Do quantum strategies always win?

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December 9, 2015

N. Anand and C. Benjamin, Quantum Information Processing, 14, 4027 (2015).

Outline

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   - Matching Pennies: Pure vs. Mixed strategies
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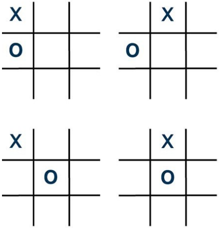
Introduction

What is a game?

A form of competitive sport or activity played according to rules.



TICK-TACK-TOE

CHESS

Applications of game theory

Game theory is a branch of Mathematics which deals with situations in which decision makers(players) interact.

Why should we study game theory?

- Biology(evolutionary game theory)
- Quantum Physics (Quantum game theory, Quantum algorithms)
- Statistical Physics (Minority games)

Basic definitions

- Players: The individuals who compete in the game i.e. whose interaction we wish to study.
- Actions : The set of all choices available to a player.
- Payoff : With each action we associate some value(a real number) such that higher values(i.e. payoff) are preferred.
- Optimal Strategy : Strategy that maximizes a player's expected payoff.

Types of games

Cooperative and non-cooperative

A game is cooperative if the players are able to form binding agreements i.e. the optimal strategy is to cooperate, players can coordinate their strategies and share the payoff.
Example of a cooperative game : Treasure Hunt- An expedition of n people have found a treasure in the mount; each pair of them can carry out one piece, but not more. How will they pair up?
Example of a non-cooperative game: Chess(Sports), Matching pennies, Penny flip

Types of games

Zero sum and Non-zero sum

If one player wins exactly the same amount the other player looses then the sum of their payoff's is zero. Since the payoff's are against each other these games are also known as non-cooperative games.
Example of a zero sum game : Matching pennies
Example of a non-zero sum game : Prisoner's dilemma

Types of games

Simultaneous and sequential

In simultaneous games players play simultaneously or say the players do not know of the other player's actions it makes the game effectively simultaneous.
Sequential games are where players play one after the another.
Example of a sequential game : Chess
Example of a simultaneous game : Matching pennies

Von-Neumann's Minimax theorem for zero sum games

Minimax via cake division

Cutter goes for nearly half the cake by electing to split the cake evenly. This amount, the maximum row minimum, is called "maximin". Cutter acts to maximize the minimum the chooser will leave him-"maximin".
Chooser looks for minimum column maximum-"minimax".

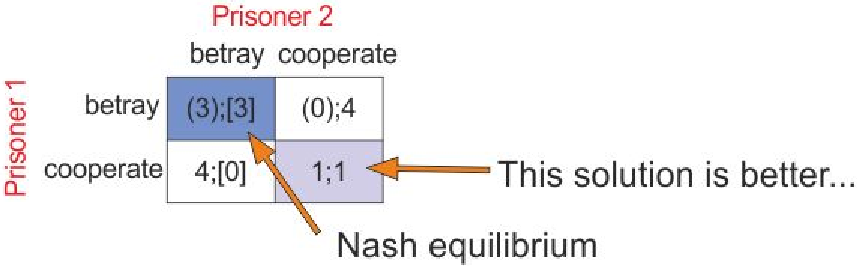
Chooser's strategies

		Choose bigger piece	Choose smaller piece
Cutter's strategies	Cut cake as evenly as possible	Half the cake minus a crumb	Half the cake plus a crumb
	Make one piece bigger than the	Small piece	Big piece

Nash Equilibrium for zero and non-zero sum games

Nash Equilibrium via Prisoner's dilemma

Nash equilibrium: A set of strategies is a Nash equilibrium if no player can do better by unilaterally changing their strategy



Matching pennies

The game is played between two players, Players A and B.
Each player has a penny and must secretly turn the penny to heads or tails.
The players then reveal their choices simultaneously.
If the pennies match both heads or both tails then player A keeps both pennies,(so wins 1 from B i.e. +1 for A and -1 for B).
If they don't match player B keeps both the pennies.

B

A		Heads	Tails
	Heads	1,-1	-1,1
	Tails	-1,1	1,-1

A zero sum, non-cooperative and simultaneous game without a fixed Nash equilibrium.

Matching pennies

	Heads	Tails
Heads	1 cent *MINIMAX*	−1 cent *MAXIMIN*
Tails	−1 cent *MAXIMIN*	1 cent *MINIMAX*

	Heads	Tails	Random
Heads	1 cent	−1 cent	0
Tails	−1 cent	1 cent	0
Random	0	0	0 *MINIMAX & MAXIMIN*

Pure vs. Mixed strategies

Pure: Playing heads or tails with certainty.
Mixed: Playing heads or tails randomly (with 50% probability for each)

Meyer's Penny flip game

The PQ penny flip was designed by David Meyer, its a close cousin of the Matching pennies game and has the following rules:
Players P and Q each have access to a single penny.
Initial state of the penny is heads(say). Each player can choose to either flip or not flip the penny.
Players cannot see the current state of the penny.
Sequence of actions :$Q \to P \to Q$
If final state is heads, Q wins else P wins

Payoff's for Meyer's Penny flip game

The payoff matrix for the game is as follows with the first entry as the payoff of P and the second is the payoff of Q.

	FF	FN	NF	NN
F	+1,-1	-1,+1	-1,+1	+1,-1
N	-1,+1	+1,-1	+1,-1	-1,+1

Since players cannot see the current state of the penny, their actions become independent of each other and so each strategy, flipping or not flipping is equally desirable.

Quantum Games - An Introduction

Quantization Rules:

Superposed initial states
Quantum entanglement of initial states.
Superposition of strategies to be used on the initial states

Quantum Penny Flip game

Quantum Penny flip game Rules:

The penny of the game is represented as a qubit(two-level system), with Heads maps to $|0>$ and tails mapped to $|1>$.
Player P does classical moves i.e. Flip (X) or not flip (I).
Player Q does quantum moves i.e. any general unitary U (say Hadamard).

Quantum vs. Classical moves

$$|0\rangle \quad \overset{\text{Q does } H}{\longmapsto} \quad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\overset{\text{P does } X \text{ or } I}{\longmapsto} \quad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\overset{\text{Q does } H}{\longmapsto} \quad |0\rangle$$

Why does Q win?

Q's quantum strategy puts the penny into the equal superposition of 'head' and 'tail'.
This state is invariant under X or I, Q always wins.

D. Mayer, Quantum Strategies, Phys. Rev. Lett. 82, 1052 (1999).

Can classical strategies win against quantum strategies?

Motivation of our work

In the quantum penny flip game we see how the quantum player can outperform the classical player.
However, is the converse at all possible?

Quantum entangled penny flip game: Introduction

Introduction

A maximally entangled state of two qubits is the "penny" of the game.
It is shared by P and Q; each allowed to make moves on only the qubit in their possession.

Moves

Sequence of actions : $Q \rightarrow P \rightarrow Q$

Rules of winning

If the final state of the game is a maximally entangled state then Q wins, If it is a separable state then P wins.
If it is a non-maximally entangled state then its a draw.

Playing the Quantum entangled penny flip game

The classical pure strategy

In this case the classical player P is allowed only the pure strategy of either flipping or not flipping his qubits.
The initial state of the system:$|\psi>=\frac{1}{\sqrt{2}}(|10>-|01>)$

Moves

Sequence of actions : $Q \rightarrow P \rightarrow Q$

Q does a Hadamard

$H \otimes I|\psi>=\frac{1}{2}(|00>-|01>-|10>-|11>)$

The classical pure strategy

P's move: To flip or not to flip

$I \otimes X \frac{1}{2}(|00> -|01> -|10> -|11>) = \frac{1}{2}(|01> -|00> -|11> -|10>)$

OR

$I \otimes I \frac{1}{2}(|00> -|01> -|10> -|11>) = \frac{1}{2}(|00> -|01> -|10> -|11>)$

Q's move: H again

$H \otimes I \frac{1}{2}(|01> -|00> -|11> -|10>) = \frac{1}{\sqrt{2}}(|11> -|00>)$

OR

$H \otimes I \frac{1}{2}(|00> -|01> -|10> -|11>) = \frac{1}{\sqrt{2}}(|00> +|11>)$

In either case Q wins.

What did we learn?

Moral

In quantum entangled penny flip game, with one player having classical pure strategy, while the other player does quantum moves gives a definite win to quantum player.

Why is this important?

The game here is about whether player Q having all quantum strategies at his hand can keep the state maximally entangled, whereas P with classical moves can or cannot reduce/destroy the entanglement.

Algorithms

Strategy is similar to an algorithm: finite # of steps to solve a problem/win a game.

The classical mixed strategy

Defining mixed strategy

P can now flip or not flip with some probability “p”.
A maximally entangled state of two qubits is the “penny” of the game.
It is shared by P and Q; each allowed to make moves on only the qubit in their possession.
Sequence of actions :$Q \to P \to Q$

Playing with classical mixed strategy

Initial state

The maximally entangled state "Penny":

$|\psi >= \frac{1}{\sqrt{2}}(|10> - |01>)$

In the form of density matrix:

$$\rho_0 = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Q's move

Q makes an unitary transformation on her part of the shared state.

$$U_{Q_1} = \begin{bmatrix} a & b^* \\ b & -a^* \end{bmatrix}.$$

The state after Q's move then is $\rho_1 = (U_{Q1} \otimes I)\rho_0(U_{Q1} \otimes I)^\dagger$.

Playing with classical mixed strategy

P's move

P now plays a mixed strategy, which entails flipping the state of his qubit with probability "p" or not flipping. The state after P's move then is: $\rho_2 = p(I \otimes X)\rho_1(I \otimes X)^\dagger + (1-p)(I \otimes I)\rho_1(I \otimes I)^\dagger$.

Q's final move

At the end Q makes her final move, which as before has to be an unitary transformation, it further could be same as her first move or different. Thus $U_{Q2} = \begin{bmatrix} \alpha & \beta^* \\ \beta & -\alpha^* \end{bmatrix}$. The state after this final move then is $\rho_3 = (U_{Q2} \otimes I)\rho_2(U_{Q2} \otimes I)^\dagger$.

Analysing the game

When Q's moves are Hadamard

To understand this case of P using mixed, lets analyse this case for Q using the familiar Hadamard transform in both steps 2 and 4. In this special case,

$$\rho_3 = \frac{1}{2}\begin{bmatrix} p & 0 & 0 & -p \\ 0 & 1-p & -1+p & 0 \\ 0 & -1+p & 1-p & 0 \\ -p & 0 & 0 & p \end{bmatrix}.$$

Is the final state entangled or separable?

To check the entanglement content of this final state we take recourse to an entanglement measure- Concurrence.
Concurrence for a two qubit density matrix ρ_3 is defined as follows- we first define a "spin-flipped" density matrix, γ as $(\sigma_y \otimes \sigma_y)\rho_3^*(\sigma_y \otimes \sigma_y)$. Then we calculate the square root of the eignevalues of the matrix $\rho_3\gamma$ (say λ_1,λ_2,λ_3,λ_4) in decreasing order. Then, Concurrence is :

$$\max\left(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\right)$$

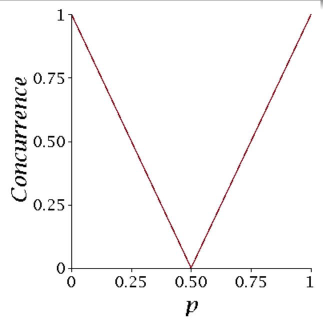


Figure : Concurrence vs p showing that entanglement vanishes at $p = 1/2$, so by P's classical moves entanglement is completely destroyed enabling him to win.

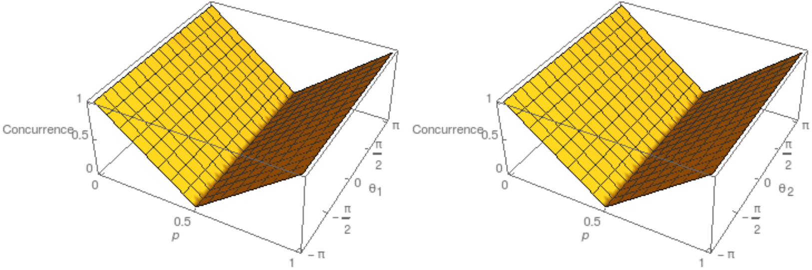
Classical random strategy wins against quantum strategy

Although the individual moves had no effect, a probabilistic move has enabled the classical player to win!!!.

Quantum entangled penny flip game: General quantum strategy versus mixed classical strategy

What if the quantum player uses a general unitary and not just a Hadamard?
Further in successive turns he does not implement the same unitary, i.e., $U_{Q_1} \neq U_{Q_2}$

$$U_{Q_i} = \begin{bmatrix} \cos(\theta_i)e^{i\phi_i} & \sin(\theta_i)e^{i\phi'_i} \\ \sin(\theta_i)e^{-i\phi'_i} & -\cos(\theta_i)e^{-i\phi_i} \end{bmatrix}, i = 1, 2.$$

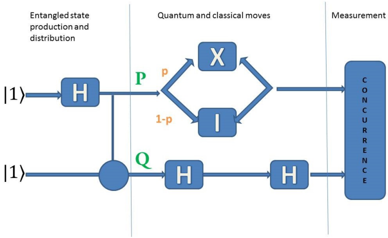


(a) Concurrence vs. θ_1, $\theta_2 = 0, \phi_1 = \pi/2, \phi_1' = 0, \phi_2 = \pi/2, \phi_2' = 0$

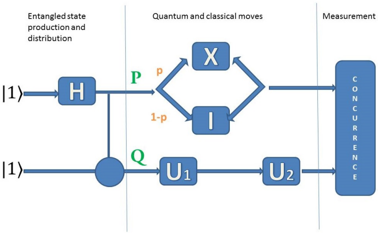
(b) Concurrence vs. θ_2, $\theta_1 = 0, \phi_1 = \pi/2, \phi_1' = 0, \phi_2 = \pi/2, \phi_2' = 0$

Figure : The Concurrence when quantum player plays a general unitary vs. classical players mixed strategy. The classical player always wins when $p = 1/2$, confirming that regardless of whether quantum player uses a Hadamard or any other unitary he always loses when classical player plays a mixed strategy of either flipping or not flipping with probability 50%.

Quantum circuit implementation



(a) Quantum player uses Hadamard.



(b) Quantum player uses a general unitary

Figure : The quantum circuit for the entangled penny flip game. M denotes measurement of entanglement content via concurrence.

Conclusion

Quantum entangled penny flip game

In a particular case where classical player uses a mixed strategy with p = “0.5”, the quantum player indeed loses as opposed to the expected win for all possible unitaries!

Meyer’s penny flip

Meyer showed that in the PQ penny flip if both players use Quantum strategies then there is no advantage. However, a player using a quantum strategy will win 100% of the time against a player using a classical strategy.

What we show

Quantum strategies are not(always) better than classical strategies.

Perspective on quantum algorithms

Quantum algorithms have been shown to be more efficient than classical algorithms, for example Shor's algorithm. We in this work put forth a counter example which demonstrates that a particular classical algorithm can outwit the previously unbeatable quantum algorithm in the entangled quantum penny flip problem. On top of that the mixed strategy works against any possible unitary as we show by simulation on a strategy space for all possible parameters.