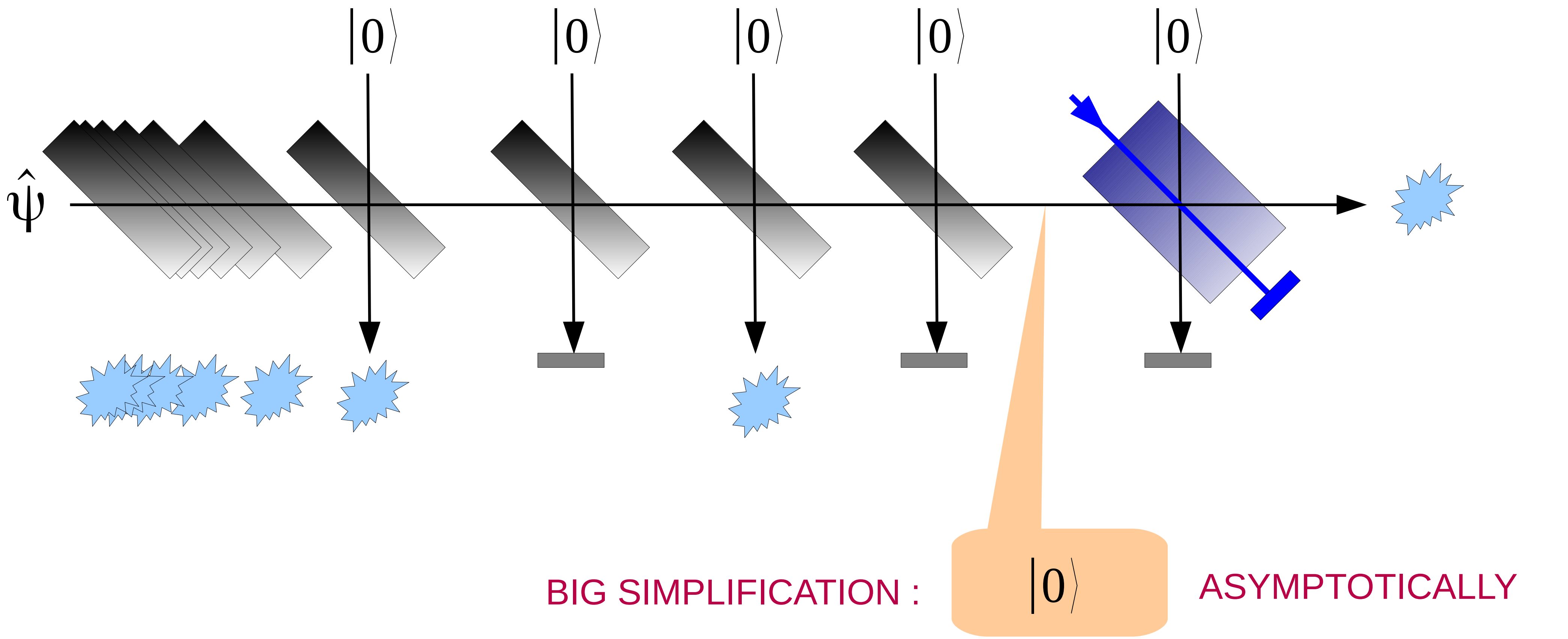
Vacuum $\Phi_0 = |0\rangle\langle 0|$ is a **fixed point** of beam splitter, hence it is sufficient to prove conjecture for quantum-limited amplifier

Complementary channel = phase-conjugating channel



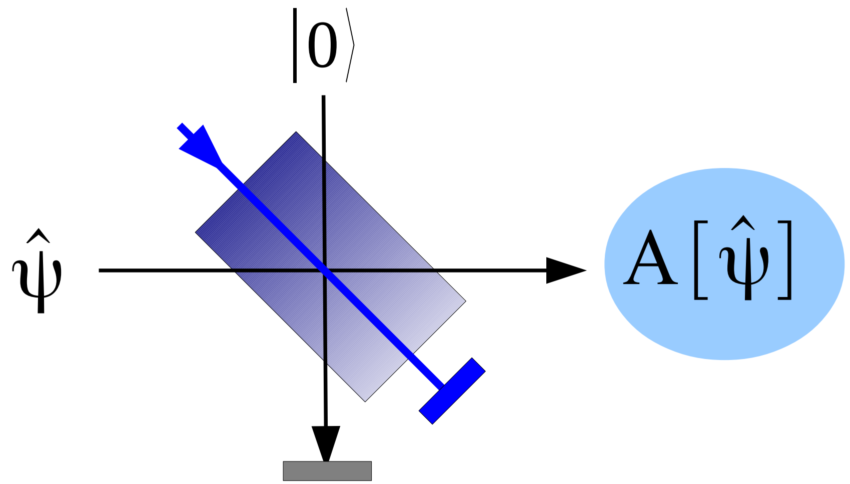
Infinite chain of isospectral states

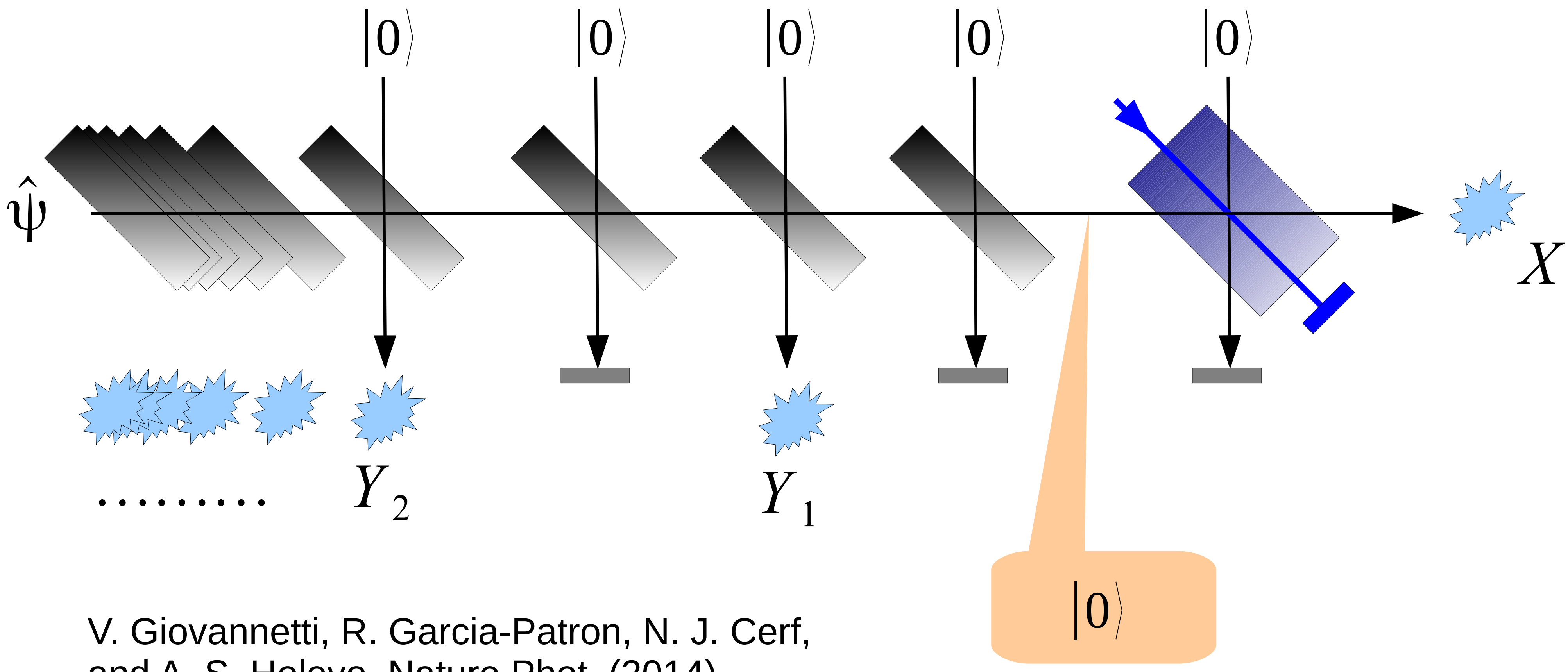




PROOF OF THE ENTROPY CONJECTURE

$$
\begin{aligned}
S\big(A[\hat{\psi}]\big) &= S\big(X, Y_1, Y_2, \cdots\big) \\
&\simeq \underbrace{S\big(X\big)}_{S(M[\Phi_0])} + \underbrace{S\big(Y_1, Y_2, \cdots\big)}_{\geq 0}
\end{aligned}
$$





V. Giovannetti, R. Garcia-Patron, N. J. Cerf, and A. S. Holevo, Nature Phot. (2014).

Generalization to other functionals

For <u>any functional</u> $F(\rho)$ that obeys the properties

- non-negative $F(\rho) \geq 0$
- unitarily invariant $F(U \rho U^{\dagger}) = F(\rho)$
- strictly concave $F\left(w \rho_1 + (1-w)\rho_2\right) \geq w F(\rho_1) + (1-w) F(\rho_2)$

$$\boxed{F\left(\mathrm{M}[\Phi_0]\right) \leq F\left(\mathrm{M}[\rho]\right) \qquad \forall \rho}$$

$$\Phi_0 = |0\rangle\langle 0|$$

vacuum input minimizes F at the output

e.g., entropy $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$

same "mechanism" as the proof for minimum entropy

Implies majorization conjecture

Consider special class of such functionals $F(\rho)$ that are written as

$$F(\rho) = \mathrm{Tr}\ f(\rho) = \sum_j f(\lambda_j)$$ where λ_j are the eigenvalues of $\hat{\rho}$

$\rightarrow \forall$ function $f(x)$ $\quad x \in [0,1]$
that is non-negative and stricly concave

$$\mathrm{Tr}\ f(\mathrm{M}[\Phi_0]) \leq \mathrm{Tr}\ f(\mathrm{M}[\rho]) \qquad \forall \rho \qquad \underline{\forall f(x)}$$

$$\Longleftrightarrow \quad \mathrm{M}[\Phi_0] \succ \mathrm{M}[\rho] \qquad \forall \rho$$

$\rightarrow \Phi_0 = |0\rangle\langle 0|$

vacuum input yields global majorizing state at the output

A. Mari, V. Giovannetti, and A. S. Holevo, Nature Comm. (2014).

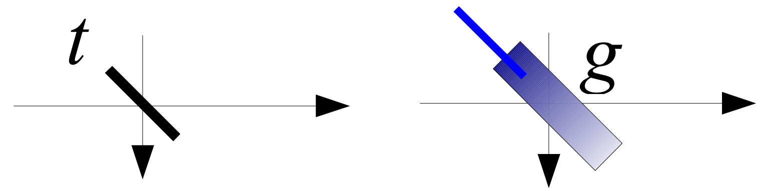
Summary

$\hat{\rho} \succ \hat{\sigma}$

- Majorization theory is ubiquitous in Gaussian quantum information

  (at the crossroads between quantum optics and quantum information)

- Majorization relations naturally arise when characterizing Gaussian optical components



- Interesting tool for solving Gaussian entropic conjectures on bosonic channels or for deriving conditions on entanglement transformations between Gaussian states

- But we probably still miss the “big picture” (?)

  … further developments are expected